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Yousuke Ohyama, Jean-Pierre Ramis and Jacques Sauloy

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The space of monodromy data for the Jimbo–Sakai family of q-difference equations

Yousuke Ohyama [1], Jean-Pierre Ramis [2] and Jacques Sauloy [3]

To the memory of our friend Hiroshi Umemura

Abstract. — We formulate a geometric Riemann–Hilbert correspondence that applies to the derivation by Jimbo and Sakai of equation q-PVI from "isomonodromy" conditions. This is a step within work in progress towards the application of q-isomonodromy and q-isoStokes to q-Painlevé.

Résumé. — Nous formulons une correspondance de Riemann–Hilbert géométrique qui s'applique à la dérivation par Jimbo et Sakai de l'équation q-PVI à partir de conditions « d'isomonodromie ». C'est une étape d'un travail en cours en vue de l'application de la q-isomonodromie et des q-isoStokes à q-Painlevé.

1. Introduction

1.1. Position of the problem

This paper is a first step towards a formulation of a Riemann–Hilbert correspondence for the q-Painlevé equations, the q-analogs of the classical differential Painlevé equations. In this very first step we limit ourselves to the case of the Jimbo–Sakai q-PVI equation [41] for fixed generic values of the "local parameters".

For the q-Painlevé equations, according to the pioneering work of Hidetaka Sakai [68], we have an exhaustive information *"on the left hand side"* of the q-analog of the Riemann–Hilbert map: the q-analogs of the *Okamoto*

(1) Department of Mathematical Sciences, Tokushima University, 2-1 Minamijyousanjima-cho, Tokushima 770-8506, Japan — ohyama@tokushima-u.ac.jp
(2) Institut de France (Académie des Sciences) and Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université Paul Sabatier (Toulouse 3), 118 route de Narbonne, 31062 Toulouse CEDEX 9, France — ramis.jean-pierre@wanadoo.fr
(3) Toulouse — jacquessauloy@gmail.com

spaces of initial conditions are open rational surfaces obtained by blowing up $\mathbf{P}^2(\mathbf{C})$ in nine points and removing some lines. But *"on the right hand side"* the q-analogs of the *character varieties* with their structure of open cubic surface are unknown. Our initial aim was to fill this gap in the q-PVI case.

We consider, just as Birkhoff did, the family of q-difference systems $\sigma_q X = AX$

$$A = A_0 + \cdots + x^n A_n \in \mathrm{Mat}_n(\mathbf{C}[x]),\qquad A_0, \ldots, A_n \in \mathrm{Mat}_n(\mathbf{C}),\; A_0, A_n \in \mathrm{GL}_n(\mathbf{C}). \tag{1.1}$$

Following Birkhoff [4], we associate to a system (1.1) a matrix M (a variant of Birkhoff connection matrix) representing in some sense (this will be commented in more detail in Subsections 1.3 and 1.4) some kind of q-analog of the monodromy data for differential equations. This map induces *an isomorphism*, the Riemann–Hilbert–Birkhoff correspondence, between the systems modulo rational gauge equivalence on one side and the "matrices of monodromy data" modulo a natural equivalence on the other side.

The Jimbo–Sakai family studied in [41] is associated to a subspace of the space:

$$\left\{A_0 + xA_1 + x^2A_2 \in \mathrm{Mat}_2(\mathbf{C}[x]) \,\middle|\, A_0, A_2 \in \mathrm{GL}_2(\mathbf{C}),\; A_1 \in \mathrm{Mat}_2(\mathbf{C})\right\}.$$

The subspace is restrained by conditions on the "local data", i.e. the conjugacy classes of A_0 and A_2 (actually their spectra, for they are assumed to be semi-simple). Sakai gave a direct description of the space of equations $\sigma_q X = \left(A_0 + xA_1 + x^2A_2\right) X$ as an open rational surface; this is what we consider as the "left hand side" of the Riemann–Hilbert correspondence. The "right hand side" is our space F of "monodromy data" modulo equivalence. We call it the *space of monodromy data* or *q-character variety*. Then we give a first geometric description of F as an algebraic surface. In particular we give an embedding of F into $(\mathbf{P}^1(\mathbf{C}))^4$.

In the second part of our paper we introduce a new tool: we call it *the Mano decomposition* and use it to get a more precise description of the algebraic variety F.

This extremely useful process was inspired to us by the paper [46] of Toshiyuki Mano. The equations that appear in the Jimbo–Sakai family can, in some sense, be *split* into q-hypergeometric components and the corresponding monodromy matrix M can be *split* into the monodromy matrices of these q-hypergeometric components. So, Mano decomposition can be understood as providing a splitting of the *global* monodromy around the four intermediate singularities into *local* monodromies around two pairs of singularities.

Mano decompositions allow us to describe parameterizations (the q-pants parameterizations) of F. They are q-analogs of the classical parametrizations of the Fricke cubic surface (the character variety of PVI) associated to a pant decomposition [33].

We tried to “identify” F among the “classical” surfaces. We only got some partial informations(1) allowing some guesses: in particular F could be a Zariski open subset of a K3 surface.

1.2. Isomonodromy and Painlevé equations

The purpose of this part is to recall briefly some classical results on Painlevé equations and linear representations. It is necessary for the detailed description of some (not so evident) analogies between the differential case and the q-difference case that we will present below.

Theory of the Painlevé differential equations has developed through two very different lines. One is the classification of second order algebraic ordinary differential equations which satisfy the Painlevé property (i.e. the movable singularities are poles). The other one is a deformation theory of linear ordinary differential equations: one asks to move the coefficients of the equation without changing its monodromy data (or more generally its generalized monodromy data). At the very beginning of the XX-th century, P. Painlevé [57] and B. Gambier initiated the first line and R. Fuchs [21] initiated the second. In the present paper everything is in the spirit of this second line.

1.2.1. Representations of the free group of rank 3 into $\mathrm{SL}_2(\mathbf{C})$. Character varieties

We present the character varieties in elementary purely algebraic terms (no differential equations here). At the end of the paragraph we will introduce some topology: fundamental groups of punctured spheres.

We denote $\Gamma_3 := \langle u_0, u_t, u_1\rangle$ the free group of rank 3 generated by the letters(2) u_0, u_t, u_1. It is identified with the free group $\langle u_0, u_t, u_1, u_\infty \mid u_0u_tu_1u_\infty = 1\rangle$ generated by u_0, u_t, u_1, u_∞ up to the relation $u_0u_tu_1u_\infty = 1$.

Let $\rho : \Gamma_3 \to \mathrm{SL}_2(\mathbf{C})$ be a linear representation. We set $M_l := \rho(u_l)$ ($l = 0, t, 1, \infty$). We denote e_l and e_l^{-1} ($l = 0, t, 1, \infty$) the eigenvalues of M_l.

(1) In particular about elliptic fibrations.

(2) The motivation for the indices $0, t, 1, \infty$ will appear at the end of this paragraph.

The representation ρ can be identified with $(M_0, M_t, M_1) \in (\mathrm{SL}_2(\mathbf{C}))^3$. Therefore the set of such representations $\mathrm{Hom}(\Gamma_3, \mathrm{SL}_2(\mathbf{C}))$ modulo the adjoint action of $\mathrm{SL}_2(\mathbf{C})$ can be identified with $(\mathrm{SL}_2(\mathbf{C}))^3/\,\mathrm{SL}_2(\mathbf{C})$ (the set of triples of matrices up to overall conjugation):

$$\mathrm{Hom}(\Gamma_3, \mathrm{SL}_2(\mathbf{C}))/\,\mathrm{SL}_2(\mathbf{C}) = (\mathrm{SL}_2(\mathbf{C}))^3\,/\,\mathrm{SL}_2(\mathbf{C});$$

$(\mathrm{SL}_2(\mathbf{C}))^3$ is a complex affine variety of dimension 9.

To a representation $\rho : \Gamma_3 \to \mathrm{SL}_2(\mathbf{C})$ we associate its seven *Fricke coordinates* (or trace coordinates), the four "parameters":

$$a_l := \mathrm{Tr}\, M_l = e_l + e_l^{-1}, \quad l = 0, t, 1, \infty$$

and the three "variables":

$$X_0 = \mathrm{Tr}\, M_1 M_t, \quad X_t = \mathrm{Tr}\, M_1 M_0, \quad X_1 = \mathrm{Tr}\, M_t M_0.$$

These seven coordinates satisfy the *Fricke relation* $F(X, a) = 0$ (cf. [45]), where:

$$\begin{aligned} F(X,a) &:= F\left((X_0, X_t, X_1); (a_0, a_t, a_1, a_\infty)\right) \\ &:= X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - A_0 X_0 - A_t X_t - A_1 X_1 + A_\infty, \end{aligned} \tag{1.2}$$

with:

$$\begin{aligned} A_i &:= a_i a_\infty + a_j a_k, \quad \text{for } i = 0, t, 1, \\ \text{and} \quad A_\infty &:= a_0 a_t a_1 a_\infty + a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - 4. \end{aligned} \tag{1.3}$$

The seven Fricke coordinates of ρ are clearly invariant by equivalence of representations. Then, using the seven Fricke coordinates, we get an algebraic map from $(\mathrm{SL}_2(\mathbf{C}))^3/\,\mathrm{SL}_2(\mathbf{C})$ to $\mathbf{C}^7$. The image is the six dimensional quartic hypersurface of $\mathbf{C}^7$ defined by the equation $F(X, a) = 0$.

We fix the parameter a and denote $S(a)$ or $S_{A_0, A_t, A_1, A_\infty}$ or(3) $S_{VI}(a)$ the cubic surface of $\mathbf{C}^3$ defined by the equation $F(X, a) = 0$. We call this surface the *character variety* of PVI.

By a theorem of Fricke, Klein and Vogt [25, 45] the equivalence class of an *irreducible* representation is completely determined by its seven Fricke coordinates.

We denote $\bar{S}(a)$ the projective completion(4) of $S(a)$ in $\mathbf{P}^3(\mathbf{C})$. The family $\{\bar{S}(a)\}_{a \in \mathbf{C}^4}$ contains all *smooth* projective cubic surfaces (up to linear transformations). The list of projective cubic surfaces was given by Schläfli [74] over a century ago. For this list we refer to [10, Table 4, p. 255]. There are 20 families of singular projective cubic surfaces. An excellent reference is [36, §3, p. 11)].

(3) For reasons that will appear in the next paragraph.

(4) As an abstract algebraic surface it is a del Pezzo surface of degree 3.

The surface $S(a)$ is *simply connected* [11]. It can be smooth or have singular points according to the values of a. The number of singular points is at most 4. Singular points of $S(a)$ appear from semi-stable representations which are of two kinds:

- Either $M_l = \pm I_2$ that is $\rho(u_l)$ belongs to the center of $\mathrm{SL}_2(\mathbf{C})$ for some $l = 0, t, 1, \infty$, hence $e_l = \pm 1$ and $a_l = \pm 2$. This case is called *the resonant case*.
- Or the representation is reducible. This condition can be translated into an algebraic condition on a, cf. [34], [43, p. 22], [48]; we have:

$$e_0\, e_t^{\pm 1} e_1^{\pm 1} e_\infty^{\pm 1} = 1 \tag{1.4}$$

  for some triple of signs.

An example of a singular cubic surface with 4 singular points is the Cayley cubic [11]. We get it for $(A_0, A_t, A_1, A_\infty) = (0, 0, 0, -4)$ (this is true either if $a = (0, 0, 0, 0)$ or if $a = (\pm 2, \pm 2, \pm 2, \pm 2)$ with product -16):

$$X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - 4 = 0. \tag{1.5}$$

We denote: $F_{X_i} := \frac{\partial F(X,a)}{\partial X_i} = X_j X_k + 2X_i - A_i$. The character variety $S_{VI}(a) = S_{A_0, A_t, A_1, A_\infty}$ is equipped with a "natural" algebraic symplectic form (Poincaré residue):

$$\omega_{VI,a} := \frac{\mathrm{d}X_t \wedge \mathrm{d}X_0}{2i\pi F_{X_1}} = \frac{\mathrm{d}X_1 \wedge \mathrm{d}X_t}{2i\pi F_{X_0}} = \frac{\mathrm{d}X_1 \wedge \mathrm{d}X_t}{2i\pi F_{X_t}} \tag{1.6}$$

We have[(5)] $\mathrm{d}F \wedge \omega_{VI,a} = -\frac{1}{2i\pi} \mathrm{d}X_0 \wedge \mathrm{d}X_t \wedge \mathrm{d}X_1$. The Poisson bracket associated to $-2i\pi\, \omega_{VI,a}$ is the *Goldman bracket* defined by: $\{X_i, X_j\} = F_{X_k}$, and circular permutations.

Let S_4^2 be the four punctured sphere. Its fundamental group $\pi_1(S_4^2)$ is isomorphic to a free group of rank 3: we can choose as generators the homotopy classes of three simple loops turning around three punctures.

Therefore we can apply the preceding results to the study of equivalence classes of representations of $\pi_1(S_4^2)$ into $\mathrm{SL}_2(\mathbf{C})$. It is a purely topological matter and the choice of the punctures is indifferent up to an homeomorphism. But in the following we will need the complex structure: $S^2 = \mathbf{P}^1(\mathbf{C})$. Then, starting from 4 arbitrary punctures, up to a Möbius transformation, we can choose as punctures $0, t, 1, \infty$ for some value of t. This explains our initial notation.

(5) The motivation for the choice of the factor $-\frac{1}{2i\pi}$ will appear in the next paragraph, cf. footnote 14.

For $t \in \mathbf{P}^1(\mathbf{C}) \setminus \{0,1,\infty\}$ we set:

$$\widetilde{\mathrm{Rep}}_t := \mathrm{Hom}(\pi_1(\mathbf{P}^1(\mathbf{C}) \setminus \{0,t,1,\infty\}), \mathrm{SL}_2(\mathbf{C}))/\mathrm{SL}_2(\mathbf{C}).$$

For small changes[6] of t, the group $\pi_1(\mathbf{P}^1(\mathbf{C}) \setminus \{0,t,1,\infty\})$ remains constant, more precisely there exist canonical isomorphisms:

$$\pi_1(\mathbf{P}^1(\mathbf{C}) \setminus \{0,t_1,1,\infty\}) \to \pi_1(\mathbf{P}^1(\mathbf{C}) \setminus \{0,t_2,1,\infty\}),$$

therefore there are canonical isomorphisms $\widetilde{\mathrm{Rep}}_{t_2} \to \widetilde{\mathrm{Rep}}_{t_1}$. Geometrically this says that the space of representations $\widetilde{\mathrm{Rep}} := \{\widetilde{\mathrm{Rep}}_t\}_{t \in \mathbf{P}^1(\mathbf{C}) \setminus \{0,1,\infty\}}$ can be interpreted as "a local system of varieties" parameterized by $t \in \mathbf{P}^1(\mathbf{C}) \setminus \{0,1,\infty\}$: the fibration $\widetilde{\mathrm{Rep}} \to \mathbf{P}^1(\mathbf{C}) \setminus \{0,1,\infty\}$ (whose fiber over t is $\widetilde{\mathrm{Rep}}_t$) has a natural flat Ehresmann connection on it [7].

Remark 1.1. — There are nice relations involving the coordinates of the gradient of F and some determinants (cf. [33, 3.9, p. 10]):

$$F_{X_1}^2 = (X_0X_t + 2X_1 - A_1)^2 = \begin{vmatrix} 2 & -a_0 & -a_1 & X_0 \\ -a_0 & 2 & X_t & -a_\infty \\ -a_1 & X_t & 2 & -a_1 \\ X_0 & -a_\infty & -a_1 & 2 \end{vmatrix} \tag{1.7}$$

and the circular permutations. Each relation is equivalent to $F = 0$.

1.2.2. Isomonodromy and PVI

We recall briefly some basics about the sixth Painlevé equation and its relation with isomonodromic families of linear Fuchsian differential equation. For more details, cf. [11][7].

The sixth Painlevé equation is:

$$\text{(PVI)} \quad \frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right); \tag{1.8}$$

$\alpha, \beta, \gamma, \delta \in \mathbf{C}$ are the parameters.

The generic solution of PVI has essential singularities and/or branch points in the points $0, 1, \infty$. These points are called *fixed singularities*. The other singularities, the *moving singularities* (so called because they depend

[6] More precisely if t remains in an open disc of the 3-punctured sphere $\mathbf{P}^1(\mathbf{C}) \setminus \{0,1,\infty\}$.

[7] We used the excellent presentation of [43].

on the initial conditions) are *poles*: it is the Painlevé property. A solution of PVI can be analytically continued to a meromorphic function on the universal covering of $\mathbf{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}$. For generic values of the integration constants and of the parameters $\alpha,\ \beta,\ \gamma,\ \delta$, it cannot be expressed via elementary or classical transcendental functions[8] . For this reason, Painlevé called these functions: "transcendantes nouvelles" (new transcendental functions).

In modern formulation, solutions of PVI parameterize isomonodromic deformations (in t) of rank two meromorphic connections over the Riemann sphere having simple poles at the 4 points $0,\ t, 1, \infty$.

We consider traceless 2×2 linear differential systems with four fuchsian singularities on the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ (parameterized by a complex variable t):

$$\frac{dY}{dz} = A(z;t)Y, \qquad A(z;t) := \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} + \frac{A_1(t)}{z-1} \tag{1.9}$$

with the residue matrices $A_l(t)$ $sl_2(\mathbf{C})$ ($l = 0, t, 1$) having $\pm\frac{\theta_l}{2}$ as eigenvalues (independantly of t). We set $\theta := (\theta_0, \theta_t, \theta_1, \theta_\infty)$: it encodes (through a *transcendental* mapping) the *local monodromy data*.

Choosing a germ of a fundamental matrix solution $\Phi(z,t)$ of the above system near some nonsingular point z_0, one has a linear monodromy representation (anti-homomorphism):

$$\rho : \pi_1(\mathbf{P}^1(\mathbf{C}) \setminus \{0, t, 1, \infty\}; z_0) \to \mathrm{SL}_2(\mathbf{C})$$

such that the analytic continuation of Φ along a loop γ (at z_0) defines another fundamental matrix solution $\Phi\,\rho(\gamma)$. The equivalence class of ρ in $\mathrm{SL}_2(\mathbf{C})$ is independant of the choice of the fundamental solution Φ. The system (1.9) is said *isomonodromic* if this conjugation class is locally constant with respect to t, or equivalently if the matrices A_l ($l = 0, t, 1$) depends on t in such a way that the monodromy of a fundamental solution $\Phi(z : t)$ does not change for small deformations of t.

A meromorphic connection can be interpreted as an equivalence class of systems modulo rational equivalence (gauge transformation). If two systems $\frac{dY}{dz} = A(z;t)Y$ and $\frac{dY}{dz} = B(z;t)Y$, satisfying the conditions (1.9), are rationally equivalent on $\mathbf{P}^1(\mathbf{C})$, that is if there exists a rational matrix P such that $B = P^{-1}AP - P^{-1}\frac{dP}{dz}$, then the two corresponding monodromy representation are equivalent. The isomonodromy property is invariant by a rational equivalence. We can speak of isomonodromic deformations of connections.

(8) It is the irreducibility property of PVI, cf. for example [11, 1.8, p. 2937].

Schlesinger [75] found that the isomonodromy condition is equivalent to having the linear differential equation[9] :

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = B(z,t)Y, \text{ with } B(z,t) := -\frac{A_t(t)}{z-t}Y. \tag{1.10}$$

We define the *Schlesinger system* as the system (1.9) and (1.10):

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = A(z,t)Y, \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = B(z,t)Y,$$

Then the isomonodromy of the system (1.9) is equivalent to the *complete integrability condition* (also called *zero curvature condition*) of the Schlesinger system:

$$\frac{\partial B}{\partial z} - \frac{\partial A}{\partial t} = [A,B]. \tag{1.11}$$

Expliciting this condition, we see that the isomonodromicity of the system (1.9) is expressed by the following equations (called the Schlesinger equations) on (A_0, A_t, A_1):

$$\frac{\mathrm{d}A_0}{\mathrm{d}t} = \frac{[A_t,A_0]}{t}, \quad \frac{\mathrm{d}A_t}{\mathrm{d}t} = \frac{[A_0,A_t]}{t} + \frac{[A_1,A_t]}{t-1}, \quad \frac{\mathrm{d}A_1}{\mathrm{d}t} = \frac{[A_t,A_1]}{t-1}. \tag{1.12}$$

These equations correspond equivalently to the integrability of the logarithmic connection in variables (z,t):

$$:= d - (A_0(t)\mathrm{d}\log z + A_t(t)\mathrm{d}\log(z-t) + A_1(t)\mathrm{d}\log(z-1)).$$

on the trivial rank two vector bundle on $\mathbf{P}^1(\mathbf{C})$.

We suppose now that the Schlesinger equations are satisfied by the matrix A of the system (1.9) and (following [38]) we will derive the non linear second order PVI for some values of the parameter (under some genericity condition on the local monodromy exponents $\pm\theta_l/2$).

We set $A_\infty := -A_0 - A_t - A_1$ and we suppose that the matrices A_l ($l = 0, t, 1, \infty$) are semi-simple. The eigenvalues of the A_l ($l = 0, t, 1, \infty$) are independant of t and we denote them by e_l, e_l^{-1}. We suppose $e_l = \pm 1$ or equivalently $\pm\theta_l \,/\, \pi\mathbf{Z}$ (non-resonance conditions).

From Schlesinger equations we get $\frac{\mathrm{d}A_\infty}{\mathrm{d}t} = 0$, therefore, up to a constant gauge transformation, we can suppose $A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}$.

We denote $[A]_{ij}$ the (i,j) entry of the matrix of the differential system (1.9). We suppose that the system is *irreducible*. Then $[A]_{12}$ is not identically 0. We have $A_0 + A_t + A_1 = -A_\infty$, therefore $[A_0 + A_t + A_1]_{12} = 0$.

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[9] We need a condition on $Y(\infty, t)$ to fix $B(z,t)$, see [70, p. 432].

Hence $z(z-t)(z-1)[A]_{12}$ is linear in z and it admits a unique zero at the point $z = q(t)$, where:

$$q(t) = -\frac{t[A_0]_{12}}{t[A_t]_{12} + [A_1]_{12}}.$$

The point $q(t)$ is *an apparent singularity* of the second order linear ODE satisfied by the first component y of any solution Y of the system (1.9). We denote:

$$p(t) := [A\,(q(t),t))]_{11} + \frac{\theta_0}{2q} + \frac{\theta_t}{2(q-t)} + \frac{\theta_1}{2(q-1)},$$

then the Schlesinger system (1.12) is equivalent to the Hamiltonian system of PVI whose (non autonomous) Hamiltonian is:

$$H_{VI}(q,p,t) := \operatorname{Tr}\left(\left(\frac{A_0(t)}{t} + \frac{A_1(t)}{t-1}\right)A_t(t)\right) - \frac{\theta_0\theta_t}{2t} - \frac{\theta_t\theta_1}{2(t-1)}.$$

(cf. [38]).

Now we can write the Hamiltonian system in PVI form with the following values for the parameters:

$$\alpha = (\theta_\infty - 1)^2 \quad \beta = -\theta_0^2, \quad \gamma = \theta_1^2, \quad \delta = 1 - \theta_t^2.$$

The Riemann–Hilbert correspondence RH is given by the monodromy map between the space of linear systems (1.9) with prescribed poles and local exponents $\pm\theta_l/2$, modulo $SL_2(\mathbf{C})$-gauge transformations, on one side (the source or "left hand side"), and the space of monodromy representations with prescribed local exponents modulo conjugation in $SL_2(\mathbf{C})$ on the other side (the target or "right hand side").

The relation with the notations introduced in 1.2.1 is:

$$e_l = e^{i\pi\theta_l}, \quad a_l = \operatorname{Tr} M_l = 2\cos\pi\theta_l.$$

The Riemann–Hilbert correspondence can be translated into a correspondence between solutions of PVI and equivalence classes of monodromy representations.

