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GENERATING THE SPIN MAPPING CLASS GROUP BY DEHN TWISTS

COMMENT ENGENDRER LE GROUPE DE
CLASSES D'HOMÉOMORPHISMES SPIN PAR
DES TWISTS DE DEHN

ABSTRACT. — Let ϕ be a $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a closed oriented surface Σ_g of genus $g \geq 4$. We determine a generating set of the stabilizer of ϕ in the mapping class group of Σ_g consisting of Dehn twists about an explicit collection of $2g + 1$ curves on Σ_g . If $g = 3$ then we determine a generating set of the stabilizer of an odd $\mathbb{Z}/4\mathbb{Z}$ -spin structure consisting of Dehn twists about a collection of 6 curves.

RÉSUMÉ. — Soit ϕ une $\mathbb{Z}/2\mathbb{Z}$ -structure de spin sur une surface fermée orientée Σ_g de genre $g \geq 4$. Nous déterminons une partie génératrice du stabilisateur de ϕ dans le groupe de classes d'homéomorphismes de Σ_g , composée de twists de Dehn autour d'un ensemble explicite de $2g + 1$ courbes de Σ_g . Lorsque $g = 3$, nous déterminons une partie génératrice du stabilisateur d'une $\mathbb{Z}/4\mathbb{Z}$ -structure de spin impaire, formée de twist de Dehn autour de 6 courbes.

Keywords: Spin mapping class group, Dehn twists, curve systems, group generators.

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1. Introduction

For some $r \geq 2$, a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on a closed surface Σ_g of genus g is a cohomology class $\phi \in H^1(UT^*_g, \mathbb{Z}/r\mathbb{Z})$ which evaluates to one on the oriented fibre of the unit tangent bundle $UT^*_g \rightarrow \Sigma_g$ of Σ_g . Such a spin structure exists for all r which divide $2g - 2$. If r is even, then it reduces to a $\mathbb{Z}/2\mathbb{Z}$ -spin structure on Σ_g .

A $\mathbb{Z}/2\mathbb{Z}$ -spin structure on Σ_g has a *parity*, either even or odd. Thus there is a notion of parity for all $\mathbb{Z}/r\mathbb{Z}$ -spin structures with r even. If ϕ, ψ are two $\mathbb{Z}/r\mathbb{Z}$ -spin structures on Σ_g so that either r is odd or r is even and the parities of ϕ, ψ coincide, then there exists an element of the mapping class group $\text{Mod}(\Sigma_g)$ of Σ_g which maps ϕ to ψ . Hence the stabilizers of ϕ and ψ in $\text{Mod}(\Sigma_g)$ are conjugate.

Spin structures naturally arise in the context of abelian differentials on Σ_g . The moduli space of such differentials decomposes into strata of differentials whose zeros are of the same order and multiplicity. Understanding the orbifold fundamental group of such strata requires some understanding of their projection to the mapping class group. If the orders of the zeros of the differentials are all multiples of the same number $r \geq 2$, then this quotient group preserves a $\mathbb{Z}/r\mathbb{Z}$ -spin structure ϕ on Σ_g . Hence the orbifold fundamental groups of components of strata relate to stabilizers $\text{Mod}(\Sigma_g)[\phi]$ of spin structures ϕ on Σ_g .

To make such a relation explicit we define

DEFINITION 1.1. — *A curve system on a closed surface Σ_g is a finite collection of smoothly embedded simple closed curves on Σ_g which are non-contractible and mutually not freely homotopic, and such that any two curves from this collection intersect transversely in at most one point.*

A curve system defines a *curve diagram* which is a finite graph whose vertices are the curves from the system and where two such vertices are connected by an edge if the curves intersect.

DEFINITION 1.2. — *A curve system on Σ_g is admissible if it decomposes Σ_g into a collection of topological disks and if its curve diagram is a tree.*

Using a construction of Thurston and Veech (see [Lei04] for a comprehensive account), admissible curve systems on Σ_g give rise to abelian differentials on Σ_g , and the component of the stratum and hence the equivalence class of a spin structure (if any) it defines can be read off explicitly from the combinatorics of the curve system. This makes it desirable to investigate the subgroup of the mapping class group generated by Dehn twists about the curves of an admissible curve system.

The main goal of this article is to present a systematic study of stabilizers of suitably chosen curves in the spin mapping class group $\text{Mod}(\Sigma_g)[\phi]$ and to use this information to build generators for this group by induction over subsurfaces. As a main application we obtain the following.

For $g \geq 3$ let C_g and V_g be the collections of $2g + 1$ nonseparating simple closed curves on a closed surface Σ_g of genus g shown in Figure 1.1.

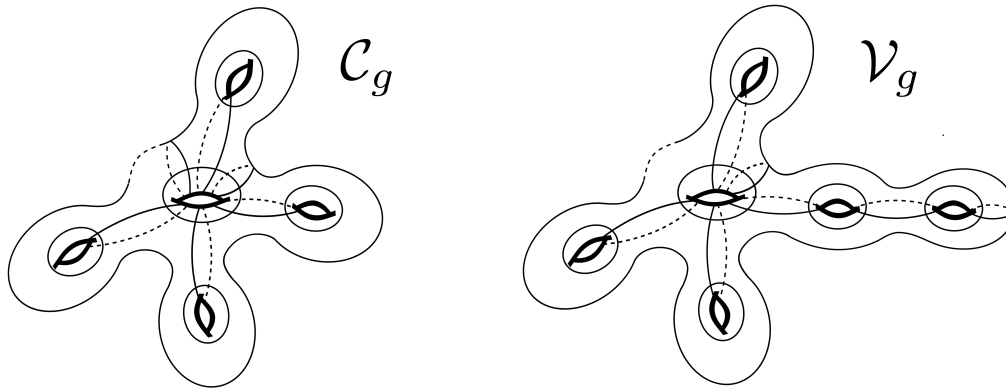


Figure 1.1.

We show

THEOREM 1.3. —

- (1) Let ϕ be an odd $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a closed surface Σ_g of genus $g \geq 3$. Then $\text{Mod}(\Sigma_g)[\phi]$ is generated by the Dehn twists about the curves from the curve system C_g .
- (2) Let ϕ be an even $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a closed surface Σ_g of genus $g \geq 4$. Then $\text{Mod}(\Sigma_g)[\phi]$ is generated by the Dehn twists about the curves from the curve system V_g .

That the spin mapping class group can be generated by finitely many finite products of Dehn twists is due to Hirose. In [Hir02] he found for any genus $g \geq 2$ a generating set for the stabilizer of an even $\mathbb{Z}/2\mathbb{Z}$ -spin structure by finitely many finite products of Dehn twists, and the stabilizer of an odd $\mathbb{Z}/2\mathbb{Z}$ -spin structure is treated in [Hir05].

For surfaces of genus $g \geq 5$, Calderon [Cal20] and Calderon and Salter [CS21] identified the image of the orbifold fundamental group of most components of strata in the mapping class group by constructing a different but equally explicit generating set for the spin mapping class group. Earlier Salter ([Sal19, Theorem 9.5]) obtained a partial result by identifying for $g \geq 5$ a finite generating set of a finite index subgroup of the spin mapping class group by Dehn twists. Walker [Wal09, Wal10] obtained some information on the image of the orbifold fundamental group of some strata of quadratic differentials in the mapping class group using completely different tools.

Theorem 1.3 does not construct generators for the stabilizer of an even $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a surface of genus $g = 2, 3$. Namely, in these cases there is no admissible curve system with the property that the Dehn twists about the curves from the system stabilize an even $\mathbb{Z}/2\mathbb{Z}$ -spin structure and such that the Dehn twists about these curves generate a finite index subgroup of the mapping class group. This corresponds to a classification result of Kontsevich and Zorich [KZ03]: There is no component of a stratum of abelian differentials with a single zero on a surface of genus 2 and even spin structure. On a surface Σ_3 of genus 3, the component of the stratum of abelian differentials with two zeros of order two and even spin structure is hyperelliptic and hence the projection of its orbifold fundamental group to $\text{Mod}(\Sigma_3)$ commutes with a hyperelliptic involution and is of infinite index.

Our results can be used to construct an explicit finite set of generators of the stabilizer of a $\mathbb{Z}/r\mathbb{Z}$ -spin structure for any $r \leq 2g - 2$ and any closed surface Σ_g , given by Dehn twists, positive powers of Dehn twists and products of Dehn twists about two simple closed curves forming a bounding pair. Potentially they can also be used inductively to find generators by Dehn twists about curves from an admissible curve system. We carry this program only out in a single case, which is the odd $\mathbb{Z}/4\mathbb{Z}$ -spin structure on a surface of genus 3.

Consider the system E_6 of simple closed curves on the surface Σ_3 of genus 3 shown in Figure 1.2 which is of particular relevance for the understanding of the stratum of abelian differentials with a single zero on Σ_3 [LM14].

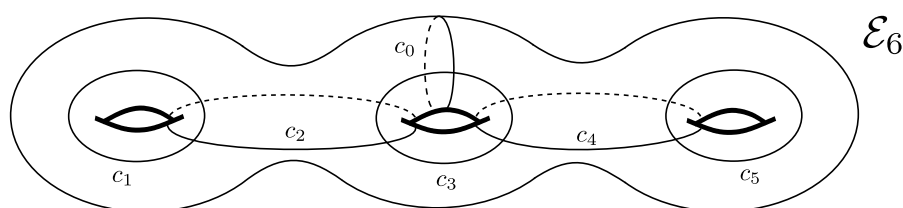


Figure 1.2.

We show

THEOREM 1.4. — *The subgroup of $\text{Mod}(\Sigma_3)$ generated by the Dehn twists about the curves from the curve system E_6 equals the stabilizer of an odd $\mathbb{Z}/4\mathbb{Z}$ -spin structure on Σ_3 .*

The strategy for the proofs of the main results is as follows.

For some $r \geq 2$ let us consider an arbitrary $\mathbb{Z}/r\mathbb{Z}$ -spin structure ϕ on a compact oriented surface S of genus $g \geq 2$, perhaps with boundary. Following [HJ89] and [Sal19], the spin structure can be viewed as a $\mathbb{Z}/r\mathbb{Z}$ -valued function on oriented closed curves on S which assumes the value one on the oriented boundary of an embedded disk in S . Changing the orientation of the curve changes the value of ϕ on the curve to its negative [HJ89, Sal19].

Define a graph CG_1^+ as follows. Vertices are nonseparating simple closed curves c on S with $\phi(c) = \pm 1$, and two such vertices d, e are connected by an edge if d, e can be realized disjointly and if furthermore, $S - (d \cup e)$ is connected. Thus CG_1^+ is a subgraph of the curve graph of S . The stabilizer $\text{Mod}(S)[\phi]$ of ϕ in the mapping class group of S acts on CG_1^+ as a group of simplicial automorphisms.

In Section 2 we show that for any $g \geq 3$ and $r \leq 2g - 2$ the graph CG_1^+ is connected. We also note that for an odd $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a surface of genus $g = 2$, this is not true. In Section 3 we verify that the action of the group $\text{Mod}(S)[\phi]$ on the graph CG_1^+ is transitive on vertices.

For a vertex c of CG_1^+ we are then led to describing the intersection of $\text{Mod}(S)[\phi]$ with the stabilizer of c in $\text{Mod}(S)$. Most important is the understanding of the intersection of $\text{Mod}(S)[\phi]$ with the so-called *disk pushing subgroup*, namely the kernel of the natural homomorphism of the stabilizer of c to the mapping class group

of the surface obtained from $S - c$ by capping off the two distinguished boundary components of $S - c$. This is also carried out in Section 3.

In Section 4 we specialize further to a $\mathbb{Z}/2\mathbb{Z}$ -spin structure ϕ . We find a presentation of $\text{Mod}(S)[\phi]$ as a quotient of a $\mathbb{Z}/2\mathbb{Z}$ -extension of the free product of two copies of the stabilizer of a vertex of \mathcal{CG}_1^+ , amalgamated over the stabilizer of an edge of \mathcal{CG}_1^+ . This is used to prove Theorem 1.3(1) with an argument by induction on the genus g of the closed surface S .

The proof of Theorem 1.3(2) uses similar methods and is contained in Section 5. A variation of these arguments yield the proof of Theorem 1.4 in Section 6.

The Appendix A contains a technical variation of the main result of Section 2 which is used in Section 5. Its proof follows along exactly the same line as the proof of the main result of Section 2.

This work is inspired by the article [Sal19] of Salter. However, aside from some simple constructions using curves and [Sal19, Proposition 4.9], our approach uses different methods.

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2. Graphs of curves with fixed spin value

In this section we consider a compact surface S of genus $g \geq 2$, with or without boundary. For a number $r \geq 2$ we introduce $\mathbb{Z}/r\mathbb{Z}$ -spin structures on S and use these structures to define various subgraphs of the curve graph of S . Of primary interest is a graph \mathcal{CG}_1 whose vertices are nonseparating simple closed curves with spin value ± 1 and where two such curves are connected by an edge if they can be realized disjointly. We then study connectedness of this graph.

Small genus of the surface may cause the graph \mathcal{CG}_1 to have few edges. This problem leads us to proceed in two steps. In Proposition 2.12 we show connectedness of \mathcal{CG}_1 for surfaces of genus $g \geq 3$ and $r = 2, 4$, taking advantage of some special properties of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ spin structures. Proposition 2.16 shows connectedness of \mathcal{CG}_1 for surfaces of genus $g \geq 4$ and all r , taking advantage of sufficiently large complexity of the underlying surface. These results are used in Section 3 to study the stabilizer of a spin structure in the mapping class group of S .

This section is divided into 5 subsections. We begin with summarizing some information on spin structures. Each of the remaining subsections is devoted to the investigation of a specific subgraph of the curve graph of S defined by a spin structure ϕ on S .

2.1. Spin structures

The following is taken from [HJ89], see [Sal19, Definition 3.1]. For its formulation, denote by ι the symplectic form on $H_1(S, \mathbb{Z})$.

DEFINITION 2.1 (Humphries–Johnson). — *For a number $r \geq 2$, a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on S is a $\mathbb{Z}/r\mathbb{Z}$ -valued function ϕ on isotopy classes of oriented simple closed curves on S with the following properties.*

- (1) (*Twist linearity*) Let c, d be oriented simple closed curves and let T_c be the left Dehn twist about c ; then

$$\phi(T_c(d)) = \phi(d) + \iota(d, c)\phi(c) \pmod{r}.$$

- (2) (*Normalization*) $\phi(\zeta) = 1$ for the oriented boundary ζ of an embedded disk $D \subset S$.

As an additional property, one obtains that whenever c^{-1} is obtained from c by reversing the orientation, then $\phi(c^{-1}) = -\phi(c)$ ([HJ89, Lemma 2.2]).

Humphries and Johnson [HJ89] (see [Sal19, Theorem 3.5]) also give an alternative description of spin structures. Namely, for some choice of a hyperbolic metric on S let UTS be the unit tangent bundle of S . It can be viewed as the quotient of the complement of the zero section in the tangent bundle of S by the multiplicative group $(0, \infty)$ and hence it does not depend on the metric.

The *Johnson lift* of a smoothly embedded oriented simple closed curve c on S is simply the closed curve in UTS which consists of all unit tangents of c defining the given orientation. The following is [HJ89, Theorem 2.1 and Theorem 2.5] as formulated in [Sal19, Theorem 3.5].

THEOREM 2.2 (Humphries–Johnson). — *Let S be a compact surface and let ζ be the oriented fibre of the unit tangent bundle $UTS \rightarrow S$. A cohomology class $\psi \in H^1(UTS, \mathbb{Z}/r\mathbb{Z})$ with $\psi(\zeta) = 1$ determines a $\mathbb{Z}/r\mathbb{Z}$ -spin structure via*

$$\alpha \mapsto \psi(\tilde{\alpha})$$

where α is an oriented simple closed curve on S and $\tilde{\alpha}$ is its Johnson lift. This determines a 1-1 correspondence between $\mathbb{Z}/r\mathbb{Z}$ -spin structures and

$$\{\psi \in H^1(UTS, \mathbb{Z}/r\mathbb{Z}) \mid \psi(\zeta) = 1\}.$$

There is another interpretation as follows; we refer to [Hai95, p. 131] for more information on this construction. Given a number $r \geq 2$ which divides $2g - 2$, an application of the Gysin sequence for the Euler class of UTS yields a short exact sequence

$$(2.1) \quad 0 \rightarrow \mathbb{Z}/r\mathbb{Z} \rightarrow H_1(UTS, \mathbb{Z}/r\mathbb{Z}) \rightarrow H_1(S, \mathbb{Z}/r\mathbb{Z}) \rightarrow 0.$$

By covering space theory, an r^{th} root of the tangent bundle of S , viewed as a complex line bundle for some fixed complex structure, is determined by a homomorphism $H_1(UTS, \mathbb{Z}/r\mathbb{Z}) \rightarrow \mathbb{Z}/r\mathbb{Z}$ whose composition with the inclusion $\mathbb{Z}/r\mathbb{Z} \rightarrow H_1(UTS, \mathbb{Z}/r\mathbb{Z})$ is the identity and therefore

PROPOSITION 2.3. — *There is a natural one-to-one correspondence between the r^{th} roots of the canonical bundle of S and splittings of the sequence (2.1).*

A $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a compact surface S of genus g with empty or connected boundary has a *parity* which is defined as follows.

A *geometric symplectic basis* for $H_1(S, \mathbb{Z})$ is a system $a_1, b_1, \dots, a_g, b_g$ of simple closed curves on S such that a_i, b_i intersect in a single point and that $a_i \cap b_i$ is disjoint from $a_j \cap b_j$ for $i \neq j$. Then the parity of the spin structure ϕ equals

$$(2.2) \quad \text{Arf}(\phi) = \sum_i (\phi(a_i) + 1) (\phi(b_i) + 1) \in \mathbb{Z}/2\mathbb{Z}.$$

This does not depend on the choice of the geometric symplectic basis.

2.2. The graph of nonseparating curves with vanishing spin value

The *curve graph* CG of S is the graph whose vertices are *essential* (that is, neither nullhomotopic nor homotopic into the boundary) simple closed curves in S and where two such curves are connected by an edge if they can be realized disjointly. We can use the spin structure ϕ to introduce various subgraphs of CG and study their properties. One of the main technical ingredients to this end is the following result of Salter [Sal19, Corollary 4.3]).