We recall that an analytic complex vector field on a complex manifold (resp. the associate flow) is called *complete* if complex solutions (flow curves) *exist for all complex time*. The very naive phase space[10] of the system associated to PVI is $\left(\mathbf{P}^1(\mathbf{C}) \setminus \{0,t,1,\infty\}\right) \times \mathbf{C}^2$. It is not a good phase space because the solutions have poles: the Painlevé flow is not complete. Using a series of blowing-ups K. Okamoto introduced a good space of initial conditions $M_{t_0}(\theta)$ at any point $t_0 \in \mathbf{C}$ [54, 56]. It is a convenient semi

[10] A solution is defined by its initial values $y(t_0)$ and $y'(t_0)$.

compactification of the naive phase space $\mathbf{C}^2$, an open rational surface(11). This surface is endowed with an algebraic symplectic structure given by the extension of the standard symplectic form(12) $dp \wedge dq$. The Okamoto variety of initial conditions at t_0 can be identified with the moduli space of meromorphic connections over the Riemann sphere(13) having simple poles at the four points $0, t_0, 1, \infty$ with local exponents $\{\pm\theta_l\}_{l=0,t,1,\infty}$.

For θ fixed, we have a fiber bundle $M(\theta) \to \mathbf{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}$: the fiber above t_0 is $M_{t_0}(\theta)$.

The naive Painlevé foliation extends to this fiber bundle. This extension is transverse to the fibers and we get a complete (symplectic) flow, the Painlevé flow. For all $t_0, t_1 \neq 0, 1, \infty$ this flow induces an analytic symplectic diffeomorphism $M_{t_0}(\theta) \to M_{t_1}(\theta)$. We get also *analytic* maps (Riemann–Hilbert maps):

$$\mathrm{RH} : M_t(\theta) \to \mathbf{S}_{P_{VI}}.$$

Such a map can be interpreted as an analytic map:

$$\mathrm{RH} : M_t(\theta) \to S_{A_0 A_t A_1 A_\infty},$$

where (using (1.3)):

$$A_i = 4\left(\cos\theta_i \cos\theta_\infty + \cos\theta_j \cos\theta_k\right), \tag{1.13}$$

where (i, j, k) is a permutation of $(0, t, 1)$. and

$$\begin{aligned} A_\infty = {} & 16(\cos\theta_0 \cos\theta_t \cos\theta_1 \cos\theta_\infty) \\ & + 4(\cos^2\theta_0 + \cos^2\theta_t + \cos^2\theta_1 + \cos^2\theta_\infty - 1). \end{aligned} \tag{1.14}$$

This map is always proper. If the cubic surface $S_{A_0 A_t A_1 A_\infty}$ is smooth, then this map is an analytic symplectic isomorphism(14). In the singular case RH is a proper map, more precisely it realizes an analytic minimal resolution of singularities of $S_{A_0 A_t A_1 A_\infty}$. Along the irreducible components of the exceptional divisor, PVI restricts to a Riccati equation(15).

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(11) A 8 point blow-up of the Hirzebruch surface Σ_2 minus an anti-canonical divisor.

(12) The pole divisor of this extension is the anticanonical divisor of a compactification of the Okamoto variety: the vertical leaves. The vertical leaves configuration is described by a Dynkin diagram: today a “good list” of the Painlevé equations is labelled by such diagrams.

(13) In the non resonant case. In the resonant case, that is if one of the θ_l is an integer, then $M_{t_0}(\theta)$ is the moduli space of *parabolic* connections [32].

(14) The pull back by RH of the symplectic form $\omega_{VI,a}$ is the standard symplectic form on the Okamoto variety of initial conditions [43, Prop. 4.3]

(15) The singular points of type A_1, A_2, A_3, D_4 on the cubic surface yield 1, 2, 3 and 4 exceptional Ricatti curves [59].

Remark 1.2. — "Pulling back" the fiber bundle $\widetilde{\text{Rep}} \to \mathbf{P}^1(\mathbf{C}) \setminus \{0,1,\infty\}$ and its connection by the Riemann–Hilbert map (i.e. keeping the base and changing the fibers through RH) yields the fiber bundle $M \to \mathbf{P}^1(\mathbf{C}) \setminus \{0,1,\infty\}$ with its PVI connection. This allows one to give an important interpretation of the non-linear monodromy of PVI using a braid group [34], [11, p. 2].

1.2.3. Iso-irregularity and the Painlevé equations. Wild character varieties

This paper is limited to fuchsian q-difference equations. The Lax pairs of irregular q-Painlevé equations are listed up by Murata [51] (see Subsection 1.4.2). We look here at the irregular differential case briefly and think in terms of q-analogies.

In order to classify *irregular* connections, the monodromy is no longer sufficient, it is necessary to introduce *generalized monodromy data* (formal monodromy, Stokes multipliers *and* links). Martinet–Ramis [47] have constructed a local *wild fundamental group*, so that germs of connections with irregular singularities can be interpreted as finite dimensional representations of this group. This construction uses *multisummability* of divergent series as an essential ingredient[16] . In the global case a group is no longer sufficient, it is necessary to introduce a wild *groupoid* [8, 59]. The generalized monodromy data are in some sense representations of this wild groupoid. The quotient of the set of these representations by the natural equivalence relation is a *wild character variety* [8, 9, 59].

In all the cases PI, PII, PIII, PIV, PV, the wild character varieties are, as in the PVI case, cubic surfaces. The interested reader will find a list of equations of these surfaces in [59, p. 19–20] and (in a nice form) in [14, Table 1, p. 2].

We consider some[17] traceless 2×2 linear differential systems with at most 3 singularities (one at least being irregular) on the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ (parameterized by a complex variable t):

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = A(z;t)Y \tag{1.15}$$

The system (1.15) parameterized by t is said *iso-irregular* if the conjugation class of generalized monodromy data (or wild monodromy representation)

[16] "Generically" Ramis k-summability is sufficient. It is the case for the Painlevé equations.

[17] cf. for explicit conditions [38, 39, 55].

is locally constant with respect to t, or equivalently if the matrices A_l ($l = 0, t, 1$) depends on t in such a way that the generalized monodromy data of a fundamental solution $\Phi(z : t)$ does not change for small deformations of t.

As in the fuchsian case, the iso-irregularity condition is equivalent to an integrability condition (Schlesinger equation) and therefore it is possible in each case to express it as a Painlevé equation, along similar lines [38, 39].

René Garnier was the first to show that, as PVI, the other Painlevé equations are given by *compatibility* conditions [22]. Later he defined a version of Stokes multipliers by *confluence* and interpreted PI and PII as *iso-Stokes* deformations [23].

1.3. The first age of q-monodromy; q-PVI according to Jimbo and Sakai

1.3.1. Riemann–Hilbert correspondence for q-difference equations

In his celebrated 1913 article [4] "The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations", Birkhoff looks for "transcendental invariants" in order to classify rational fuchsian q-difference equations or systems. The systems have the form:

$$Y(qx) = A(x)Y(x),$$

x a complex variable, q a complex number such that $0 < |q| < 1$ and $A(x)$ an invertible $n \times n$ matrix of rational functions (so the unknown Y is a vector of functions).

Rational equivalence is induced by gauge transformations $Z = QY$, $Q \in \mathrm{GL}_n(\mathbf{C}(x))$, so that Z is a candidate solution of $Z(qx) = B(x)Z(x)$, where:

$$B(x) := Q(qx)A(x)Q(x)^{-1} \text{ is declared rationally equivalent to } A(x).$$

The problem of classification is not changed if A and B are replaced respectively by fA and fB with f any scalar function, so one can as well (and Birkhoff does) assume that A is polynomial:

$$A = A_0 + \cdots + A_\mu x^\mu, \quad A_0, \ldots, A_\mu \in \mathrm{Mat}_n(\mathbf{C}).$$

Birkhoff moreover assumes that $A_0, A_\mu \in \mathrm{GL}_n(\mathbf{C})$ (this means in essence that 0 and ∞ are regular singularities) and implicitly(18) that A_0, A_μ are

(18) Almost no assumption or definition is explicit in [4], and many conclusions are not either. However *the heart of the matter* is dealt with.

semi-simple and "non-resonant" (such details will be explained Section 3). Their eigenvalues, seen as elements of $\mathbf{C}^* \pmod{q^{\mathbf{Z}}}$, are considered as *exponents* at 0 and ∞ and should encode the local monodromies there.

Fuchs–Frobenius type algorithms yield local fundamental solutions $Y^{(0)}$ and $Y^{(\infty)}$, made up of multivalued functions. *Birkhoff connection matrix* is then $P := (Y^{(\infty)})^{-1}Y^{(0)}$. The main results of Birkhoff (in the part devoted to q-difference equations) are then that:

(1) The local exponents being fixed, P classifies A up to rational equivalence.
(2) P can be characterized by $\mu n^2 + 1$ "characteristic constants", the transcendental invariants looked for.

The second statement comes from the fact that P is almost q-invariant (the defect comes from the multivaluedness) so its elements can almost be identified to elliptic functions and those are very much controlled by their zeroes and poles. More precisely, each coefficient of P has μ zeroes by which it is determined up to a constant; this altogether yields $(\mu + 1)n^2$ degrees of freedom, but taking in account gauge freedom reduces this dimension to $\mu n^2 + 1$, see the Master: [4, §20]. (Also see, in the case $\mu = n = 2$, Remark 5.2 at the beginning of Subsection 5.1.)

Birkhoff's paper has some drawbacks:

- Contrary to the case of differential equations, multivaluedness can (and should) be avoided.
- The problem is solved only under generically true assumptions.
- Irregular equations are not considered.

As for the first two drawbacks, see Part 1.4.1 below. As for the third one, Birkhoff himself with his student Guenther made a decisive step in [5], but the sequel had to wait for seven decades, see Part 1.4.2. However, the main question from our point of view is: in what sense does Birkhoff connection matrix encode monodromy?

1.3.2. q-analogues of Painlevé equations

The search for q-analogues of classical special functions has been a flourishing industry in the best part of twentieth century. Some physicists have been specially(!) interested in discrete analogues of Painlevé functions, see e.g. [61]. One way to specify them was *confinement of singularities*, invented by Grammatikos, Ramani and Papageorgiou see [26]. It seems that it can be considered as a sensible discrete analogue of the Painlevé property, which was the guiding criterion of Painlevé himself. However, this did not lead to the discovery of a q-analog to PVI.

In [41], Jimbo and Sakai adapted the isomonodromy approach to the q-difference setting. They considered a family of order 2 degree 2 systems:

$$Y(qx) = A_t(x)Y(x), \quad A_t(x) = A_0(t) + A_1(t)x + A_2(t)x^2,$$
$$t,\ A_0(t), A_2(t) \quad \mathrm{GL}_2(\mathbf{C}),\ A_1(t) \quad \mathrm{Mat}_2(\mathbf{C}),$$

with conditions similar to those imposed by Birkhoff in [4].

Then they imposed that the family has constant local data, i.e. that $A_0(t)$ and $A_2(t)$ have constant eigenvalues[19]. To express isomonodromy, they consider Birkhoff connection matrix as depending on t and, in a bold step, assume that it is q-constant:

$$t, x\ ,\ P(qt, x) = P(t, x).$$

They deduce a "Lax pair", some kind of integrability condition analogous to (1.11): it is the system (1.16) herebelow. From this they derive a nonlinear q-difference equation they consider as the adequate analogue of PVI. Applying the usual test to support such a claim[20], they go to the "continuous limit" $q \to 1$ and show how to recover classical PVI equation.

The succesful attack of Jimbo and Sakai was very influential. Any attempt at a theory of monodromy for q-difference equations should use it as a touchstone. For the sake of completeness, we now give a description of their model in their own notations.

Connection preserving deformation and q-P_{VI}. We review here the q-analogue of the sixth Painlevé equation obtained by Jimbo and Sakai in [41].

We denote $y = y(t), z = z(t)$, $\bar{y} = y(qt), \bar{z} = z(qt)$. We take a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 as complex parameters of the equation. The q-analogue q-P_{VI} of the sixth Painlevé equation considered here is:

$$\frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z} - b_1t)(\bar{z} - b_2t)}{(\bar{z} - b_3)(\bar{z} - b_4)}, \quad \frac{z\bar{z}}{b_3b_4} = \frac{(y - a_1t)(y - a_2t)}{(y - a_3)(y - a_4)} \quad \text{with} \quad \frac{b_1b_2}{b_3b_4} = q\frac{a_1a_2}{a_3a_4}.$$

Jimbo and Sakai derive q-P_{VI} from *connection preserving deformation* of a fuchsian linear q-difference equation with rank two and order two. What is

(19) Actually there is a subtle twist in the case of A_0, but this does not matter here; also, the singularities (zeroes of $\det A(x)$) are subject to some similar condition.

(20) This is a standard process in the history of q-analogues. For a very detailed (and unusually rigorous) such study, see [72] which tackles the case of q-hypergeometric equations under the name of "confluence". We shall not further delve into these matters in the present work.

called here “connection preserving deformation” is a compatibility condition of the following two q-difference equations:

$$\begin{aligned} Y(qx,t) &= A(x,t)Y(x,t), \\ Y(x,qt) &= B(x,t)Y(x,t). \end{aligned} \tag{1.16}$$

The system above (1.16) can be best understood by writing $Y_t(x) := Y(x,t)$ and $A_t(x) := A(x,t)$. The first relation says that we have a family (parameterized by t) of q-difference equations; the second relation states a gauge equivalence of Y_t with Y_{qt}. The system (1.16) is compatible if and only if:

$$A(x,qt)B(x,t) = B(qx,t)A(x,t).$$

By the compatibility condition, the Birkhoff connection matrix $P(t)$ is a “quasi-constant”, i.e. $P(qt) = P(t)$: this is Jimbo and Sakai interpretation of q-isomonodromy.

Jimbo and Sakai set the rank two matrices $A(x,t)$ and $B(x,t)$ as

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2$$

and

$$B(x,t) = \frac{x}{(x-a_1qt)(x-a_2qt)}(xI + B_0(t))\cdot$$

We assume that $A_2 = \mathrm{Diag}(\sigma_1,\sigma_2)$ and that the eigenvalues of $A_0(t)$ are $\rho_1 = \theta_1 t, \rho_2 = \theta_2 t$. We set $\det A(x,t) = \sigma_1\sigma_2(x-x_1)(x-x_2)(x-x_3)(x-x_4)$. We define the parameters a_j and b_k by:

$$x_1 = a_1t, \qquad x_2 = a_2t, \qquad x_3 = a_3, \qquad x_4 = a_4,$$

$$b_1 = \frac{a_1a_2}{\rho_1}, \quad b_2 = \frac{a_1a_2}{\rho_2}, \quad b_3 = \frac{1}{\sigma_1 q}, \quad b_4 = \frac{1}{\sigma_2}.$$

We take variables $y = y(t)$, $z_i = z_i(t)$ ($i = 1,2$) such that

$$A_{12}(y,t) = 0, \quad A_{11}(y,t) = \sigma_1 z_1, \quad A_{22}(y,t) = z_2,$$

We set a variable z in such a way that:

$$z_2 = \sigma_1\sigma_2 qz(y-a_3).$$

Then we obtain q-P_{VI} by the compatibility condition

$$A(x,qt)B(x,t) = B(qx,t)A(x,t).$$

1.3.3. Does Birkhoff connection matrix encode monodromy?

Since in some sense Birkhoff connection matrix connects solutions at 0 and at ∞, it is comparable to connection matrices of the classical theory (those related to analytic continuation of solution along pathes, see for instance [35]). So it was generally felt that it should relate to the monodromy of the q-difference system if *that* could be defined somehow. However it was not clear what was the topology underlying it. So for some time the confirmations of the monodromy interpretation were indirect. Actually they came from Galois theory.

Without going in any detail, let us say that differential Galois theory, as created by Picard and Vessiot, attaches to a differential equation or system with rational coefficients a linear algebraic group G. This is related to the monodromy group M in the following ways:

(1) In all cases, M naturally embeds into G: $M \subset G$.
(2) In case of a fuchsian differential equation, G is the Zariski closure of M (Schlesinger density theorem).

In [19], Etingof proved a q-analogue of Schlesinger density theorem with Birkhoff connection matrix in the role of monodromy in the following way. He assumes that the rational system $Y(qx) = A(x)Y(x)$, $A \in \mathrm{GL}_n(\mathbf{C}(x))$ is such that $A(0) = A(\infty) = I_n$, the identity matrix; this means in essence that 0 and ∞ are not merely regular singularities, as in Birkhoff's paper, but *ordinary* points (indeed, it is the case when the equation can be solved with power series, without the need of special transcendental functions). In this case, the connection matrix $P(x)$ as built by Birkhoff is uniform over $\mathbf{C}^*$ and truly elliptic. On the other hand, in the mean time (since Picard and Vessiot), differential Galois theory had been extended to difference equations over fields more general than the complex numbers, so that there is a linear algebraic group G attached to the equation. Then the *values* $P(a)^{-1}P(b)$, where defined, generate a subgroup M of G which is Zariski-dense in G. So it would be a natural conjecture that the q-analogue of the monodromy group is the group generated by all the $P(a)^{-1}P(b)$.

Van der Put and Singer then extended in [60] this result to the case of general fuchsian systems. However, difficulties appear that are not present in the classical case of differential equations. First, the natural field of constants in the q-different setting is the field of meromorphic functions over $\mathbf{C}^*$ that are q-invariant: $f(qx) = f(x)$. This field can be identified with a field of elliptic functions (see the end of Subsection 2.2). It is not algebraically closed, which is a severe drawback for Picard–Vessiot theory. Second, natural solutions to basic q-difference equations have bad multiplicative properties. For instance, writing e_c a non trivial solution of the constant scalar equation $f(qx) = cf(x)$ ($c \in \mathbf{C}^*$), it is not possible to impose that $e_c e_d = e_{cd}$ or even that $e_c e_d / e_{cd} \in \mathbf{C}^*$. To overcome these difficulties, van der Put and Singer introduced symbolic solutions. The theory then develops nicely, in particular (to stick to our monodromy-headed point of view[21]) it does contain a Schlesinger type density theorem for general fuchsian equations.

So in some sense it was established that Birkhoff connection matrix has something to do with monodromy. However the transcendental point of view of Birkhoff seemed partly abandonned.

1.4. The second age of q-monodromy

Since the end of the last century (and millenium), mainly under the influence of the second author, transcendental methods in the theory of q-difference equations (including Galois theory) have relied on the use of theta functions. This is related to the fact that $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$, as a Riemann surface, can be seen as an elliptic curve; and solutions of q-difference systems as sections of holomorphic vector bundles over $\mathbf{E}_q$.

1.4.1. Uniform solutions to q-difference equations

In the work of Praagman on formal classification of difference and q-difference operators [58] appears an argument based on the fact that every holomorphic vector bundle on the elliptic curve $\mathbf{E}_q$ is meromorphically trivial. An easy consequence of this fact is that any rational q-difference system:

$$Y(qx) = A(x)Y(x), \quad A \in \mathrm{GL}_n(\mathbf{C}(x)), \tag{1.17}$$

admits a "full complement" (that is a system of maximal possible rank) of solutions meromorphic over $\mathbf{C}^*$. Therefore, contrary to the case of differential equations *it is not necessary to use multivalued functions*. From our point of view (Riemann–Hilbert correspondence), this means that what we consider as monodromy should not be related on ambiguity of analytic continuation.

In [72], the third author gave a concrete content to this result by solving explicitly fuchsian systems in a way similar to the Fuchs–Frobenius method for differential equations. This was applied to the Riemann–Hilbert correspondence and also to Galois theory in [73]. In the latter paper, the Galois group was defined by tannakian means and a Schlesinger density theorem similar to those quoted above was proved. Moreover, as a bonus rewarding the use of "true" (not symbolic) functions, a very precise meaning could

(21) Note however that the theory expounded in [60] has many more advantages, including a tannakian interpretation and a description of the universal Galois group for fuchsian equations.

be given to the degeneracy ("continuous limit"), when $q \to 1$, towards monodromy and towards differential Galois theory. In particular, the values $P(a)^{-1}P(b)$ degenerate, when $q \to 1$, into monodromy matrices of the differential system.

However some undue complications in the computations led to the idea that Birkhoff connection matrix mixes, in some sense, local monodromies at 0 and ∞ with monodromy at the "intermediate singularities" (those in $\mathbf{C}^*$). We find it relevant to explain this point in some detail, because the way we define and use monodromy in the present work is directly related to it.

We suppose that $A(0), A(\infty) \in \mathrm{GL}_n(\mathbf{C})$, which, as already noted, means that the above system is fuchsian at 0 and ∞ (it can indeed be characterized by the fact that solutions satisfy some kind of moderate growth condition, [72]). Then there exist constant invertible matrices[22] $A^{(0)}, A^{(\infty)} \in \mathrm{GL}_n(\mathbf{C})$ such that:

$$A(x) = M^{(0)}(qx)A^{(0)}(M^{(0)}(x))^{-1}$$
$$\text{and} \quad A^{(\infty)}(x) = M^{(\infty)}(qx)A^{(\infty)}(M^{(\infty)}(x))^{-1},$$

where $M^{(0)} \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$ and $M^{(\infty)} \in \mathrm{GL}_n(\mathbf{C}(\{1/x\}))$. This implies that one can look at fundamental solutions of (1.17) in the form:

$$Y^{(0)} = M^{(0)}e_{A^{(0)}} \quad \text{and} \quad Y^{(\infty)} = M^{(\infty)}e_{A^{(\infty)}},$$

where $e_{A^{(0)}}$, $e_{A^{(\infty)}}$ are respectively solutions of the systems *with constant coefficients*

$$Y(qx) = A^{(0)}Y(x), \quad \text{resp.} \quad Y(qx) = A^{(\infty)}Y(x).$$

Birkhoff (along with his predecessors) solves those systems using multivalued functions such as $x^{\ln c/\ln q}$ (where c is an eigenvalue of $A^{(0)}$, resp. $A^{(\infty)}$); van der Put and Singer use a symbol e_c; and Sauloy uses $\theta_q(x)/\theta_q(cx)$ (the theta function θ_q will be precisely defined later)[23]. Then Birkhoff connection matrix writes:

$$P := (Y^{(\infty)})^{-1}Y^{(0)} = (e_{A^{(\infty)}})^{-1}Me_{A^{(0)}}, \quad \text{where} \quad M := (M^{(\infty)})^{-1}M^{(0)}.$$

It comes out that $e_{A^{(0)}}, e_{A^{(\infty)}}$ really encode the local monodromies at 0 and ∞ and the corresponding local Galois groups can be directly computed from them[24]. And it has been verified in many contexts that M indeed encodes

(22) Generically $A^{(0)} = A(0)$ and $A^{(\infty)} = A(\infty)$, but this is not the case if there are "resonancies".

(23) Functions described here suffice in the generic case that $A^{(0)}, A^{(\infty)}$ are semi-simple. Otherwise, one also introduces "q-logarithms", see Part 2.4.2.

(24) The local Galois groups were independently found by Baranovsky and Ginzburg [2] in the context of loop groups.

the monodromy at intermediate singularities. In this paper we define a space of monodromy data for the Jimbo–Sakai family using M instead of P.

Remark 1.3. — Birkhoff matrix P still plays an important role, since it directly relates to solutions. For instance, in [53], the first author computes it for basic hypergeometric equations; also see [65], where Roques uses it to study Galois groups.

1.4.2. Irregular q-difference equations and other q-Painlevé equations: Murata's list

As we said before, Birkhoff and Guenther had led a first attack at irregular q-difference equations in [5], but that part of the theory remained dormant for quite a long time. In [63], the second and third authors along with Changgui Zhang defined a q-analog of Stokes phenomenon and applied it to Riemann–Hilbert correspondence for irregular q-difference equations. This was further used for Galois theory in [62].

On the other hand, Murata, in [51], extended the work of Jimbo and Sakai to various degeneracies of q-PVI related to families of irregular equations.

It is natural to envision an application of the tools of [62, 63] to extend the methods and results of the present paper to Murata's list. A first attempt was sketched in Anton Eloy's thesis [16]. We hope to pursue this goal in a near future.

1.4.3. Families, moduli

In all versions of Riemann–Hilbert correspondence for q-difference equations during the second age, moduli problems and behaviour of continuous families were not properly adressed. In the present work, we fix the local data: obviously this should give rise to a fibering of some global space of monodromy data over a space of local monodromy data. An attempt at this appears in already quoted Eloy's thesis, but most of the work is yet to be done. We also hope to pursue this goal in a near future.

1.4.4. "Intermediate" singularities

One of the successes of classical Riemann–Hilbert theory lies in the ability to decompose global phenomena into local ones, in particular, to define local monodromies, local Galois groups, etc. From the beginning, it has seemed very difficult to do something similar for q-difference equations.

One aspect of the problem is that the obvious singularities other than 0 and ∞, i.e. the poles of $A(x)$ and those of $A(x)^{-1}$ in $\mathbf{C}$, are not really local: they are moved under the action of the dilatation operator $x \to qx$. Therefore it seems that they should be replaced either by the corresponding q-spirals (discrete spirals of the form $aq^{\mathbf{Z}}$); or by the corresponding points in $\mathbf{E}_q$.

In [73], reduction of the global Galois group to local contributions was accomplished only in the trivial case of an abelian Galois group(25) . But the first significant progress in this direction (understanding local contributions) was accomplished much later by Roques in [65]. He used the Lie algebra instead of the Galois group to take in account the local contribution of the connection matrix (via the residue of its logarithmic derivative) at the only intermediate singularity of a "basic" hypergeometric (i.e. q-hypergeometric) equation. In a somewhat different vein (sheaf theoretic approach), Roques and the third author gave in [67] a cohomological interpretation of the rigidity index defined by Sakai and Yamaguchi in [69]. There, the local contributions of intermediate singularities to an Euler characteristics can be measured.

In the present paper, a new technique is developped under the name of "Mano decomposition" (as it has its roots in Mano's paper [46]) which in some sense allows us to localize the monodromy at *pairs of points*. We put great hopes in this process for future progress.

1.5. Contents of this paper

Section 2 is devoted to general notations and conventions, along with some basic tools for dealing with q-difference equations.

In Section 3, we state and prove a variant of Riemann–Hilbert correspondence from [4] but using the matrix M described above in Part 1.4.1. We apply it first to a criterion of reducibility, second to the case of "hypergeometric" systems (actually a slightly more general class allowing for some degeneracies).

In Section 4 we define the Jimbo–Sakai family, thus formalising the objects studied in [41]. We introduce the space F of its monodromy data, defined as the space of rational equivalence classes of such equations but translated through our Riemann–Hilbert correspondence; we do this for fixed local monodromy data (exponents at 0 and ∞) and also fixed singular set. Then we give a first geometric description of F as an algebraic surface. This

(25) In that case, it boils down to "class field theory over $\mathbf{E}_q$", as described in Serre's book [77].

part (Subsections 4.2 to 4.5) has the character of a preliminary exploration, collecting as much information as possible in order to be later able to identify our surface, which will be done to some extent conjecturally in Sections 6 and 7. Thus for instance we give a close look to incidence relations in Subsection 4.3.

Section 5 deals with a new process inspired by the paper [46] of Mano. This allows to decompose the monodromy matrix M of a system in the Jimbo–Sakai family into the product $M = PQ$ of two hypergeometric monodromy matrices, while distributing the four singularities of M among P and Q. The proof is very detailed because it involves some new objects, techniques and tools which we hope will be handy in the future. The main results in this part are the existence Theorem 5.13, and the gauge freedom and normal forms (Propositions 5.9 and 5.10).

In Section 6 we apply Mano decomposition to obtain a more precise description of the space F as an algebraic *fibered* surface. We do that under the same assumptions as Jimbo and Sakai, plus some more that are generically true and that seem reasonable; actually, they are essentially the same as those underlying similar works in the classical case of differential equations related to Painlevé and isomonodromy.

Section 7 is devoted to a larger picture and tries to formulate analogies between the character varieties and their dynamics in the differential and in the q-difference case. By nature, it is partly conjectural.

In conclusive Section 8 we describe some interesting open problems and perspectives.

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2. Tools

2.1. General notations

Here are some standard notations of general use:

- $\mathbf{C}(x)$ is the field of rational fractions over $\mathbf{C}$.
- $\mathbf{C}(\{x\})$, the field of meromorphic germs at 0 (or Laurent[26] convergent power series), is the quotient field of $\mathbf{C}\{x\}$, the ring of holomorphic germs at 0 (i.e. convergent power series).
- $\mathbf{C}((x))$, the field of Laurent formal power series, is the quotient field of $\mathbf{C}[[x]]$, the ring of formal power series.
- $\mathbf{C}\{1/x\}, \mathbf{C}(\{1/x\}), \mathbf{C}[[1/x]], \mathbf{C}((1/x))$ are similarly defined replacing x by $1/x$.
- $M(\Omega)$ is the ring of meromorphic functions on the open subset Ω of a Riemann surface (thus a field if Ω is a domain). Most of the time, Ω will be a domain of the Riemann sphere $\mathbf{S}$ or of the elliptic curve $\mathbf{E}_q$ defined further below.
- $\mathbf{S}$, the Riemann sphere and its open subsets $\mathbf{C} = \mathbf{S} \setminus \{\infty\}$ and $\mathbf{C}_\infty := \mathbf{S} \setminus \{0\}$.
- Mat_n, $\mathrm{Mat}_{m,n}$, GL_n are spaces of square, resp. rectangular matrices and the linear group; $\mathrm{D}_n(\mathbf{C})$ $\mathrm{GL}_n(\mathbf{C})$ is the subgroup of diagonal invertible matrices.
- $\mathrm{Diag}(a_1, \ldots, a_n)$, $\mathrm{Sp}(A)$ respectively denote a diagonal matrix and the spectrum of an arbitrary matrix A. Most of the time we consider the spectrum as a *multiset*, i.e. its elements have multiplicities.

2.2. q-notations

Here are some notations related to q but of general interest:

- q is a complex number such that $0 < |q| < 1$.
- σ_q is the q-dilatation operator $f(z) \to f(qz)$.
- $C_q := \{z \quad \mathbf{C} \mid |q| < |z|\ 6\ 1\}$, the *fundamental annulus*.
- For $x \quad \mathbf{C}$, we write $R(x) \quad C_q$ its unique representative modulo $q^{\mathbf{Z}}$.

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[26] We shall sometimes (as here) understand Laurent power series to have bounded below exponents, whence the form $_{n > n_0}$ for some $n_0 \quad \mathbf{Z}$; and sometimes not. The context should make it clear.

- $\mathbf{E}_q$ is $\mathbf{C}^*/q^{\mathbf{Z}}$ either seen as a group or, more frequently, as a Riemann surface (a complex torus, or "elliptic curve"). Indeed, the compositum of the canonical projection $\mathbf{C}^* \to \mathbf{E}_q$ with the map $z \mapsto e^{2i\pi z}$ is a covering map between Riemann surfaces and also a group morphism with kernel $\mathbf{Z} + \mathbf{Z}\tau$, where $q = e^{2i\pi\tau}$, whence an identification of $\mathbf{E}_q$ with $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$.
- The canonical projection $\pi : \mathbf{C}^* \to \mathbf{E}_q$ is also denoted $a \mapsto \bar{a}$. It is bijective from C_q to $\mathbf{E}_q$.
- We write $[a; q] := aq^{\mathbf{Z}}$ the discrete logarithmic q-spiral $\pi^{-1}(\bar{a})$.
- $(a; q)_n = \prod_{0 \leqslant i < n}(1 - aq^i)$ and $(a; q)_\infty := \prod_{n \geqslant 0}(1 - aq^n)$ are the q-Pochhammer symbols.
- These notations are "collectivized" as follows:

$$[a_1, \ldots, a_m; q] := \bigcup_{i=1}^{m} [a_i; q],$$

$$(a_1, \ldots, a_m; q)_n := \prod_{i=1}^{m} (a_i; q)_n, \quad (a_1, \ldots, a_m; q)_\infty := \prod_{i=1}^{m} (a_i; q)_\infty.$$

The operator σ_q acts naturally on the field $\mathcal{M}(\mathbf{C}^*)$, the subfield of "constants":

$$\mathcal{M}(\mathbf{C}^*)^{\sigma_q} := \{f \in \mathcal{M}(\mathbf{C}^*) \mid \sigma_q f = f\}$$

has a natural identification with the field of elliptic functions $\mathcal{M}(\mathbf{E}_q)$; any $f \in \mathcal{M}(\mathbf{E}_q)$ can at will be seen as a meromorphic function on $\mathbf{E}_q$; as a meromorphic function on $\mathbf{C}^*$ such that $f(qx) = f(x)$; or as a meromorphic function on $\mathbf{C}$ with $(\mathbf{Z} + \mathbf{Z}\tau)$-periodicity.

A convention for notations of congruences. Since in all the text most congruences in $\mathbf{C}^*$ are modulo $q^{\mathbf{Z}}$, we shall systematically (when $a, b \in \mathbf{C}^*$) write $a \equiv b$ for $a \equiv b \pmod{q^{\mathbf{Z}}}$.

2.3. Some functions

The main one is the following theta function:

$$\theta_q(x) := \sum_{n \in \mathbf{Z}} q^{n(n-1)/2} x^n.$$

It is holomorphic over $\mathbf{C}^*$ and satisfies the functional equations:

$$\theta_q(qx) = \frac{1}{x}\,\theta_q(x) = \theta_q(1/x).$$

Thanks to *Jacobi's triple product formula*:

$$\theta_q(x) = (q; q)_\infty (-x; q)_\infty (-q/x; q)_\infty,$$

it has simple zeroes over $[-1;q]$ and nowhere else, which we summarize[27] by:

$$\mathrm{div}_{\mathbf{C}^*}(\theta_q) = \sum_{a \in [-1;q]} [a].$$

Since this divisor is σ_q-invariant, or because θ_q can be seen as a section of a line bundle over $\mathbf{E}_q$, we can also write:

$$\mathrm{div}_{\mathbf{E}_q}(\theta_q) = [\overline{-1}].$$

For every $c \in \mathbf{C}^*$, the function $e_{q,c}(x) := \theta_q(x/c)/\theta_q(x)$ is a non trivial meromorphic solution of the q-difference equation $\sigma_q f = cf$ such that $\mathrm{div}_{\mathbf{E}_q}(e_{q,c}) = [\overline{-c}] - [\overline{-1}]$. (One could as well use instead the function $\theta_q(x)/\theta_q(cx)$.)

2.4. Some first order equations

2.4.1. Equations $\sigma_q f = uf$

Using Laurent series expansions, one proves easily the following facts:

- The equation $\sigma_q f = cx^k f$, $c \in \mathbf{C}^*$, $k \in \mathbf{Z}$, has non trivial solutions in $\mathbf{C}(\{x\})$ if, and only if, $k = 0$ and $c = q^m$, $m \in \mathbf{Z}$; and then these solutions are the elements of $\mathbf{C}^* x^m$.
- The equation $\sigma_q f = cx^k f$, $c \in \mathbf{C}^*$, $k \in \mathbf{Z}$, has non trivial solutions in $\mathcal{O}(\mathbf{C}^*)$ if, and only if $k < 0$ or $k = 0$ and $c = q^m$, $m \in \mathbf{Z}$. In the last case, these solutions are the elements of $\mathbf{C}^* x^m$.

The second statement can be completed as follows. Let $c \in \mathbf{C}^*$ and $k \in \mathbf{N}^*$. Then the solutions of $\sigma_q f = cx^{-k} f$ in $\mathcal{O}(\mathbf{C}^*)$ form a $\mathbf{C}$-vector space of dimension k. Using the theory of theta functions, one can moreover prove that any non trivial solution can be written:

$$f = \text{constant} \times \theta_q(x/x_1) \cdots \theta_q(x/x_k), \quad x_1 \cdots x_k = c.$$

Thus $\mathrm{div}_{\mathbf{E}_q}(f)$ is an effective divisor of degree k and evaluation $\overline{(-1)^k c} \in \mathbf{E}_q$, i.e.:

$$\mathrm{div}_{\mathbf{E}_q}(f) = [\alpha_1] + \cdots + [\alpha_k],$$
$$\text{where } \alpha_1, \ldots, \alpha_k \in \mathbf{E}_q \text{ and } \alpha_1 + \cdots + \alpha_k = \overline{(-1)^k c}.$$

(27) We write $\sum m_i[x_i]$ the divisors on a Riemann surface X, where the $m_i \in \mathbf{Z}$ and the $x_i \in X$; and $\mathrm{div}_X(f)$ the divisor of a function f on X or of a section of a line bundle (when this divisor is defined). Note that if X is non compact, the support of a divisor is not necessarily finite.

2.4.2. q-logarithms

For $E = \mathbf{C}\{x\}, \mathcal{O}(\mathbf{C}^*), \mathcal{O}(\mathbf{C}^*, 0), \ldots$ there is a so-called "q-De Rham complex":

$$0 \longrightarrow \mathbf{C} \longrightarrow E \xrightarrow{\sigma_q - 1} E \longrightarrow \mathbf{C} \longrightarrow 0.$$

The meaning of the left side is that q-constants (i.e. solutions of $\sigma_q f = f$) are true constants. The right side map $E \to \mathbf{C}$ sends $\sum a_n x^n$ to a_0. It is related to so-called *q-logarithms*, i.e. solutions of $\sigma_q f - f = 1$ (more generally of $\sigma_q f - f = c \in \mathbf{C}^*$).

Any solution $f \in \mathcal{M}(\mathbf{C}^*, 0)$ of $\sigma_q f - f = c \in \mathbf{C}^*$ can be uniquely extended to $\mathcal{M}(\mathbf{C}^*)$ and it has poles, as can be seen for instance by integration along the boundary of the fundamental annulus $\mathcal{C}_q$ (and using Cauchy formula). The simplest solutions are obtained as follows; let:

$$l_q(x) := x\frac{\theta_q'(x)}{\theta_q(x)}.$$

Then $\sigma_q l_q - l_q = -1$ and l_q has simple poles over $[-1; q]$ and nowhere else. More generally (and more precisely), the solutions of $\sigma_q f - f = c \in \mathbf{C}^*$ having one simple pole modulo $q^{\mathbf{Z}}$ are the $-cl_q(x/a) + b$, $a \in \mathbf{C}^*$, $b \in \mathbf{C}$.

More generally, we shall repeatedly use the following fact:

Lemma 2.1. — *Let $c \in \mathbf{C}^*$ and set $\phi(x) := \theta_q\left(\frac{x}{c}\right)$ and $\psi(x) := x\phi'(x) = \frac{x}{c}\theta_q'\left(\frac{x}{c}\right)$. Then, if $f, g \in \mathcal{O}(\mathbf{C}^*)$ are such that $\sigma_q f = \frac{c}{x} f$ and $\sigma_q g = \frac{c}{x}(g - f)$, then we have $f = \alpha\phi$ and $g = \alpha\psi + \beta\phi$ for some $\alpha, \beta \in \mathbf{C}$.*

Proof. — The function f/ϕ must be elliptic with at most simple poles, so it must be constant: $f = \alpha\phi$. If $f = 0$, a similar argument applies to g. Otherwise, g/f must be a q-logarithm with at most a single pole modulo $q^{\mathbf{Z}}$ and we use the remark preceding the lemma.

2.5. Gauge transformations

Let K be any of the fields over which σ_q can be defined and let $A, B \in \mathrm{GL}_n(K)$. Then, formally, if X is a column vector solution of the system:

$$\sigma_q X = AX, \tag{2.1}$$

then one gets a solution $Y = FX$ of $\sigma_q Y = BY$ if:

$$B = F[A] := (\sigma_q F)AF^{-1}. \tag{2.2}$$

We shall symbolize this relation by the diagram:

$$A \xrightarrow{F} B.$$

Indeed, F can be seen as a morphism (actually an isomorphism) from A to B in some category. It is easy to check that $A \stackrel{I_n}{\to} A$ and that from $A \stackrel{F}{\to} B$ and $B \stackrel{G}{\to} C$ one can infer $A \stackrel{GF}{\to} C$.

2.6. Local reduction for fuchsian equations

Assumptions, definitions and results are stated here at 0; the corresponding facts at ∞ are obtained by substituting $1/x$ for x. Detailed statements and proofs are given in [72].

Let A $\mathrm{GL}_n(\mathbf{C}(\{x\}))$ be such that $A(0)$ $\mathrm{GL}_n(\mathbf{C})$, meaning that $A(x)$ is well defined at $x = 0$ and that its value is invertible. We also say that A is *regular* at 0. Thus actually A $\mathrm{GL}_n(\mathbf{C}\{x\})$.

Definition 2.2. — *We say that A is* non resonant *at* 0 *if* $\mathrm{Sp}\, A(0) \cap q^{\mathbf{N}}\ \mathrm{Sp}\, A(0) =$ *; said otherwise, for every c, d* $\mathrm{Sp}\, A(0)$*, if $c \equiv d$, then $c = d$.*

Proposition 2.3. —

(i) *Let A* $\mathrm{GL}_n(\mathbf{C}(\{x\}))$ *be such that $A(0)$* $\mathrm{GL}_n(\mathbf{C})$*. Then there exists F* $\mathrm{GL}_n(\mathbf{C}(x))$ *such that $B := F[A]$* $\mathrm{GL}_n(\mathbf{C}(\{x\}))$ *and B is non resonant at* 0*.*
(ii) *Let A* $\mathrm{GL}_n(\mathbf{C}(\{x\}))$ *be non resonant at* 0*. Then there exists a unique F* $\mathrm{GL}_n(\mathbf{C}\{x\})$ *such that $F(0) = I_n$ and $A = F[A(0)]$.*

We shall use the following variant of the second statement:

Corollary 2.4. — *If $A(0) = CRC^{-1}$, C* $\mathrm{GL}_n(\mathbf{C})$*, $R = \mathrm{Diag}(\rho_1, \ldots, \rho_n)$ and if $\rho_i \equiv \rho_j$ for $i = j$ (so $A(0)$ is at the same time non resonant and semi simple), then there is a unique F* $\mathrm{GL}_n(\mathbf{C}\{x\})$ *such that $F(0) = C$ and $A = F[R]$.*

Normal forms

Combining the two statements in the proposition, we get:

Corollary 2.5. — *Let A* $\mathrm{GL}_n(\mathbf{C}(\{x\}))$ *be such that $A(0)$* $\mathrm{GL}_n(\mathbf{C})$*. Then there exists F* $\mathrm{GL}_n(\mathbf{C}(\{x\}))$ *and $A^{(0)}$* $\mathrm{GL}_n(\mathbf{C})$ *such that $A = F[A^{(0)}]$. Moreover, $A^{(0)}$ can be taken such that* $\mathrm{Sp}(A^{(0)})$ *$\mathcal{C}_q$ (the fundamental annulus); it is then unique up to conjugacy.*

Indeed, more generally, if $A_1, A_2 \in \mathrm{GL}_n(\mathbf{C})$ have all their eigenvalues in the fundamental annulus, then:

$$(A_2 = F[A_1], F \in \mathrm{GL}_n(\mathbf{C}(\{x\}))) \Longrightarrow F \in \mathrm{GL}_n(\mathbf{C}).$$

(Without the assumption on eigenvalues, one could still deduce that F is a Laurent polynomial.)

2.7. Singularities of meromorphic matrices

Let $M \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^*))$. We call *singularities* of M its poles as well as the poles of M^{-1}. The singular locus is written Σ_M:

$$\Sigma_M := \{\text{Poles of } M\} \cup \{\text{Poles of } M^{-1}\}.$$

Writing $\tilde{M} := {}^t\mathrm{com}(M)$ the transpose of the comatrix of M, we have, by Cramer's relations:

$$M\tilde{M} = \tilde{M}M = (\det M)I_n,$$

whence:

$$\Sigma_M := \{\text{Poles of } M\} \cup \{\text{Zeroes of } \det M\}.$$

In particular, if $M \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^*)) \cap \mathrm{Mat}_n(\mathcal{O}(\mathbf{C}^*))$, then Σ_M is the set of zeroes of $\det M$, and we can speak of multiplicity: the *multiplicity* of a singularity is its multiplicity as a zero of the non trivial holomorphic function $\det M$.

2.8. Birkhoff factorisation of analytic matrices

The *preliminary theorem* of Birkhoff [4, p. 266–267], stated in the basic case of a single simple contour, is the following:

Theorem 2.6. — *Let C a simple closed analytic curve on $\mathbf{S}$ separating 0 from ∞ and let $D_0 \ni 0$, $D_\infty \ni \infty$ the connected components of $\mathbf{S} \setminus C$. Let $M(x)$ an analytic invertible matrix in a neighborhood of C (i.e. $x \mapsto M(x)$ is analytic with values in $\mathrm{GL}_n(\mathbf{C})$). Then there exists open neighborhoods V_0 of $\overline{D_0}$ and V_∞ of $\overline{D_\infty}$ and analytic matrices M_0 on V_0 and M_∞ on V_∞ such that:*

(i) *$M_0 = M_\infty M$ in a neighborhood of C contained in $V_0 \cap V_\infty$.*
(ii) *M_0 is regular (i.e. holomorphic with holomorphic inverse) over V_0.*
(iii) *M_∞ is regular over $V_\infty \setminus \{\infty\}$ and holomorphic at ∞.*

Note however that the last condition cannot be sharpened, one cannot in general require that $M_\infty(\infty) \in \mathrm{GL}_n(\mathbf{C})$. It is easy to state variants, where M_0 and M_∞, resp. M_0 and M_0^{-1} exchange their positions, etc.

We shall apply the theorem with $M \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$. Then the relations $M_0 = M_\infty M$ and $M_\infty = M_0 M^{-1}$ automatically allow for an invertible meromorphic extension of M_0 and M_∞ over $\mathbf{C}$, and we simply write $M_0, M_\infty \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$. Moreover, the regularity conditions in the conclusion of the theorem then say that M_0 has the same singularities as M over $V_\infty \setminus \{\infty\}$ and that M_∞ as the same singularities as M over V_0. In particular:

Corollary 2.7. — *Let $M \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C})) \cap \mathrm{Mat}_n(\mathcal{O}(\mathbf{C}))$ with singular locus $\Sigma = \det^{-1}(0)$. We assume that $\det M$ has only simple zeroes. Let C as in the preliminary theorem (so C does not meet Σ) and write $\Sigma_0 := \Sigma \cap D_0$, $\Sigma_\infty := \Sigma \cap D_\infty$. Then one has a factorisation $M = M_0^{-1} M_\infty$ over $\mathbf{C}$, with:*

1. *M_0 is regular over $\mathbf{C} \setminus \Sigma_\infty$, M_0^{-1} is holomorphic over $\mathbf{C}$ and $\det M_0^{-1}$ has simple zeroes over Σ_∞.*
2. *M_∞ is regular over $\mathbf{C} \setminus \Sigma_0$, holomorphic over $\mathbf{S} \setminus \{0\}$ and $\det M_\infty$ has simple zeroes over Σ_0.*

2.9. Rational classification of fuchsian systems

As in the theory of differential equations, one of the main problems is the rational classification of rational systems. We say that $A, B \in \mathrm{GL}_n(\mathbf{C}(x))$ are *globally* or *rationally equivalent* if there exists a rational gauge transformation $F \in \mathrm{GL}_n(\mathbf{C}(x))$ such that $B = F[A]$. This is plainly an equivalence relation.

Again as in the theory of differential equations, the first step towards global classification is local classification. The weaker equivalence relation induced by gauge transformations $F \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$, resp. $F \in \mathrm{GL}_n(\mathbf{C}(\{1/x\}))$, is called *local analytic*[(28)] *equivalence at* 0, resp. at ∞.

As already noted in Subsection 2.5, it will be convenient to denote gauge transformations by diagrams:

$$B = F[A] := (\sigma_q F) A F^{-1} \text{ is denoted } A \xrightarrow{F} B.$$

The reason is that there is a more general notion of (rational or local analytic) morphism from $A \in \mathrm{GL}_n(\mathbf{C}(x))$ to $B \in \mathrm{GL}_p(\mathbf{C}(x))$, defined as a rectangular $p \times n$ (rational or local analytic) matrix F such that $(\sigma_q F) A = BF$. Gauge

(28) Since this work is restricted to fuchsian systems, we have no use for *formal* classification.

transformations then correspond to (rational or local analytic) isomorphisms. Thus for instance we can compose gauge transformations $A \xrightarrow{F} B$ and $B \xrightarrow{G} C$ to obtain $A \xrightarrow{GF} C$, meaning that $(GF)[A] = G[F[A]]$. We also can use commutative diagrams, invert arrows, etc.