LEMMA 2.4 (Salter). — *Let S be an embedded one-holed torus. Then there exists a simple closed curve c with $\phi(c) = 0$.*

Denote by CG_0 the complete subgraph of the curve graph whose vertex set consists of nonseparating curves c with $\phi(c) = 0$. Note that this is well defined, that is, it is independent of the choice of an orientation of c . As a fairly easy consequence of Lemma 2.4 we obtain

LEMMA 2.5. — *Let ϕ be a spin structure on a closed surface of genus $g \geq 3$. Then CG_0 is connected.*

Proof. — We use the following result of Masur–Schleimer [MS06], see [Put08, Theorem 1.2]. Let SG be the complete subgraph whose vertex set consists of *separating* simple closed curves; then SG is connected. Note that this requires that $g \geq 3$.

Let a, b be vertices of CG_0 . Choose simple closed curves \hat{a}, \hat{b} which intersect a, b in a single point; such curves exist since a, b are nonseparating. Then the boundary c, d of a tubular neighborhood of $a \cap \hat{a}$ and $b \cap \hat{b}$, respectively, is a separating simple closed curve which decomposes S into a one-holed torus containing a, b and a surface of genus $g - 1 \geq 2$ with boundary.

Connect c to d by an edge path $(c_i)_{0 \leq i \leq k}$ in SG (here $c = c_0$ and $d = c_k$). Construct inductively an edge path (a_i) in CG_0 connecting $a = a_0$ to $b = a_k$ such that for each i , a_i is disjoint from c_i , as follows. Put $a_0 = a$ and assume that we constructed already such a path for some $j < k$. Then a_j is disjoint from c_j .

If a_j also is disjoint from c_{j+1} then define $a_{j+1} = a_j$. Otherwise a_j is contained in the same component of $S - c_j$ as c_{j+1} . Choose a one-holed torus T in $S - c_j$. Such a torus exists since c_j decomposes S into two surfaces of positive genus with connected boundary. By Lemma 2.4, this torus contains a nonseparating simple closed curve

a_{j+1} with $\phi(a_{j+1}) = 0$, and this curve is disjoint from both a_j and c_{j+1} . This yields the induction step. \square

Remark 2.6. — The proof of Lemma 2.5 extends with a bit more care to compact surfaces of genus at least 3 with connected boundary. We expect that the Lemma also holds true for $g = 2$.

2.3. The graph of nonseparating curves with spin value ± 1 on a surface of genus 2

Define \mathcal{CG}_1 to be the complete subgraph of \mathcal{CG} of all nonseparating simple closed curves c on S with $\phi(c) = \pm 1$. Note that this condition does not depend on the orientation of c and hence it is indeed a condition on the vertices of \mathcal{CG} . In this subsection we discuss the special case $g = 2$.

PROPOSITION 2.7. — *Let ϕ be an odd $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a closed surface S of genus 2. Then any two simple closed nonseparating curves c, d on S with $\phi(c) = \phi(d) = 1$ intersect.*

Proof. — Let ϕ be a $\mathbb{Z}/2\mathbb{Z}$ -spin structure on S . Let c be a nonseparating simple closed curve on S with $\phi(c) = 1$. Assume that there is a nonseparating simple closed curve d with $\phi(d) = 1$ which is disjoint from c . As a surface of genus two does not admit bounding pairs, the surface $S - (c \cup d)$ is a four-holed sphere. Thus there exists a simple closed separating curve e which decomposes S into two one-holed tori T_1, T_2 such that $c \subset T_1, d \subset T_2$.

Denoting by ι the mod two homological intersection form on $H_1(S, \mathbb{Z}/2\mathbb{Z})$, there are two nonseparating simple closed curves $v \subset T_1, w \subset T_2$ so that

$$(2.3) \quad \iota(v, c) = 1 = \iota(w, d) \text{ and } \iota(w, c) = \iota(v, d) = 0.$$

The curves $a_1 = c, b_1 = v, a_2 = d, b_2 = w$ define a geometric symplectic basis for $H_1(S, \mathbb{Z})$. Since $\phi(a_1) = \phi(a_2) = 1$, the formula (2.2) for the Arf invariant shows that ϕ is even as claimed. \square

2.4. $\mathbb{Z}/r\mathbb{Z}$ -spin structures for $r = 2, 4$ on a surface of genus $g \geq 3$

In this subsection we study the graph \mathcal{CG}_1 for a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on a surface of genus $g \geq 3$ for $r = 2, 4$. We begin with evoking a result of Salter [Sal19].

Namely, let c, d be disjoint simple closed curves on the compact surface S . Let ϵ be an embedded arc in S connecting c to d whose interior is disjoint from $c \cup d$. A regular neighborhood ν of $c \cup \epsilon \cup d$ is homeomorphic to a three-holed sphere. Two of the boundary components of ν are the curves c, d up to homotopy. We choose an orientation of c, d in such a way that ν lies to the left. The third boundary component $c +_\epsilon d$, oriented in such a way that ν is to its right, satisfies $[c +_\epsilon d] = [c] + [d]$ where $[c]$ denotes the homology class of the oriented curve c . The following is [Sal19, Lemma 3.13].

LEMMA 2.8 (Salter). — $\phi(c +_\epsilon d) = \phi(c) + \phi(d) + 1$.

As a consequence, if $r = 2, 4$ then the boundary of any embedded pair of pants $P \subset S$ contains a simple closed curve c with $\phi(c) = \pm 1$. To use this fact for our purpose we introduce another graph related to simple closed curves on surfaces.

DEFINITION 2.9. — *Let S be a compact surface of genus $g \geq 2$. The graph of nonseparating pairs of pants NS is the graph whose vertices are pairs of pants in S whose boundary consists of three pairwise distinct nonseparating simple closed curves and where two such pair of pants are connected by an edge if their intersection consists of precisely one boundary component.*

By the preceding remark, the graph NS can be used to find paths in the graph CG_1 provided we can show that it is connected. To this end we evoke an observation of Putman ([Put08, Lemma 2.1]) which we refer to as the *Putman trick* in the sequel.

LEMMA 2.10 (Putman). — *Let G be a graph which admits a vertex transitive isometric action of a finitely generated group Γ and let v be a vertex of G . If for each element s of a finite generating set S of Γ , the vertex v can be connected to sv by an edge path in G , then G is connected.*

We apply the Putman trick to show

LEMMA 2.11. — *The graph of nonseparating pairs of pants is connected.*

Proof. — If $P \subset S$ is a nonseparating pair of pants, then $S - P$ is a connected surface of genus $g - 2$ with three distinguished boundary components. Thus the pure mapping class group $P\text{Mod}(S)$ of S acts transitively on the vertices of NS . As a consequence, it suffices to show that there exists a generating set S of $P\text{Mod}(S)$ and a nonseparating pair of pants $P \in NS$ which can be connected to its image $\psi(P)$ by an edge path in NS for every element $\psi \in S$.

Now $P\text{Mod}(S)$ can be generated by Dehn twists T_{c_i} about the collection of simple closed curves c_0, \dots, c_k shown in Figure 2.1 (see [FM12, Section 4.4]).

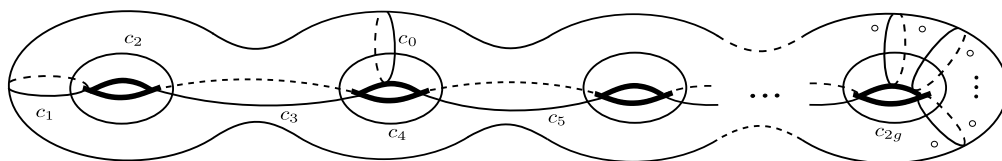


Figure 2.1.

Furthermore, the simple closed curves c_0, c_1, c_3 are nonseparating and bound a pair of pants P . This pair of pants is stabilized by all elements of S with the exception of the Dehn twists about the simple closed curves c_2 and c_4 .

As the genus of S is at least 3, for $i = 2, 4$ the complement \widehat{S} in S of the union of P with c_i is a surface of genus $g - 2 \geq 1$ with two distinguished boundary components, one of which, say the curve c , is a boundary component of P .

The surface \widehat{S} contains a nonseparating simple closed curve d . As in Lemma 2.8, choose an embedded arc $\epsilon \subset \widehat{S}$ connecting c to d . A small neighborhood of $c \cup \epsilon \cup d$

is a nonseparating pair of pants \widehat{P} whose intersection with P equals the curve c . As \widehat{P} is disjoint from c_i , it is left fixed by the Dehn twist ψ_i about c_i and hence $P, \widehat{P}, \psi_i(P)$ is a path of length three in NS connecting P to $\psi_i(P)$. Lemma 2.11 now is an immediate consequence of Lemma 2.10. \square

The following is now an easy consequence of Lemma 2.11.

PROPOSITION 2.12. — *Let $r = 2, 4$ and let ϕ be a $\mathbb{Z}/r\mathbb{Z}$ spin structure on a compact surface S of genus $g \geq 3$, with or without boundary. Then the graph CG_1 is connected.*

Proof. — Let c, d be nonseparating simple closed curves with $\phi(c) = \pm 1, \phi(d) = \pm 1$. Choose nonseparating pairs of pants P, Q containing c, d in their boundary. By Lemma 2.11, we can connect P to Q by a path in the graph NS , say the path (P_i) with $P_0 = P$ and $P_k = Q$.

For each i let c_i be a boundary component of P_i with $\phi(c_i) = \pm 1$ and such that $c_0 = c, c_k = d$. Then for each i , either $c_i = c_{i+1}$ or c_i and c_{i+1} are disjoint. Thus (c_i) is a path in CG_1 connecting c to d . This shows the proposition. \square

2.5. $\mathbb{Z}/r\mathbb{Z}$ -spin structures on a surface of genus $g \geq 4$

In this subsection we investigate the graph CG_1 on a surface of genus $g \geq 4$ for an arbitrary $r \geq 2$. To show connectedness we use the following auxiliary graph PS . The vertices of PS are pairs of disjoint separating curves (c, d) which each decompose S into a surface of genus $g - 1$ and a one-holed torus. Thus $S - (c, d)$ is the disjoint union of two one-holed tori and a surface of genus $g - 2$. Two such pairs (c_1, d_1) and (c_2, d_2) are connected by an edge if up to renaming, $c_1 = c_2$ and d_2 is disjoint from c_1, d_1 . Then $S - (c_1, d_1, d_2)$ is the disjoint union of a surface of genus $g - 3$ with at least three holes and three one-holed tori. In particular, the graph PS is only defined if the genus of S is at least three.

We use the Putman trick to show

LEMMA 2.13. — *For a compact surface S of genus $g \geq 4$, perhaps with boundary, the graph PS is a connected $\text{Mod}(S)$ -graph.*

Proof. — The mapping class group $\text{Mod}(S)$ of the surface S clearly acts on PS , furthermore this action is vertex transitive. Namely, for any two vertices (a_1, b_1) and (a_2, b_2) of PS , the complement $S - (a_i, b_i)$ is the union of two one-holed tori and a surface of genus $g - 2$ with $k + 2$ boundary components where $k \geq 0$ is the number of boundary components of S . Hence there exists $\phi \in \text{Mod}(S)$ with $\phi(a_1, b_1) = (a_2, b_2)$.

Consider again the curve system H shown in Figure 2.1 with the property that the Dehn twists about these curves generate the mapping class group. Choose a pair of disjoint separating simple closed curves (a, b) which decompose S into a surface of genus $g - 1$ and a one-holed torus $X(a), X(b)$ and such that a curve $c \in H$ intersects at most one of the curves a, b . If it intersects one of the curves a, b , then this intersection consists of precisely two points. For example, we can choose a to be the boundary of a small neighborhood of $c_1 - c_2$, and b to be the boundary of a small neighborhood of $c_5 - c_6$.

Now let $c \subset H$ and let T_c be the left Dehn twist about c . If c is disjoint from $a \cap b$, then $T_c(a, b) = (a, b)$ and there is nothing to show. Thus assume that c intersects a .

The image $T_c(a)$ of a is a separating simple closed curve contained in a small neighborhood Y of $X(a) \cap c$. By assumption on c , this surface is a two-holed torus disjoint from b . As $g \geq 4$, the genus of $S - (Y \cup X(b))$ is at least one and hence there is a separating curve $e \subset S - (Y \cup X(b))$ which decomposes $S - (Y \cup X(b))$ into a one-holed torus and a surface S' . But this means that (a, b) can be connected to $T_c(a, b) = (T_c a, b)$ by the edge path $(a, b) \cup (e, b) \cup (T_c a, b)$. As the roles of a and b can be exchanged, the lemma now follows from the Putman trick. \square

We shall use another auxiliary graph which is defined as follows.

DEFINITION 2.14. — *Let S be a compact surface of genus $g \geq 1$ with two distinguished boundary components A_1, A_2 . The nonseparating arc graph is the graph whose vertices are isotopy classes of embedded arcs in S connecting A_1 to A_2 . The endpoints of an arc may move freely along the boundary circles A_1, A_2 in such an isotopy class. Two such arcs ϵ_1, ϵ_2 are connected by an edge if ϵ_1, ϵ_2 are disjoint and $S - (\epsilon_1 \cup \epsilon_2)$ is connected.*

We apply the Putman trick to show

LEMMA 2.15. — *The nonseparating arc graph $A(A_1, A_2)$ on a compact surface S of genus $g \geq 1$ with two distinguished boundary components A_1, A_2 is connected.*

Proof. — Clearly the pure mapping class group $P\text{Mod}(S)$ of S acts transitively on the vertices of $A(A_1, A_2)$, so it suffices to show that there exists a generating set S of $P\text{Mod}(S)$ and an arc $\epsilon \in A(A_1, A_2)$ which can be connected to its image $\psi(\epsilon)$ by an edge path in $A(A_1, A_2)$ for every element $\psi \in S$.

There exists two disjoint arcs ϵ_1, ϵ_2 connecting A_1 to A_2 such that $\epsilon_1 \cup \epsilon_2$ projects to an essential nonseparating simple closed curve in the surface obtained from S by capping off the boundary components A_1, A_2 . Furthermore, we may assume that ϵ_1 intersects one of the curves shown in Figure 2.1, say the curve c_1 , in a single point and is disjoint from the remaining curves, and c_1 is disjoint from ϵ_2 .

Then $T_{c_i} \epsilon_1 = \epsilon_1$ for $i \geq 2$, and ϵ_1 can be connected to $T_{c_1}(\epsilon_1)$ by the edge path $\epsilon_1, \epsilon_2, T_{c_1} \epsilon_1$. By the Putman trick this implies that $A(A_1, A_2)$ is connected. \square

We are now ready to show

PROPOSITION 2.16. — *Let ϕ be an r -spin structure ($r \geq 2$) on a compact surface S of genus $g \geq 4$. Then the graph \mathcal{CG}_1 is connected.*

Proof. — Let S be a compact surface of genus $g \geq 2$ and consider the graph \mathcal{PS} . To each of its vertices, viewed as a disjoint pair (c, d) of separating simple closed curves, we associate in a non-deterministic way a vertex (c, d) of \mathcal{CG}_1 as follows.

Denote by $\mathbb{T}_{c, d}$ the one-holed torus bounded by c, d . If one of the tori $\mathbb{T}_{c, d}$ contains a simple closed curve a with $\phi(a) = \pm 1$ then define $(c, d) = a$.

Now assume that none of the tori $\mathbb{T}_{c, d}$ contains a simple closed curve a with $\phi(a) = \pm 1$. By Lemma 2.4, there are simple closed nonseparating curves $a \subset \mathbb{T}_{c, b}$ $\mathbb{T}_{d, b}$ so that $\phi(a) = 0 = \phi(b)$. Since the tori $\mathbb{T}_{c, d}$ are disjoint, the pair (a, b) is nonseparating, that is, $S - (a \cup b)$ is connected. Choose an embedded arc ϵ in S

connecting a to b . By Lemma 2.8, the curve $(c, d) = a +_\epsilon b$ satisfies $\phi(a +_\epsilon b) = \pm 1$, furthermore it is nonseparating.

Let a be any vertex of \mathcal{CG}_1 and let b be any simple closed curve which intersects a in a single point. Such a curve exists since a is nonseparating. Then a tubular neighborhood of $a \cap b$ is a torus containing a . Let c be the boundary curve of this torus and choose a second separating simple closed curve d so that $(c, d) \in PS$.

Let $e \in \mathcal{CG}_1$ be another vertex. Construct as above a vertex $(p, q) \in PS$ so that e is contained in the one-holed torus cut out by p . Connect (c, d) to (p, q) by an edge path $(c_i, d_i)_{0 \leq i \leq k}$ in PS . We use this edge path to construct an edge path $(a_j) \in \mathcal{CG}_1$ connecting a to e which passes through suitable choices a_{j_i} ($i \leq k$) of the curves (c_i, d_i) .

Define $a_0 = a$ and by induction, let us assume that we constructed already the path $(a_j)_{0 \leq j \leq j_i}$ for some $i \geq 0$. We distinguish two cases.

Case 1. — One of the tori (c_i, d_i) contains a curve f with $\phi(f) = \pm 1$.

By construction, in this case we may assume by renaming that $f = a_{j_i} \in (c_i, d_i)$.

If $(c_i, d_i) \in \{c_{i+1}, d_{i+1}\}$ then define $a_{j_{i+1}} = a_{j_i} = f \in (c_{i+1}, d_{i+1})$ and note that this is consistent with the requirements for the induction step.

Thus we may assume now that $(c_i, d_i) \notin \{c_{i+1}, d_{i+1}\}$. If one of the tori (c_{i+1}, d_{i+1}) , say the torus (c_{i+1}, d_{i+1}) , contains a curve h with $\phi(h) = \pm 1$, then as (c_i, d_i) is disjoint from (c_{i+1}, d_{i+1}) , the curve h is disjoint from a_{j_i} and we can define $a_{j_{i+1}} = h \in (c_{i+1}, d_{i+1})$.

Thus assume that neither (c_{i+1}, d_{i+1}) nor (c_{i+1}, d_{i+1}) contains such a curve. Since (c_i, d_i) and (c_{i+1}, d_{i+1}) are pairwise disjoint, we can find an embedded arc ϵ in $S - (c_i, d_i)$ connecting a simple closed curve $u \in (c_{i+1}, d_{i+1})$ with $\phi(u) = 0$ to a curve $h \in (c_{i+1}, d_{i+1})$ with $\phi(h) = 0$. We then can define $a_{j_{i+1}} = u +_\epsilon h \in (c_{i+1}, d_{i+1}) = a_{j_{i+1}}$.

Case 2. — None of the tori (c_i, d_i) contains a curve f with $\phi(f) = \pm 1$.

In this case there are simple closed curves $f \in (c_i, d_i)$, $h \in (c_{i+1}, d_{i+1})$ with $\phi(f) = \phi(h) = 0$, and there is an embedded arc ϵ connecting f to h so that

$$a_{j_i} \in (c_i, d_i) = f +_\epsilon h.$$

Assume without loss of generality that $d_i = d_{i+1}$.