Definition 2.8. — *The system* $A \in \mathrm{GL}_n(\mathbf{C}(x))$ *is said to be* strictly fuchsian *at* 0*, resp. at* ∞*, if* $A(0) \in \mathrm{GL}_n(\mathbf{C})$*, resp.* $A(\infty) \in \mathrm{GL}_n(\mathbf{C})$*. It is said to be* fuchsian *at* 0*, resp. at* ∞*, if it is locally analytically equivalent at* 0*, resp. at* ∞*, to a system which is strictly fuchsian at* 0*, resp. at* ∞*.*

The following was proved in [72, 2.1]:

Proposition 2.9. — *If* $A \in \mathrm{GL}_n(\mathbf{C}(x))$ *is fuchsian at* 0 *and at* ∞*, it is rationally equivalent to a system which is strictly fuchsian at* 0 *and at* ∞*.*

Note that for every gauge transformation F and every $f \in \mathbf{C}(x)^*$, one has:

$$F[A] = B \Longrightarrow F[fA] = fB \quad \text{so that} \quad A \sim B \Longrightarrow fA \sim fB$$

for any of the above equivalence relations. Thus, for the rational classification of rational systems, we may and shall restrict to the case that A is a polynomial matrix which is invertible as a rational matrix: $A \in \mathrm{GL}_n(\mathbf{C}(x)) \cap \mathrm{Mat}_n(\mathbf{C}[x])$. For fuchsian systems, this can be made more precise:

Lemma 2.10. — *Let* $B \in \mathrm{GL}_n(\mathbf{C}(x))$ *be strictly fuchsian at* 0 *and at* ∞ *and let* f *the lcm of all the denominators of its coefficients, so that* $A := fB$ *is polynomial:*

$$A = A_0 + xA_1 + \cdots + x^{\mu} A_{\mu} \in \mathrm{Mat}_n(\mathbf{C}[x]), \mu \in \mathbf{N}, A_{\mu} \neq 0.$$

and the gcd of the coefficients of A *is* 1*. Then* $A_0, A_{\mu} \in \mathrm{GL}_n(\mathbf{C})$*.*

Proof. — Since $A(0) \in \mathrm{GL}_n(\mathbf{C})$, we see that $f(0) \neq 0$, so that $A_0 \in \mathrm{GL}_n(\mathbf{C})$. At infinity, $B \equiv B(\infty)$, so $A \equiv Cx^N B(\infty)$, where CX^N is the leading term of f. Thus $N = \mu$ and $A_{\mu} = CB(\infty)$.

3. A Birkhoff type classification theorem

Birkhoff classification theorem in [4] is a form of Riemann–Hilbert correspondence for q-difference equations. For reasons explained in Part 3.1.3 (see also Part 1.4.1), we use a variant of Birkhoff connection matrix (our matrix M introduced in Corollary 3.5).

So from now on, we assume, just as Birkhoff did, that A has the form:

$$A = A_0 + xA_1 + \cdots + x^{\mu}A_{\mu} \quad \mathrm{Mat}_n(\mathbf{C}[x]), \quad \mu \quad \mathbf{N}, A_0, A_{\mu} \quad \mathrm{GL}_n(\mathbf{C}).$$

We consider as *local data* the conjugacy classes of A_0, A_{μ} (this is for 0 and ∞) and the zeroes of det $A(x)$ (this is for so-called *intermediate singularities*). We do classification for *fixed local data*. We intend, in a future work, to describe the space of monodromy data as fibered above a base, the space of possible local data.

In a first version of the theorem (Theorem 3.7), we add nonresonancy assumptions that are generically satisfied (these are the same assumptions as in [4] and also those taken by Jimbo and Sakai in [41]). Then we give a more general and slightly less precise version (Theorem 3.8) which we shall need in a special case.

Remark 3.1. — Readers interested mainly in character varieties of q-Painlevé equations should skip the proofs in this section (they are standard q-difference technology) and concentrate on the constructions and on the statements about them.

3.1. Classification theorem for non resonant systems

Here we add the following hypotheses:

- A_0 is non resonant in the following strong sense:

$$\mathrm{Sp}\, A_0 = \{\rho_1, \ldots, \rho_n\} \quad \mathbf{C} \quad \text{and} \quad i = j = \quad \rho_i \equiv \rho_j .$$

- A_{μ} is non resonant in the strong sense:

$$\mathrm{Sp}\, A_{\mu} = \{\sigma_1, \ldots, \sigma_n\} \quad \mathbf{C} \quad \text{and} \quad i = j = \quad \sigma_i \equiv \sigma_j .$$

Remark 3.2. — Non resonancy in the "weak" sense would allow for multiple eigenvalues (see Definition 2.2). This weaker property can always be achieved up to rational gauge transformation (Proposition 2.3). Actually, any fuchsian $A(x)$ is rationally equivalent to some strictly fuchsian B such that all the eigenvalues of $B(0)$ are in C_q. Strong non resonancy defined here is equivalent to weak non resonancy plus separability (all eigenvalues distinct).

Obviously, A_0 and A_{μ} are then semisimple. We shall set:

$$R := \mathrm{Diag}(\rho_1, \ldots, \rho_n), \quad S := \mathrm{Diag}(\sigma_1, \ldots, \sigma_n),$$

so that A_0 and R are conjugate, and the same for A_{μ} and S. Note that, with those notations, we implicitly fixed an order on the spectra.

From the given form $A = A_0 + \cdots + x^\mu A_\mu$, we draw:

$$\det A(x) = \sigma_1 \cdots \sigma_n (x - x_1) \cdots (x - x_N), \quad N := n\mu, \quad x_1, \ldots, x_N \quad \mathbf{C} \ ,$$

subject to Fuchs relation:

$$x_1 \cdots x_N = (-1)^N \frac{\rho_1 \cdots \rho_n}{\sigma_1 \cdots \sigma_n}$$

We shall add one more strong non resonancy condition:

- $k = l = \quad x_k \equiv x_l$.

3.1.1. Local reductions

In this section, we consider R, S and $\underline{x} := \{x_1, \ldots, x_N\}$ as fixed and subject to the above strong non resonancy conditions and also to Fuchs relation.

Let $E_{R,S,\underline{x}}$ the set of matrices $A = A_0 + \cdots + x^\mu A_\mu$ with all $A_i \quad \mathrm{Mat}_n(\mathbf{C})$ and such that:

- A_0 is conjugate to R;
- A_μ is conjugate to S;
- $\det A(x)$ vanishes at the $x_i \quad \underline{x}$.

Together, those conditions imply that $\det A(x) = \sigma_1 \cdots \sigma_n (x - x_1) \cdots (x - x_N)$. (Recall that $\deg \det A = n\mu = N$.)

We denote the equivalence relation induced on $E_{R,S,\underline{x}}$ by rational equivalence. *We intend to describe the quotient set $E_{R,S,\underline{x}}/\ $.* This is the meaning of Birkho□'s interpretation of the Riemann–Hilbert problem.

Lemma 3.3. —

(i) *Let $C \quad \mathrm{GL}_n(\mathbf{C})$ such that $A_0 = CRC^{-1}$. Then there exists a unique $M_0 \quad \mathrm{GL}_n(\mathbf{C}\{x\})$ such that $M_0(0) = C$ and $M_0[R] = A$. (Recall these notations were introduced in Subsection 2.5).*

(ii) *Let $D \quad \mathrm{GL}_n(\mathbf{C})$ such that $A_\mu = DSD^{-1}$. Then there exists a unique $M_\infty \quad \mathrm{GL}_n(\mathbf{C}\{1/x\})$ such that $M_\infty(\infty) = D$ and $M_\infty[Sx^\mu] = A$.*

(iii) *Let C , D alternative choices for the conjugating matrices C, D and M_0, M_∞ the resulting gauge transformations as in* (i), (ii). *Then there exist constant diagonal $n \times n$ matrices Γ, Δ such that $C = C\Gamma$, $D = D\Delta$; and then $M_0 = M_0\Gamma$, $M_\infty = M_\infty\Delta$.*

Proof. — Statements (i) and (ii) were proved in Subsection 2.7.

Proof of (iii). — $\Gamma := C^{-1}C$ commutes with R and $\Delta := D^{-1}D$ with S, so they are diagonal. Then $M_0\Gamma$ and $M_\infty\Delta$ satisfy the adequate relations, so by unicity they are respectively equal to M_0, M_∞.

All this can be summarized by the following commutative diagram:

$$\begin{array}{ccccc} R & & & & Sx^{\mu} \\ & \overset{M_0}{\searrow} & & \overset{M_\infty}{\swarrow} & \\ \Gamma\,\Big| & & A & & \Big|\,\Delta \\ & \underset{M_0}{\nearrow} & & \underset{M_\infty}{\nwarrow} & \\ R & & & & Sx^{\mu} \end{array}$$

Proposition 3.4 (Properties of M_0 and M_∞). —

(i) $M_0 \in \mathrm{GL}_n(\mathbf{C}\{x\})$ *admits a unique extension* $M_0 \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$ *such that:*
  - M_0 *has simple poles over* $\underline{x}q^{-\mathbf{N}}$ *(and nowhere else);*
  - M_0^{-1} *is holomorphic all over* $\mathbf{C}$*.*

(ii) $M_\infty \in \mathrm{GL}_n(\mathbf{C}\{1/x\})$ *admits a unique extension* $M_\infty \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}_\infty))$ *such that:*
  - M_∞ *is holomorphic all over* $\mathbf{C}_\infty$*;*
  - M_∞^{-1} *has simple poles over* $\underline{x}q^{\mathbf{N}}$ *(and nowhere else).*

Proof. —

(i). — We use the arrow $R \xrightarrow{M_0} A$, i.e. the equality $A = M_0[R] = (\sigma_q M_0)RM_0^{-1}$ first in the clearly equivalent forms: $M_0 = A^{-1}(\sigma_q M_0)R$ and $M_0^{-1} = R^{-1}(\sigma_q M_0^{-1})A$.

The second relation allows us to extend M_0^{-1}, which is initially defined and holomorphic over some open disk $\mathring{\mathrm{D}}(0,r)$, $r > 0$, to $\mathring{\mathrm{D}}(0,q^{-1}r)$, where $\left|q^{-1}r\right| = |q|^{-1}r > r$ since $|q| < 1$. Iterating, we get a holomorphic extension to $\mathbf{C}$.

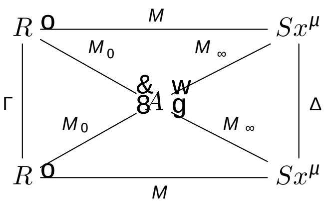
The first relation shows that on any open disk $\mathring{\mathrm{D}}(0,r)$, M_0 has the same poles as $\sigma_q M_0$, i.e. those of M_0 over the smaller disk $\mathring{\mathrm{D}}(0,qr)$; but one must add the poles of A^{-1} if any. Iterating yields the conclusion.

One could also argue using only the determinant of the second relation: $\frac{\sigma_q \det M_0^{-1}}{\det M_0^{-1}} = \frac{\det A}{\rho_1 \cdots \rho_n}$.

(ii). — Similarly, the arrow $Sx^{\mu} \xrightarrow{M_\infty} A$, i.e. the equality $A = M_\infty[Sx^{\mu}] = (\sigma_q M_\infty)(Sx^{\mu})M_\infty^{-1}$ translate into $\sigma_q M_\infty = AM_\infty(Sx^{\mu})^{-1}$ and the argument goes on the same lines (here we use σ_q to expand disks centered at ∞).

3.1.2. Connection matrix and Riemann–Hilbert–Birkhoff correspondence

This section can be best understood with the help of the following commutative diagram:



For instance the south-west and south-east diagonal arrows respectively mean that $M_0[R] = A$ and that $M_\infty[Sx^\mu] = A$, so that:

$$\left((M_0)^{-1}M_\infty\right)[Sx^\mu] = R,$$

and similarly for M_0 and M_∞. All this can be read on the diagram.

Corollary 3.5. — *Set* $M := M_0^{-1}M_\infty \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$. *Then:*

- $\sigma_q M = RM(Sx^\mu)^{-1}$.
- M *is holomorphic all over* $\mathbf{C}$.
- M^{-1} *has simple poles over* $[\underline{x};q] = \underline{x}q^{\mathbf{Z}}$ *(and nowhere else); equivalently,* $\det M$ *has simple zeroes over* $[\underline{x};q]$ *(and nowhere else).*

We shall write $F_{R,S,\underline{x}}$ the set of such matrices:

$$F_{R,S,\underline{x}} := \left\{ M \in \mathrm{Mat}_n(\mathcal{O}(\mathbf{C})) \;\middle|\; \begin{array}{l}\sigma_q M = RM(Sx^\mu)^{-1} \text{ and all zeroes of} \\ \det M \text{ are simple and lay over } [\underline{x};q]\end{array} \right\}.$$

Note that this set actually depends only of the image of $\underline{x}$ in $\mathbf{E}_q$, not on $\underline{x} \in \mathbf{C}$ itself.

Gauge freedom. We saw that, A being given, M_0 and M_∞ are uniquely determined up to the right action of the group $\mathrm{D}_n(\mathbf{C}) \subset \mathrm{GL}_n(\mathbf{C})$ of diagonal matrices. From the relations $M_0 \leftarrow M_0\Gamma$, $M_\infty \leftarrow M_\infty\Delta$, $\Gamma, \Delta \in \mathrm{D}_n(\mathbf{C})$, we deduce that $M \leftarrow \Gamma^{-1}M\Delta$. We are thus led to introduce the following right action of $\mathrm{D}_n(\mathbf{C}) \times \mathrm{D}_n(\mathbf{C})$ on $F_{R,S,\underline{x}}$:

$$M^{(\Gamma,\Delta)} := \Gamma^{-1}M\Delta.$$

The reader may check that this is indeed a right action[29] and that $F_{R,S,\underline{x}}$ is stable under this action. We shall write $M \mapsto M^{(\Gamma,\Delta)}$ the corresponding

[29] Later in the text, we shall rather use the left action $M \to \Gamma M \Delta^{-1}$. Of course, the equivalence classes (orbits) are the same.

equivalence relation on $F_{R,S,\underline{x}}$ and $F_{R,S,\underline{x}}/\sim$ the quotient of $F_{R,S,\underline{x}}$ under this action and equivalence relation. As a consequence, we see that we have constructed a well defined map:

$$E_{R,S,\underline{x}} \longrightarrow F_{R,S,\underline{x}}/\sim,$$

mapping A to the equivalence class of M.

From now on, we shall write:

$$\mathcal{E}_{R,S,\underline{x}} := E_{R,S,\underline{x}}/\sim \quad \text{and} \quad \mathcal{F}_{R,S,\underline{x}} := F_{R,S,\underline{x}}/\sim.$$

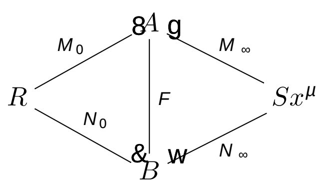
Moreover, if no confusion is to be feared, we shall frequently omit the indication of local data and abreviate:

$$\mathcal{E} := \mathcal{E}_{R,S,\underline{x}} \quad \text{and} \quad \mathcal{F} := \mathcal{F}_{R,S,\underline{x}}.$$

Proposition 3.6. — *The above map goes to the quotient and defines a "Riemann–Hilbert–Birkhoff correspondence":*

$$\mathcal{E}_{R,S,\underline{x}} = E_{R,S,\underline{x}}/\sim \;\longrightarrow\; \mathcal{F}_{R,S,\underline{x}} = F_{R,S,\underline{x}}/\sim. \tag{3.1}$$

Proof. — Let $B = B_0 + \cdots + B_\mu x^\mu \in E_{R,S,\underline{x}}$, (so that $B_0 \sim R$, $B_\mu \sim S$ and $\det B$ has simple zeroes at $\underline{x}$) and assume that $B = F[A]$, $F \in \mathrm{GL}_n(\mathbf{C}(x))$. We have a commutative diagram:



Let $\Gamma := N_0^{-1} F M_0 \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$ and $\Delta := N_\infty^{-1} F M_\infty \in \mathrm{GL}_n(\mathbf{C}(\{1/x\}))$. Then $\Gamma = (\gamma_{i,j})_{1 \leqslant i,j \leqslant n}$ is an automorphism of R and $\Delta = (\delta_{i,j})_{1 \leqslant i,j \leqslant n}$ is an automorphism of Sx^μ, so that:

$$\Gamma[R] = R \iff \sigma_q \Gamma R = R \Gamma \iff \forall i,j = 1,\ldots,n\,,\ \sigma_q \gamma_{i,j} = \frac{\rho_i}{\rho_j} \gamma_{i,j}.$$

Since $\gamma_{i,j} \in \mathbf{C}(\{x\})$ and $\rho_i/\rho_j \notin q^{\mathbf{Z}}$ for $i \neq j$, we conclude that $\gamma_{i,j} = 0$ for $i \neq j$ and that $\gamma_{i,i} \in \mathbf{C}^*$ (it cannot be 0 since Γ is invertible) so at last $\Gamma \in \mathrm{D}_n(\mathbf{C})$. A similar argument works for Δ (the scalar x^μ factor gets simplified at once).

Now, from $FM_0 = N_0 \Gamma$ and $FM_\infty = N_\infty \Delta$ we draw:

$$\begin{aligned} M = M_0^{-1} M_\infty &= (FM_0)^{-1} F M_\infty \\ &= (N_0 \Gamma)^{-1} N_\infty \Delta = \Gamma^{-1} N_0^{-1} N_\infty \Delta = \Gamma^{-1} N \Delta \end{aligned}$$

as expected.

Theorem 3.7 ("Riemann–Hilbert–Birkhoff correspondence", first version). — *The map* (3.1) *defined in Proposition 3.6 is bijective.*

Proof. — The proof comes in two parts.

Injectivity. — Using the usual notations, let $A \in E_{R,S,\underline{x}}$, resp. $B \in E_{R,S,\underline{x}}$, have image the class of $M = M_0^{-1}M_\infty$, resp. the class of $N = N_0^{-1}N_\infty$ in $F_{R,S,\underline{x}}/\sim$ and assume these images are the same, that is $M \sim N$, so that $M = \Gamma^{-1}N\Delta$ where $\Gamma, \Delta \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$. Then:

$$M_0^{-1}M_\infty = \Gamma^{-1}N_0^{-1}N_\infty\Delta \quad \Longrightarrow \quad N_0\Gamma M_0^{-1} = N_\infty\Delta M_\infty^{-1}.$$

Call the latter matrix F. Then:

$$F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C})) \cap \mathrm{GL}_n(\mathcal{M}(\mathbf{C}_\infty)) = \mathrm{GL}_n(\mathcal{M}(\mathbf{S})) = \mathrm{GL}_n(\mathbf{C}(x)).$$

On the other hand, we have a commutative diagram:

$$\begin{array}{ccccc} R & \xrightarrow{M_0} & A & \xleftarrow{M_\infty} & Sx^\mu \\ \Big\downarrow \Gamma & & \vdots F & & \Big\downarrow \Delta \\ R & \xrightarrow{N_0} & B & \xleftarrow{N_\infty} & Sx^\mu \end{array}$$

in which $F = N_0\Gamma M_0^{-1} = N_\infty\Delta M_\infty^{-1}$ is, by force, an isomorphism, i.e. $F[A] = B$, so that the classes of A and B in $E_{R,S,\underline{x}}/\sim$ are the same, which concludes the proof of injectivity. Note that the relation $F[A] = B$ can also be deduced by direct computation:

$$\begin{aligned} FM_0 = N_0\Gamma \quad &\Longrightarrow \quad \sigma_q(FM_0)R(FM_0)^{-1} = \sigma_q(N_0\Gamma)R(N_0\Gamma)^{-1} \\ &\Longrightarrow \quad \sigma_q F A F^{-1} = B. \end{aligned}$$

Surjectivity. — $R, S, \underline{x}$ being given (and satisfying Fuchs relation), let $M \in F_{R,S,\underline{x}}$. We draw on the Riemann sphere $\mathbf{S}$ a closed analytic curve separating $\underline{x}$ from $q\underline{x}$, so that, with the notations of Subsection 2.8, $\Sigma_0 = q^{\mathbf{N}}\underline{x}$ and $\Sigma_\infty = q^{-\mathbf{N}}\underline{x}$. Using Birkhoff factorisation theorem and in particular Corollary 2.7, we obtain a decomposition $M = M_0^{-1}M_\infty$, where:

- M_0 is regular (i.e. holomorphic, invertible with holomorphic inverse) on $\mathbf{C} \setminus q^{-\mathbf{N}}\underline{x}$, with simple poles on $q^{-\mathbf{N}}\underline{x}$.
- M_0^{-1} is holomorphic over $\mathbf{C}$ and $\det M_0$ has simple zeroes $q^{-\mathbf{N}}\underline{x}$.
- M_∞ is regular on $\mathbf{C}^* \setminus q^{\mathbf{N}}\underline{x}$, it is holomorphic over $q^{\mathbf{N}}\underline{x}$ and $\det M_\infty$ has simple zeroes there. It is also holomorphic at ∞, although it cannot be required to be regular there.
- M_∞^{-1} is holomorphic over $\mathbf{C}^*$ and meromorphic at ∞.

From the condition $\sigma_q M = RM(Sx^\mu)^{-1}$, expressed by the arrow $R \overset{M}{\leftarrow} Sx^\mu$, we see that $M_0[R] = M_\infty[Sx^\mu]$. Call A this matrix, whence a diagram:

$$R \xrightarrow{M_0} A \xleftarrow{M_\infty} Sx^\mu$$

We want to show that $A \in E_{R,S,\underline{x}}$ and that its class in $\mathcal{E}_{R,S,\underline{x}} = E_{R,S,\underline{x}}/\sim$ is the preimage of the class of M in $\mathcal{F}_{R,S,\underline{x}} = F_{R,S,\underline{x}}/\sim$. Clearly:

$$A \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C})) \cap \mathrm{GL}_n(\mathcal{M}(\mathbf{C}_\infty)) = \mathrm{GL}_n(\mathcal{M}(\mathbf{S})) = \mathrm{GL}_n(\mathbf{C}(x)).$$

Actually, more can be said. From the listed properties of M_0 and M_∞, one gets that A is holomorphic over $\mathbf{C}$ and meromorphic at ∞, whence polynomial:

$$A = A_0 + \cdots + x^d A_d, \quad A_0, \ldots, A_d \in \mathrm{Mat}_n(\mathbf{C}),\ A_d \neq 0.$$

Here $A_0 = A(0) = CRC^{-1} \in \mathrm{GL}_n(\mathbf{C})$, where $C = M_0(0) \in \mathrm{GL}_n(\mathbf{C})$.

From the given relations, we also see that $a(x) := \det A(x)$ has simple zeroes at $\underline{x}$ and nowhere else. Thus $a(x) = s(x - x_1)\cdots(x - x_N)$ for some $s \in \mathbf{C}^*$. Then $a(0) = s(-1)^N x_1 \cdots x_N$, but also $a(0) = \det A_0 = \det CRC^{-1} = \det R$, so by Fuchs relation $s = \det S$. Now we use the relation $A = M_\infty[Sx^\mu]$; taking the determinant and setting $f := \det M_\infty \in \mathbf{C}(\{1/x\})^*$, we draw:

$$\frac{\sigma_q f}{f} = \frac{a}{sx^N} = \prod_{i=1}^{N}(1 - x_i/x) \implies f = \phi \prod_{i=1}^{N} \frac{1}{(x_i/x; q)_\infty},$$

where ϕ is elliptic. But at the same time, $\phi \in \mathbf{C}(\{1/x\})^*$, so that actually $\phi \in \mathbf{C}^*$. This implies that $f(\infty) = \phi$, so that M_∞ is regular at ∞ (which Birkhoff factorisation did not automatically imply). Setting $D := M_\infty(\infty) \in \mathrm{GL}_n(\mathbf{C})$, we see that $A = M_\infty[Sx^\mu]$ is asymptotic to $(DSD^{-1})x^\mu$ at ∞. Since it is also asymptotic to $x^d A_d$, we get that $d = \mu$ and $A_d = DSD^{-1} \in \mathrm{GL}_n(\mathbf{C})$. It is then immediate that $A \in E_{R,S,\underline{x}}$ and the fact that its class in $\mathcal{E}_{R,S,\underline{x}} = E_{R,S,\underline{x}}/\sim$ is the antecedent of the class of M in $\mathcal{F}_{R,S,\underline{x}} = F_{R,S,\underline{x}}/\sim$ follows from the various equalities we found out during the computation (i.e. the construction of M goes through the M_0 and M_∞ used in the proof).

3.1.3. Comparison with Birkhoff classification

Recall the notation $e_{q,c}$ at the end of 2.3. Let $e_R := \mathrm{Diag}(e_{q,\rho_1}, \ldots, e_{q,\rho_n})$, so that $\sigma_q e_R = Re_R = e_R R$. Then $X^{(0)} := M_0 e_{q,R}$ is a solution of the system $\sigma_q X = AX$ which maybe considered "local at 0".

Let likewise $e_S := \mathrm{Diag}(e_{q,\sigma_1}, \ldots, e_{q,\sigma_n})$, so that $\sigma_q e_S = Se_S = e_S S$. Then $X^{(\infty)} := M_\infty e_{q,S}\, \theta_q{}^{-\mu}$ is a solution of the system $\sigma_q X = AX$ which maybe considered "local at ∞".