Let us in addition assume for the moment that the arc ϵ is disjoint from (c_{i+1}, d_{i+1}) . If furthermore there exists a simple closed curve $u \in (c_{i+1}, d_{i+1})$ with $\phi(u) = \pm 1$, then this curve is a choice for (c_{i+1}, d_{i+1}) which is disjoint from a_{j_i} and we are done.

Otherwise cut S open along the simple closed curve $h \in (c_{i+1}, d_{i+1})$ and let H_1, H_2 be the two boundary components of $S - h$. By renaming, assume without loss of generality that ϵ connects the boundary component H_1 to the curve f , i.e. it leaves the curve h from the side corresponding to H_1 . Now note that $M = S - h - \epsilon - (c_i, d_i)$ is a connected surface of genus $g - 2 \geq 2$ with two distinguished boundary circles, one of which is the curve H_2 , and $M \in (c_{i+1}, d_{i+1})$. Therefore there exists an embedded arc $\epsilon \in M$ connecting H_2 to a simple closed curve $u \in (c_{i+1}, d_{i+1})$ with $\phi(u) = 0$. Define $a_{j_{i+1}} = h +_\epsilon u$ and note that this definition is consistent with all requirements. This construction completes the induction step under the additional assumption that the arc ϵ is disjoint from (c_{i+1}, d_{i+1}) .

We are left with the case that ϵ is *not* disjoint from (c_{i+1}, d_{i+1}) . Cut S open along $f \cap h$ and note that the resulting surface Z has genus $g - 2 \geq 2$ and four distinguished

boundary components, say the components F_1, F_2, H_1, H_2 . Assume that ϵ connects F_1 to H_1 .

Consider the nonseparating arc graph $A(F_1, H_1)$ in Z of arcs connecting F_1 to H_1 . By Lemma 2.15, this graph is connected. Let ϵ_i be a path in $A(F_1, H_1)$ which connects ϵ to an arc ϵ_i disjoint from c_{i+1} . For any two consecutive of such arcs, say the arcs $\epsilon_j, \epsilon_{j+1}$, the surface $Z - (\epsilon_j \cup \epsilon_{j+1})$ is connected and hence we can find a disjoint arc δ_j connecting F_2 to H_2 . The curves $f +_{\epsilon_j} h, f +_{\delta_j} h, f +_{\epsilon_{j+1}} h$ are disjoint and yield a path in CG_1 connecting $f +_{\epsilon} h$ to a curve $f +_{\epsilon} h$ which is disjoint from c_{i+1} . We then can apply the construction for the case that the arc connecting f to h is disjoint from c_{i+1} . This completes the proof of the Proposition 2.16. \square

For technical reasons we need a stronger version of Proposition 2.12 and Proposition 2.16. Consider a $\mathbb{Z}/r\mathbb{Z}$ -spin structure ϕ on a compact surface S of genus g (with or without boundary) for an arbitrary number $r \geq 2$. We introduce another graph CG_1^+ as follows. The vertices of CG_1^+ coincide with the vertices of CG_1 . Any two such vertices c, d are connected by an edge if c, d are disjoint and if furthermore $S - (c \cup d)$ is connected. Thus CG_1^+ is obtained from CG_1 by removing some of the edges. In particular, if CG_1^+ is connected then the same holds true for CG_1 . We use connectedness of CG_1 to establish connectedness of CG_1^+ .

LEMMA 2.17. — *If the genus g of S is at least 3 then the graph CG_1^+ is connected provided that CG_1 is connected.*

Proof. — Let $c, d \in CG_1$ be two vertices which are connected by an edge in CG_1 and which are not connected by an edge in CG_1^+ . This means that c, d are disjoint, and $S - (c \cup d)$ is disconnected. We have to show that c, d can be connected in CG_1^+ by an edge path.

To this end recall that c, d are nonseparating and therefore the disconnected surface $S - (c \cup d)$ has two connected components S_1, S_2 . The surface S_1 has genus $g_1 \geq 1$ and at least two boundary components, and the surface S_2 has genus $g_2 = g - g_1 - 1 \geq 0$ and at least two boundary components.

Choose a simple closed curve $d_i \subset S_i$ ($i = 1, 2$) which bounds with $c \cup d$ a pair of pants P_i . Write $\partial_i = S_i - P_i$; the genus of ∂_i equals g_i . Glue P_1 to P_2 along $c \cup d$ so that the resulting surface ∂_0 is a two-holed torus containing $c \cup d$ in its interior. Choose a nonseparating simple closed curve $e \subset \partial_0$ which intersects both c, d in a single point. Since $\phi(c) = \pm 1$ we have $\phi(T_c e) = \phi(e) \pm 1$ where T_c is the left Dehn twist about c . Thus via replacing e by $T_c^k e$ for a suitable choice of $k \in \mathbb{Z}$ we may assume that $\phi(e) = 1$. In other words, we may assume that e is a vertex of CG_1 .

Assume for the moment that $g_2 \geq 1$. By Lemma 2.4, there exist simple closed curves $a \subset \partial_1, b \subset \partial_2$ with $\phi(a) = \phi(b) = 0$. Connect a to b by an embedded arc ϵ which is disjoint from $c \cup e$ (and crosses through the curve d). The curve $a +_{\epsilon} b$ satisfies $\phi(a +_{\epsilon} b) = 1$, and it is disjoint from both c and e . Moreover, the surfaces $S - (c \cup a +_{\epsilon} b)$ and $S - (e \cup a +_{\epsilon} b)$ are connected. As a consequence, c can be connected to e by an edge path in CG_1^+ of length two which passes through $a +_{\epsilon} b$.

By symmetry of this construction, e can also be connected to d by an edge path in CG_1^+ and hence c can be connected to d by such a path. This completes the proof in the case that the genus g_2 of S_2 is positive.

If the genus of S_2 vanishes then the genus of S_1 equals $g_1 = g - 1 \geq 2$. Any nonseparating simple closed curve in S_1 forms with both c, d a nonseparating pair. To find such a curve e with $\phi(e) = 1$, note that S_1 contains two disjoint one-holed tori T_1, T_2 , and by Lemma 2.4, there are embedded simple closed curves $a_i \in T_i$ which satisfy $\phi(a_i) = 0$. Then for any arc ϵ in S_1 connecting a_1 to a_2 , the curve $e = a_1 +_\epsilon a_2$ is nonseparating, and it is connected with both c, d by an edge in \mathcal{CG}_1^+ . This is what we wanted to show. \square

Proposition 2.12, Proposition 2.16 and Lemma 2.17 together show

COROLLARY 2.18. — *Let ϕ be a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on a closed surface of genus $g \geq 3$. Then the graph \mathcal{CG}_1^+ is connected.*

3. The action of $\text{Mod}(S)[\phi]$ on geometrically defined graphs

In this section we consider an arbitrary $\mathbb{Z}/r\mathbb{Z}$ -spin structure ϕ on a compact surface S of genus $g \geq 3$, possibly with boundary, for some number $r \geq 2$. Our goal is to gain some information on the stabilizer $\text{Mod}(S)[\phi]$ of ϕ through its action on the graph \mathcal{CG}_1^+ introduced in Section 2.

We begin with some information on the stabilizer of a spin structure ϕ on a compact surface S with boundary. Fix a boundary component C of S . Denote by $P_C \text{Mod}(S)$ the subgroup of the mapping class group $\text{Mod}(S)$ of S which fixes the boundary component C . Note that as we allow that a mapping class in $P_C \text{Mod}(S)$ exchanges boundary components of S different from C , the group $P_C \text{Mod}(S)$ coincides with the pure mapping class group of S only if the boundary of S consists of one or two components.

Write $P_C \text{Mod}(S)[\phi]$ to denote the stabilizer of ϕ in $P_C \text{Mod}(S)$. This is a subgroup of $P_C \text{Mod}(S)$ of finite index. Let \tilde{S} be the surface obtained from S by attaching a disk to C . There is an embedding $S \hookrightarrow \tilde{S}$ which induces a surjective homomorphism

$$: P_C \text{Mod}(S) \twoheadrightarrow \text{Mod}(\tilde{S}).$$

By a result of Johnson, extending earlier work of Birman (see [FM12, Section 4.2.5]), there is an exact sequence

$$(3.1) \quad 1 \rightarrow \mathbb{Z} \rightarrow \ker(\iota) \rightarrow \pi_1(\tilde{S}) \rightarrow 1$$

where \mathbb{Z} is the infinite cyclic central subgroup of $P_C \text{Mod}(S)$ generated by the Dehn twists about C and where $\pi_1(\tilde{S})$ is a so-called point pushing group.

For the formulation of the following Lemma 3.1, recall that the integral homology $H_1(\tilde{S}, \mathbb{Z})$ of a compact surface \tilde{S} of genus $g \geq 2$, possibly with boundary, is a free abelian group \mathbb{Z}^h for some $h \geq 4$. In fact, $h = 2g$ if the boundary of \tilde{S} is empty or connected, and in this case this group is generated by the homology classes of nonseparating simple closed curves on \tilde{S} . If the boundary of \tilde{S} is disconnected, then it is still true that $H_1(\tilde{S}, \mathbb{Z})$ is generated by simple closed possibly peripheral curves.

For $m \geq 1$ let $\pi_m = \pi_1(S)$ be the subgroup defined by the exact sequence

$$0 \rightarrow \pi_m \rightarrow \pi_1(S) \rightarrow H_1(S, \mathbb{Z}/m\mathbb{Z}) \rightarrow 0.$$

If $\zeta : \pi_1(S) \rightarrow H_1(S, \mathbb{Z})$ denotes the natural surjective projection, then $\pi_1^{-1}(m)$ is the preimage under ζ of the lattice in $H_1(S, \mathbb{Z})$ generated by m times the simple loop generators, and it is a subgroup of $\pi_1(S)$ of finite index. Using the notations from the previous paragraph we have

LEMMA 3.1. — *Assume that the boundary circle C is equipped with the orientation induced from the orientation of S .*

- (1) *If $\phi(C) = -1$ then $(\ker \pi_C \text{Mod}(S)[\phi]) = \pi_1^{-1}(m)$.*
 (2) *If $\phi(C) = 1$, then $(\ker \pi_C \text{Mod}(S)[\phi]) = \pi_1^{-1}(m)$ where $m = r/2$ if r is even, and $m = r$ otherwise.*

Proof. — Choose a basepoint p for $\pi_1^{-1}(m)$ in the interior of the attached disk. Let α be a simple nonseparating loop through the basepoint p . Up to homotopy, the oriented boundary of a tubular neighborhood of α consists of two simple closed curves c_1, c_2 which enclose the circle C . In other words, together with C the curves c_1, c_2 bound a pair of pants P in S . We equip the curves c_i with the orientation as boundary curves of P .

By [Sal19, Proposition 3.8], we have

$$(3.2) \quad \phi(C) + \phi(c_1) + \phi(c_2) = -1$$

and hence if $\phi(C) = -1$ then $\phi(c_1) + \phi(c_2) = 0$.

Let as before T_d be the left Dehn twist about a simple closed curve d . Let $\beta \subset S$ be an oriented simple closed curve which crosses through the pair of pants P . As c_1, c_2 are disjoint, we have $\iota(T_{c_2}^{-1}(\beta), c_1) = \iota(\beta, c_1)$ and therefore Definition 2.1 shows that

$$(3.3) \quad \begin{aligned} \phi(T_{c_1}T_{c_2}^{-1}(\beta)) &= \phi(T_{c_2}^{-1}(\beta)) + \iota(\beta, c_1)\phi(c_1) \\ &= \phi(\beta) + \iota(\beta, c_1)\phi(c_1) - \iota(\beta, c_2)\phi(c_2). \end{aligned}$$

On the other hand, as $c_1 + c_2$ is homologous to the boundary curve C , the homological intersection number fulfills $\iota(\beta, c_1 + c_2) = 0$. Hence from (3.2) we conclude that if $\phi(C) = -1$ then $\phi(T_{c_1}T_{c_2}^{-1}(\beta)) = \phi(\beta)$. Since β was an arbitrary simple closed curve, this shows that $T_{c_1}T_{c_2}^{-1} \in \pi_C \text{Mod}(S)[\phi]$. But $T_{c_1}T_{c_2}^{-1} \in \pi_C \text{Mod}(S)$ is just the point-pushing map about α and therefore α is contained in $(\ker \pi_C \text{Mod}(S)[\phi])$. We refer to [FM12] for a comprehensive discussion of the various versions of the Birman exact sequence.

As the point pushing group $\pi_1^{-1}(m)$ is generated by point pushing maps along simple nonseparating loops, this shows the part (1) of the Lemma 3.1.

To show the part (2) of the Lemma 3.1, assume now that $\phi(C) = 1$. Equation (3.2) shows that $\phi(c_1) + \phi(c_2) = -2$ and hence by formula (3.3) we have

$$\phi(T_{c_1}T_{c_2}^{-1}(\beta)) = \phi(\beta) + \iota(\beta, c_1)\phi(c_1) + \iota(\beta, c_2)(\phi(c_1) + 2).$$

Now let us assume that the oriented simple closed curve β crosses a single time through c_1 , say when it enters P . Then $\iota(\beta, c_1) = -1, \iota(\beta, c_2) = 1$ and hence

$$(3.4) \quad \phi(T_{c_1}T_{c_2}^{-1}(\beta)) = \phi(\beta) - \phi(c_1) + \phi(c_1) + 2 = \phi(\beta) + 2.$$

Using this formula $r/2$ times if r is even, and r times if r is odd, we conclude that the point pushing map about α is not contained in $\text{Mod}(S)[\phi]$, but it is the

case for its $r/2^{\text{th}}$ power or r^{th} power, respectively. Namely, putting $m = r/2$ if r is even and $m = r$ otherwise, it follows from the above discussion that we have $\phi((T_{c_1}T_{c_2}^{-1})^m(\beta)) = \phi(\beta)$ for every simple closed curve β which either is disjoint from P or which crosses through P precisely once. As such curves span the first homology of S , we conclude that the pull-back of ϕ under $(T_{c_1}T_{c_2}^{-1})^m$ coincides with ϕ on a collection of simple closed curves which span $H_1(S, \mathbb{Z})$. [HJ89, Corollary 2.6] then shows that indeed, $(T_{c_1}T_{c_2}^{-1})^m \in P_C \text{Mod}(S)[\phi]$. Moreover, by equation (3.4), we know that $(T_{c_1}T_{c_2}^{-1})^k \in P_C \text{Mod}(S)[\phi]$ if k is not a multiple of m .

On the other hand, by [Sal19, Lemma 3.15], Dehn twists about separating simple closed curves in S are contained in $\text{Mod}(S)[\phi]$. As the commutator subgroup of $\pi_1(\Sigma_g)$ is generated by simple closed separating curves, and for each such curve α both Dehn twists T_{c_1}, T_{c_2} about the boundary curves of a tubular neighborhood of α as above are contained in $P_C \text{Mod}(S)[\phi]$, this yields the part (2) of Lemma 3.1. \square

Consider again an arbitrary compact surface S of genus $g \geq 2$, equipped with a $\mathbb{Z}/r\mathbb{Z}$ -spin structure ϕ for some $r \geq 2$. We use Lemma 3.1 to analyze the action of $\text{Mod}(S)[\phi]$ on the graph \mathcal{CG}_1^+ . We begin with the investigation of the stabilizer of a vertex c of \mathcal{CG}_1^+ in $\text{Mod}(S)[\phi]$. As $\text{Mod}(S)[\phi]$ is a subgroup of $\text{Mod}(S)$ of finite index, the stabilizer $\text{Stab}(c)[\phi]$ of c in $\text{Mod}(S)[\phi]$ is a subgroup of finite index of the stabilizer $\text{Stab}(c)$ of c in $\text{Mod}(S)$.

The group $\text{Stab}(c)$ can be described as follows. Cut S open along c . The result is a surface Σ_{g-2} of genus $g - 1$ with two distinguished boundary components C_1, C_2 . These components are equipped with an orientation as subsets of the oriented boundary of Σ_{g-2} . To simplify notations, let $\text{Mod}(\Sigma_{g-2})$ be the subgroup of the mapping class group of Σ_{g-2} which preserves the subset $C_1 \cup C_2$ of the boundary. We allow that an element of $\text{Mod}(\Sigma_{g-2})$ exchanges C_1 and C_2 . The stabilizer $\text{Stab}(c)$ of c in the mapping class group $\text{Mod}(S)$ of S can be identified with the quotient of the group $\text{Mod}(\Sigma_{g-2})$ by the relation $T_{C_1}T_{C_2}^{-1} = 1$ where T_{C_i} denotes the left Dehn twist about the boundary circle C_i ([FM12, Theorem 3.18]). In short, we have

$$\text{Stab}(c) = \text{Mod}(\Sigma_{g-2}) / \mathbb{Z}.$$

The infinite cyclic subgroup of $\text{Stab}(c)$ generated by the Dehn twist about c is central. The quotient group $\text{Stab}(c)/\mathbb{Z}$ can naturally be identified with the mapping class group $\text{Mod}(\Sigma_{g-2})$ of a surface Σ_{g-2} of genus $g - 1$ with two punctures and perhaps with boundary if the boundary of S is non-trivial. We refer to [FM12] for a comprehensive discussion of these facts.

Let Σ_{g-2} be the surface obtained from Σ_{g-2} by forgetting the punctures. Alternatively, Σ_{g-2} is obtained from Σ_{g-2} by attaching a disk to each boundary component. The group $\text{Mod}(\Sigma_{g-2}) = \text{Stab}(c)/\mathbb{Z}$ fits into the *Birman exact sequence*

$$(3.5) \quad 1 \rightarrow \pi_1(C(\Sigma_{g-2}, 2)) \xrightarrow{\rho} \text{Stab}(c)/\mathbb{Z} \rightarrow \text{Mod}(\Sigma_{g-2}) \rightarrow 1$$

where $\pi_1(C(\Sigma_{g-2}, 2))$ is the *surface braid group*, that is, the fundamental group of the configuration space of two unordered distinct points in Σ_{g-2} . In particular, $\pi_1(C(\Sigma_{g-2}, 2))$ is a normal subgroup of $\text{Stab}(c)/\mathbb{Z} = \text{Mod}(\Sigma_{g-2})$.

The surjective homomorphism

$$\theta : \text{Stab}(c) \rightarrow \text{Stab}(c)/\mathbb{Z} = \text{Mod}(\Sigma_{g-2})$$

restricts to a homomorphism $\text{Stab}(c)[\phi] \rightarrow \text{Mod}(\Sigma_2)$. The next proposition gives some first information on its image under the assumption that ϕ is a $\mathbb{Z}/2\mathbb{Z}$ -spin structure and $\phi(c) = 1$.