Birkhoff connection matrix[30] is then defined as:

$$P := X_0^{-1} X_\infty \in \mathrm{GL}_n(M(\mathbf{C}^*))^{\sigma_q} = \mathrm{GL}_n(M(\mathbf{E}_q)).$$

In our notations, $P = e_R^{-1} M e_{q,S}\, \theta_q{}^{-\mu}$. We think of M as freed from the local contributions at ∞ present in P. In the interpretation of Birkhoff connection matrix as encoding the monodromy, we think of M as encoding specifically the "intermediate" monodromy related to the singularities of A on $\mathbf{C}^*$. As explained in Part 1.4.1, this was shown to be necessary in Galois theory [73] and we think it to be useful here as well.

3.2. A more general classification theorem

The previous result, Theorem 3.7, is only valid for semi-simple local data R and S (equivalently, $A(0)$ and $A(\infty)$). Here we relax this assumption. In the next version, Theorem 3.8, we take the following local data:

- $R, S \in \mathrm{GL}_n(\mathbf{C})$ such that $\mathrm{Sp}\, R, \mathrm{Sp}\, S \subset C_q$;
- $x_1, \ldots, x_N \in \mathbf{C}^*$ such that $i \neq j \Rightarrow x_i \not\equiv x_j$,
- and moreover subject to Fuchs relation: $x_1 \cdots x_N = (-1)^N \frac{\det R}{\det S}$.

We define the set $E_{R,S,\underline{x}}$ and its equivalence relation exactly like at the beginning of Part 3.1.1; and the set $F_{R,S,\underline{x}}$ in the same way as just after Corollary 3.5. But we take the equivalences $M \sim M^{(\Gamma,\Delta)}$ among those induced by matrices $\Gamma, \Delta \in \mathrm{GL}_n(\mathbf{C})$ such that Γ commutes with R and Δ commutes with S:

$$M, N \in F_{R,S,\underline{x}},\ M \sim N \underset{\mathrm{def}}{\Longleftrightarrow} \exists \Gamma, \Delta \in \mathrm{GL}_n(\mathbf{C}) : \begin{cases} N = M^{(\Gamma,\Delta)} := \Gamma^{-1} M \Delta, \\ [\Gamma, R] = [\Delta, S] = 0. \end{cases}$$

(Under the assumptions of Part 3.1.1, this boils down to the previous definition.)

Theorem 3.8 ("Riemann–Hilbert–Birkhoff correspondence", second version). — *There is a natural bijection:*

$$\mathcal{E}_{R,S,\underline{x}} = E_{R,S,\underline{x}}/\sim \ \longrightarrow\ \mathcal{F}_{R,S,\underline{x}} = F_{R,S,\underline{x}}/\sim .$$

[30] The original definition of Birkhoff (taken up by Jimbo and Sakai) involved multivalued choices as solutions of the elementary equations $\sigma_q e_c = c e_c$, resp. $\sigma_q f = x^{-1} f$, such as $x^{\log_q c}$, resp. $q^{-\log_q x(\log_q x - 1)/2}$. This does not impact the present discussion.

Proof. — We only sketch the modifications to the proof of the first version. According to Subsection 2.6, we can write $A = M^{(0)}[R] = M^{(\infty)}[Sx^{\mu}]$, but these gauge matrices are not unique (and respectively have poles at $0, \infty$).

The polarity properties of $M^{(0)}$ and $M^{(\infty)}$ on $\mathbf{C}^*$ are exactly the same as before, because we only used the fact that these matrices were regular (i.e. holomorphic with holomorphic inverse) in a punctured neighborhood of 0, resp. ∞. The matrix $M := (M^{(0)})^{-1}M^{(\infty)}$ belongs to $F_{R,S,\underline{x}}$. We thus obtain a correspondence between $E_{R,S,\underline{x}}$ and $F_{R,S,\underline{x}}$, but not a mapping in either direction.

Let $N^{(0)}$, $N^{(\infty)}$ another choice of gauge transformations realizing the same reductions. Then $\Gamma := (M^{(0)})^{-1}N^{(0)}$ is such that $\Gamma[R] = R$. Using Subsection 2.6 one shows that $\Gamma \in \mathrm{GL}_n(\mathbf{C})$ and $[\Gamma, R] = 0$. Similarly for Δ. This gives an injective map $\mathcal{E}_{R,S,\underline{x}} = E_{R,S,\underline{x}}/\sim \; \to \; \mathcal{F}_{R,S,\underline{x}} = F_{R,S,\underline{x}}/\sim$. The rest of the argument does not change.

Remark 3.9. — Local data R and S play a symmetric role in the following sense: $M \in F_{R,S,\underline{x}} \Leftrightarrow {}^tM \in F_{{}^tS,{}^tR,\underline{x}}$; also the equivalence relations on these two sets correspond to each other. This observation allows one to shorten some case studies.

3.3. Reducibility criteria

We shall have need for the possibility of determining if a system $\sigma_q X = AX$ is reducible by looking at its image (R, M, S) by the Riemann–Hilbert–Birkhoff correspondence. We give such a criterion in the generic case, the general case can be tackled similarly but the formulation would be more complicated and we do not need it here.

Theorem 3.10. — *We assume strong non resonancy as above, i.e. the n eigenvalues of R, resp. of S, are pairwise non congruent modulo $q^{\mathbf{Z}}$. Then the system $\sigma_q X = AX$ is reducible if, and only if some matrix M' obtained from M by permutation of lines and columns is block triangular.*

Corollary 3.11. — *If $n = 2$, under the same generic assumptions, reducibility is equivalent to: M has a zero coefficient.*

To prove the theorem we use a tannakian criterion based on the Galois theory as expounded for instance in [62]. Recall that to the system $\sigma_q X = AX$ is attached a Galois *groupoid* with base $\{0, \infty\}$ and its canonical representation. It can be realized as $(G(0), G(0, \infty), G(\infty))$ operating on $(V(0), V(\infty))$, i.e. $G(0)$, resp. $G(\infty)$ is an algebraic group of automorphisms of the linear space $V(0)$, resp. $V(\infty)$; and $G(0,\infty)$ is a set of isomorphisms $V(0) \to V(\infty)$. By Galois correspondence, reducibility of the system is equivalent to reducibility of the representation, i.e. to the existence of non trivial (non zero and non whole) subspaces V_0 of $V(0)$ and V_∞ of $V(\infty)$ such that $G(0)$ leaves V_0 stable, $G(\infty)$ leaves V_∞ stable and $G(0,\infty)$ sends V_0 to V_∞.

We can and will take $V(0) = V(\infty) = \mathbf{C}^n$. After [62] we introduce:

(1) The subgroup G_0 of $G(0) \subset \mathrm{GL}_n(\mathbf{C})$ made up of all diagonal matrices $\mathrm{Diag}(\phi(\rho_1), \ldots, \phi(\rho_n))$ where $\phi : \mathbf{C}^* \to \mathbf{C}^*$ is a group morphism such that $\phi(q) = 1$;
(2) the subgroup G_∞ of $G(\infty) \subset \mathrm{GL}_n(\mathbf{C})$ built similarly, except that we also allow invertible scalar matrices(31) and of course all resulting products;
(3) the subset $G_{0,\infty}$ of $G(0,\infty)$ made up of all values $M(x)^{-1}$ at regular points (i.e. where $M(x)$ is invertible).

A preliminary fact is that these three components are included in the corresponding components of the Galois groupoid. This is analogous to the classical fact that the monodromy group of a (complex, linear, analytic) differential equation is included in its Galois group. Then a Schlesinger type density theorem states that the whole Galois groupoid is the Zariski closure of the subgroupoid generated by these three components (two local components and a connection component).

Non congruent elements of $\mathbf{C}^*$ can be separated by morphisms of the above kind. Under our strong non resonancy assumption, this implies that the subspaces of $V(0) = \mathbf{C}^n$ stable under G_0 are exactly those generated by a subset of the canonical basis; and the same at ∞. It follows that the canonical representation is reducible if and only if there are two non trivial (non empty, non whole) subsets B_0, B_∞ of the canonical basis such that all invertible values $M(x)$ send $\mathrm{Vect}(B_\infty)$ isomorphically to $\mathrm{Vect}(B_0)$. The criterion of the theorem is just a rephrasing of that fact.

3.4. The hypergeometric class

This is the case(32) $n = 2$, $\mu = 1$, $N = 2$, $x_1 x_2 = \det R / \det S$, $x_1/x_2 \notin q^{\mathbf{Z}}$. As in the case of ordinary differential equations, we shall (generically) find *rigidity*, i.e. these local data being fixed, there are no continuous moduli.

(31) This accounts for the non trivial “slope” μ in Sx^μ.

(32) We shall abusively call q-hypergeometric the systems classified herebelow, without checking if they really come from a q-hypergeometric equation. By [66] this is generically true but certainly false if the system is reducible, i.e., from the above study, if at least one coefficient of M vanishes.

We may assume that each R, S take one of the following forms, respectively called *generic*, *trivial* and *logarithmic*:

$$R = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}, \quad \rho_1 = \rho_2, \text{ or } \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \text{ or } \begin{pmatrix} \rho & \rho \\ 0 & \rho \end{pmatrix}$$

$$S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \sigma_1 = \sigma_2, \text{ or } \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \text{ or } \begin{pmatrix} \sigma & \sigma \\ 0 & \sigma \end{pmatrix}$$

where $\rho_1, \rho_2, \rho, \sigma_1, \sigma_2, \sigma \in \mathbf{C}_q$. Also we shall write for short $E := E_{R,S,\underline{x}}$ and $F := F_{R,S,\underline{x}}$; and also $\mathcal{E} := E/\sim$ and $\mathcal{F} := F/\sim$.

Note that, from the relations $x_1 x_2 = \det R/\det S$ and $\sigma_q(\det M) = (\det R/\det S)\det M$, we draw, using Subsection 2.4, that $\det M$ vanishes at x_1 if, and only if, it vanishes at x_2, so we need use only one of these conditions to test whether $M \in F$.

Also note that cases 1, 2 and 3 herebelow are special in that x_1, x_2 are imposed by R, S (other values would mean that E is empty); and that cases 1 and 3 do not fall under our assumptions for x_1, x_2 (but we all the same describe $\mathcal{E}$ and $\mathcal{F}$).

Case 1, trivial/trivial: $R = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$, $S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$. — Here $E = \{(\rho + \sigma x) I_2\}$ and *a fortiori* $\mathcal{E}$ is a singleton. Also $\det A(x) = (\rho + \sigma x)^2$ so $x_1 = x_2 = -\rho/\sigma$ so we are not within the assumptions of our theorem. Actually, one sees easily that matrices $M \in F$ have the form $\theta_q\left(\frac{\sigma}{\rho} x\right) C$, $C \in \mathrm{GL}_2(\mathbf{C})$ being arbitrary, with equivalences $M \sim \Gamma^{-1} M \Delta$ for arbitrary $\Gamma, \Delta \in \mathrm{GL}_2(\mathbf{C})$ so $\mathcal{F}$ is also a singleton.

Case 2, trivial/generic: $R = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$, $S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$. — Here, $E = \left\{\text{conjugates of } \rho I_2 + x S = \begin{pmatrix} \rho + \sigma_1 x & 0 \\ 0 & \rho + \sigma_2 x \end{pmatrix}\right\}$ (conjugacy by $\mathrm{GL}_2(\mathbf{C})$), so $\mathcal{E}$ is a singleton. We have $\det A(x) = (\rho + \sigma_1 x)(\rho + \sigma_2 x)$ so the only non void possibility (up to reindexing) is $x_i = -\rho/\sigma_i$, $i = 1, 2$; we are under the assumptions of our theorem. Matrices $M \in F$ have the form

$$M = \begin{pmatrix} \alpha_{1,1}\, \theta_q(\frac{\sigma_1}{\rho} x) & \alpha_{1,2}\, \theta_q(\frac{\sigma_2}{\rho} x) \\ \alpha_{2,1}\, \theta_q(\frac{\sigma_1}{\rho} x) & \alpha_{2,2}\, \theta_q(\frac{\sigma_2}{\rho} x) \end{pmatrix} = C T(x)$$

$$\text{where } C \in \mathrm{Mat}_2(\mathbf{C}) \text{ and } T(x) := \begin{pmatrix} \theta_q(\frac{\sigma_1}{\rho} x) & 0 \\ 0 & \theta_q(\frac{\sigma_2}{\rho} x) \end{pmatrix}.$$

Since $\det T$ vanishes at x_1, x_2 but not identically, we get that $E = \{C T(x) \mid C \in \mathrm{GL}_2(\mathbf{C})\}$. The action of Γ, Δ comes here with arbitrary Γ so $\mathcal{F}$ is also a singleton.

Case 3, trivial/logarithmic: $R = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$, $S = \begin{pmatrix} \sigma & \sigma \\ 0 & \sigma \end{pmatrix}$. — Here, $E = \{\text{conjugates of } \rho I_2 + x S\}$ (conjugacy by $\mathrm{GL}_2(\mathbf{C})$), so $\mathcal{E}$ is a singleton.

We have $\det A(x) = (\rho + \sigma x)^2$ so the only non void possibility is $x_1 = x_2 = -\rho/\sigma$; we are not under the assumptions of our theorem. Yet we go on! Coefficients of matrices $(m_{i,j}) \in F$ must satisfy the functional equations

$$\left\{\begin{array}{l} \sigma_q m_{i,1} = \frac{\rho}{\sigma x} m_{i,1}, \\ \sigma_q m_{i,2} = \frac{\rho}{\sigma x}(m_{i,2} - m_{i,1}), \end{array}\right. \quad \text{for } i = 1, 2,$$

which we solve using Lemma 2.1. More precisely, holomorphic solutions of $\sigma_q f = \frac{\rho}{\sigma x} f$ have the form $f = \alpha\,\theta_q(\frac{\sigma}{\rho}x)$ and then holomorphic solutions of $\sigma_q g = \frac{\rho}{\sigma x}(g - f)$ have the form $g = \alpha\frac{\sigma}{\rho}x\theta_q(\frac{\sigma}{\rho}x) + \beta\,\theta_q(\frac{\sigma}{\rho}x)$. Therefore, matrices $M \in F$ have the form

$$\begin{aligned} M &= \begin{pmatrix} \alpha_1\,\theta_q(\frac{\sigma}{\rho}x) & \alpha_1\frac{\sigma}{\rho}x\theta_q(\frac{\sigma}{\rho}x) + \beta_1\,\theta_q(\frac{\sigma}{\rho}x) \\ \alpha_2\,\theta_q(\frac{\sigma}{\rho}x) & \alpha_2\frac{\sigma}{\rho}x\theta_q(\frac{\sigma}{\rho}x) + \beta_2\,\theta_q(\frac{\sigma}{\rho}x) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} T(x), \quad \text{where } T(x) := \begin{pmatrix} \theta_q(\frac{\sigma_1}{\rho}x) & \frac{\sigma}{\rho}x\theta_q(\frac{\sigma}{\rho}x) \\ 0 & \theta_q(\frac{\sigma_2}{\rho}x) \end{pmatrix}. \end{aligned}$$

Since $\det T$ is non trivial but vanishes at $x_1 = x_2$, we get that $F = \{CT(x) \mid C \in \mathrm{GL}_2(\mathbf{C})\}$. The action of Γ, Δ comes here with arbitrary Γ so F is again a singleton.

In cases 4, 5 and 6 (those truly of interest), the space E and its quotient $\overline{E}$ are more complicated to study directly so our technology comes handy.

Case 4, generic/generic: $R = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$, $S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$. — Matrices $M \in F$ have the form

$$M = \begin{pmatrix} \alpha_{1,1}\,\theta_q(\frac{\sigma_1}{\rho_1}x) & \alpha_{1,2}\,\theta_q(\frac{\sigma_2}{\rho_1}x) \\ \alpha_{2,1}\,\theta_q(\frac{\sigma_1}{\rho_2}x) & \alpha_{2,2}\,\theta_q(\frac{\sigma_2}{\rho_2}x) \end{pmatrix}.$$

Such a matrix is completely determined by the quadruple $(\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2}) \in \mathbf{C}^4$. Since the functions $\theta_q(\frac{\sigma_1}{\rho_1}x)\,\theta_q(\frac{\sigma_2}{\rho_2}x)$ and $\theta_q(\frac{\sigma_1}{\rho_2}x)\,\theta_q(\frac{\sigma_2}{\rho_1}x)$ are linearly independent, the condition that $\det M$ does not vanish identically is equivalent to $(\alpha_{1,1}\alpha_{2,2}, \alpha_{1,2}\alpha_{2,1}) = (0, 0)$. On the other hand, the gauge freedom on F is expressed by the fact that arbitrary invertible diagonal matrices $\mathrm{Diag}(\gamma_1, \gamma_2), \mathrm{Diag}(\delta_1, \delta_2)$ act on M, so that:

$$(\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2}) \mapsto \left(\frac{\delta_1}{\gamma_1}\alpha_{1,1}, \frac{\delta_2}{\gamma_1}\alpha_{1,2}, \frac{\delta_1}{\gamma_2}\alpha_{2,1}, \frac{\delta_2}{\gamma_2}\alpha_{2,2}\right)$$

for all $\delta_1, \delta_2, \gamma_1, \gamma_2 \in \mathbf{C}^*$, which in turn implies that:

$$(\alpha_{1,1}\alpha_{2,2}, \alpha_{1,2}\alpha_{2,1}) \mapsto \frac{\delta_1\delta_2}{\gamma_1\gamma_2}(\alpha_{1,1}\alpha_{2,2}, \alpha_{1,2}\alpha_{2,1}).$$

We thus obtain a well defined map $M \to \alpha(M) := (\alpha_{1,1}\alpha_{2,2} : \alpha_{1,2}\alpha_{2,1})$ from $F/\sim$ to $\mathbf{P}^1(\mathbf{C})$. We shall see that α is "almost injective". On the other hand, the condition $\det M(x_1) = 0$ (equivalently $\det M(x_2) = 0$) says that the image of this map is reduced to a single point. To make this more precise while legible, we identify $\mathbf{P}^1(\mathbf{C})$ with $\mathbf{C}\quad\{\infty\}$ and $(a_1 : a_2)\quad\mathbf{P}^1(\mathbf{C})$ with a_1/a_2. Also we introduce the q-elliptic function:

$$\Phi(x) := \frac{\theta_q(\frac{\sigma_1}{\rho_1}x)\,\theta_q(\frac{\sigma_2}{\rho_2}x)}{\theta_q(\frac{\sigma_1}{\rho_2}x)\,\theta_q(\frac{\sigma_2}{\rho_1}x)}\cdot$$

Then $\det M(x_1) = 0 \Leftrightarrow \alpha(M) = \frac{1}{\Phi(x_1)}\cdot$ The apparent dissymmetry of this condition with respect with x_1, x_2 disappears if one notes that Φ admits an involution under which $x_1 \leftrightarrow x_2$:

$$\Phi(x\ ) = \Phi(x\ ) \qquad x\,x \equiv \frac{\rho_1\rho_2}{\sigma_1\sigma_2} \text{ or } x = x \quad .$$

In the last step, we want to recover the quadruple $(\alpha_{i,j})$ (up to the gauge action by Γ, Δ) from the point $\alpha(M) = \frac{1}{\Phi(x_1)}$ in $\mathbf{P}^1(\mathbf{C})$. A small computation shows that for $\alpha(M) = 0, \infty$, that is for $\alpha(M)\quad\mathbf{C}$, the preimage is unique; while for $\alpha(M) = 0$ there are three preimages: $(0, 1, 1, 0)$, $(0, 1, 1, 1)$ and $(1, 1, 1, 0)$; and for $\alpha(M) = \infty$, there are three preimages: $(1, 0, 0, 1)$, $(1, 1, 0, 1)$ and $(1, 0, 1, 1)$. The conclusion is that the nature of $F = F/\quad$ depends on the element $\Phi(x_1) = \Phi(x_2)\quad\mathbf{P}^1(\mathbf{C})$, which is totally determined by the local data. If this element is 0 or ∞, the space F has three elements; otherwise (thus generically) it is a singleton.

Remark 3.12. — The special cases $\Phi(x_1) = \Phi(x_2)\quad\{0, \infty\}$ correspond to those when one of the coeﬃcients $\alpha_{i,j}$ vanishes, i.e., by Corollary 3.11, to the case of a reducible system.

Case 5, generic/logarithmic: $R = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$, $S = \begin{pmatrix} \sigma & \sigma \\ 0 & \sigma \end{pmatrix}$. — The functional equation for $M = (m_{i,j})$ is:

$$\sigma_q M = \begin{pmatrix} \frac{\rho_1}{\sigma x} m_{1,1} & \frac{\rho_1}{\sigma x}(m_{1,2} - m_{1,1}) \\ \frac{\rho_2}{\sigma x} m_{2,1} & \frac{\rho_2}{\sigma x}(m_{2,2} - m_{2,1}) \end{pmatrix},$$

so using Part 2.4.2 more or less as in case 3, we get:

$$\begin{aligned} M &= \begin{pmatrix} \alpha_1\,\theta_q(\frac{\sigma}{\rho_1}x) & \alpha_1\frac{\sigma}{\rho_1}x\theta_q(\frac{\sigma}{\rho_1}x) + \beta_1\,\theta_q(\frac{\sigma}{\rho_1}x) \\ \alpha_2\,\theta_q(\frac{\sigma}{\rho_2}x) & \alpha_2\frac{\sigma}{\rho_2}x\theta_q(\frac{\sigma}{\rho_2}x) + \beta_2\,\theta_q(\frac{\sigma}{\rho_2}x) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1\phi_1 & \alpha_1 x\phi_1 + \beta_1\phi_1 \\ \alpha_2\phi_2 & \alpha_2 x\phi_2 + \beta_2\phi_2 \end{pmatrix}, \end{aligned}$$

where $\phi_i(x) := \theta_q(\frac{\sigma}{\rho_i}x)$, $i = 1, 2$. The space of interest (parameterizing F) is the space of quadruples $(\alpha_1, \beta_1, \alpha_2, \beta_2)\quad\mathbf{C}^4$. The gauge action is by diagonal matrices $\Gamma = \mathrm{Diag}(\gamma_1, \gamma_2)$ and by unipotent matrices $\Delta = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$ (the omitted scalar factor in Δ can be accounted for in the action of Γ). The corresponding allowed transformations for quadruples are best described seing $L_i := (\alpha_i, \beta_i)$, $i = 1, 2$ as lines; and the α- and β-part respectively as columns C_1, C_2. The operations then are dilatations of lines $L_i \leftarrow \gamma_i L_i$, γ_i $\mathbf{C}$; and transvection $C_2 \leftarrow C_2 + \delta C_1$. So we find that:

$$\begin{aligned}\det M &= (\alpha_1\beta_2 - \alpha_2\beta_1)\phi_1\phi_2 + \alpha_1\alpha_2 x(\phi_2\phi_1 - \phi_1\phi_2) \\ &= x\phi_1\phi_2 \left(\frac{\alpha_1\beta_2 - \alpha_2\beta_1}{x} + \alpha_1\alpha_2\left(\frac{\phi_2}{\phi_2} - \frac{\phi_1}{\phi_1}\right)\right),\end{aligned}$$

i.e. $\det M/(x\phi_1\phi_2)$ is the logarithmic derivative of $x^{\alpha_1\beta_2 - \alpha_2\beta_1}(\frac{\varphi_2}{\varphi_1})^{\alpha_1\alpha_2}$, from which we draw that $\det M$ vanishes identically if, and only if $\alpha_1\beta_2 - \alpha_2\beta_1 = \alpha_1\alpha_2 = 0$. This bad set within $\mathbf{C}^4$ has three components: $\alpha_1 = \alpha_2 = 0$, $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 0$. Each of these three components is invariant under the gauge (Γ, Δ) action.

We must now check the condition $\det M(x_1) = 0$ while staying within the good part of $\mathbf{C}^4$; the latter is the union of three disjoint components, each invariant under the gauge action, and we discuss[33] the corresponding cases.