PROPOSITION 3.2. — *Let ϕ be a $\mathbb{Z}/2\mathbb{Z}$ -spin structure on S and let c be a simple closed curve with $\phi(c) = 1$. Then $\rho(\pi_1(PC(\Sigma_2))) = \theta(\text{Stab}(c)[\phi])$.*

Proof. — Let $\pi_1(PC(\Sigma_2))$ be the intersection of the fibre of the Birman exact sequence (3.5) with the subgroup of $\text{Mod}(\Sigma_2)$ which fixes each of the two distinguished punctures. Following [FM12, Section 4.2.5], the group $\pi_1(PC(\Sigma_2))$ can be described as follows.

Let C_1, C_2 be the distinguished boundary components of the surface $\Sigma^2 = S - c$. Let Σ^1 be the surface obtained from Σ^2 by attaching a disk to the boundary circle C_1 . Let $P\text{Stab}(c)$ and $P\text{Mod}(\Sigma^2)$ be the index two subgroup of $\text{Stab}(c)$ and $\text{Mod}(\Sigma^2)$ which preserves each of the two boundary components C_1, C_2 of $S - c$. The inclusion $\Sigma^2 \hookrightarrow \Sigma^1$ induces a surjective homomorphism

$$: P\text{Stab}(c)/\mathbb{Z} \rightarrow P_{C_2}\text{Mod}(\Sigma^1)/\mathbb{Z}$$

where as before $P_{C_2}\text{Mod}(\Sigma^1)$ is required to fix the boundary component C_2 of Σ^1 and where the group \mathbb{Z} acts as the group of Dehn twists about c and about C_2 . The kernel $\ker(\rho)$ of this homomorphism is isomorphic to $\pi_1(\Sigma^1)$ (see [FM12] for more information on this version of the Birman exact sequence).

The spin structure ϕ pulls back to a spin structure $\hat{\phi}$ on Σ^2 . Since ϕ is a $\mathbb{Z}/2\mathbb{Z}$ -spin structure on S and $\phi(c) = 1$, the value of $\hat{\phi}$ on each of the two boundary circles C_1, C_2 coincides with the value of a spin structure on the boundary of an embedded disk. This implies that $\hat{\phi}$ induces a spin structure ϕ on Σ^1 . Or, equivalently, $\hat{\phi}$ is the pull-back of a spin structure ϕ on Σ^1 via the inclusion $\Sigma^2 \hookrightarrow \Sigma^1$. By Lemma 3.1, the group $\ker(\rho) = \pi_1(\Sigma^1)$ stabilizes $\hat{\phi}$, that is, we have $\ker(\rho) \subset \text{Mod}(\Sigma^2)[\hat{\phi}]$.

Apply Lemma 3.1 a second time to the homomorphism $P_{C_2}\text{Mod}(\Sigma^1)/\mathbb{Z} \rightarrow \text{Mod}(\Sigma)$ where Σ is obtained from Σ^1 by attaching a disk to C_2 . As the group $\pi_1(PC(\Sigma_2))$ can be described as the quotient by its center \mathbb{Z}^2 of the kernel of the homomorphism $P\text{Mod}(\Sigma^2) \rightarrow \text{Mod}(\Sigma)$ which is obtained by applying the Birman exact sequence twice, first to a map which caps off the boundary component C_1 , followed by the map which caps off C_2 , this shows that $\pi_1(PC(\Sigma_2)) = \theta(\text{Stab}(c)[\phi])$. As exchanging C_1 and C_2 also preserves $\hat{\phi}$ the Proposition 3.2 follows. \square

We are now ready to give a complete description of the stabilizer in $\text{Mod}(S)[\phi]$ of a nonseparating simple closed curve c on S with $\phi(c) = 1$ where as before, ϕ is a $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a compact surface S of genus $g \geq 3$, with empty or connected boundary.

Cut S open along c and write $\Sigma^2 = S - c$. The spin structure ϕ of S pulls back to a $\mathbb{Z}/2\mathbb{Z}$ -spin structure $\hat{\phi}$ on Σ^2 . Denote as before by Σ^1 the surface of genus $g - 1$ with empty or connected boundary obtained from Σ^2 by capping off the two distinguished boundary components. We have

PROPOSITION 3.3. — *The $\mathbb{Z}/2\mathbb{Z}$ -spin structure ϕ on S induces a $\mathbb{Z}/2\mathbb{Z}$ -spin structure ϕ_c on Σ^1 whose parity coincides with the parity of ϕ . If $\rho : \text{Stab}(c)/\mathbb{Z} \rightarrow \text{Mod}(\Sigma)$*

denotes the surjective homomorphism induced by the inclusion $S - c \hookrightarrow S$ then

$$\pi^{-1} \text{Mod}(g)[\phi_c] = \text{Stab}(c)[\phi]/\mathbb{Z}.$$

Proof. — As ϕ is a $\mathbb{Z}/2\mathbb{Z}$ -spin structure, the value of ϕ on a boundary circle of $S - c$ corresponding to a copy of c coincides with the value of a $\mathbb{Z}/2\mathbb{Z}$ -spin structure on the boundary of a disk. Thus ϕ induces a spin structure ϕ_c on Σ_c .

To compare the parities of the spin structures ϕ and ϕ_c , assume that Σ_c is obtained from $S - c$ by attaching disks D_1, D_2 to the two boundary components of S which correspond to the two copies of c . Choose a geometric symplectic basis $a_1, b_1, \dots, a_{g-1}, b_{g-1}$ for Σ_c , consisting of simple closed oriented curves which do not intersect the disks D_1, D_2 . Then $a_1, b_1, \dots, a_{g-1}, b_{g-1}$ can be viewed as a system of curves in $\Sigma^2 = \Sigma - (D_1 \cup D_2)$ which maps to a curve system with the same properties in S by the map $\Sigma^2 \hookrightarrow S$. This curve system can be extended to a geometric symplectic basis for S containing the curve c , equipped with any orientation. As $\phi(c) = 1$ we have $\phi(c) + 1 = 0$. The claim now follows from the fact that $\phi_c(u) = \phi(\hat{u})$ for $u \in \{a_1, b_1, \dots, a_{g-1}, b_{g-1}\}$ where \hat{u} is the image of u under the inclusion $\Sigma^2 \hookrightarrow S$, together with the formula (2.2) for the Arf invariant.

We are left with showing that $\text{Stab}(c)[\phi]/\mathbb{Z} = \pi^{-1} \text{Mod}(g)[\phi_c]$. Observe first that as ϕ_c is induced from ϕ , we have $\text{Stab}(c)[\phi]/\mathbb{Z} \subset \pi^{-1} \text{Mod}(g)[\phi_c]$.

To show that in fact equality holds let Σ_2 be the surface obtained from $S - c$ by replacing the boundary components by punctures. The group $\text{Stab}(c)[\phi]/\mathbb{Z}$ can be identified with a subgroup Γ_c of $\text{Mod}(\Sigma_2)$. We view the punctures of Σ_2 as marked points p_1, p_2 in Σ_2 .

Let θ be any diffeomorphism of Σ_2 which preserves ϕ_c . Then θ is isotopic to a diffeomorphism of Σ_2 which equals the identity on a disk D containing both points p_1, p_2 . Thus θ lifts to a diffeomorphism θ of Σ_2 which preserves the pull-back of ϕ_c to a spin structure on Σ_2 .

The boundary circle ∂D of D can be viewed as a simple closed curve in $S - c$. Via the projection $S - c \rightarrow S$ which identifies the two distinguished boundary components of $S - c$, the curve ∂D projects to a separating simple closed curve in S which decomposes S into a one-holed torus T containing c and a surface of genus $g - 1$ with connected boundary. The diffeomorphism θ lifts to a diffeomorphism of S which is the identity on T .

Then ϕ is a spin structure on S which defines the same function on $H_1(S, \mathbb{Z})$ as ϕ . Using once more the result of Humphries and Johnson [HJ89] (see [Sal19, Theorem 3.9]), this implies that Γ_c stabilizes ϕ . As Γ_c projects to the mapping class of Σ_c defined by the diffeomorphism θ , this shows surjectivity of the homomorphism $\pi^{-1} \text{Mod}(g)[\phi_c] \rightarrow \text{Stab}(c)[\phi]/\mathbb{Z}$.

On the other hand, by Proposition 3.2 the kernel of the homomorphism $\pi^{-1} \text{Mod}(g)[\phi_c] \rightarrow \text{Stab}(c)[\phi]/\mathbb{Z}$ is also contained in $\text{Stab}(c)[\phi]/\mathbb{Z}$. Together this completes the proof of the Proposition 3.3. \square

The next observation uses [Sal19, Proposition 4.9]. For its formulation, recall from Section 2 the definition of the graph \mathcal{CG}_1^+ . Its vertices are nonseparating simple closed curves with prescribed value ± 1 of the spin structure. The graph \mathcal{CG}_1^+ is well defined if the genus g of S is at least two although it may not have edges.

Note that in the statement of Proposition 3.4, we allow that the surface S has non-empty boundary, and we consider $\mathbb{Z}/r\mathbb{Z}$ -spin structures where r may be larger than $2g - 2$. This is crucial for an inductive approach towards higher spin structures via cutting surfaces open along separating simple closed curves, and it is used in the proof of Theorem 1.4.

PROPOSITION 3.4. — *Let ϕ be a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on a compact surface S of genus $g \geq 2$ with empty or connected boundary. Then for any two directed edges e_1, e_2 of the graph CG_1^+ there exists a mapping class $\zeta \in \text{Mod}(S)[\phi]$ with $\zeta(e_1) = e_2$. In particular, the action of $\text{Mod}(S)[\phi]$ on CG_1^+ is vertex transitive.*

Proof. — The proof consists of an adjustment of the argument in the proof of [Sal19, Proposition 4.9].

Recall that a geometric symplectic basis for S is a set $\{a_1, b_1, \dots, a_{2g}, b_{2g}\}$ of simple closed curves on S such that a_i, b_i intersect in a single point, and a_i, b_i is disjoint from a_j, b_j for $j \neq i$.

A vertex of CG_1^+ is a simple closed curve c on S with $\phi(c) = \pm 1$. In the sequel we always orient such a vertex c in such a way that $\phi(c) = 1$. For a given directed edge e of CG_1^+ with ordered endpoints c, d , we aim at constructing a geometric symplectic basis $B(e)$ such that $a_1 = c, a_2 = d, \phi(a_i) = 0$ for $i \geq 3$, $\phi(b_i) = 0$ for $i \leq g - 1$ and $\phi(b_g) = 0$ or 1 as predicted by the parity of ϕ . If such a basis $B(e_1), B(e_2)$ can be found for any two directed edges e_1, e_2 of CG_1^+ with ordered endpoints c_1, d_1 and c_2, d_2 , then there exists a diffeomorphism ζ of S which maps $B(e_1)$ to $B(e_2)$ and maps c_1, d_1 to c_2, d_2 . The pullback $\zeta^* \phi$ of ϕ is a spin structure on S whose values on $B(e_1)$ coincide with the values of ϕ . By a result of Humphries and Johnson [HJ89], see [Sal19, Theorem 3.9], this implies that $\zeta^* \phi = \phi$ and hence the isotopy class of ζ is contained in $\text{Mod}(S)[\phi]$ and maps the directed edge e_1 to the directed edge e_2 .

To simplify further, choose any geometric symplectic basis

$$B = \{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$$

for S with $\alpha_1 = c, \alpha_2 = d$. A small tubular neighborhood of α_i, β_i is a one-holed torus T_i embedded in S . By Lemma 2.4, for all $i \geq 3$ we may replace α_i by an oriented simple closed curve in T_i , again denoted by α_i , which satisfies $\phi(\alpha_i) = 0$.

Assume that β_i ($i = 1, 2$) is oriented in such a way that $\iota(\beta_i, \alpha_i) = 1$ where ι is the symplectic form. As $\phi(T_{\alpha_i}(\beta_i)) = \phi(\beta_i) + 1$, via perhaps replacing β_i by its image under a suitably chosen power of a Dehn twist about α_i we may assume that $\phi(\beta_i) = 0$. Therefore for the construction of a geometric symplectic basis $B(e)$ with the required properties, it suffices to modify successively the curves β_i ($i \geq 3$) while keeping α_j ($j \geq 1$) and β_k for $k < i$ fixed such that ϕ assumes the prescribed values on the modified curves.

We follow the proof of [Sal19, Proposition 4.9]. For $1 \leq i \leq g$ let δ_i be the boundary curve of the torus T_i which is a small tubular neighborhood of α_i, β_i , equipped with the orientation as an oriented boundary circle of $S - T_i$ ($i \geq 1$). By homological coherence ([Sal19, Proposition 3.8]), we have $\phi(\delta_i) = 1$ for all i .

Thus if ϵ is an embedded arc in S connecting β_3 to δ_1 whose interior is disjoint from α_3 and all δ_j for $j = 3$, then $\phi(\beta_3 + \epsilon \delta_1) = \phi(\beta_3) + 2$. Moreover, $\beta_3 + \epsilon \delta_1$ is disjoint from δ_j for all $j = 3$.

A small tubular neighborhood of $\alpha_3 \cap (\beta_3 + \epsilon \delta_1)$ is a one-holed torus \widehat{T}_3 disjoint from the tori T_i for $i = 3$. Thus we can repeat this construction with an arc connecting $\beta_3 + \epsilon \delta_1$ to δ_1 which is disjoint from α_3 and whose interior is disjoint from all δ_j for $j = 3$. Repeating further if necessary, we can find a simple closed curve β_3 intersecting α_3 in a single point and disjoint from the curves δ_j for $j = 3$ so that $\phi(\beta_3) \in \{0, 1\}$.

Let δ_3 be the boundary of a tubular neighborhood of $\alpha_3 \cap \beta_3$. Then δ_3 is disjoint from all the curves δ_j for $j = 3$. As in the proof of [Sal19, Proposition 4.9], repeat this procedure with the curve β_4 and the curves $\delta_1, \delta_2, \delta_3, \dots, \delta_g$. In finitely many steps we can change the geometric symplectic basis B to a geometric symplectic basis $B = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \dots, \alpha_g, \beta_g\}$ which fulfills $\phi(\beta_j) = 0$ or 1 for all $3 \leq j \leq g$.

It remains to further alter β_j for $3 \leq j \leq g - 1$ to a nonseparating simple closed curve β_j with $\phi(\beta_j) = 0$, and to alter β_g to a simple closed curve β_g with $\phi(\beta_g) = 0$ or 1 depending on the parity of the $\mathbb{Z}/r\mathbb{Z}$ -spin structure ϕ . This construction is carried out in detail in the proof of [Sal19, Proposition 4.9] and will not be presented here as it would require the introduction of a significant amount of new notation. It takes place in a subsurface of S of genus $g - 2$ which is disjoint from $\alpha_1, \beta_1, \alpha_2, \beta_2$ and contains α_i, β_i for $3 \leq i \leq g$. The resulting geometric symplectic basis has the properties we are looking for. \square

Remark 3.5. — The proof of Proposition 3.4 can also be used to show the following. Under the assumption of the proposition, let $c, d \subset S$ be two nonseparating simple closed curves with $\phi(c) = \phi(d) = 0$; then there exists some $\zeta \in \text{Mod}(S)[\phi]$ with $\zeta(c) = d$. In fact, this case is more explicitly covered by [Sal19, Proposition 4.2 and Proposition 4.9].

The next statement is an extension of Proposition 3.4 to surfaces with more than one boundary component under some restrictions on the spin structure.

COROLLARY 3.6. — *Let ϕ be a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on a compact surface S of genus $g \geq 2$ with non-empty boundary which is induced from a spin structure ϕ on a compact surface Σ of genus g with empty or connected boundary by an inclusion $S \hookrightarrow \Sigma$ which maps each boundary component of S to the boundary of an embedded disk in Σ . Then for any two vertices c, d of CG_1^+ there exists a mapping class $\zeta \in \text{Mod}(S)[\phi]$ with $\zeta(c) = d$. In particular, the action of $\text{Mod}(S)[\phi]$ is transitive on the vertices of CG_1^+ .*

Proof. — Let $\iota : S \hookrightarrow \Sigma$ be the natural embedding. Let c, d be vertices of the graph CG_1^+ for the spin structure ϕ on S . Then c, d are nonseparating simple closed curves and hence their images $\iota(c), \iota(d)$ are nonseparating simple closed curves on Σ . Furthermore, as ϕ is the pull-back of a spin structure ϕ on Σ , we have $\phi(\iota(c)) = \phi(\iota(d)) = 1$.

By Proposition 3.4, there exists a mapping class $\theta \in \text{Mod}(\Sigma)(\phi)$ which maps $\iota(c)$ to $\iota(d)$. We can choose a diffeomorphism of Σ representing θ which equals the identity on each component of $\Sigma - S$. Thus there exists a lift ζ of θ to a mapping class of S . This mapping class is contained in $\text{Mod}(S)[\phi]$, and it maps the simple closed curve c to a simple closed curve d whose image under ι is isotopic to $\iota(d)$.

Using once more the Birman exact sequence, this implies that there exists a mapping class β in the kernel of the homomorphism $\text{Mod}(S) \rightarrow \text{Mod}(\Sigma)$ which maps d to \bar{d} . But by an iterated application of Lemma 3.1, this kernel is contained in $\text{Mod}(S)[\phi]$ and hence c can be mapped to d by an element of $\text{Mod}(S)[\phi]$. \square

The *augmented Teichmüller space* $\bar{T}(S)$ of the compact surface S is the union of the Teichmüller space with so-called *boundary strata*. Each of these boundary strata is defined by a non-empty system \mathcal{C} of pairwise disjoint essential simple closed curves. The stratum defined by such a curve system can be thought of as the Teichmüller space of the surface obtained from S by shrinking each component of \mathcal{C} to a node. In other words, such a stratum is a complex manifold which is naturally biholomorphic to the Teichmüller space of the surface obtained by cutting S open along the components of \mathcal{C} and replacing each boundary component of the resulting bordered surface by a puncture.

Using Fenchel Nielsen coordinates, the augmented Teichmüller space can be equipped with a natural topology. For this topology, the usual Teichmüller space embeds into $\bar{T}(S)$ as an open dense subset. Furthermore, the inclusion of the Teichmüller space of a punctured surface defined by the curve system \mathcal{C} onto a boundary stratum of $\bar{T}(S)$ also is an embedding. We refer to [Wol10] for an detailed description and for a discussion of the following

THEOREM 3.7. — *The augmented Teichmüller space $\bar{T}(S)$ is a non locally compact stratified space. The mapping class group $\text{Mod}(S)$ of S acts on $\bar{T}(S)$, with quotient the Deligne Mumford compactification of the moduli space of curves of genus g .*

Fix again a $\mathbb{Z}/2\mathbb{Z}$ -spin structure ϕ on a surface S of genus $g \geq 2$. Define the *spin Teichmüller space* $T_{\text{spin}}(S)$ to be the Teichmüller space of S together with this spin structure. The group $\text{Mod}(S)[\phi]$ acts on $T_{\text{spin}}(S)$ as a group of biholomorphic transformations, with quotient the *spin moduli space* $\mathcal{M}_\phi = T_{\text{spin}}(S)/\text{Mod}(S)[\phi]$.

We can define an augmented spin Teichmüller space $\bar{T}_{\text{spin}}(S)$ as the union of spin Teichmüller space with all strata of augmented Teichmüller space which are defined by systems of nonseparating simple closed curves c on S with $\phi(c) = 1$. Equipped with the subspace topology, this is a subspace of $\bar{T}(S)$ which is invariant under the action of the spin mapping class group. As a corollary of the discussion in this section, we have

COROLLARY 3.8. — *The quotient $\bar{T}_{\text{spin}}(S)/\text{Mod}(S)[\phi]$ is a partial bordification of the spin moduli space $T_{\text{spin}}(S)/\text{Mod}(S)[\phi]$. Its boundary contains the spin moduli space of the same parity on a surface of genus $g - 1$ with two marked points (punctures) as an open dense subset.*

Remark 3.9. — Corollary 3.8 can be thought of as describing a specific subset of a Deligne Mumford compactification of the moduli space of curves with a fixed spin structure. Such a Deligne Mumford compactification was constructed by Cornalba [Cor89].

4. Structure of the spin mapping class group of odd parity

The goal of this section is to prove the Theorem 1.3(1).

We begin with some additional information on the spin mapping class group. Fix a $\mathbb{Z}/r\mathbb{Z}$ -spin structure ϕ on a closed surface Σ_g of genus g for some $r \geq 2$. For a simple closed curve c on Σ_g with $\phi(c) = \pm 1$, this spin structure restricts to a spin structure on the surface Σ_{g-1}^2 of genus $g - 1$ with two boundary circles c_1, c_2 obtained by cutting Σ_g open along c . We denote this spin structure again by ϕ . Define the group Σ_{g-1}^2 to be the following quotient of the spin mapping class group $\text{Mod}(\Sigma_{g-1}^2)[\phi]$.

The group $\text{Mod}(\Sigma_{g-1}^2)[\phi]$ contains a rank two free abelian central subgroup generated by the r^{th} powers of the left Dehn twists T_{c_1}, T_{c_2} about the boundary circles c_1, c_2 of Σ_{g-1}^2 . Define $\Sigma_{g-1}^2 = \text{Mod}(\Sigma_{g-1}^2)[\phi]/\mathbb{Z}$ where the infinite cyclic subgroup \mathbb{Z} is generated by $T_{c_1}^r T_{c_2}^{-r}$. Then Σ_{g-1}^2 is isomorphic to the stabilizer in $\text{Mod}(\Sigma_g)[\phi]$ of the curve c . Note that up to isomorphism, the group Σ_{g-1}^2 does not depend on the vertex $c \in CG_1$. Namely, by Proposition 3.4, the stabilizers in $\text{Mod}(\Sigma_g)[\phi]$ of nonseparating simple closed curves c with $\phi(c) = \pm 1$ are all conjugate and hence isomorphic.

Observe that the group Σ_{g-1}^2 is an infinite cyclic central extension of a finite index subgroup of the mapping class group of a surface $\Sigma_{g-1,2}$ of genus $g - 1$ with two punctures. Thus it makes sense to talk about its action on isotopy classes of essential curves on the surfaces $\Sigma_{g-1,2}$ and Σ_{g-1}^2 . The map $\Sigma_{g-1}^2 \rightarrow \Sigma_{g-1,2}$ which contracts each boundary component to a puncture defines a bijection on such isotopy classes.

We have

PROPOSITION 4.1. — *Let ϕ be a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on a closed surface Σ_g of genus $g \geq 3$. There is a commutative diagram*

$$(4.1) \quad \begin{array}{ccc} \Sigma_{g-1}^2 & \xrightarrow{\iota_1} & \Sigma_{g-1}^2 \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z} \\ & \searrow \iota_2 & \downarrow \rho \\ & & \text{Mod}(\Sigma_g)[\phi] \end{array}$$

where the homomorphisms ι_1, ι_2 are inclusions, and the homomorphism ρ is surjective. The subgroup A of Σ_{g-1}^2 is the stabilizer in Σ_{g-1}^2 of a nonseparating simple closed curve d on Σ_{g-1}^2 with $\phi(d) = \pm 1$. The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\Sigma_{g-1}^2 \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z}$ by exchanging the two factors, and it acts as an automorphism on A .

Proof. — Fix a pair of nonseparating simple closed disjoint curves c, d on Σ_g with $\phi(c) = \phi(d) = \pm 1$ which are connected by an edge in the graph CG_1^+ , that is, so that $\Sigma_g - (c \cup d)$ is connected. Let $\Sigma_c, \Sigma_d \subset \text{Mod}(\Sigma_g)[\phi]$ be the stabilizers of c, d in the spin mapping class group of Σ_g . By Corollary 3.6, these groups are naturally isomorphic to the group Σ_{g-1}^2 , and they intersect in the index two subgroup $A = \Sigma_c \cap \Sigma_d$ of the stabilizer of $c \cup d$ in $\text{Mod}(\Sigma_g)[\phi]$ consisting of all elements which preserve both c, d individually. The full stabilizer of $c \cup d$ in $\text{Mod}(\Sigma_g)[\phi]$ is a $\mathbb{Z}/2\mathbb{Z}$ extension of $\Sigma_c \cap \Sigma_d$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts as involution on $A = \Sigma_c \cap \Sigma_d$ exchanging c and d . This involution extends to an involution of $\Sigma_c \cup \Sigma_d$ exchanging the two subgroups Σ_c, Σ_d .

By the universal property of free amalgamated products, there is a homomorphism

$$\rho: \pi_1(\Sigma_g) \rtimes_{\mathbb{Z}/2\mathbb{Z}} \text{Mod}(\Sigma_g)[\phi].$$

All we need to show is that ρ is surjective, that is, that $\rho(\pi_1(\Sigma_g)) = \text{Mod}(\Sigma_g)[\phi]$.

As $\text{Mod}(\Sigma_g)[\phi]$ acts transitively on the vertices of the graph CG_1^+ , for this it suffices to show that its subgroup $\rho(\pi_1(\Sigma_g))$ acts transitively on the vertices of CG_1^+ as well. Namely, by construction, the stabilizer of the vertex c of CG_1^+ in $\rho(\pi_1(\Sigma_g))$ coincides with its stabilizer in $\text{Mod}(\Sigma_g)[\phi]$. As $\rho(\pi_1(\Sigma_g))$ is a subgroup of $\text{Mod}(\Sigma_g)[\phi]$, this then implies equality.

To show transitivity of the action of $\rho(\pi_1(\Sigma_g))$ on the vertices of CG_1^+ let $v \in CG_1^+$ be any vertex. By Corollary 2.18, the graph CG_1^+ is connected and hence we can find an edge path $(c_i) \subset CG_1^+$ connecting $c_0 = c$ to $c_k = v$. We also may assume that $c_1 = d$.

By the assumption $\phi(d) = \pm 1$, for one of the two boundary components d_1, d_2 of $\Sigma_g - d$, equipped with the orientation as a boundary component of $\Sigma_g - d$, say the component d_1 , we have $\phi(d_1) = -1$. Thus we can attach a disk D to c_1 and obtain a surface Σ with spin structure ϕ which induces the spin structure ϕ on $\Sigma_g - d$. As a consequence, the restriction of ϕ to $\Sigma_g - d$ fulfills the hypothesis in Corollary 3.6. As $c = c_0$ and c_2 are nonseparating simple closed curves in $\Sigma_g - d$ with $\phi(c) = \phi(c_2) = \pm 1$, Corollary 3.6 shows that there exists an element $\gamma_1 \in \rho(\pi_1(\Sigma_g))$ such that $\gamma_1(c) = c_2$. Then the stabilizer of c_2 in $\text{Mod}(\Sigma_g)[\phi]$ equals $\pi_1(c_2)^{-1}$ and hence it is contained in $\rho(\pi_1(\Sigma_g))$. Thus we can apply Corollary 3.6 to $\pi_1(c_2)^{-1}$ and find an element $\gamma_2 \in \rho(\pi_1(\Sigma_g))$ which maps c_1 to c_3 . Proceeding inductively and using the fact that c is conjugate to d in $\rho(\pi_1(\Sigma_g))$ by the generator of the subgroup $\mathbb{Z}/2\mathbb{Z}$, this completes the proof of the Proposition 4.1. \square

Recall from the introduction the definition of an admissible curve system on a closed surface Σ_g of genus $g \geq 2$. The mapping class group of Σ_g naturally acts on the family of all admissible curve systems on Σ_g . Recall also that the curve diagram of an admissible curve system is a finite tree.

Since the curve diagram of an admissible curve system C is connected, each curve $c \in C$ intersects at least one other simple closed curve on Σ_g transversely in a single point and hence it is nonseparating.

We need some technical information on admissible curve systems. To this end let C be any admissible curve system on an oriented surface S . We require that the boundary of S is empty, but we allow for the moment that S has punctures. For admissibility, we require that all complementary components of C are either topological disks or once punctured topological disks.

The union $\{c \mid c \in C\}$ is an embedded graph G in S whose vertices are the intersection points between the curves from C . Choose a basepoint $x \in G$ which is contained in the interior of an edge of G . This edge is contained in a simple closed curve $c_0 \in C$ which defines a distinguished vertex v_0 in the curve diagram of C .

Construct inductively a family L of homotopy classes of loops in G based at x as follows. Let L_0 be the family consisting of the two based loops which go once around the simple closed curve $c_0 \in C$ containing x in either direction. Assume by induction that for some $k \geq 1$ we defined a system of based loops L_{k-1} . Let $\{c_{k_1}, \dots, c_{k_s}\} \subset C$

be the curves in C whose distance in the curve diagram to the distinguished vertex v_0 equals k . Define

$$L_k = \{T_{c_{ku}}^{\pm 1} d \mid u \leq s, d \in L_{k-1}\}$$

and let $L = L_b$ where $b \geq 1$ is the maximal distance of a vertex in the curve diagram of C to the distinguished vertex v_0 .

The following appears implicitly in [PV96] and explicitly as [Sal19, Lemma 9.3].

LEMMA 4.2. — *The loops from the system L generate the fundamental group $\pi_1(S, x)$ of S .*

As a consequence we obtain (see [Sal19, Lemma 9.4]).

LEMMA 4.3. — *Let C be an admissible curve system on a surface S , possibly with punctures. Let p be a puncture of S and assume that there are two curves $c_1, c_2 \in C$ which bound a once punctured annulus, with p as puncture. Then the subgroup of $\text{Mod}(S)$ generated by the Dehn twists about the curves from the curve system C contains the kernel of the homomorphism $\text{Mod}(S) \rightarrow \text{Mod}(\Sigma)$ where Σ is obtained from S by forgetting p .*

For a closed surface Σ_g of genus $g \geq 2$ consider the system S_g of $3g - 2$ simple closed curve on Σ_g shown in Figure 4.1.

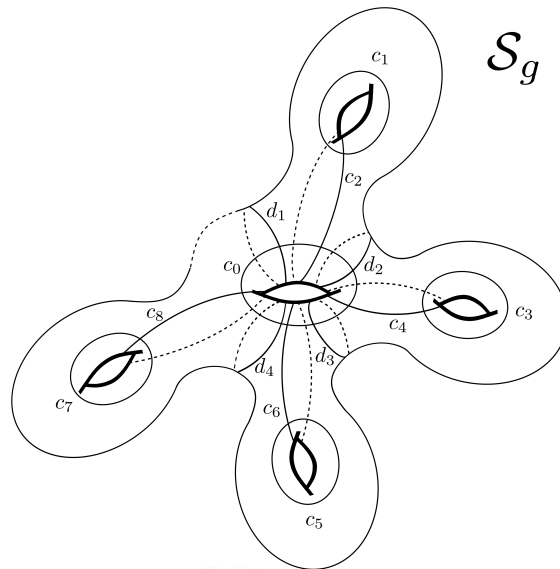


Figure 4.1.

The following is well known but hard to locate in the literature.

LEMMA 4.4. — *For $g = 2$, Dehn twists about the curves from the curve system S_g generate the stabilizer of an odd spin structure in $\text{Mod}(\Sigma_2)$.*

Proof. — S_2 is just a chain of 4 curves which are invariant under the hyperelliptic involution. The subgroup of $\text{Mod}(\Sigma_2)$ generated by the Dehn twists about these

curves fixes one of the Weierstrass points, that is, one of the fixed points of the hyperelliptic involution.

Replacing the Weierstrass point by a small disk shows that we may view \mathcal{M}_g as a subgroup of the quotient by its center of the *symmetric mapping class group* of a surface of genus 2 with connected boundary, which equals the braid group in 5 strands ([FM12, Theorem 9.2]). This group is known to be generated by the Dehn twists about the curves from the curve system S_2 (see [FM12, Section 9.1]) and hence

$\text{Mod}(\Sigma_2)$ equals the stabilizer of one of the Weierstrass points. This Weierstrass point defines a $\mathbb{Z}/2\mathbb{Z}$ -spin structure on Σ_2 whose stabilizer in $\text{Mod}(\Sigma_2)$ equals the stabilizer of the Weierstrass point. This shows the Lemma 4.4. \square

LEMMA 4.5. — *The Dehn twists about the curves from the system S_g preserve an odd $\mathbb{Z}/2\mathbb{Z}$ -spin structure on Σ_g .*

Proof. — There exists a cyclic subgroup G of the diffeomorphism group of Σ_g of order $g-1$ which preserves S_g and acts freely on Σ_g as a group of rotations about the center curve c_0 . The group G cyclically permutes the complementary components of S_g .

As a consequence, the curve system S_g descends to a curve system on a closed surface Σ_2 of genus 2. The curve diagram of this system is just a line segment of length 4 and hence the Dehn twists about these curves preserve an odd spin structure on Σ_2 (see Lemma 4.4). This spin structure lifts to a spin structure on Σ_g which is invariant under the Dehn twist about the curves from S_g . The parity of this spin structure is odd, as can also easily be checked explicitly using the formula (2.2). This is what we wanted to show. \square

We use Lemma 4.3 and Proposition 4.1 to show

PROPOSITION 4.6. — *Let ϕ be an odd $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a surface Σ_g of genus $g \geq 2$. Then the group $\text{Mod}(\Sigma_g)[\phi]$ is generated by the Dehn twists about the curves from the curve system S_g .*

Proof. — Lemma 4.5 shows that the subgroup \mathcal{M}_g of $\text{Mod}(\Sigma_g)$ generated by the Dehn twists about the curves from the curve system S_g is a subgroup of $\text{Mod}(\Sigma_g)[\phi]$. We have to show that it coincides with $\text{Mod}(\Sigma_g)[\phi]$.

We proceed by induction on the genus. By Lemma 4.4, the claim of the proposition holds true for $g=2$. Thus let us assume that the proposition is known for some $g-1 \geq 2$. Consider the curve system S_g on a surface of genus g . Using the labeling from Figure 4.1, let a_1 be the simple closed curve on Σ_g which intersects the curve c_1 in a single point and is disjoint from any other curve from S_g . We know that $\phi(a_1) = 1$. We aim at showing that $\text{Stab}(a_1) = \text{Mod}(\Sigma_g)[\phi] \text{Stab}(a_1)$.

To this end cut Σ_g open along a_1 . The resulting surface is a surface Σ_{g-1}^2 of genus $g-1$ with two boundary components. Replace these two boundary components by punctures and let $\Sigma_{g-1,2}$ be the resulting twice punctured surface. As before, the spin structure ϕ descends to a spin structure, again denoted by ϕ , on the surface Σ_{g-1} obtained by closing the punctures, and to a spin structure on $\Sigma_{g-1,2}$. The curve system S_g descends to the curve system S_{g-1} on Σ_{g-1} .

By induction hypothesis, the Dehn twists about the curves from the curve system S_{g-1} generate the spin mapping class group $\text{Mod}(\Sigma_{g-1})[\phi]$. On the other hand, we

can apply Lemma 4.3 to each of the two punctures of $\Sigma_{g-1,2}$ as each of these two punctures is contained in a once punctured annulus bounded by two curves from the restriction of S_g to $\Sigma_{g-1,2}$. We conclude that the point pushing maps about these punctures are contained in the group $\text{Stab}(a_1)$. As a consequence, the group

$\text{Stab}(a_1)$ surjects onto the index two subgroup of $\text{Mod}(\Sigma_{g-1,2})[\phi]$ which fixes each of the two punctures.

We have to show that there also is an element of $\text{Stab}(a_1)$ which exchanges the two boundary components of $\Sigma_g - a_1$. For this it suffices to find an element of $\text{Mod}(\Sigma_g)$ which fixes the curves c_1, c_2 and exchanges d_1, d_2 .

If $g = 3$ then consider the hyperelliptic involution of the surface Σ_2 obtained by cutting Σ_3 open along the simple closed curve a_1 and removing the punctures. This element can be represented as an explicit word in the Dehn twists about the curves c_2, c_0, c_4, c_3 (or, rather, their projection to Σ_2). The mapping class ψ , viewed as an element of the mapping class group of Σ_3 , preserves the curves c_i and exchanges d_1 and d_2 .

For $g \geq 4$ the same argument can be used. Namely, the element ψ still acts as an involution on Σ_g which preserves the curves c_1, c_2 and exchanges d_1 and d_2 . However this involution does not preserve the curve system S_g .

To summarize, we showed so far that $\text{Stab}(a_1)$ surjects onto $\text{Stab}(a_1)[\phi]/\mathbb{Z}$. Thus to show that $\text{Stab}(a_1) = \text{Mod}(\Sigma_g)[\phi] \text{Stab}(a_1)$ it suffices to show that $\text{Stab}(a_1)$ contains the square $T_{a_1}^2$ of the Dehn twist about a_1 . For an application of Proposition 4.1, we have to show furthermore that $\text{Stab}(a_1)$ contains an involution which exchanges the curve a_1 with a curve disjoint from a_1 . We show first that $\text{Stab}(a_1)$ contains an involution which maps a_1 to a_2 .