$\alpha_1\alpha_2 = 0$. — Up to Γ-action, we may assume that $\alpha_1 = \alpha_2 = 1$ and up to Δ-action, we may assume that $\beta_1 = 0$, whence (writing β instead of β_2):

$$M = \begin{pmatrix} \phi_1 & x\phi_1 \\ \phi_2 & x\phi_2 + \beta\phi_2 \end{pmatrix}$$
$$= \quad \det M(x_1) = \beta(\phi_1\phi_2)(x_1) + x_1(\phi_2\phi_1 - \phi_1\phi_2)(x_1).$$

We have three possibilities:

(1) If $(\phi_1\phi_2)(x_1) = 0$ there is a unique β such that $\det M(x_1) = 0$. There is exactly one corresponding class in F.
(2) If $\phi_1(x_1) = 0$, then $\phi_1(x_1) = 0$ (because θ_q only has simple zeroes) and $\phi_2(x_1) = 0$ (because of non resonancy). There is no such corresponding class.
(3) If $\phi_2(x_1) = 0$, same conclusion for symmetric reasons.

Subset $\alpha_1 = 0$ $\alpha_2\beta_1 = 0$. — We may suppose $\alpha_2 = \beta_1 = 1$ and $\beta_2 = 0$, so $M = \begin{pmatrix} 0 & x\varphi_1 \\ \varphi_2 & x\varphi_2 \end{pmatrix}$. If $(\phi_1\phi_2)(x_1) = 0$, this yields one class; otherwise none.

Subset $\alpha_2 = 0$ $\alpha_1\beta_2 = 0$. — Same conclusion for symmetric reasons.

We now summarize the discussion in intelligible form:

- If $(\phi_1\phi_2)(x_1) = 0$, then F is a singleton.
- If $(\phi_1\phi_2)(x_1) = 0$, that is, if $\{x_1, x_2\} = \left\{-\frac{\rho_1}{\sigma}, -\frac{\rho_2}{\sigma}\right\}$, then F has two elements respectively corresponding to the classes $\alpha_1 = 0$ and $\alpha_2 = 0$. Representatives have been described above.

[33] In essence, we are looking for normal forms.

Case 6, logarithmic/logarithmic: $R = \begin{pmatrix} \rho & \rho \\ 0 & \rho \end{pmatrix}$, $S = \begin{pmatrix} \sigma & \sigma \\ 0 & \sigma \end{pmatrix}$. — Under our assumptions, F is isomorphic to $\mathbf{C}$. We just give normal forms leaving to the interested reader to provide the necessary arguments. Let $\phi(x) := \theta_q(\frac{\sigma}{\rho}x)$ and $\psi := x\phi'$. Then each class admits a unique representative of the form $\begin{pmatrix} -\psi & m_0+\lambda\varphi \\ \varphi & \psi \end{pmatrix}$, $\lambda \in \mathbf{C}$, where $m_0 \in \mathcal{O}(\mathbf{C}^*)$ is a particular solution of the functional equation $\sigma_q m = \frac{\rho}{\sigma x}(m - \phi + 2\psi)$. We have no simple explicit formula for m but it can be proven that for any $c \in \mathbf{C}^*$ and $g \in \mathcal{O}(\mathbf{C}^*)$ there exists $f \in \mathcal{O}(\mathbf{C}^*)$ such that $\sigma_q f = \frac{c}{x}(f + g)$. One way of obtaining f is to iterate the operator $f \to -g + \frac{x}{c}\sigma_q f$.

4. The Jimbo–Sakai family (I)

The Jimbo–Sakai family studied in [41] is a path inside a subspace of the space:

$$\left\{ A_0 + xA_1 + x^2A_2 \in \mathrm{Mat}_2(\mathbf{C}[x]) \;\middle|\; A_0, A_2 \in \mathrm{GL}_2(\mathbf{C}) \right\}.$$

The subspace is restrained by conditions on the local monodromy data at 0 and ∞ and also by conditions on intermediate singularities. Sakai gave a direct description of the space of equations $\sigma_q X = \left(A_0 + xA_1 + x^2A_2\right) X$ as an algebraic surface (indeed, a rational surface); this is what we consider as the "left hand side" of the Riemann–Hilbert–Birkhoff correspondence. In this section, we introduce the corresponding "right hand side", the space of monodromy data.

We shall, here and in all this paper, consider the local data at 0 and ∞ (denoted R and S) and the intermediate singularities (denoted $\underline{x}$) as fixed.

4.1. Definitions and assumptions

In this section we model the family studied by Jimbo and Sakai in [41]: $n = 2$ and $\mu = 2$, whence $N = 4$. The local data are $R := \mathrm{Diag}(\rho_1, \rho_2)$, $S := \mathrm{Diag}(\sigma_1, \sigma_2)$ (thus R and S are the exponents at 0 and ∞); and $\underline{x} := \{x_1, x_2, x_3, x_4\}$ (the so-called "intermediate singularities", i.e. those in $\mathbf{C}^*$). We assume *Fuchs relation* and *strong non resonancy*(34) in the following form:

(FR) $x_1x_2x_3x_4 \equiv \frac{\rho_1\rho_2}{\sigma_1\sigma_2}$;
(NR) $\frac{\rho_1}{\rho_2}, \frac{\sigma_1}{\sigma_2} \notin q^{\mathbf{Z}}$ and for $1 \leqslant k < l \leqslant 4$, $\frac{x_k}{x_l} \notin q^{\mathbf{Z}}$.

(34) The usual non resonancy condition would only require that $\frac{\rho_1}{\rho_2}$ and $\frac{\sigma_1}{\sigma_2}$ do not belong to $q^{\mathbf{Z}\setminus\{0\}}$, i.e. equalities $\rho_1 = \rho_2$ or $\sigma_1 = \sigma_2$ would be allowed. Although life is simpler with strong non resonancy, our results probably extend to the more general case.

4.1.1. Riemann–Hilbert–Birkhoff correspondence

Left hand side of the RHB correspondence. — Our primary objects of interest are the space of corresponding systems and its quotient by the gauge equivalence relation:

$$\mathcal{E}_{R,S,\underline{x}} := \left\{ A = A_0 + xA_1 + x^2A_2 \in \mathrm{Mat}_2(\mathbf{C}[x]) \;\middle|\; \begin{array}{l} A_0 \text{ is conjugate to } R, \\ A_2 \text{ is conjugate to } S, \\ \det A(x) \text{ vanishes on } \underline{x} \end{array} \right\},$$

$$E_{R,S,\underline{x}} := \frac{\mathcal{E}_{R,S,\underline{x}}}{\sim}.$$

Note that, for any $A \in \mathcal{E}_{R,S,\underline{x}}$, one has

$$\begin{aligned} \det A(x) &= (\det A_2)(x - x_1)(x - x_2)(x - x_3)(x - x_4) \\ &= \sigma_1\sigma_2(x - x_1)(x - x_2)(x - x_3)(x - x_4) \\ &= (\det A_0)(1 - x/x_1)(1 - x/x_2)(1 - x/x_3)(1 - x/x_4) \\ &= \rho_1\rho_2(1 - x/x_1)(1 - x/x_2)(1 - x/x_3)(1 - x/x_4). \end{aligned}$$

Right hand side of the RHB correspondence. — Our secondary objects of interest are dictated by the classification theorems stated in the previous section, i.e. the space of corresponding "monodromy" matrices M and its quotient by the equivalence relation defined there:

$$\mathcal{F}_{R,S,\underline{x}} := \left\{ M \in \mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*)) \;\middle|\; \begin{array}{l} \sigma_q M = RM(Sx^2)^{-1}, \\ \det M \neq 0, \\ \det M \text{ vanishes on } \underline{x} \end{array} \right\}, \quad F_{R,S,\underline{x}} := \frac{\mathcal{F}_{R,S,\underline{x}}}{\sim}.$$

Note that, for any $M \in \mathcal{F}_{R,S,\underline{x}}$, one has $\sigma_q(\det M) = \frac{\rho_1\rho_2}{\sigma_1\sigma_2}x^{-4}(\det M)$, so that

$$\det M = C\,\theta_q(-x/x_1)\,\theta_q(-x/x_2)\,\theta_q(-x/x_3)\,\theta_q(-x/x_4) \text{ for some } C \in \mathbf{C}^*,$$

and therefore it has simple zeroes at $[\underline{x}; q]$ and nowhere else, entailing that M^{-1} has simple poles at $[\underline{x}; q]$ and nowhere else.

Recall the equivalence relation on $\mathcal{F}_{R,S,\underline{x}}$ (here $\mathrm{D}_2(\mathbf{C})$ denotes the group of invertible diagonal 2×2 matrices):

$$M, N \in \mathcal{F}_{R,S,\underline{x}},\ M \sim N \underset{\text{def}}{\iff} \exists \Gamma, \Delta \in \mathrm{D}_2(\mathbf{C}) : N = \Gamma^{-1}M\Delta.$$

We shall write for short $\mathcal{F} := \mathcal{F}_{R,S,\underline{x}}$ and $F := F_{R,S,\underline{x}}$. In Section 3, we have described a natural bijective Riemann–Hilbert correspondence $E \longleftrightarrow F$. In the rest of this section (from Subsection 4.2 on) we attempt at a description of F as an *algebraic surface*, at least under some generically satisfied assumptions. More complete results will come in Section 6.

4.1.2. Reducibility in the Jimbo–Sakai family

First we discuss reducibility in the Jimbo–Sakai family. This will be helpful later in order to understand many exceptional cases, singularities, etc. Applying Corollary 3.11 of Theorem 3.10, we see that matrices $M = (m_{i,j})_{i,j=1,2} \in F$ corresponding to reducible systems in E are those such that $m_{i,j} = 0$ for some $i,j \in \{1,2\}$. Note that this property is invariant under the *left* action by diagonal matrices $M \to \Gamma M \Delta^{-1}$ (from now and for simplicity, on this replaces the previous *right* action $M \to \Gamma^{-1} M \Delta$ without any unwanted logical consequence).

Definition 4.1. — *We say that $M \in F$ is* reducible *if it corresponds to a reducible system. (Since the local data R, S are fixed, this makes sense.) We then say that the class of M in F is reducible. (By the invariance stated above, this also makes sense.)*

We discuss the case that $m_{1,1} = 0$, the other ones being entirely similar. If $m_{1,1} = 0$, then $\det M = -m_{1,2} m_{2,1} = 0$ and it must vanish over $\underline{x}$. But each of $m_{1,2}, m_{2,1}$ has *a priori* two q-spirals of simple zeroes, the union of all four of them being the whole of $[\underline{x}; q]$. This implies that $m_{1,2} = c x^r \theta_q(-x/x_k) \theta_q(-x/x_l)$ and $m_{2,1} = d x^s \theta_q(-x/x_m) \theta_q(-x/x_n)$ for some $c, d \in \mathbf{C}$, some $r, s \in \mathbf{Z}$ and some "splitting" $\{1,2,3,4\} = \{k,l\} \cup \{m,n\}$. Since $\sigma_q m_{i,j} = (\rho_i/\sigma_j) m_{i,j}$, this in turn implies that $x_k x_l \equiv \rho_1/\sigma_2$ and $x_m x_n \equiv \rho_2/\sigma_1$.

Conversely, if these congruences are satisfied, we can produce matrices $M \in F$ with $m_{1,1} = 0$, $m_{1,2} = c x^r \theta_q(-x/x_k) \theta_q(-x/x_l)$, $m_{2,1} = d x^s \theta_q(-x/x_m) \theta_q(-x/x_n)$ and an arbitrary $m_{2,2}$ such that $\sigma_q m_{2,2} = (\rho_2/\sigma_2) m_{2,2}$. Alternatively, we can take $m_{2,2} = 0$ and an arbitrary $m_{1,1}$ such that $\sigma_q m_{1,1} = (\rho_1/\sigma_1) m_{1,1}$.

Definition 4.2. — *We say that there is a* splitting *of Fuchs relation* (FR) *if there is a permutation (i,j,k,l) of $(1,2,3,4)$ and a permutation (m,n) of $(1,2)$ such that:*

$$x_i x_j \equiv \frac{\rho_m}{\sigma_n} \quad \text{and} \quad x_k x_l \equiv \frac{\rho_n}{\sigma_m}.$$

Note that, because of (FR)*, these two congruence relations are actually equivalent.*

The general conclusion is as follows:

Theorem 4.3. — *The space E contains reducible systems (equivalently, the space F contains reducible matrices) if, and only if there exist a "splitting" of Fuchs relation.*

4.2. $\mathcal{F}$ as an algebraic surface: heuristics

We begin by some heuristic considerations. Let $M := (m_{i,j})_{i,j=1,2} \in \mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*))$. Then:

$$M \in F \iff \begin{cases} \sigma_q m_{i,j} = \frac{\rho_i}{\sigma_j} x^{-2} m_{i,j}, \quad i,j = 1,2, \\ m_{1,1}m_{2,2} = m_{1,2}m_{2,1}, \\ m_{1,1}m_{2,2} - m_{1,2}m_{2,1} \text{ vanishes on } \underline{x}. \end{cases}$$

The space $\tilde{F}$ of all matrices $M \in \mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*))$ such that $\sigma_q M = RM(Sx^2)^{-1}$ is the product of the four spaces defined by $\sigma_q m_{i,j} = \frac{\rho_i}{\sigma_j} x^{-2} m_{i,j}$, $i,j = 1,2$ thus it is a linear space of dimension 8 (see Subsection 2.4).

Condition $m_{1,1}m_{2,2} = m_{1,2}m_{2,1}$ defines a dense Zariski open subset of $\tilde{F}$. The four conditions $(m_{1,1}m_{2,2} - m_{1,2}m_{2,1})(x_i) = 0$ actually represent *three* independent conditions (because of (FR)), so F has dimension 5. (Beware however that F is *not* dense or Zariski-dense in $\tilde{F}$.)

Now, if $M := (m_{i,j}), N := (n_{i,j}) \in \mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*))$:

$$\begin{aligned} M \sim N \iff & n_{i,j} = \frac{\delta_j}{\gamma_i} m_{i,j}, i,j = 1,2, \text{ for some } \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbf{C}^* \\ \iff & n_{i,j} = \lambda_{i,j} m_{i,j}, i,j = 1,2, \text{ for some } \lambda_{i,j} \in \mathbf{C}^* \\ & \qquad \text{such that } \lambda_{1,1}\lambda_{2,2} = \lambda_{1,2}\lambda_{2,1}. \end{aligned}$$

Thus $\mathcal{F}$ is actually the quotient of F by the free action of a 3 dimensional torus: therefore $\mathcal{F}$ has dimension 2, it is a surface.

What follows (in this section) rests on the observation that the above conditions for $M \in F$ can be nicely expressed in terms of the diagonal and antidiagonal products $m_{1,1}m_{2,2}$ and $m_{1,2}m_{2,1}$, and similarly for $M \sim N$ where $N := (n_{i,j})$. So, for M, N as above in $\mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*))$, we write:

$$f_1 := m_{1,1}m_{2,2}, \quad f_2 := m_{1,2}m_{2,1} \quad g_1 := n_{1,1}n_{2,2}, \quad g_2 := n_{1,2}n_{2,1}.$$

Then we observe:

- If $M \in \tilde{F}$, then f_1, f_2 are solutions of the same equation $\sigma_q f = \frac{\rho_1\rho_2}{\sigma_1\sigma_2} x^{-4} f$, which defines in $\mathcal{O}(\mathbf{C}^*)$ a linear space W of dimension 4. This space will be studied in Subsection 4.3.
- Let $M \in \tilde{F}$; then $M \in F$ if and only if $f_1 = f_2$ and $f_1(x_i) = f_2(x_i)$, $i = 1, \ldots, 4$; actually, under (FR), three of these four conditions are enough.
- If $M \sim N$, then $(g_1, g_2) = \lambda(f_1, f_2)$ for some $\lambda \in \mathbf{C}^*$; actually, with the above notations, $\lambda = \lambda_{1,1}\lambda_{2,2} = \lambda_{1,2}\lambda_{2,1}$.

So there is a natural map $M \to (f_1, f_2)$ from $\mathbb{F}$ to $W \times W$ which goes to the quotient as:

$$F \longrightarrow \frac{W \times W}{\mathbf{C}}, \text{class of } M$$
$$\longrightarrow \text{class of } (m_{1,1}m_{2,2}, m_{1,2}m_{2,1}).$$

In the following subsections we shall find out what exactly the image G of this map is and how far it is from injective. Then we shall describe G in algebro-geometrical terms. Here, as in Section 6, the idea is that we have explicit finite dimensional spaces of functions and that our (seemingly) transcendental conditions can be expressed in the language of linear and multilinear algebra. In order to do so, we need again to discuss some spaces of solutions of elementary q-difference equations.

4.3. Spaces of functions again

For $k \in \mathbf{N}$ and $a \in \mathbf{C}$, let:

$$V_{k,a} := \{f \in O(\mathbf{C}) \mid \sigma_q f = ax^{-k} f\},$$

a $\mathbf{C}$-linear space of dimension k (see Subection 2.4). An explicit basis is, for instance, the family of all $\theta_q(x/\alpha)^k$ where $\alpha^k = a$. All elements of $V_{k,a}$ have the form $\lambda\,\theta_q(x/\alpha_1)\cdots\theta_q(x/\alpha_k)$ where $\lambda \in \mathbf{C}$ and $\alpha_1 \cdots \alpha_k = a$. We write $V_{k,a} := V_{k,a} \setminus \{0\}$.

Let $k, l \in \mathbf{N}$ and $a, b \in \mathbf{C}$. There is a natural map

$$V_{k,a} \times V_{l,b} \longrightarrow V_{k+l,ab},$$
$$(f, g) \longrightarrow fg.$$

We study it in case $k = l = 2$, writing it:

$$p_{a,b} : V_{2,a} \times V_{2,b} \longrightarrow V_{4,ab}.$$

Proposition 4.4. —

(i) *Let $a \equiv b$. Then the image of $p_{a,b}$ in $V_{4,ab}$ is a homogeneous quadric hypersurface, its equation is $XT - YZ = 0$ in some coordinate system.*

(ii) *Let $b = q^k a$, $k \in \mathbf{Z}$. Then the image is a hyperplane.*

Proof. —

(i). — Let $\alpha, \beta \in \mathbf{C}$ such that $\alpha^2 = a$, $\beta^2 = b$, so that $(u_1, u_2) := (\theta_q(x/\alpha)^2, \theta_q(-x/\alpha)^2)$ is a basis of $V_{2,a}$ and $(v_1, v_2) := (\theta_q(x/\beta)^2, \theta_q(-x/\beta)^2)$ is a basis of $V_{2,b}$. Then $(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$ is a basis of $V_{4,ab}$ (this can be checked either by arguing on the zeroes or by using the following proposition, which is independent). Written in this basis, the image of $V_{2,a} \times V_{2,b}$ in $V_{4,ab}$ is:

$$\{(x_1y_1, x_1y_2, x_2y_1, x_2y_2) \mid x_1, x_2, y_1, y_2 \in \mathbf{C}\} \\ = \{(X, Y, Z, T) \in \mathbf{C}^4 \mid XT - YZ = 0\}.$$

(ii). — Let (u_1, u_2) any basis of $V_{2,a}$, so that $(x^k u_1, x^k u_2)$ is a basis of $V_{2,b}$. Then $(x^k u_1^2, x^k u_1u_2, x^k u_2^2)$ is a free system in $V_{4,ab}$. Complete it into a basis. Written in this basis, the image of $V_{2,a} \times V_{2,b}$ in $V_{4,ab}$ is: is:

$$\{(x_1y_1, x_1y_2 + x_2y_1, x_2y_2, 0) \mid x_1, x_2, y_1, y_2 \in \mathbf{C}\} = \mathbf{C}^3 \times \{0\}.$$

Proposition 4.5. — *Let $f \in V_{2,a}$, $g \in V_{2,b}$ so that $fg \in V_{4,ab}$. If $a \equiv b$, the preimage of fg is: $p_{a,b}^{-1}(fg) = \{(\lambda^{-1}f, \lambda g) \mid \lambda \in \mathbf{C}^*\}$. If $b = q^k a$, $k \in \mathbf{Z}$, the preimage is: $p_{a,b}^{-1}(fg) = \{(\lambda^{-1}f, \lambda g) \mid \lambda \in \mathbf{C}^*\} \cup \{(\lambda^{-1}x^{-k}g, \lambda x^k f) \mid \lambda \in \mathbf{C}^*\}$.*

Proof. — Let f, g as in the statement and $f_1 \in V_{2,a}$, $g_1 \in V_{2,b}$ such that $f_1g_1 = fg$. Then $f_1, g_1 = 0$. We have an equality of elliptic functions: $\frac{f_1}{f} = \frac{g}{g_1}$. If there is cancellation of zeroes, our elliptic function is constant (order 1 is impossible), which yields the first part of the preimage $\{(\lambda^{-1}f, \lambda g) \mid \lambda \in \mathbf{C}^*\}$.

If there is no cancellation of zeroes, these are elliptic functions with order 2 and they have the same divisor of zeroes. So $h := g/f_1$ satisfies the relation $\sigma_q h = (b/a)h$ and it has neither zeroes nor poles on $\mathbf{C}^*$. Since $h = 0$, this is only possible if $b/a = q^k$ for some $k \in \mathbf{N}$ and $h = \lambda x^k$ for some $\lambda \in \mathbf{C}^*$, whence the conclusion.

Remark 4.6. — We see that in all cases there is a $\mathbf{C}^*$-action on $V_{2,a} \times V_{2,b}$, given by $(\lambda, (f, g)) \to (\lambda^{-1}f, \lambda g)$. In case that $b = q^k a$, there is moreover an involution $(f, g) \to (x^{-k}g, x^k f)$. The latter does not commute to the former (the involution conjugates λ with λ^{-1}) and we eventually get the action of the corresponding semi-direct product $\mathbf{C}^* \rtimes (\mathbf{Z}/2\mathbf{Z})$.

Corollary 4.7. — *We suppose that $a \equiv b$. Then the planes included in the quadric hypersurface $\Sigma := \operatorname{Im} p_{a,b} \subset V_{4,ab}$ are either of the form $fV_{2,b}$ for some $f \in V_{2,a} \setminus \{0\}$, or of the form $gV_{2,a}$ for some $g \in V_{2,b} \setminus \{0\}$.*

Proof. — It is clear that such sets $fV_{2,b}$, $gV_{2,a}$ are indeed planes included in Σ. Conversely, let $f_1, f_2 \in V_{2,a} \setminus \{0\}$ and $g_1, g_2 \in V_{2,b} \setminus \{0\}$ be such that $\operatorname{Vect}(f_1g_1, f_2g_2) \subset \Sigma$. We shall prove that either f_1, f_2 are colinear, or g_1, g_2 are; this will entail our conclusion. So assume for instance that f_1, f_2 are not colinear. By assumption:

$$\begin{aligned} f_1 g_1 + f_2 g_2 &= fg && \text{for some } f \in V_{2,a}, g \in V_{2,b} \\ &= (\alpha_1 f_1 + \alpha_2 f_2) g && \text{for some } \alpha_1, \alpha_2 \in \mathbf{C}, \end{aligned}$$

whence $f_1(g_1 - \alpha_1 g) = f_2(\alpha_2 g - g_2)$, so that, by Proposition 4.5 (and the assumption that f_1, f_2 are not colinear), $g_1 = \alpha_1 g$ and $g_2 = \alpha_2 g$, i.e. g_1, g_2 are colinear.

Corollary 4.8. — *Let $c = a_1 b_1 = a_2 b_2$, so that $\Sigma_1 := \operatorname{Im} p_{a_1,b_1} \subset V_{4,c}$ and $\Sigma_2 := \operatorname{Im} p_{a_2,b_2} \subset V_{4,c}$. In the generic situation that the two decompositions of c are "essentially inequivalent", i.e. none of a_1/a_2, a_1/b_2, b_1/a_2, b_1/b_2 belongs to $q^{\mathbf{Z}}$, the intersection $\Sigma_1 \cap \Sigma_2$ cannot contain a plane.*

Proof. — Indeed, by the previous corollary, this would imply the existence of a non trivial common factor, i.e. a non zero element in one of the spaces $V_{2,a_1} \cap x^k V_{2,a_2}$, $V_{2,a_1} \cap x^k V_{2,b_2}$, $V_{2,b_1} \cap x^k V_{2,a_2}$, $V_{2,b_1} \cap x^k V_{2,b_2}$, for some $k \in \mathbf{Z}$.