To this end consider again first the case $g = 3$. The curve system S_3 contains a curves system $E_6 \subset S_3$ obtained from S_3 by deleting the curve d_2 . This is the curve system shown in Figure 1.2 in the introduction. By [Mat00, Theorem 1.4], there exists an explicit word $c(E_6)$ in the Dehn twists about the curves from the system E_6 , the image of the so-called *Garside element* of the Artin group of type E_6 , which acts as a reflection on the curve diagram of E_6 exchanging the curves c_1 and c_3 . Then this reflection exchanges a_1 and a_2 and hence it has the desired properties.

As before, this reasoning extends to any $g \geq 4$. Namely, the element $c(E_6)$, viewed as an element of the mapping class group of Σ_g , still acts as an involution on Σ_g which exchanges a_1 and a_2 and preserves the subsurface of Σ_g filled by the curves $c_1, c_2, c_0, c_4, c_3, d_2$.

For an application of Proposition 4.1, we are left with showing that the square of the Dehn twist about a_1 is contained in $\text{Stab}(a_1)$. By the above discussion, we know that

$\text{Stab}(a_1)$ surjects onto $\text{Mod}(\Sigma_{g-1,2})[\phi]$. In particular, $\text{Stab}(a_1)$ contains $T_{a_2}^2$, viewed as an element of $\text{Stab}(a_1) \subset \text{Mod}(\Sigma_g)$. Since a_1 is the image of a_2 under an involution contained in $\text{Stab}(a_1)$, it follows that $T_{a_1}^2 \in \text{Stab}(a_1)$.

To summarize, we showed that $\text{Stab}(a_1) = \text{Mod}(\Sigma_g)[\phi] \text{Stab}(a_1)$, furthermore $\text{Stab}(a_1)$ contains an involution which exchanges a_1 and a_2 . Proposition 4.1 now shows that $\text{Stab}(a_1) = \text{Mod}(\Sigma_g)[\phi]$. This completes the proof of the Proposition 4.6. \square

We use Proposition 4.6 as the base case for the proof of Theorem 1.3 from the introduction. The curve system C_g is shown in Figure 1.1 in the introduction. Note that we have $C_3 = S_3$.

THEOREM 4.7. — *Let ϕ be an odd $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a surface Σ_g of genus $g \geq 3$. Then the group $\text{Mod}(\Sigma_g)[\phi]$ is generated by the Dehn twists about the curves from the curve system C_g .*

Proof. — The curve system C_g is obtained from the curve system S_g by deleting the curves d_3, \dots, d_{g-1} . Let Γ be the subgroup of $\text{Mod}(\Sigma_g)[\phi]$ generated by the Dehn twists about the curves from the curve system C_g . By Proposition 4.6, it suffices to show that the Dehn twists T_{d_i} for $i = 3, \dots, g-1$ are contained in Γ . Moreover, as $D_3 = C_3$, we may assume that $g \geq 4$.

Let a_i be the simple closed curve which intersects c_{2i-1} in a single point and does not intersect any other curve from S_g . We first claim that $T_{a_1}^2 \in \Gamma$.

To show the claim consider the subsurface Σ_2^1 of Σ_g which is filled by the curves $a_1, c_0, c_1, c_2, d_1, d_2$. This is a surface of genus 2 with connected boundary. The curves d_1, d_2 bound a one-holed annulus containing the boundary circle C of Σ_2^1 .

By homological coherence ([Sal19, Proposition 3.8]), we have $\phi(C) = 1$. Thus the spin structure ϕ descends to a spin structure on Σ_2^1 , on the surface $\Sigma_{2,1}$ obtained from Σ_2^1 by replacing the boundary component by a puncture and on the surface Σ_2 obtained from $\Sigma_{2,1}$ by forgetting the puncture, again denoted by ϕ . The curves of the curve system C_g which are contained in Σ_2^1 define a curve system F on Σ_2^1 which descends to a curve system on Σ_2 . The curve diagram of this system is just a line segment of length 4. By Lemma 4.4, the group generated by Dehn twists about the curves from F projects onto $\text{Mod}(\Sigma_2)[\phi]$.

On the other hand, F also contains two simple closed curves which enclose the boundary component of Σ_2^1 . It now follows from Lemma 4.3 that the subgroup of $\text{Mod}(\Sigma_{2,1})$ generated by the Dehn twists about the curves from F equals $\text{Mod}(\Sigma_{2,1})[\phi]$. In particular, this group contains $T_{a_1}^2$ and therefore $T_{a_1}^2 \in \Gamma$.

We claim next that $T_{a_2}^2 \in \Gamma$. To this end consider the subsurface Σ_3^1 of Σ_g which is filled by the system of curves $G = \{c_1, c_2, c_0, c_4, c_3, d_1, d_2, d_3\}$. This is a surface of genus 3 with connected boundary. The curves d_1, d_3 bound a one-holed annulus containing the boundary circle A of Σ_3^1 .

The subsurface Σ_3^1 of Σ_g contains the curves $c_1, c_2, c_0, c_4, c_3, d_2$ whose curve diagram is the Dynkin diagram of type E_6 (see Figure 1.2 in the introduction). There is an involution of Σ_3^1 which fixes the curves c_0, d_2 and exchanges c_2, c_4 and a_1, a_2 . By [Mat00, Theorem 1.4], this involution is contained in the subgroup of the mapping class group of Σ_3^1 which is generated by the Dehn twists about the curves $c_1, c_2, c_0, c_4, c_3, d_2$. As a consequence, there is an element of $\text{Mod}(\Sigma_3^1)$ which exchanges a_1 and a_2 . This implies that $T_{a_2}^2 \in \Gamma$.

By the chain relation for Dehn twists of surfaces (see [FM12, p. 108]), we have $(T_{a_2}^2 T_{c_3} T_{c_4})^3 = T_{d_2} T_{d_3}$. Since $T_{d_2} \in \Gamma$, we conclude that $T_{d_3} \in \Gamma$.

Now repeat this argument, replacing the curves c_j by c_{j+2} and the curve a_i by a_{i+1} where the first step discussed above is the case $i = 1$. In finitely many such steps we find that indeed $T_{d_i} \in \Gamma$ for all i . This is what we wanted to show. \square

5. Structure of the spin mapping class group of even parity

The goal of this section is to prove the Theorem 1.3(2). Our strategy is to reduce this result to the Theorem 1.3(1) by a change of parity construction.

Consider for the moment an arbitrary $\mathbb{Z}/r\mathbb{Z}$ -spin structures ϕ on a compact surface S of genus $g \geq 4$. In the Appendix A we introduce a graph CG_2^+ whose vertices are ordered pairs (a, b) of nonseparating simple closed curves which intersect in a single point and hence they fill a one-holed torus $T(a, b)$. Furthermore, it is required that $\phi(a) = 2$ and $\phi(b) = 0$. The spin structure on S restricts to a spin structure $\hat{\phi}$ on $(a, b) = S - T(a, b)$.

By homological coherence ([Sal19, Proposition 3.5]), if we orient the boundary circle c of (a, b) as the oriented boundary of (a, b) then we have $\phi(c) = 1$. Thus if $r = 2$ then ϕ descends to a spin structure $\hat{\phi}$ on the surface (a, b) obtained from (a, b) by capping off the boundary. This spin structure $\hat{\phi}$ has a parity, either even or odd.

LEMMA 5.1. — *A $\mathbb{Z}/2\mathbb{Z}$ -spin structure ϕ on S induces a $\mathbb{Z}/2\mathbb{Z}$ -spin structure $\hat{\phi}$ on the surface (a, b) whose parity is opposite to the parity of ϕ .*

Proof. — Choose a geometric symplectic basis $a_1, b_1, \dots, a_{g-1}, b_{g-1}$ for (a, b) . This basis then lifts to a curve system on the surface $(a, b) = S - T(a, b)$. Using the inclusion $(a, b) \subset S$, this basis can be extended to a geometric symplectic basis of S by adding a, b . As $\phi(a) = \phi(b) = 0$, the parity of ϕ is opposite to the parity of $\hat{\phi}$. □

The next observation is an analog of Proposition 3.4. Note that we only require $g \geq 3$ here.

PROPOSITION 5.2. — *Let ϕ be a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on a compact surface S of genus $g \geq 3$ with empty or connected boundary. Then for any two vertices c, d of the graph CG_2^+ there exists a mapping class $\zeta \in \text{Mod}(S)[\phi]$ with $\zeta(c) = d$. In particular, the action of $\text{Mod}(S)[\phi]$ is transitive on the vertices of the graph CG_2^+ .*

Proof. — The proof is very similar to the proof of Proposition 3.4 and will be omitted. □

Consider again a $\mathbb{Z}/r\mathbb{Z}$ -spin structure ϕ on a closed surface Σ_g of genus $g \geq 3$. Let c be a separating simple closed curve on Σ_g which is the boundary of a small neighborhood of a vertex $(a, b) \in CG_2^+$. Then c decomposes Σ_g into a one holed torus T_1 and a surface Σ_{g-1}^1 of genus $g - 1$ with connected boundary. The spin structure restricts to a spin structure on T_1 . If r is even then this spin structure has a parity, and this parity is odd.

Since c is separating, the group $\text{Mod}(\Sigma_{g-1}^1)[\phi] \times \text{Mod}(T_1)[\phi]$ contains a rank two free abelian central subgroup generated by the left Dehn twists T_{c_1}, T_{c_2} about the boundary circles c_1, c_2 of Σ_{g-1}^1, T_1 . Define

$$\Sigma_{g-1,2}^2 = \text{Mod}(\Sigma_{g-1}^1)[\phi] \times \text{Mod}(T_1)[\phi] / \mathbb{Z}$$

where the infinite cyclic subgroup \mathbb{Z} is generated by $T_{c_1} T_{c_2}^{-1}$. Then $\Sigma_{g-1,2}^2$ is isomorphic to the stabilizer in $\text{Mod}(\Sigma_g)[\phi]$ of the curve c . Note that up to isomorphism, the group

$\mathbb{Z}/2\mathbb{Z}$ does not depend on c since by Proposition 5.2, the stabilizers in $\text{Mod}(S_g)[\phi]$ of vertices of CG_2^+ are all conjugate and hence isomorphic.

Observe that the group $\mathbb{Z}/2\mathbb{Z}$ is an infinite cyclic central extension of the product of a finite index subgroup of the mapping class group of a surface $S_{g-1,1}$ of genus $g - 1$ with one puncture and a once punctured torus $T_{1,1}$. Thus it makes sense to talk about its action on isotopy classes of essential curves on the surfaces $S_{g-1,1}$ and $T_{1,1}$. The map $S_{g-1,1} \amalg T_{1,1} \rightarrow S_{g-1,1} \amalg T_{1,1}$ which contracts each boundary component to a puncture defines a bijection on such isotopy classes.

The following observation is the analog of Proposition 4.1.

PROPOSITION 5.3. — *Let ϕ be a $\mathbb{Z}/r\mathbb{Z}$ -spin structure on a closed surface S_g of genus $g \geq 4$. There is a commutative diagram*

$$(5.1) \quad \begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\iota_1} & \mathbb{Z}/2\mathbb{Z} \rtimes A \\ & \searrow \iota_2 & \downarrow \rho \\ & & \text{Mod}(S_g)[\phi] \end{array}$$

where the homomorphisms ι_1, ι_2 are inclusions, and the homomorphism ρ is surjective. The subgroup A of $\mathbb{Z}/2\mathbb{Z} \rtimes A$ is the stabilizer in $\mathbb{Z}/2\mathbb{Z}$ of a separating simple closed curve d on S_{g-1} which is defined by a vertex of the graph CG_2^+ . The curve d decomposes S_{g-1} into a one-holed torus and a surface of genus $g - 2$ with two boundary components. The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}/2\mathbb{Z} \rtimes A$ by exchanging the two factors, and it acts as an automorphism on A .

Proof. — Fix a pair of vertices of the graph CG_2^+ which are connected by an edge. These two vertices then determine a pair of disjoint separating simple closed curves c, d on S_g which cut from S_g a one-holed torus each. These tori are disjoint. Let $\mathbb{Z}/2\mathbb{Z}_c, \mathbb{Z}/2\mathbb{Z}_d \subset \text{Mod}(S_g)[\phi]$ be the stabilizers of c, d in the spin mapping class group of S_g . By Corollary 3.6, these groups are naturally isomorphic to the group $\mathbb{Z}/2\mathbb{Z}$, and they intersect in the index two subgroup $A = \mathbb{Z}/2\mathbb{Z}_c \cap \mathbb{Z}/2\mathbb{Z}_d$ of the stabilizer of $c = d$ in $\text{Mod}(S_g)[\phi]$ consisting of all elements which preserve both c, d individually. The full stabilizer of $c = d$ in $\text{Mod}(S_g)[\phi]$ is a $\mathbb{Z}/2\mathbb{Z}$ extension of A , where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts as involution on A exchanging c and d . This involution extends to an involution of $\mathbb{Z}/2\mathbb{Z}_c \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z}_d$ exchanging the two subgroups $\mathbb{Z}/2\mathbb{Z}_c, \mathbb{Z}/2\mathbb{Z}_d$.

By the universal property of free amalgamated products, there is a homomorphism

$$\rho: \mathbb{Z}/2\mathbb{Z}_c \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z}_d \rightarrow \text{Mod}(S_g)[\phi].$$

All we need to show is that ρ is surjective, that is, that $\rho(\mathbb{Z}/2\mathbb{Z}_c \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z}_d) = \text{Mod}(S_g)[\phi]$.

As $\text{Mod}(S_g)[\phi]$ acts transitively on the vertices of the graph CG_2^+ , for this it suffices to show that its subgroup $\rho(\mathbb{Z}/2\mathbb{Z}_c \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z}_d)$ acts transitively on the vertices of CG_2^+ as well. Namely, by construction, the stabilizer of the vertex c of CG_2^+ in $\rho(\mathbb{Z}/2\mathbb{Z}_c \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z}_d)$ coincides with its stabilizer in $\text{Mod}(S_g)[\phi]$. As $\rho(\mathbb{Z}/2\mathbb{Z}_c \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z}_d)$ is a subgroup of $\text{Mod}(S_g)[\phi]$, this then implies equality.

To show transitivity of the action of $\rho(\mathbb{Z}/2\mathbb{Z}_c \rtimes A \rtimes \mathbb{Z}/2\mathbb{Z}_d)$ on the vertices of CG_2^+ let $v \in CG_2^+$ be any vertex. By Proposition A.3, the graph CG_2^+ is connected and hence we can find

an edge path $(c_i) \subset CG_2^+$ connecting $c_0 = c$ to $c_k = v$. We also may assume that $c_1 = d$.

By Proposition 5.2, there exists an element $\rho_1 \in \rho(\Gamma)$ such that $\rho_1(c_0) = c_2$. Then the stabilizer of c_2 in $\text{Mod}(S_g)[\phi]$ equals $\rho_1 \circ \rho_1^{-1}$ and hence it is contained in $\rho(\Gamma)$. Thus we can apply Corollary 3.6 to $\rho_1 \circ \rho_1^{-1}$ and find an element $\rho_2 \in \rho(\Gamma)$ which maps c_1 to c_3 . Proceeding inductively and using the fact that c is conjugate to d in $\rho(\Gamma)$ by the generator of the subgroup $\mathbb{Z}/2\mathbb{Z}$, this completes the proof of the Proposition 5.3. \square

For a surface S of genus $g \geq 3$ consider the following system U_g of $3g - 2$ simple closed curve on S .

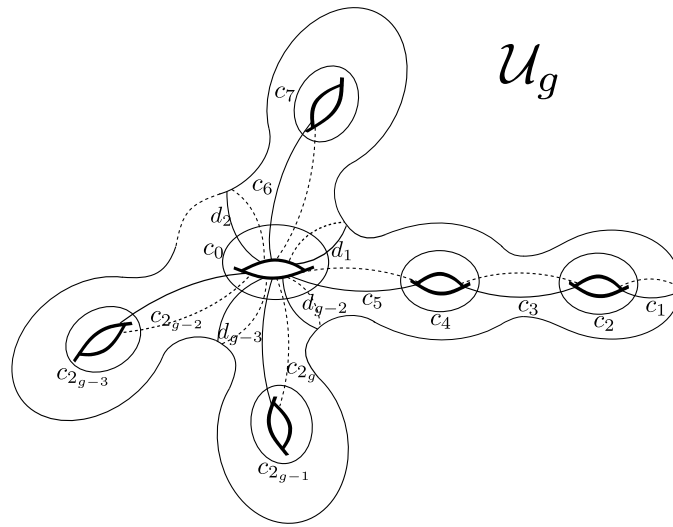


Figure 5.1.

Note that for $g = 3$, the system S_g is just a chain of 7 curves which are invariant under a hyperelliptic involution. It follows from Lemma 4.5 and Lemma 5.1 that the Dehn twists about these curves preserve an even $\mathbb{Z}/2\mathbb{Z}$ -spin structure on S_g .

We use Lemma 4.3 and Proposition 4.1 to show

PROPOSITION 5.4. — *Let ϕ be an even $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a surface S_g of genus $g \geq 4$. Then the group $\text{Mod}(S_g)[\phi]$ is generated by the Dehn twists about the curves from the curve system U_g .*

Proof. — We observed above that the subgroup of $\text{Mod}(S_g)$ generated by the Dehn twist about the curves from the curve system U_g is a subgroup of $\text{Mod}(S_g)[\phi]$. We have to show that it coincides with $\text{Mod}(S_g)[\phi]$.

To this end we proceed by induction on the genus, beginning with genus 4. Let a be the separating simple closed curve which intersects c_3 in two points and is disjoint from the remaining curves from the system U_4 . It decomposes U_4 into a one holed torus T_1 containing the curves c_1, c_2 , and a surface S_3 of genus 3 with connected boundary which contains the curve system S_3 . As we are looking at a $\mathbb{Z}/2\mathbb{Z}$ -spin structure we know that the pair (c_1, c_2) and hence the curve a defines a vertex in

CG_2^+ . The spin structure ϕ induces a spin structure on Σ_3^1 and Σ_1^1 , again denoted by ϕ . It also induces a spin structure on the closed surface Σ_3 of genus 3 obtained from Σ_3^1 by capping off the boundary, again denoted by ϕ .

It is well known that the mapping class group of one holed tori is generated by a pair of Dehn twists about simple closed curves which intersect in a single point. Thus we have $\text{Mod}(\Sigma_1^1) = \text{Stab}(a)$.

On the other hand, by Proposition 4.6, the Dehn twists about the curves from the system S_3 generate the spin mapping class group $\text{Mod}(\Sigma_3)[\phi]$ of Σ_3 . Thus the projection of $\text{Mod}(\Sigma_3^1)[\phi]$ to $\text{Mod}(\Sigma_3)[\phi]$ is surjective.

To apply Proposition 5.3 we have to show that the point pushing group of $\text{Mod}(\Sigma_{3,1})[\phi]$ is contained in the projection of $\text{Stab}(a)$. We use once more Lemma 4.3 to this end.

Consider the curves c_0, c_7, c_6, d_1, c_5 which define a curve system on the surface $\Sigma_{3,1}$ whose curve diagram is the Dynkin diagram D_5 . By [Mat00, Theorem 1.5], there exists an explicit word in the Dehn twists about these curves which defines the product $T_{a_4}^3 T_{a_4}$ where a_4 is a simple closed curve in Σ_3^1 which intersects c_4 in a single point and is disjoint from all other curves and where a_4 is the simple closed curve which bounds together with a_4 a once punctured annulus in Σ_3^1 .

On the other hand, the chain relation [FM12] yields that $T_{a_4} T_{a_4} = (T_{c_1} T_{c_2} T_{c_3})^4$. Since T_{a_4}, T_{a_4} commute we deduce that

$$(T_{a_4})^{-2} T_{a_4}^{-2} T_{a_4}^3 T_{a_4} = T_{a_4} T_{a_4}^{-1}.$$

As a consequence, the group $\text{Stab}(a)$ contains the point pushing map $T_{a_4} T_{a_4}^{-1}$ about the based loop α in Σ_3 which is homotopic to the common projection of a_4, a_4 . Lemma 4.3 now shows that $\text{Stab}(a)$ contains indeed the point pushing group of $\text{Mod}(\Sigma_{3,1})[\phi]$.

By Proposition 5.3, we are left with finding an element ψ which maps a to a curve disjoint from a . However, the curve system U_4 contains a subsystem consisting of the curves c_i ($i = 0, \dots, 7$). The Dehn twists about these curves are well known to generate the stabilizer of a Weierstrass point in the *hyperelliptic mapping class group*, that is, the subgroup of the mapping class group which commutes with a hyperelliptic involution ([FM12] and compare Lemma 4.4). This group is isomorphic to the quotient of the Artin braid group in $2g + 1$ strands by its center, and it contains an element ψ which maps a to a disjoint curve, e.g. the boundary of a small neighborhood of $c_0 - c_5$. The proposition for $g = 4$ now follows from Proposition 5.3.

By induction, let us now assume that the Proposition 5.4 is known for some $g - 1 \geq 4$. Consider the curve system U_g on a surface of genus g . Using the labeling from Figure 5.1, let a_7 be the simple closed curve on Σ_g which intersects the curve c_7 in a single point and is disjoint from any other curve from U_g . We know that $\phi(a_7) = 1$. We aim at showing that $\text{Stab}(a_7) = \text{Mod}(\Sigma_g)[\phi] \text{Stab}(a_7)$.

To this end cut Σ_g open along a_7 . The resulting surface is a surface Σ_{g-1}^2 of genus $g - 1$ with two boundary components. Replace these two boundary components by punctures and let $\Sigma_{g-1,2}$ be the resulting twice punctured surface. As before, the spin structure ϕ descends to a spin structure, again denoted by ϕ , on the surface Σ_{g-1}

obtained from $\Sigma_{g-1,2}$ by closing the punctures, and to a spin structure on $\Sigma_{g-1,2}$. The curve system U_g descends to the curve system U_{g-1} on Σ_{g-1} .

By the induction hypothesis, the Dehn twists about the curves from the curve system U_{g-1} generate the spin mapping class group $\text{Mod}(\Sigma_{g-1})[\phi]$. On the other hand, we can apply Lemma 4.3 to each of the two punctures of $\Sigma_{g-1,2}$ as each of these two punctures is contained in a once punctured annulus bounded by two curves from the restriction of U_g to $\Sigma_{g-1,2}$. We conclude that the point pushing maps about these punctures are contained in the group $\text{Stab}(a_7)$. As a consequence, the group $\text{Stab}(a_7)$ surjects onto $\text{Mod}(\Sigma_{g-1,2})[\phi]$.

To summarize, we showed so far that $\text{Stab}(a_7)$ surjects onto $\text{Stab}(a_7)[\phi]/\mathbb{Z}$ where \mathbb{Z} is the intersection of $\text{Mod}(\Sigma_g)[\phi]$ with the infinite cyclic group of Dehn twists about a_7 . Thus to show that $\text{Stab}(a_7) = \text{Mod}(\Sigma_g)[\phi] \cdot \text{Stab}(a_7)$ it suffices to show that $\text{Stab}(a_7)$ contains the square $T_{a_7}^2$ of the Dehn twist about a_7 . We also have to find as well as an involution which exchanges a_7 with a simple closed curve disjoint from a_7 .

To find an involution as required, consider first the case $g = 4$. The curve system U_4 contains a curve system $E_6 \subset U_4$ consisting of the curves $c_7, c_6, c_0, d_1, c_5, c_4$. By [Mat00, Theorem 1.4], there exists an explicit word $c(E_6)$ in the Dehn twists about the curves from the system E_6 , the image of the so-called *Garside element* of the Artin group of type E_6 , which acts as a reflection on the curve diagram of E_6 exchanging the curves c_7 and c_4 . Then this reflection maps a_7 to a disjoint curve a_7 and hence it has the desired properties.

This reasoning extends to any $g \geq 5$. Namely, the element $c(E_6)$, viewed as an element of the mapping class group of Σ_g , still acts as an involution on Σ_g which maps a_7 to a disjoint curve a_7 and preserves the subsurface of Σ_g filled by the curves $c_7, c_6, c_0, d_1, c_5, c_4$. Thus there always exists an involution which maps a_7 to a disjoint curve a_7 .

For an application of Proposition 4.1, we are left with showing that the square of the Dehn twist about a_7 is contained in $\text{Stab}(a_7)$. By the above discussion, we know that $\text{Stab}(a_7)$ surjects onto $\text{Mod}(\Sigma_{g-1,2})[\phi]$. In particular, $\text{Stab}(a_7)$ contains $T_{a_7}^2$, viewed as an element of $\text{Stab}(a_7) \subset \text{Mod}(\Sigma_g)$. Since a_7 is the image of a_7 under an involution contained in $\text{Stab}(a_7)$, it follows that $T_{a_7}^2 \in \text{Stab}(a_7)$.

To summarize, we showed that $\text{Stab}(a_7) = \text{Mod}(\Sigma_g)[\phi] \cdot \text{Stab}(a_7)$, furthermore $\text{Stab}(a_7)$ contains an involution which exchanges a_7 and a_7 . Proposition 4.1 now shows that $\text{Stab}(a_7) = \text{Mod}(\Sigma_g)[\phi]$. This completes the proof of the Proposition 5.4. \square

We use Proposition 4.6 as the base case for the proof of the second part of Theorem 1.3 from the introduction. The curve system V_g is defined as in the Theorem 1.3. Note that we have $V_3 = U_3$.

THEOREM 5.5. — *Let ϕ be an even $\mathbb{Z}/2\mathbb{Z}$ -spin structure on a surface Σ_g of genus $g \geq 4$. Then the group $\text{Mod}(\Sigma_g)[\phi]$ is generated by the Dehn twists about the curves from the curve system V_g .*

Proof. — The curve system V_g is obtained from the curve system U_g by deleting the curves d_2, \dots, d_{g-2} .

Let \mathcal{H} be the subgroup of $\text{Mod}(\Sigma_g)[\phi]$ generated by the Dehn twists about the curves from the curve system V_g . By Proposition 4.6, it suffices to show that the Dehn twists T_{d_i} for $i = 2, \dots, g-2$ are contained in \mathcal{H} .

To see that $T_{d_{g-2}} \in \mathcal{H}$, note that d_{g-2} is the image of d_1 under the hyperelliptic involution of the surface of genus 3 with connected boundary filled by the curves $d_1, d_{g-2}, c_0, c_1, c_2, c_3, c_4$.

Consider the surface S filled by $c_1, \dots, c_6, d_1, d_2, d_{g-2}$. This is a surface of genus 4 with connected boundary. The union of the system V_g with the curve d_{g-2} intersects S in a curve system of type U_4 . By Proposition 5.4 and what we have proved so far, the stabilizer of the surface S in the group \mathcal{H} surjects onto the spin mapping class group of the surface obtained from S by capping off the boundary. In particular, if we denote by e_1, e_3 the nonseparating simple closed curves which intersect c_4 in a single point, do not intersect any other curve and form a bounding pair, then $T_{e_1} T_{e_2}^{-1} \in \mathcal{H}$.

Now by [Mat00, Theorem 1.4], the stabilizer in \mathcal{H} of the surface of genus 3 with two boundary components obtained from S by removing the one-holed torus T filled by c_1, c_2 contains a half-twist which exchanges the two boundary components of the surface. Let S' be the surface obtained from $S - T$ by replacing two boundary components by punctures. Lemma 4.3, applied to the Dehn twists about the curves e_1, e_2 enclosing the boundary component of T , shows that the subgroup of the mapping class group of S' which is the point pushing group of the puncture corresponding to the boundary of T is contained in the projection of the stabilizer of $S - T$ in \mathcal{H} . But then the same holds true for the point pushing group of the second puncture of S' .

This shows that we have $T_{d_2} T_{d_{g-2}}^{-1} \in \mathcal{H}$. As $T_{d_{g-2}} \in \mathcal{H}$, we conclude that the same holds true for T_{d_2} . To generate the remaining twists about the curves d_i we argue as in the proof of Theorem 4.7, using the Dehn twists T_{d_1} and T_{d_2} . \square

6. Generating the $\mathbb{Z}/4\mathbb{Z}$ -spin mapping class group in genus 3

The goal of this section is to prove Theorem 1.4 from the introduction. Our strategy is similar to the strategy used in Section 4. We first introduce one more graph of curves which will be useful to this end.

Consider an odd $\mathbb{Z}/2\mathbb{Z}$ -spin structure ϕ on a surface Σ_3 of genus 3. A separating simple closed curve a on Σ_3 decomposes Σ_3 into a one-holed torus T and a surface Σ_2^1 of genus 2 with connected boundary. By homological coherence ([Sal19, Proposition 3.15]), we have $\phi(a) = 1$. In particular, ϕ induces a spin structure on the surface Σ_2^1 which has a parity. Define a to be *odd* if this parity is odd. Note that a vertex of the graph CG_2^+ defined in the appendix and used in Section 5 defines a separating simple closed curve which is *even*, that is, it is not odd.

Let OS be the graph whose vertices are odd separating simple closed curves on (Σ_3, ϕ) and where two such curves are connected by an edge if they are disjoint. Let \mathcal{H} be a $\mathbb{Z}/4\mathbb{Z}$ -spin structure on Σ_3 whose $\mathbb{Z}/2\mathbb{Z}$ -reduction equals ϕ . The stabilizer $\text{Mod}(\Sigma_3)[\phi]$ and its subgroup $\text{Mod}(\Sigma_3)[\mathcal{H}]$ act on OS as a group of simplicial automorphisms. The following observation is similar to Proposition 3.4. It uses some special properties of $\mathbb{Z}/4\mathbb{Z}$ -spin structures.

LEMMA 6.1. —

- (1) *The group $\text{Mod}(\Sigma_3)[\pm 1]$ acts transitively on the vertices of OS .*
- (2) *Let $a \in OS$ be any vertex. Then the stabilizer of a in $\text{Mod}(\Sigma_3)[\pm 1]$ acts transitively on the edges of OS issuing from a .*

Proof. — A vertex a of OS decomposes Σ_3 into a one-holed torus T and a surface $\Sigma_3 - T$ of genus 2 with connected boundary and odd spin structure. Since the parity of the spin structure of ϕ on Σ_3 is odd, the torus T contains a simple closed curve c with $\phi(c) = 1$ and hence $\langle c \rangle = \pm 1$. Via perhaps changing the orientation of c we may assume that $\langle c \rangle = 1$, furthermore there is a simple closed curve d in T which intersects c in a single point and satisfies $\langle d \rangle = 0$.

By homological coherence ([Sal19, Proposition 3.15]), if we orient a as the oriented boundary of the surface $V = \Sigma_3 - T$ then we have $\langle a \rangle = 1$. Since the spin structure induced on V is odd, a geometric symplectic basis for V consists of simple closed curves a_1, b_1, a_2, b_2 with $\phi(a_1) = 1$ and hence $\langle a_1 \rangle = \pm 1$ (up to ordering). A tubular neighborhood T' of $a_1 \cup b_1$ is an embedded bordered torus in V . Choose an orientation for a_1 so that $\langle a_1 \rangle = 1$. After perhaps replacing b_1 by its image under a multiple of a Dehn twist about a_1 we may assume that $\langle b_1 \rangle = 0$.

Consider the pair of curves a_2, b_2 . Since the spin structure on V is odd, we have $\phi(a_2) = \phi(b_2) = 0$ and hence $\langle a_2 \rangle, \langle b_2 \rangle \in \{0, 2\}$. Our goal is to modify a_2, b_2 so that $\langle \cdot \rangle$ vanishes on the modified curves. Thus assume without loss of generality that $\langle a_2 \rangle = 2$. Connect a_2 to the boundary curve a of V by an embedded arc ϵ which is disjoint from T' and b_2 , and connect b_2 to the boundary δ of T' by an embedded arc η which is disjoint from ϵ and a_2 . Since $\langle a \rangle = 1$ for the orientation as a boundary curve of V , we obtain that $\langle a_2 + \epsilon a \rangle = 0$, furthermore this curve is disjoint from T' and intersects b_2 in a single point. Replace a_2 by $a_2 + \epsilon a$. Similarly, if $\langle b_2 \rangle = 2$ then we replace b_2 by $b_2 + \eta \delta$. This process yields a geometric symplectic basis for Σ_3 consisting of simple closed curves disjoint from a .

Given any other odd separating curve a' on Σ_3 we can find in the same way a geometric symplectic basis for Σ_3 consisting of curves disjoint from a' . Then there is a mapping class which maps a to a' and identifies the geometric symplectic bases in such a way that the values of $\langle \cdot \rangle$ on these curves match up. By the result of Humphries and Johnson [HJ89], this implies that this mapping class is contained in $\text{Mod}(\Sigma_3)[\pm 1]$. In other words, there is an element of $\text{Mod}(\Sigma_3)[\pm 1]$ which maps a to a' . This shows the part (1) of the Lemma 6.1.

The proof of the part (2) of the lemma is completely analogous but easier and will be omitted. \square

LEMMA 6.2. — *The graph OS is connected.*

Proof. — Consider the curve system \mathcal{C}_3 on the surface Σ_3 . Using the labels for the curves shown in Figure 4.1, there is an odd separating simple closed curve a which intersects the curve c_2 in two points and is disjoint from the remaining curves from the system \mathcal{C}_3 . Using the Putman trick, Theorem 4.7 and the first part of Lemma 6.1, all we need to show is that the curve a can be connected to $T_{c_2}(a)$ by an edge path in OS .

However, the curve a which intersects the curve c_4 in two points and is disjoint from the remaining curves from the system \mathcal{C}_3 is separating and odd, and it is disjoint from both a and $T_{c_2}(a)$. Thus $a, a, T_{c_2}(a)$ is an edge path in \mathcal{OS} which connects a to $T_{c_2}(a)$. \square

Using the labels from Figure 1.2 from the introduction, let d be the separating simple closed curve on Σ_3 which intersects the curve c_2 in two points and is disjoint from the remaining curves from the system E_6 . We show

LEMMA 6.3. — *The subgroup Γ of $\text{Mod}(\Sigma_3)$ which is generated by the Dehn twists about the curves from the curve system E_6 equals the stabilizer $\text{Mod}(\Sigma_3)[\Gamma]$ of an odd $\mathbb{Z}/4\mathbb{Z}$ -spin structure σ on Σ_3 if and only if its intersection with $\text{Stab}(d)$ coincides with $\text{Stab}(d) \cap \text{Mod}(\Sigma_3)[\Gamma]$.*

Proof. — Since Γ is a subgroup of $\text{Mod}(\Sigma_3)[\Gamma]$, the condition is clearly necessary, so we have to show sufficiency. Thus assume that $\text{Stab}(d) = \text{Mod}(\Sigma_3)[\Gamma] \cap \text{Stab}(d)$.

Consider again the graph \mathcal{OS} . Lemma 6.2 shows that \mathcal{OS} is connected. Moreover, by Lemma 6.1, the group $\text{Mod}(\Sigma_3)[\Gamma]$ acts transitively on the directed edges of \mathcal{OS} as a group of simplicial automorphisms. The curve d is odd and hence a vertex of \mathcal{OS} .

By [Mat00, Theorem 1.4], the group Γ contains an involution which induces a reflection in the curve diagram of the curve system E_6 at the edge connecting the vertices c_0 and c_3 . It maps the simple closed curve d to the separating simple closed curve d which intersects c_4 in two points and is disjoint from all other curves from the system. Since d is odd, the same is true for d .

We use this as follows. Let e be any vertex of \mathcal{OS} and let $d = d_0, d_1, d_2, \dots, d_m = e$ be an edge path in \mathcal{OS} which connects d to e . We may assume that $d_1 = d$. Since there exists an element of Γ which maps d to d , the stabilizer of d in Γ is conjugate to the stabilizer of d and hence by our assumption, it coincides with the stabilizer of d in $\text{Mod}(\Sigma_3)[\Gamma]$. In particular, by the part (2) of Lemma 6.1, there exists an element of Γ which fixes d and maps d_0 to d_2 . Arguing inductively as in the proof of Proposition 4.1, we conclude that Γ acts transitively on the odd separating curves in Σ_3 . As Γ is a subgroup of $\text{Mod}(\Sigma_3)[\Gamma]$ and furthermore the stabilizer of a vertex in \mathcal{OS} coincides with its stabilizer in $\text{Mod}(\Sigma_3)[\Gamma]$, it has to coincide with $\text{Mod}(\Sigma_3)[\Gamma]$. The Lemma 6.3 follows. \square

Our final goal is to show that the group Γ fulfills the assumption in Lemma 6.3, which completes the proof of Theorem 1.4. We proceed in two steps.

Let a_1, a_5 be the nonseparating simple closed curves on Σ_3 which intersect c_1, c_5 in a single point and are disjoint from the remaining curves from the system E_6 . We have $(a_j) = \pm 1$, in particular, by [Sal19, Lemma 3.13], the intersection of $\text{Mod}(\Sigma_3)[\Gamma]$ with the infinite cyclic group of Dehn twists about the curve a_j is generated by $T_{a_j}^4$.