Our intended application is with

$$(a_1, b_1) := (\rho_1/\sigma_1, \rho_2/\sigma_2), \quad \text{resp.} \quad (a_2, b_2) := (\rho_1/\sigma_2, \rho_2/\sigma_1),$$

thus defining two quadric hypersurfaces

$$\Sigma_1 := \operatorname{Im} p_{a_1,b_1} \subset W \quad \text{and} \quad \Sigma_2 := \operatorname{Im} p_{a_2,b_2} \subset W$$

of the *same* linear space

$$W := V_{4,c}, \quad \text{where} \quad c := (\rho_1\rho_2)/(\sigma_1\sigma_2) = a_1 b_1 = a_2 b_2.$$

We shall also write

$$W_{i,j} := V_{2,\rho_i/\sigma_j}.$$

By the non resonancy assumption (NR) we are then in the generic situation mentioned in the above corollary (the two decompositions of c are "essentially inequivalent"). As a consequence, the intersection surface can be explicitly described; we leave it to the reader to check that:

Proposition 4.9. — *Recall that $(a_1, b_1) := (\rho_1/\sigma_1, \rho_2/\sigma_2)$ and $(a_2, b_2) := (\rho_1/\sigma_2, \rho_2/\sigma_1)$. Then:*

$$\Sigma_1 \cap \Sigma_2 = \left\{ \lambda\, \theta_q(x/\alpha)\, \theta_q(x/\alpha')\, \theta_q(x/\beta)\, \theta_q(x/\beta') \;\middle|\; \begin{array}{l} \lambda \in \mathbf{C}^*, \alpha, \alpha', \beta, \beta' \in \mathbf{C}^*, \\ \alpha\alpha' = a_1,\ \beta\beta' = b_1, \\ \alpha\beta = a_2,\ \alpha'\beta' = b_2 \end{array} \right\}.$$

It can be parameterized by λ, α by taking:

$$\alpha' := a_1/\alpha, \ \beta := a_2/\alpha, \ \text{and } \beta' := b_1\alpha/a_2 = b_2\alpha/a_1 = b_1 b_2 \alpha/c = c\alpha/(a_1 a_2).$$

In a second step, we shall consider the *projective* quadric surfaces, the respective images $\tilde{\Sigma}_1$ of Σ_1 and $\tilde{\Sigma}_2$ of Σ_2 in $\mathbf{P}(W)$.

4.3.1. Incidence relations of planes of Σ_1, Σ_2

The motivation for the following study was explained in Subsection 1.5. Recall that $(a_1, b_1) := (\rho_1/\sigma_1, \rho_2/\sigma_2)$ and $(a_2, b_2) := (\rho_1/\sigma_2, \rho_2/\sigma_1)$.

First we note the following facts:

- Two distinct planes in W intersect along a line if, and only if, they are contained in a common hyperplane (equivalently: their sum is a hyperplane). This is because the ambient space W has dimension 4. In this situation, we shall say that the two planes are *incident*.
- A line of W has the form $D_{\alpha,\beta,\gamma,\delta} := \mathbf{C}\,\theta_q(x/\alpha)\,\theta_q(x/\beta)\,\theta_q(x/\gamma) \times \theta_q(x/\delta)$ for some fixed $\alpha, \beta, \gamma, \delta \in \mathbf{C}^*$ such that $\alpha\beta\gamma\delta = c$.
- Write H_x the hyperplane $\{f \in W \mid f(x) = 0\}$ (of course, $H_{qx} = H_x$). The only such hyperplanes containing $D_{\alpha,\beta,\gamma,\delta}$ are H_α, H_β, H_γ and H_δ.

Combining these facts with Corollary 4.7, we get the following:

Proposition 4.10. — *All pairs of incident planes $P_1 \in \Sigma_1$, $P_2 \in \Sigma_2$ are obtained by the following process. Let $\alpha, \beta, \gamma, \delta \in \mathbf{C}^*$ such that:*

$$\alpha\beta = a_1, \quad \alpha\gamma = a_2, \quad \gamma\delta = b_1, \quad \beta\delta = b_2.$$

(These conditions of course imply $\alpha\beta\gamma\delta = c$.) Then one of the following four possibilities holds:

- $P_1 = \theta_q(x/\alpha)\,\theta_q(x/\beta)W_{2,2}$ *and* $P_2 = \theta_q(x/\alpha)\,\theta_q(x/\gamma)W_{2,1}$*; then* $P_1 + P_2 = H_\alpha$*;*
- $P_1 = \theta_q(x/\alpha)\,\theta_q(x/\beta)W_{2,2}$ *and* $P_2 = \theta_q(x/\beta)\,\theta_q(x/\delta)W_{1,2}$*; then* $P_1 + P_2 = H_\beta$*;*
- $P_1 = \theta_q(x/\gamma)\,\theta_q(x/\delta)W_{1,1}$ *and* $P_2 = \theta_q(x/\alpha)\,\theta_q(x/\gamma)W_{2,1}$*; then* $P_1 + P_2 = H_\gamma$*;*
- $P_1 = \theta_q(x/\gamma)\,\theta_q(x/\delta)W_{1,1}$ *and* $P_2 = \theta_q(x/\beta)\,\theta_q(x/\delta)W_{1,2}$*; then* $P_1 + P_2 = H_\delta$.

In all cases, $P_1 \cap P_2 = D_{\alpha,\beta,\gamma,\delta}$.

We see that those hyperplanes of W which cut Σ_1 and Σ_2 along planes are of a very special nature: they have the form H_x defined above. Among their incidence properties, we see that through a generic line in W pass four such hyperplanes and each of them gives rise to two pairs of planes (one in each of Σ_1, Σ_2) having a common intersection line; we say a bit more about that at the end of Part 4.3.2 herebelow; and we return to them in Part 4.3.3.

4.3.2. Incidence relations of lines of $\tilde{\Sigma}_1, \tilde{\Sigma}_2$

We shall write $\tilde{D}_{\alpha,\beta,\gamma,\delta}$ $\mathbf{P}(W)$ the image of line $D_{\alpha,\beta,\gamma,\delta}$ (thus a point), $\tilde{H}_x$ $\mathbf{P}(W)$ the image of hyperplane H_x (thus a projective plane) and $\tilde{P}_i$ $\mathbf{P}(W)$ the image of plane P_i (thus a projective line).

Corollary 4.11. — *All pairs of intersecting lines* Δ_1 $\tilde{\Sigma}_1$, Δ_2 $\tilde{\Sigma}_2$ *are obtained by taking* $\Delta_i := \tilde{P}_i$, $i = 1, 2$*, with the above construction. For a given* α $\mathbf{C}$ *, parameters* β, γ, δ $\mathbf{C}$ *are uniquely determined. The process defines four lines (two in each of* $\tilde{\Sigma}_1, \tilde{\Sigma}_2$*). All four lines meet at the point* $\tilde{D}_{\alpha,\beta,\gamma,\delta}$.

From what we said hereabove at the end of Part 4.3.1, we draw that through a generic point of $\mathbf{P}(W)$ pass four planes that cut $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ along two lines each, all the four lines corresponding to one such special plane having one common point. In the application to our problem detailed in Subsections 4.4 and 4.5, we consider the particular point $\tilde{L}$ $\mathbf{P}(W)$ image of the line

$$L := \mathrm{Vect}\,(\theta_q(-x/x_1)\,\theta_q(-x/x_2)\,\theta_q(-x/x_3)\,\theta_q(-x/x_4)) \quad W,$$

i.e. the intersection of the four hyperplanes H_{x_i}. It seems likely that the sixteen corresponding lines in $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$ are related somehow to the "special fibers" we shall encounter in our analysis in Section 6.

4.3.3. The family of hyperplanes $(H_x)_{x}$ $\mathbf{C}$

As noted at the end of Part 4.3.1, the hyperplanes H_x of W, x $\mathbf{C}$, seem to play a special role. They have interesting properties which we feel to be relevant in understanding the geometry of F. Although we have not been able to exploit them completely, we summarize here some of these properties.

Recall W is the set of holomorphic solutions of some equation $\sigma_q f = cx^{-4} f$. Let W the dual of the space W. Each H_x can be seen as a point of $\mathbf{P}(W)$, which is isomorphic to $\mathbf{P}^3(\mathbf{C})$. For f W, the functional equation $\sigma_q f = cx^{-4} f$ implies that $f(qx) = 0 \Leftrightarrow f(x) = 0$, i.e. $H_{qx} = H_x$. Therefore we get an holomorphic map $\bar{x} \to H_x$ from $\mathbf{E}_q$ to $\mathbf{P}(W)$. That map is injective. Indeed, if $x \not\equiv y$ one can easily build f W such that $f(x) = 0 = f(y)$, so that $H_x = H_y$. It follows that the image $\{H_x \mid x$ $\mathbf{C}\}$ is a holomorphic curve isomorphic to $\mathbf{E}_q$, hence an elliptic curve. We write it $\mathbf{E}_q$.

To understand $\mathbf{E}_q$ as an embedded algebraic curve, note that any three distinct points on it generate a (projective) plane which cuts the curve at a fourth point (the latter maybe not distinct from the three former ones). Let indeed $a_1, a_2, a_3 \in \mathbf{C}^*$ pairwise non congruent defining H_{a_1}, H_{a_2}, H_{a_3} our three distinct points on $\mathbf{E}_q$. Let $x_4 \in \mathbf{C}^*$ be chosen such that $a_1 a_2 a_3 a_4 \equiv c$. Then $H_{a_1} \cap H_{a_2} \cap H_{a_3} \cap H_{a_4}$ is the line generated by $\theta_q(-x/a_1)\,\theta_q(-x/a_2)\,\theta_q(-x/a_3)\,\theta_q(-x/a_4)$; while, for any other choice of a_4 the intersection would be $\{0\}$.

To find equations for $\mathbf{E}_q$ in $\mathbf{P}(W)$, we proceed as follows. Recall that the affine algebra of W (the algebra of polynomials on W) is the symmetric algebra on its dual, i.e. $\mathbf{S}^\bullet(W)$. We look for a homogeneous ideal in $\mathbf{S}^\bullet(W)$. Selecting a basis (f_1, f_2, f_3, f_4) of W gives an identification of $\mathbf{S}^\bullet(W)$ with $\mathbf{C}[X_1, X_2, X_3, X_4]$. We map the homogeneous component $\mathbf{C}[X_1, X_2, X_3, X_4]_d$ into $W^{(d)} := \{f \in O(\mathbf{C}^*) \mid \sigma_q f = c^d x^{-4d} f\}$ by sending $F(X_1, X_2, X_3, X_4)$ to $F(f_1, f_2, f_3, f_4)$. Then F is an equation of $\mathbf{E}_q$ if, and only if, $F(f_1, f_2, f_3, f_4)$ vanishes at all $x \in \mathbf{C}^*$, i.e. if F is in the kernel of $\mathbf{C}[X_1, X_2, X_3, X_4]_d \to W^{(d)}$. The source of this map has dimension $(d+1)(d+2)(d+3)/6$, while its target has dimension $4d$. Therefore, for $d > 2$, the kernel is non trivial. For instance, taking $d = 2$, we find two independent quadratic forms vanishing on $\mathbf{E}_q$. Therefore this curve is, at any rate, a component of the intersection of two quadric surfaces. We do not know if it is the only component.

One way to find the quadratic forms (i.e. the degree 2 component of the ideal of $\mathbf{E}_q$) is to choose pairwise non congruent $a_1, a_2, a_3, a_4 \in \mathbf{C}^*$ such that $a_1 a_2 a_3 a_4 \equiv c$; and then $f_1, f_2, f_3, f_4 \in W$ such that $f_i(a_j) = \delta_{i,j}$ (easy using theta functions). Then we require that $\sum_{1 \leqslant i \leqslant j \leqslant 4} c_{i,j} f_i f_j$ vanish at all a_i at order 2 (in $W^{(2)}$ this implies that it is 0). Evaluation at a_i yields $c_{i,i} = 0$. Then we are left with four linear conditions (vanishing of the derivatives) on the six coordinates $c_{i,j}$, $1 \leqslant i < j \leqslant 4$.

4.4. F as a degree 2 covering of a quadric surface

Let $W_1 := V_{2,\frac{\rho_1}{\sigma_1}} \times V_{2,\frac{\rho_2}{\sigma_2}}$ and $W_2 := V_{2,\frac{\rho_1}{\sigma_2}} \times V_{2,\frac{\rho_2}{\sigma_1}}$. As seen in 4.2, $\tilde{F}$ is identified with $W_1 \times W_2$ by $(m_{i,j}) \mapsto ((m_{1,1}, m_{2,2}), (m_{1,2}, m_{2,1}))$.

Let $W := V_{4,\frac{\rho_1\rho_2}{\sigma_1\sigma_2}}$. We have defined in 4.3 multiplication maps $W_1 \to W$ and $W_2 \to W$. Composing, we get a map:

$$\begin{aligned} \tilde{F} &\longrightarrow W \times W, \\ M = (m_{i,j}) &\longmapsto (m_{1,1}m_{2,2}, m_{1,2}m_{2,1}), \end{aligned}$$

the image of which is (by Proposition 4.4) $\Sigma_1 \times \Sigma_2$, where Σ_i, image of $W_i \to W$ for $i = 1, 2$, is a homogeneous quadric hypersurface of the four dimensional $\mathbf{C}$-linear space W. Thus we have a surjective mapping:

$$\tilde{F} \longrightarrow \Sigma_1 \times \Sigma_2,$$
$$(m_{i,j}) \longrightarrow (f_1, f_2) := (m_{1,1}m_{2,2}, m_{1,2}m_{2,1}).$$

The condition: $\det M = 0$ (on elements of F) translates to: $f_1 = f_2$. The condition: $\det M$ vanishes on $\underline{x}$ translates to: $f_1 - f_2$ belongs to the intersection of the four hyperplanes $\mathrm{Ker}(f \to f(x_i))$. However, because of (FR), these are really three independent linear conditions and that intersection is the line

$$L := \mathrm{Vect}\,(\theta_q(-x/x_1)\,\theta_q(-x/x_2)\,\theta_q(-x/x_3)\,\theta_q(-x/x_4)) \subset W.$$

We write $L^* := L \setminus \{0\}$ and deduce a surjective map:

$$F \to G := \{(f_1, f_2) \in \Sigma_1 \times \Sigma_2 \mid f_1 - f_2 \in L^*\}.$$

The torus action on F (i.e. the diagonal action of $\mathbf{C}^*$) corresponds in G to the obvious $\mathbf{C}^*$-action $(\lambda, (f_1, f_2)) \to (\lambda f_1, \lambda f_2)$. Thus we eventually get a surjective map:

$$\mathcal{F} \to \mathcal{G} := \frac{G}{\mathbf{C}^*}.$$

We shall now formulate assumptions on the local data R, S and $\underline{x}$ under which this map is bijective. Note however that these assumptions are generically satisfied[35]. There are actually two causes of non injectivity. One of them comes from the second case of Proposition 4.5; the other comes from the fact that Proposition 4.5 adresses only the case of non zero functions.

The former cause is taken care by the assumption (nicknamed "special condition"):

(SC) $\frac{\rho_1}{\sigma_1} \not\equiv \frac{\rho_2}{\sigma_2}$ and $\frac{\rho_2}{\sigma_1} \not\equiv \frac{\rho_1}{\sigma_2}$.

This assumption guarantees that the first part of Proposition 4.5 can be applied.

As for the latter cause, let $f = m_{1,1}m_{2,2} \in W$ (the case of $m_{1,2}m_{2,1}$ is similar) and suppose that also $f = m'_{1,1}m'_{2,2}$ (with obvious notations). If $f \neq 0$, we know from Proposition 4.5 that, under assumption (SC), $(m'_{1,1}, m'_{2,2}) = (cm_{1,1}, c^{-1}m_{2,2})$ for some $c \in \mathbf{C}^*$. From this, one easily deduces that, if $m_{i,j} = 0$, then the class of $M = (m_{i,j})$ in $\mathcal{F}$ is alone in the preimage of its image. So defects of injectivity are only possible if there are matrices such that at least one $m_{i,j} = 0$. But this is the case of reducibility discussed in Theorem 4.3. Now recall the terminology introduced in Definition 4.2.

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[35] And that moreover we shall relax them in Section 6 where a quite different approach is followed.

Proposition 4.12. — *If* (SC) *holds and there is no splitting of* (FR)*, the map $F \to G$ is bijective.*

Proof. — Immediate from the above discussion and Theorem 4.3.

Now we give a preliminary crude description of F and G. This will be refined in the next Subsection 4.5.

For that, we project G to Σ_1 by $(f_1, f_2) \to f_1$. The preimage of f_1 is $\{f_1\} \times ((f_1 + L) \cap \Sigma_2)$. Assume that $f_1 \notin \Sigma_1 \cap \Sigma_2$. Then the punctured affine line $f_1 + L$ meets Σ_2 at two points, or at a double point in case of tangency (that is, if f_1 is in the critical locus of the projection). We dare not call theorem the following, because of the difficulty of stating precise assumptions; but it is plainly true "in general". Note that, as a (non closed) algebraic surface, G is endowed with the Zariski topology inherited from $\mathbf{P}(W^2)$.

Fact 4.13. — Under the projection $(f_1, f_2) \to f_1$, an open dense subset of the space G is a degree 2 ramified covering of a quasi-projective quadric surface (an open dense subset of the image of Σ_1 in $\mathbf{P}(W)$).

If we try to make more precise where the projection fails to be a covering, we are led to define two loci: the set $\Sigma_1 \cap \Sigma_2$ on the one hand; and the set of those $f_1 \in \Sigma_1$ such that $f_1 + L$ is tangent to Σ_2. *These two loci are generically not the same*. Indeed, it is a general fact that the locus of points on Σ_2 where L is a tangent direction is included in a hyperplane[36]. By symmetry reason, if the two loci were generically the same, $\Sigma_1 \cap \Sigma_2$ would be included in two distinct hyperplanes, so it would be plane, in contradiction with Corollary 4.8.

We shall not pursue here this *first approach* to the geometry of F based on deducing incidence relations from theta relations. Hereafter we propose a more algebraic approach, but our main attack will come later in Sections 6 and 7 where we can use our main tool, Mano decomposition.

4.5. Embedding of F into $(\mathbf{P}^1(\mathbf{C}))^4$

We will give a description of F as a (non closed) surface inside $(\mathbf{P}^1(\mathbf{C}))^4$. More precisely, it is a *first attempt* towards such a description. We will assume some (imprecise ...) genericity hypothesis and our description is partly heuristic, in the "old italian algebraic geometry" style.

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[36] If B is a nondegenerate symmetric bilinear form on a space V and Σ is the quadric hypersurface $B(x,x) = 0$, then the set of points of Σ where the direction $\mathrm{Vect}(u)$ is tangent is (except if $B(u,u) = 0$) the intersection of Σ with the hyperplane $B(x,u) = 0$. Here $B(u,u) = 0$ because a generator of the line L is $u := \theta_q(-x/x_1)\,\theta_q(-x/x_2)\,\theta_q(-x/x_3)\,\theta_q(-x/x_4) \in \Sigma_2$.

We have an isomorphism $F \to G$, therefore we have a description of the surface F as a (non closed) algebraic surface of the projective space associated to W^2 (which is isomorphic to $\mathbf{P}^7(\mathbf{C})$).

We set:

$$\overline{G} := \{(f_1, f_2) \in \Sigma_1 \times \Sigma_2 \mid f_1 - f_2 \in L\} \quad \text{and} \quad \overline{\mathsf{G}} := \frac{\overline{G}}{\mathbf{C}}.$$

Thus G is open and Zariski dense in the projective surface $\overline{\mathsf{G}}$.

We denote by $\mathbf{P}(W)$ the projective space associated to W and $\widetilde{\Sigma}_1$, $\widetilde{\Sigma}_2$, $\widetilde{L}$ the respective images of Σ_1, Σ_2, L in $\mathbf{P}(W)$; $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ are quadric surfaces and $\widetilde{L}$ is a point. We consider the tangent cones to $\widetilde{\Sigma}_1$, resp. $\widetilde{\Sigma}_2$, directed from the point $\widetilde{L}$, that we denote by $C(\widetilde{L}, \widetilde{\Sigma}_1)$, resp. $C(\widetilde{L}, \widetilde{\Sigma}_2)$; they are quadratic cones. The intersections $Q_1 := \widetilde{\Sigma}_1 \cap C(\widetilde{L}, \widetilde{\Sigma}_2)$, $Q_2 := \widetilde{\Sigma}_2 \cap C(\widetilde{L}, \widetilde{\Sigma}_1)$ and $Q_\infty := \widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2$ are quadratic curves.

We denote $\varpi_1 : \mathsf{G} \to \widetilde{\Sigma}_1$ the projection induced by $(f_1, f_2) \mapsto f_1$. We have the following description of the fibration of G by ϖ_1.

- The image of ϖ_1 is $\widetilde{\Sigma}_1 \setminus (Q_1 \cap Q_\infty)$. Generically $Q_1 \cap Q_\infty$ is a finite set (at most 16 points).
- Above $\widetilde{\Sigma}_1 \setminus (Q_1 \cup Q_\infty)$ there are exactly two points of G.
- Above $Q_1 \setminus Q_\infty$ there is exactly one point of G.
- Above $\widetilde{\Sigma}_1 \setminus Q_\infty$ we have a degree two covering ramified on $Q_1 \setminus Q_\infty$.

We have similar properties for $\varpi_2 : \mathsf{G} \to \widetilde{\Sigma}_1$, the projection induced by $(f_1, f_2) \mapsto f_2$.

The maps ϖ_1, ϖ_2 extend repectively into maps $\overline{\varpi}_1 : \overline{\mathsf{G}} \to \widetilde{\Sigma}_1$ and $\overline{\varpi}_2 : \overline{\mathsf{G}} \to \widetilde{\Sigma}_2$.

The quadric surface $\widetilde{\Sigma}_1$ is a bi-ruled surface: $\widetilde{\Sigma}_1 \simeq \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$. If D is a line of one of the two families, then D intersects the quadric $C(\widetilde{L}, \widetilde{\Sigma}_1)$ at most two points and generically at two points. Then Q_1 can be considered as a curve of bidegree $(2, 2)$ in $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$. In order to simplify the notations we will identify $\widetilde{\Sigma}_1$ and $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$. In particular we will interpret $\overline{\varpi}_1$ as a map from $\overline{\mathsf{G}}$ to $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$. We have a similar description for $\widetilde{\Sigma}_2$.

Proposition 4.14. —

(i) *The map $\overline{\varpi}_1$ is a double ramified covering of $(\mathbf{P}^1(\mathbf{C}))^2$ branched over Q_1.*

(ii) *There exists a projective curve $\Gamma \in \overline{\mathsf{G}}$ such that $\overline{\varpi}_1(\Gamma) = Q_\infty$ and $\overline{\mathsf{G}} \setminus \Gamma = \mathsf{G}$.*

Proof. —

(i). — Follows easily from the description at the beginning of this paragraph.

(ii). — We denote $\iota : \overline{G} \to \overline{G}$ the involution associated to the covering.

We verify that the closure in $\overline{G}$ of the intersection $G \cap \varpi^{-1}(Q \ \setminus (Q \ \cap Q \ ))$ is a curve Γ. Then $\varpi^{-1}(Q \ ) = \Gamma \quad \iota(\Gamma)$. We verify that $\varpi(\Gamma) = Q$. We have $G = \overline{G} \setminus \Gamma$.

Let D_0 be a line of one of the families $\left(D_0 = \{x\} \times \mathbf{P}^1(\mathbf{C})\right)$. Then $\overline{\varpi}_1$ induces a double covering $\overline{\varpi}_1^{-1}(D_0) \to D_0$ ramified above $D_0 \cap Q_1$, that is generically above 2 points. If D_0 is tangent to Q_1, then we get a ramification above a unique point.

The map: $(\overline{\varpi}_1, \overline{\varpi}_2) : \overline{G} \to (\mathbf{P}^1(\mathbf{C}))^4$ is clearly regular (i.e. a morphism of algebraic varieties) and injective.

Proposition 4.15. — [37] *The map:*

$$(\overline{\varpi}_1, \overline{\varpi}_2) : \overline{G} \to (\mathbf{P}^1(\mathbf{C}))^4$$

realizes an embedding of $\overline{G}$ into $(\mathbf{P}^1(\mathbf{C}))^4$.

Then F can be interpreted as a closed algebraic surface of $(\mathbf{P}^1(\mathbf{C}))^4$ minus a closed curve.

Later on, using Mano decomposition, we will give a more precise description of the surface F. The problem of an explicit relation between our two descriptions will be tackled in Part 7.2.3.

5. Mano decomposition

This extremely useful process was inspired to us by the paper [46] of Toshiyuki Mano. However, we shall give a more precise statement and a direct proof of the particular property of interest here. We show that degree 2 order 2 equations (the ones that appear in the Jimbo–Sakai family, that is in the linear isomonodromic model of the q-Painlevé equation) can, in some sense, be *split* into q-hypergeometric components[38] .