LEMMA 6.4. — *For $j = 1, 5$, the group Γ contains $T_{a_j}^4$.*

Proof. — Consider the subsystem D_5^j ($j = 1, 5$) obtained from the curve system E_6 by removing the curve c_j . By [Mat00, Theorem 1.3(d)], the mapping class $T_{a_j}^4$ can be represented as an explicit word in the Dehn twists about the curves from this curve system. Thus we have $T_{a_j}^4 \in \Gamma$. \square

Lemma 6.4 is used in the proof of the final step towards Theorem 1.4.

LEMMA 6.5. — *The stabilizer in $\text{Mod}(\Sigma_3)$ of the curve d coincides with the stabilizer of d in $\text{Mod}(\Sigma_3)[\mathbb{Z}]$.*

Proof. — Let T be the one-holed torus component of $\Sigma_3 - d$. The stabilizer $\text{Stab}(d)[\mathbb{Z}]$ of d in $\text{Mod}(\Sigma_3)[\mathbb{Z}]$ is the quotient of the product of two subgroups G_1, G_2 by an infinite cyclic central subgroup. The group G_1 is the group of all isotopy classes of diffeomorphisms of Σ_3 which fix the bordered surface $S = \Sigma_3 - T$ pointwise and preserve the spin structure σ . It is isomorphic to the subgroup of the mapping class group of a one-holed torus which preserves the spin structure σ . The group G_2 is the group of all isotopy classes of diffeomorphisms of Σ_3 which fix T pointwise and preserve the spin structure σ . The center of $\text{Stab}(d)[\mathbb{Z}]$ is generated by a Dehn twist T_d about d .

Consider the curve system $A_4 \subset E_6$ which consists of the curves c_0, c_3, c_4, c_5 . It is contained in the subsurface $\Sigma_2^1 = \Sigma_3 - T$ of Σ_3 of genus 2 which is bounded by d . The Dehn twists about these curves generate a subgroup $A(A_4)$ of G_2 which is isomorphic to the braid group in five strands (see [FM12] or [Mat00] for the last statement). By [Mat00, Theorem 1.4], the Dehn twist T_d can be represented as an explicit word in the Dehn twists about the curves from the curve system $A(A_4)$. In particular, we have $T_d \in A(A_4)$.

Let $\Sigma_{2,1}$ be the surface obtained from $\Sigma_2^1 = \Sigma_3 - T$ by replacing the boundary component by a puncture, and let Σ_2 be obtained from $\Sigma_{2,1}$ by forgetting the puncture. Let ϕ be the $\mathbb{Z}/2\mathbb{Z}$ -reduction of the spin structure σ . The spin structure ϕ induces an odd spin structure on $\Sigma_{2,1}$ and Σ_2 , again denoted by ϕ . By Lemma 4.4, the subgroup $A(A_4)$ of G_1 surjects onto the spin mapping class group $\text{Mod}(\Sigma_2)$. Consequently the restriction of the puncture forgetful homomorphism $G_2 \rightarrow \text{Mod}(\Sigma_2)[\phi]$ to G_2 is surjective.

By homological coherence, if we orient d as the oriented boundary of the surface $\Sigma_3 - T$, then we have $\langle d \rangle = 1$. Thus by Lemma 3.1, the intersection of the point pushing group $\pi_1(\Sigma_2)$ with the stabilizer of σ in $\text{Mod}(\Sigma_{2,1})$ is the preimage of the sublattice $\langle \sigma \rangle$ of $H_1(\Sigma_2, \mathbb{Z})$ generated by squares of primitive homology classes of oriented simple closed curves under the natural homomorphism $\pi_1(\Sigma_2) \rightarrow H_1(\Sigma_2, \mathbb{Z})$. Or, equivalently, it equals the kernel of the surjective homomorphism $\pi_1(\Sigma_2) \rightarrow H_1(\Sigma_2, \mathbb{Z}/2\mathbb{Z})$. In particular, $\text{Mod}(\Sigma_{2,1})[\mathbb{Z}] \cap \pi_1(\Sigma_2)$ contains the commutator subgroup of $\pi_1(\Sigma_2)$.

We claim first that the square of the point pushing map along a simple closed curve α with $\langle \alpha \rangle = \pm 1$ is contained in $\text{Mod}(\Sigma_{2,1})[\mathbb{Z}]$. To this end note that as $\langle \alpha \rangle = \pm 1$ if and only if we have $\phi(\alpha) = 1$ where ϕ is the $\mathbb{Z}/2\mathbb{Z}$ -reduction of σ , the group $\text{Mod}(\Sigma_2)[\phi]$ and hence $\text{Mod}(\Sigma_{2,1})[\mathbb{Z}]$ acts transitively on these curves. Thus by equivariance, it suffices to verify this claim for a single such curve.

Consider again the simple closed curve $a_5 \subset \Sigma_{2,1}$ with $\langle a_5 \rangle = \pm 1$ which intersects c_5 in a single point and is disjoint from all other curves from the curve system E_6 . Let a be the simple closed curve which bounds with a_5 and the boundary circle C of $\Sigma_{2,1}$ a pair of pants, that is, a_5 and a bound a holed annulus in Σ_2^1 . By the chain relation in the mapping class group (see [FM12]), we have

$$(T_{c_0} T_{c_3} T_{c_4})^6 = T_{a_5} T_a = \zeta.$$

On the other hand, Lemma 6.4 shows that $T_{a_5}^4 \in \mathcal{G}$. As T_{a_5} and T_a commute, we have $T_{a_5}^{-4} \zeta^2 = T_{a_5}^{-2} T_a^2$, and this is just the square of the point pushing transformation (via replacing the boundary circle C by a puncture) along a_5 . Thus the square of the point pushing transformation about a_5 is contained in \mathcal{G} , which is what we wanted to show.

Now the sublattice $H_1(\Sigma_2, \mathbb{Z})$ is additively generated by elements of the form $2b$ where b is an oriented simple closed curve with $\phi(b) = 1$ and hence we conclude that $\pi_1(\Sigma_2)$ surjects onto \mathbb{Z} .

We are left with showing that the point pushing map along any element in the commutator subgroup of $\pi_1(\Sigma_2)$ is contained in \mathcal{G} . The commutator subgroup of $\pi_1(\Sigma_2)$ is generated by separating simple closed curves. Note that $\text{Mod}(\Sigma_2)[\phi]$ acts transitively on separating simple closed curves in Σ_2 . Namely, as the parity of ϕ is odd, the formula (2.2) for the Arf invariant shows that any separating simple closed curve c on Σ_2 decomposes Σ_2 into two one holed tori T_1, T_2 such that up to exchanging T_1 and T_2 , there is a geometric symplectic basis α_1, β_1 for T_1 with $\phi(\alpha_1) = 1, \phi(\beta_1) = 0$, and a geometric symplectic basis α_2, β_2 for T_2 with $\phi(\alpha_2) = \phi(\beta_2) = 0$. Then transitivity of the action of $\text{Mod}(\Sigma_2)[\phi]$ on separating simple closed curves follows once again from [HJ89]. Thus it suffices to show the following: there exists a separating simple closed curve e in Σ_2 such that the point pushing map along e in Σ_2 is contained in \mathcal{G} .

Now by [Mat00, Theorem 1.4], the Dehn twist about the separating simple closed curve d which intersects c_4 in two points and is disjoint from the remaining curves from E_6 is contained in \mathcal{G} . This separating curve is odd in the sense described above. The second separating curve which bounds together with the boundary circle C and d a pair of pants is the boundary of a tubular neighborhood of $c_0 \cup c_1$. As the Dehn twists about c_0, c_1 are contained in \mathcal{G} , the same holds true for the Dehn twist about that curve. We conclude that the point pushing maps about separating simple closed curves is contained in \mathcal{G} .

To summarize, the quotient of G_2 by the infinite cyclic group of Dehn twists about the boundary curve d contains a generating set for the point pushing subgroup of G_2/\mathbb{Z} and hence it contains this point pushing subgroup. As G_2 surjects onto the quotient G_2/\mathbb{Z} by the point pushing subgroup, we conclude that \mathcal{G} surjects onto G_2/\mathbb{Z} . But \mathcal{G} contains the infinite cyclic center of G_2 and hence $\mathcal{G} = G_2$.

To complete the proof of the Lemma 6.5 we are left with showing that the subgroup G_1 of $\text{Mod}(\Sigma_3)[\]$ is contained in \mathcal{G} . But conjugation by the involution in \mathcal{G} which acts as an involution on the curve diagram of the curve system E_6 and exchanges c_1 and c_5 and c_2 and c_4 maps G_1 to a subgroup of G_2 and hence to a subgroup of \mathcal{G} . Thus $G_1 \subset \mathcal{G}$ as well and we conclude that indeed, $\text{Stab}(d) = \text{Mod}(\Sigma_3)[\] \text{Stab}(d)$. \square

Remark 6.6. — Theorem 1.4 classifies connected components of the preimage in the Teichmüller space of abelian differentials of the odd component of the stratum of abelian differentials on a surface Σ_3 of genus 3 with a single zero. Those components correspond precisely to odd $\mathbb{Z}/4\mathbb{Z}$ -spin structures on Σ_3 .

Remark 6.7. — The results in this article give a general recipe for finding generators of spin mapping class groups. This recipe is motivated by the recent work on compactifications of strata of abelian differentials in [BCG⁺18] and the goal to obtain a topological interpretation of this compactification.

Appendix A. Additional graphs of nonseparating curves with fixed spin value

In this appendix we complement the main result in Section 2 by studying connectedness of some additional geometrically defined graphs related to spin structures. The proofs do not use new ideas. We use the assumptions and notations from Section 2.

We begin with adding more constraints to the graph CG_1^+ . Define a graph CG_1^{++} as follows. Vertices of CG_1^{++} are ordered pairs (c, d) of nonseparating simple closed curves c, d such that $\phi(c) = \pm 1, \phi(d) = 0$ and that c, d intersect in a single point. Then $c \cup d$ fills a one-holed torus $T(c, d) \subset S$. Two such pairs $(c, d), (c', d')$ are connected by an edge if and only if the tori $T(c, d)$ and $T(c', d')$ are disjoint. We use Corollary 2.18 to show

LEMMA A.1. — *For $g \geq 4$ the graph CG_1^{++} is connected.*

Proof. — Let $(a, b), (c, d)$ be two vertices in the graph CG_1^{++} . Then a, c are vertices in the graph CG_1^+ . Connect $a = a_0$ to $c = a_k$ by an edge path (a_i) in CG_1^+ ; this is possible by Corollary 2.18. Our goal is to construct inductively a path $(c_j, d_j) \in CG_1^{++}$ connecting (a, b) to (c, d) which passes through vertices (c_{j_i}, d_{j_i}) with $c_{j_i} = a_i$.

To this end observe that if the curve b is disjoint from a_1 , then we can find a curve \hat{d}_1 which intersects a_1 in a single point and is disjoint from (a, b) . In particular, $a \cup b$ is disjoint from $c_1 \cup \hat{d}_1$.

We can not expect in general that $\phi(\hat{d}_1) = 0$. However, as before, there exists some $k \in \mathbb{Z}$ such that $\phi(T_{a_1}^k(\hat{d}_1)) = 0$. Define $c_1 = a_1$ and $d_1 = T_{c_1}^k(\hat{d}_1)$ and note that d_1 is disjoint from $a \cup b$ and intersects c_1 in a single point. Thus the pair (c_1, d_1) is a vertex in CG_1^{++} which is connected to (a, b) by an edge.

Let us now assume that b is not disjoint from a_1 . Since b intersects a in a single point, it determines a vertex in the nonseparating arc graph $A(A_1, A_2)$ of $S - a$; here A_1, A_2 are the two boundary components of $S - a$ which glue back to a . Denote this arc by b_0 .

Connect b_0 to an arc b disjoint from a_1 by an edge path (b_i) in $A(A_1, A_2)$. This is possible by Lemma 2.15. Cut $S - a$ open along b_0 . The result is a surface of genus $g - 1 \geq 3$ with connected boundary, and $S - (b \cup b_1)$ is a surface of genus $g - 2 \geq 2$ with two boundary components.

A surface of genus at least 2 contains a nonseparating curve u with $\phi(u) = 1$, and in fact it contains a pair $(u, v) \in CG_1^{++}$. In other words, there exists a vertex of CG_1^{++} which is disjoint from a, b, b_1 . Connect (a, b) to (a, b_1) by the edge path $(a, b) \cup (u, v) \cup (a, b_1)$ and proceed by induction. \square

Define a graph D as follows. Vertices are ordered pairs (x, y) where x is a vertex in CG_1^{++} and where y is a disjoint simple closed nonseparating curve with $\phi(y) = 0$. Two such pairs are connected by an edge if they can be realized disjointly. The following observation is a straightforward application of Lemma A.1 and the tools used so far. Its proof will be omitted.

LEMMA A.2. — *For $g \geq 4$ the graph D is connected.*

Define now a graph CG_2^+ as follows. Vertices are pairs (x, y) where x is a nonseparating simple closed curve on S with $\phi(x) = 2$ and where y is a simple closed curve

with $\phi(y) = 0$ intersecting x in a single point. Two such vertices are connected by an edge of length one if and only if they can be realized disjointly.

We use the above constructions to show

PROPOSITION A.3. — *For $g \geq 4$ the graph \mathcal{CG}_2^+ is connected.*

Proof. — Given a pair of disjoint simple closed curves (c, d) with $\phi(c) = \pm 1$ and $\phi(d) = 0$, cut S open along c, d and denote the boundary components of the resulting surface by C_1, C_2, D_1, D_2 . For one of the two choices of C_1, C_2 , say for C_1 , the curve $c +_\epsilon d$ defined by any embedded arc ϵ connecting C_1 to either of D_1, D_2 satisfies $\phi(c +_\epsilon d) = \pm 2$.

As a consequence, to any vertex $(c, d) \in D$ we can associate in a non-deterministic way a vertex in \mathcal{CG}_2^+ by replacing the simple closed curve a with $\phi(a) = \pm 1$ in the pair which defines a vertex of \mathcal{CG}_1^{++} to the simple closed curve component of the pair which defines a vertex in D .

Adjacent vertices may not give rise to disjoint curves, but this issue can be resolved using a path in the nonseparating arc graph. Using the fact that the surface obtained by removing from S a torus and cutting the resulting surface open along a nonseparating simple closed curve has genus at least 2, we find for any two such arcs a disjoint curve e with $\phi(e) = \pm 1$. Connect b to this curve with a disjoint arc. \square

BIBLIOGRAPHY

- [BCG⁺18] Matt Bainbridge, Dawei Chen, Quentin Gendron, Samuel Grushevsky, and Martin Möller, *Compactification of strata of abelian differentials*, *Duke Math. J.* **167** (2018), no. 12, 2347–2416. 1655
- [Cal20] Aaron Calderon, *Connected components of strata of abelian differentials over Teichmüller space*, *Comment. Math. Helv.* **95** (2020), no. 2, 361–420. 1621
- [Cor89] Maurizio Cornalba, *Moduli of curves and theta characteristics*, *Proceedings of the first college on Riemann surfaces held in Trieste, Italy, November 9–December 18, 1987*, World Scientific, 1989, pp. 560–589. 1639
- [CS21] Aaron Calderon and Nick Salter, *Higher spin mapping class groups and strata of abelian differentials over Teichmüller space*, <https://arxiv.org/abs/1906.03515>, 2021. 1621
- [FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, *Princeton Mathematical Series*, vol. 49, Princeton University Press, 2012. 1627, 1632, 1633, 1634, 1635, 1643, 1645, 1649, 1654
- [Hai95] Richard M. Hain, *Torelli groups and geometry of the moduli space of curves*, *Current topics in complex algebraic geometry* (Herbert Clemens et al., eds.), *Mathematical Sciences Research Institute Publications*, vol. 28, Cambridge University Press, 1995, pp. 97–143. 1624
- [Hir02] Susumu Hirose, *On diffeomorphisms over surfaces trivially embedded in the 4-sphere*, *Algebr. Geom. Topol.* **2** (2002), 791–824. 1621, 1623
- [Hir05] ———, *Surfaces in the complex projective plane and their mapping class groups*, *Algebr. Geom. Topol.* **5** (2005), 577–613. 1621, 1623
- [HJ89] Stephen P. Humphries and Dennis Johnson, *A generalization of winding number functions on surfaces*, *Proc. Lond. Math. Soc.* **58** (1989), no. 2, 366–386. 1622, 1624, 1634, 1636, 1637, 1652, 1655

- [KZ03] Maxim Kontsevich and Anton Zorich, *Connected components of the moduli space of Abelian differentials with prescribed singularities*, *Invent. Math.* **153** (2003), no. 3, 631–678. 1621
- [Lei04] Christopher J. Leininger, *On groups generated by two positive multi-twists: Teichmüller curves and Lehmer’s number*, *Geom. Topol.* **8** (2004), 1301–1359. 1620
- [LM14] Eduard Looijenga and Gabriele Mondello, *The fine structure of the moduli space of abelian differentials in genus 3*, *Geom. Dedicata* **169** (2014), 109–128. 1622
- [Mat00] Makoto Matsumoto, *A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities*, *Math. Ann.* **316** (2000), no. 3, 401–418. 1644, 1645, 1649, 1650, 1651, 1653, 1654, 1655
- [MS06] Howard Masur and Saul Schleimer, *The pants complex has only one end. Proceedings of the programme “Spaces of Kleinian groups and hyperbolic 3-manifolds”, Cambridge, UK, July 21–August 15, 2003*, *Spaces of Kleinian groups*, London Mathematical Society Lecture Note Series, vol. 329, Cambridge University Press, 2006, pp. 209–218. 1625
- [Put08] Andrew Putman, *A note on connectivity of certain complexes associated to surfaces*, *Enseign. Math.* **54** (2008), no. 3-4, 287–301. 1625, 1627
- [PV96] Bernard Perron and Jean-Pierre Vannier, *Groupe de monodromie géométrique des singularités simples*, *Math. Ann.* **306** (1996), no. 2, 231–245. 1642
- [Sal19] Nick Salter, *Monodromy and vanishing cycles in toric surfaces*, *Invent. Math.* **216** (2019), no. 1, 153–213. 1621, 1622, 1623, 1624, 1625, 1626, 1633, 1634, 1636, 1637, 1638, 1642, 1645, 1646, 1651, 1652, 1653
- [Wal09] Katharine C. Walker, *Connected components of strata of quadratic differentials over Teichmüller space*, *Geom. Dedicata* **142** (2009), 47–60. 1621
- [Wal10] ———, *Quotient groups of fundamental groups of certain strata of the moduli space of quadratic differentials*, *Geom. Topol.* **14** (2010), no. 2, 1129–1164. 1621
- [Wol10] Scott A. Wolpert, *Families of Riemann surfaces and Weil–Petersson geometry*, *CBMS Regional Conference Series in Mathematics*, vol. 113, American Mathematical Society, 2010. 1639

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