[37] Stricly speaking it is partly conjectural: cf. the introduction of this paragraph.

[38] As noted in a footnote at the beginning of Subsection 3.4, we abusively term q-hypergeometric all order 2 degree 1 systems, although the reducible ones cannot be such by [64, 65].

We saw at the end of Section 3.4 that the set of classes of systems of q-hypergeometric type, as seen through their monodromy data, admitted a nice geometric classification. So Mano decomposition allows for an enhanced geometric classification of the Jimbo–Sakai family. This will be done in Section 6.

Mano decomposition can also be understood as providing a splitting of the *global* monodromy around the four intermediate singularities into *local* monodromies around two pairs of singularities(39) . A discussion of what we accomplish here appears in our concluding Section 7, where a geometrical interpretation ("surgery of pants") is provided.

Since the objects and processes here seem to be new, and since their study is full of special cases, we tried to present it as clearly and cleanly as possible; all the more since these special cases seem to have some geometric meaning. The main result, synthetizing our rather lengthy discussion, is Theorem 5.13, stated at the beginning of Subsection 5.4.

5.1. Setting, general facts

Remember our general setting with generic local data(40) $\rho_1, \rho_2 \quad \mathbf{C}$ (exponents at 0), $\sigma_1, \sigma_2 \quad \mathbf{C}$ (exponents at ∞), $x_1, x_2, x_3, x_4 \quad \mathbf{C}$ ("intermediate" singularities) subject to the following conditions:

(FR) Fuchs relation: $x_1 x_2 x_3 x_4 \equiv \rho_1 \rho_2 / \sigma_1 \sigma_2$.
(NR) Non resonancy: $\rho_1/\rho_2, \sigma_1/\sigma_2 \quad q^{\mathbf{Z}}$ and for 1 6 $k < l$ 6 4, $x_k/x_l \quad q^{\mathbf{Z}}$.

Now we select a particular pair of singularities among x_1, x_2, x_3, x_4, with the idea of partially "localize" the monodromy around that pair. Let the indexing be chosen such that $\{x_1, x_2\}$ is the selected pair. We shall need the following supplementary condition:

(NS) Non splitting: for all $i, j = 1, 2$, $\rho_i/\sigma_j \equiv x_1 x_2$.

Lemma 5.1. — *Assuming* (FR) *and* (NR)*, there always is an indexing of* x_1, x_2, x_3, x_4 *such that* (NS) *holds.*

(39) We expressed in Part 1.4.4 our opinion that defining local monodromies and local Galois groups at intermediate singularities is one of the most important open problems in modern q-di□erence theory. This will require a more general version of Mano decomposition. Extension to higher degrees should be easy along the same lines, but extension to higher orders (polynomial matrices with coe□cients in $\mathrm{Mat}_n(\mathbf{C})$) seems more di□cult.

(40) Note however that, these generic local data being given, *all* equations of the corresponding class will be shown to admit a Mano decomposition.

Proof. — We leave to the reader the simple combinatorial proof of this fact (hint: there are three splittings of $\{x_1, x_2, x_3, x_4\}$ in two pairs, while there are only two splittings of $\rho_1\rho_2/\sigma_1\sigma_2$ as a product of two fractions).

Note that, because of (FR), condition (NS) is equivalent to: for all $i, j = 1, 2$, $\rho_i/\sigma_j \equiv x_3x_4$. However, beyond the partition of $\{x_1, x_2, x_3, x_4\}$ into $\{x_1, x_2\}$ and $\{x_3, x_4\}$, the two components do not play a symmetric role: $\{x_1, x_2\}$ will be related to the left factor and $\{x_3, x_4\}$ to the right one. (Thus there is a kind of "dual" decomposition obtained by permuting the roles of these two pairs.)

We let as usual $R := \begin{pmatrix}\rho_1 & 0\\ 0 & \rho_2\end{pmatrix}$ and $S := \begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}$. Also we write $\underline{x} := \{x_1, x_2, x_3, x_4\}$. Then we consider a matrix $M \in \mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*))$ such that:

(1) $\sigma_q M = RM(Sx^2)^{-1}$,
(2) $\det M = 0$,
(3) $\det M$ vanishes at $\underline{x}$.

Clearly we have $\det M \in \mathcal{M}(\mathbf{C}^*)$ and

$$\frac{\sigma_q(\det M)}{\det M} = \frac{\rho_1\rho_2}{\sigma_1\sigma_2 x^4} = \frac{x_1x_2x_3x_4}{x^4},$$

so that by Subsection 2.4, we deduce from the third condition that $\det M$ vanishes at all points of $[x_1, x_2, x_3, x_4; q]$ with multiplicity one, and nowhere else.

Remark 5.2. — Write $M := \begin{pmatrix}m_{1,1} & m_{1,2}\\ m_{2,1} & m_{2,2}\end{pmatrix}$, so that $m_{i,j} \in \mathcal{O}(\mathbf{C}^*)$ and $\sigma_q m_{i,j} = (\rho_i/\sigma_j)x^{-2}m_{i,j}$. From Subsection 2.4 we thus have for $i, j = 1, 2$ $m_{i,j} = \lambda_{i,j}\,\theta_q(x/\alpha_{i,j})\,\theta_q(x/\beta_{i,j})$, where $\lambda_{i,j} \in \mathbf{C}^*$ and $\alpha_{i,j}\beta_{i,j} = \rho_i/\sigma_j$. This is the "encoding" used by Birkhoff in [4] (actually, the "characteristic constants" alluded to in Part 1.3.1). However, we shall not use that form in the present section.

5.1.1. A projective invariant

For $k = 1, 2, 3, 4$, $M(x_k) = 0$ (otherwise, $\det M$ would have a multiple zero at x_k). Let $\begin{pmatrix}f_k\\ g_k\end{pmatrix}$ a non zero column of $M(x_k)$ and $\begin{pmatrix}f_k\\ g_k\end{pmatrix}$ the other column, so that $\begin{pmatrix}f_k\\ g_k\end{pmatrix} \in \mathbf{C}\begin{pmatrix}f_k\\ g_k\end{pmatrix}$. In particular, $f_k = 0$ means that $M(x_k) = \left(\begin{smallmatrix}0 & 0\\ & \end{smallmatrix}\right)$ and $g_k = 0$ means that $M(x_k) = \left(\begin{smallmatrix} & \\ 0 & 0\end{smallmatrix}\right)$.

Lemma 5.3. —

(i) *One cannot have* $f_1g_2 = f_2g_1 = 0$.

(ii) *The ratio* $(f_1g_2 : f_2g_1) \in \mathbf{P}^1(\mathbf{C})$ *is well defined from* M*, independently of the particular choices of non-zero columns.*

(iii) *This ratio is invariant by the group action of diagonal matrices* $M \to \Gamma M \Delta^{-1}$*,* $\Gamma, \Delta \in D_2(\mathbf{C})$*.*

Proof. —

(i). — We first prove that one cannot have $f_1 = f_2 = 0$ nor $g_1 = g_2 = 0$. We prove only the first impossibility, the second one is similar. So assume that $f_1 = f_2 = 0$. Then $M(x_1)$ and $M(x_2)$ have the form $\left(\begin{smallmatrix} 0 & 0 \\ & \end{smallmatrix}\right)$, i.e. both $m_{1,1}$ and $m_{1,2}$ vanish at x_1 and x_2. Since $\det M = 0$, they cannot both be null, say $m_{1,j} = 0$. Then by Subsection 2.4, the fact that $\sigma_q m_{1,j} = (\rho_1/\sigma_j)x^{-2}m_{1,j}$ implies that $x_1x_2 \equiv \rho_1/\sigma_j$, contradicting (NS).

Now to the point: assume for instance that $f_1 = 0$ (the case $g_2 = 0$ being similar). Since by the previous statement $f_2 = 0$, the assumption $f_2g_1 = 0$ would entail $g_1 = 0$, contradicting the definition of $\begin{pmatrix} f_k \\ g_k \end{pmatrix}$ as a non zero column.

(ii). — If for instance $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = 0$, then $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \lambda \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$ with $\lambda = 0$, whence $(f_1g_2, f_2g_1) = \lambda(f_1g_2, f_2g_1)$, so that $(f_1g_2 : f_2g_1) = (f_1g_2 : f_2g_1)$.

(iii). — Let $M := \Gamma M \Delta^{-1}$ and let j_1, j_2 the indexes of the selected nonzero columns. With obvious notations, $f_i = \frac{\gamma_1}{\delta_{j_i}} f_i$ and $g_i = \frac{\gamma_2}{\delta_{j_i}} g_i$, whence $f_1g_2 = \frac{\gamma_1\gamma_2}{\delta_{j_1}\delta_{j_2}} f_1g_2$ and $f_2g_1 = \frac{\gamma_1\gamma_2}{\delta_{j_1}\delta_{j_2}} f_2g_1$. The conclusion follows.

We thus obtain a well defined map[41]

$$\Pi : F \to \mathbf{P}^1(\mathbf{C}),$$

which goes to the quotient as

$$\Pi : \mathsf{F} \to \mathbf{P}^1(\mathbf{C})$$

(the shorthand notations $F := F_{R,S,\underline{x}}$ and $\mathsf{F} := \mathsf{F}_{R,S,\underline{x}}$ were introduced in Part 4.1.1). These maps (denoted by the same letter, which, hopefully, will cause no confusion) will come back in full glory when we describe F and F as fibered spaces.

We leave to the reader the proof of the following, which gives in generic cases a necessary and sufficient condition for two ordered pairs to give the same point in $\mathbf{P}^1(\mathbf{C})$.

Lemma 5.4. — *Let* $f_1, f_2, g_1, g_2 \in \mathbf{C}$ *and* $p_1, p_2, q_1, q_2 \in \mathbf{C}$*. Then:*

$$\frac{p_1q_2}{p_2q_1} = \frac{f_1g_2}{f_2g_1} \quad \exists \lambda, \mu \in \mathbf{C} : \begin{pmatrix} f_i \\ g_i \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} p_i \\ q_i \end{pmatrix}, i = 1, 2 .$$

[41] This map is obviously related to our selection of the pair $\{x_1, x_2\}$ and it should properly be denoted $\Pi_{1,2}$ but we don't need to do that for the moment; see Section 7.

5.1.2. Two special fibers of the projective invariant Π

The fibers[42] $\Pi^{-1}(0), \Pi^{-1}(\infty)$ F have special significance. For example, they contain all reducible matrices. Indeed, if M F is reducible, i.e. (Definition 4.1) if it corresponds to a reducible system in E, then one of its coe cients vanishes (cf. Part 4.1.2) and $M(x_1)$, $M(x_2)$ acquire one of the forms $({}^{0\,0})$, $({}_{0\,0})$. Then $f_1 f_2 g_1 g_2 = 0$, so that $\Pi(M) = 0$ or $\Pi(M) = \infty$.

However, the converse is not true. We briefly describe irreducible matrices M in $\Pi^{-1}(0)$ $\Pi^{-1}(\infty)$. This means that $f_1 f_2 g_1 g_2 = 0$:

$$\Pi^{-1}(0) = (f_1 = 0) \quad (g_2 = 0) \text{ and } \Pi^{-1}(\infty) = (f_2 = 0) \quad (g_1 = 0).$$

We treat the case $f_1 = 0$, the other ones being entirely similar. To simplify the discussion, we shall assume that F contains no reducible matrices, i.e. that there is no splitting of (FR) (Theorem 4.3). A more general situation is tackled by other means in Section 6.3.

If $f_1 = 0$, then $M(x_1) = ({}^{0\,0})$. Since by assumption of irreducibility $m_{1,1}, m_{1,2} = 0$, we see that these coe cients have the forms $m_{1,1} = c_1\,\theta_q(-x/x_1)\,\theta_q(-x/x_1)$, $m_{1,2} = c_2\,\theta_q(-x/x_1)\,\theta_q(-x/x_1)$, with arbitrary c_1, c_2 $\mathbf{C}$ and x_1, x_1 $\mathbf{C}$ such that $x_1 x_1 = \rho_1/\sigma_1$, $x_1 x_1 = \rho_1/\sigma_2$.

Up to the action $M \to \Gamma M \Delta^{-1}$ by diagonal matrices $\Gamma := \mathrm{Diag}(\gamma_1, \gamma_2)$, $\Delta := \mathrm{Diag}(\delta_1, \delta_2)$, we may and shall take $c_1 = c_2 = 1$. Then $m_{2,1}$ $V_{2,\frac{\rho_2}{\sigma_1}}$ and $m_{2,2}$ $V_{2,\frac{\rho_2}{\sigma_2}}$ are arbitrary non zero elements, except for the conditions on $\det M$ (vanishing at $\underline{x}$ but not everywhere). The remaining gauge freedom (while retaining the form $c_1 = c_2 = 1$) requires $\gamma_1 = \delta_1 = \delta_2$, i.e. only $\mathbf{C}$ -action on $(m_{2,1}, m_{2,2})$ is allowed. So we shall identify the image of $f_1 = 0$ in F to a subset of $\frac{V_{2,\frac{\rho_2}{\sigma_1}} \times V_{2,\frac{\rho_2}{\sigma_2}}}{\mathbf{C}}$.

Non triviality of $\det M$. We have equivalences:

$$\begin{aligned} \det M = 0 \qquad & m_{2,1}\,\theta_q(-x/x_1)\,\theta_q(-x/x_1) = m_{2,2}\,\theta_q(-x/x_1)\,\theta_q(-x/x_1) \\ & m_{2,1}\,\theta_q(-x/x_1) = m_{2,2}\,\theta_q(-x/x_1). \end{aligned}$$

Since $x_1 \equiv x_1$ (this follows from (NR)), we conclude that $m_{2,1} = c\,\theta_q(-x/x_1)\,\theta_q(-x/x_2)$ and $m_{2,2} = d\,\theta_q(-x/x_1)\,\theta_q(-x/x_2)$, where c, d $\mathbf{C}$ and $x_1 x_2 = \rho_2/\sigma_1$, $x_1 x_2 = \rho_2/\sigma_2$. Actually, $x_2 = x_1\rho_2/\rho_1 = x_2$; and $\det M = 0 \Leftrightarrow c = d$. So we see that matrices M $(f_1 = 0)$ such that $\det M = 0$ correspond to a $\mathbf{C}$ -line in $V_{2,\frac{\rho_2}{\sigma_1}} \times V_{2,\frac{\rho_2}{\sigma_2}}$ and their classes in F to a unique point in the associated projective space.

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[42] We shall find in Subsection 6.3 that there are two more special fibers.

Vanishing of $\det M$ **at** $\underline{x}$**.** Vanishing of $\det M$ at x_1 is ensured by the above chosen form. Vanishing at x_2, x_3, x_4 then amounts to *two* linear conditions (this is because $\sigma_q(\det M)/\det M = x_1 x_2 x_3 x_4$):

$$\left\{\begin{aligned} m_{2,1}(x_2)\,\theta_q(-x_2/x_1)\,\theta_q(-x_2/x_1) &= m_{2,2}(x_2)\,\theta_q(-x_2/x_1)\,\theta_q(-x_2/x_1),\\ m_{2,1}(x_3)\,\theta_q(-x_3/x_1)\,\theta_q(-x_3/x_1) &= m_{2,2}(x_3)\,\theta_q(-x_3/x_1)\,\theta_q(-x_3/x_1).\end{aligned}\right.$$

Simplifications are allowed by (NR) and yield:

$$\left\{\begin{aligned} m_{2,1}(x_2)\,\theta_q(-x_2/x_1) &= m_{2,2}(x_2)\,\theta_q(-x_2/x_1),\\ m_{2,1}(x_3)\,\theta_q(-x_3/x_1) &= m_{2,2}(x_3)\,\theta_q(-x_3/x_1).\end{aligned}\right.$$

Each of the two linear conditions is non trivial (because, again by (NR), $x_1 \equiv x_1$), whence defines a hyperplane in the 4-dimensional product space $V_{2,\frac{\rho_2}{\sigma_1}} \times V_{2,\frac{\rho_2}{\sigma_2}}$. These hyperplanes are distinct: indeed, using non splitting, one can find $f \in V_{2,\frac{\rho_2}{\sigma_1}}$ and $g \in V_{2,\frac{\rho_2}{\sigma_2}}$, each of them vanishing at x_2 but not at x_3; then, for a proper choice of $c, d \in \mathbf{C}^2$, the pair $(m_{2,1}, m_{2,2}) := (cf, dg)$ belongs to the first hyperplane but not to the second one. Therefore their intersection is a plane containing the line $\det M = 0$. Going to the associated projective space yields a line. So we conclude that the image of the component $(f_1 = 0)$ is a projective line minus a point, i.e. an *affine* line $\mathbf{C}$.

The fiber $\Pi^{-1}(0)$ has the two components $f_1 = 0$ and $g_2 = 0$ each isomorphic to the line $\mathbf{C}$ (this meaning that the obvious bijections are biregular in the algebro-geometric sense). Matrices in the intersection of these components have the form:

$$M = \begin{pmatrix} c_{1,1}\,\theta_q(-x/x_1)\,\theta_q(-x/x_1) & c_{1,2}\,\theta_q(-x/x_1)\,\theta_q(-x/x_1)\\ c_{2,1}\,\theta_q(-x/x_2)\,\theta_q(-x/x_2) & c_{2,2}\,\theta_q(-x/x_2)\,\theta_q(-x/x_2)\end{pmatrix}$$

for some arbitrary $c_{i,j} \in \mathbf{C}$ and $x_i, x_{\bar i} \in \mathbf{C}$ determined by obvious conditions. The Γ, Δ-action allows one to take three of the $c_{i,j}$ with value 1 and the fourth is then determined by the vanishing of $\det M$ at x_3, so our two projective lines intersect at exacly one point, corresponding to the double degeneracy $f_1 = g_2 = 0$.

Theorem 5.5. — *Assume non splitting of* (FR) *(i.e. all classes in F are irreducible). The special fibers* $\Pi^{-1}(0)$ *and* $\Pi^{-1}(\infty)$ *of* $\Pi : F \to \mathbf{P}^1(\mathbf{C})$ *are both made of two affine lines intersecting at one point.*

The study of the other fibers will be done in Section 6.

5.1.3. An elliptic function

Along with the projective invariant Π, a central role will be played by the following auxiliary function[43]:

$$\Phi(\xi) := \frac{\theta_q(\frac{x_1}{\rho_1}\xi)\,\theta_q(\frac{x_2}{\rho_2}\xi)}{\theta_q(\frac{x_1}{\rho_2}\xi)\,\theta_q(\frac{x_2}{\rho_1}\xi)}.$$

It is readily verified that Φ is an elliptic function on $\mathbf{C}^*$, i.e. that $\Phi(q\xi) = \Phi(\xi)$, so, after the conventions of Subsection 2.2, we may and will see it as a mapping (denoted the same) $\Phi : \mathbf{E}_q \to \mathbf{P}^1(\mathbf{C})$.

Under assumption (NR) there is no cancellation of zeroes between the numerator and denominator (which are both holomorphic over $\mathbf{C}^*$), so the elliptic function Φ has degree 2. Also:

$$\xi_1, \xi_2 \in \mathbf{C}^*,\ \xi_1\xi_2 \equiv \frac{\rho_1\rho_2}{x_1x_2} \Longrightarrow \Phi(\xi_1) = \Phi(\xi_2). \tag{5.1}$$

(Argument: by ellipticity of Φ, one may assume equality in the premise; and then direct calculation yields the result.) More precisely, we get:

Proposition 5.6. —

(i) *The elliptic function Φ realizes a degree 2 ramified covering $\mathbf{E}_q \to \mathbf{P}^1(\mathbf{C})$ with 4 critical values.*
(ii) *Generic fibers have the form $\{\overline{\xi_1}, \overline{\xi_1}\}$, where $\xi_1, \xi_2 \in C_q$, $\xi_1\xi_2 \equiv \frac{\rho_1\rho_2}{x_1x_2}$, $\xi_1 \not\equiv \xi_2$.*
(iii) *The four singular fibers have the form $\{\bar{\xi}\}$, where $\xi \in C_q$, $\xi^2 \equiv \frac{\rho_1\rho_2}{x_1x_2}$.*

Also note that the divisor of Φ depends only on the local data:

$$\mathrm{div}_{\mathbf{E}_q}(\Phi) = \left[\pi\left(-\frac{\rho_1}{x_1}\right)\right] + \left[\pi\left(-\frac{\rho_2}{x_2}\right)\right] - \left[\pi\left(-\frac{\rho_1}{x_2}\right)\right] - \left[\pi\left(-\frac{\rho_2}{x_1}\right)\right].$$

(Recall that $\pi : \mathbf{C}^* \to \mathbf{E}_q$ is the canonical projection.) Up to a non zero constant, Φ is determined by this divisor.

5.1.4. An involution of $\mathbf{E}_q$

Here is the geometrical meaning of Φ. The relation $\xi_1 \leftrightarrow \xi_2$ whenever $\xi_1\xi_2 \equiv \frac{\rho_1\rho_2}{x_1x_2}$ defines an involution on $\mathbf{E}_q$, and the mapping $\Phi : \mathbf{E}_q \to \mathbf{P}^1(\mathbf{C})$ defines a quotient of $\mathbf{E}_q$ by this involution. Up to a dilatation in $\mathbf{P}^1(\mathbf{C})$, this is the only realisation of this quotient with the following special fibers:

$$\text{Fiber at } 0 = \left\{\pi\left(-\frac{\rho_1}{x_1}\right), \pi\left(-\frac{\rho_2}{x_2}\right)\right\}, \quad \text{Fiber at } \infty = \left\{\pi\left(-\frac{\rho_1}{x_2}\right), \pi\left(-\frac{\rho_2}{x_1}\right)\right\}.$$

[43] This should really be denoted $\Phi_{1,2}$, see footnote 41 page 1178.

5.2. Definition, gauge freedom

Definition 5.7. — *A* Mano decomposition $M = PQ$ with factor C $\mathrm{GL}_2(\mathbf{C})$ *is given by* P, Q $\mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*))$ *such that* $\sigma_q P = RP(Cx)^{-1}$ *and* $\det P$ *vanishes at* x_1, x_2*. (It is understood that* x_1, x_2 *have previously been chosen and we do not mention them in the terminology.)*

It follows immediately from the definition, the assumptions on M and the properties recalled in 2.4 that:

- $\det P = 0$ and $\det P$ vanishes at $[x_1, x_2; q]$ with simple zeroes and nowhere else,
- $\det Q = 0$ and $\det Q$ vanishes at $[x_3, x_4; q]$ with simple zeroes and nowhere else,
- $\sigma_q Q = CQ(Sx)^{-1}$,
- $\frac{\sigma_q(\det P)}{\det P} = \frac{\rho_1\rho_2}{(\det C)x^2}$,
- $\frac{\sigma_q(\det Q)}{\det Q} = \frac{\det C}{\sigma_1\sigma_2 x^2}$,
- $\det C \equiv (\rho_1\rho_2)/(x_1x_2)$.

5.2.1. First consequences

Since we expect $\det P$ to have simple zeroes at x_1, x_2, we can, for $i = 1, 2$, choose a non zero column $\binom{p_i}{q_i}$ of $P(x_i)$; and the other column $\binom{p_i'}{q_i'}$ then necessarily belongs to $\mathbf{C}\binom{p_i}{q_i}$. Arguments similar to those used in the proof of Lemma 5.3 yield the following.

Lemma 5.8. —

(i) *One cannot have* $p_1q_2 = p_2q_1 = 0$.

(ii) *The ratio* $(p_1q_2 : p_2q_1)$ $\mathbf{P}^1(\mathbf{C})$ *is well defined from* P*, independently of the particular choices of non-zero columns.*

The "invariant" $(p_1q_2 : p_2q_1)$ will be related to $\Pi(M)$ in Proposition 5.16 and to the values of the elliptic function Φ in Lemma 5.12.

5.2.2. Gauge freedom

Proposition 5.9 (Gauge freedom). — *Let* $M = PQ$ *a Mano decomposition with factor* C.

(i) *Let* Λ $\mathrm{GL}_2(\mathcal{O}(\mathbf{C}^*))$ *be such that* $\Lambda[C]$ $\mathrm{GL}_2(\mathbf{C})$*. Let* $P' := P\Lambda^{-1}$*,* $Q' := \Lambda Q$ *and* $C' := \Lambda[C]$*. Then* $M = P'Q'$ *is a Mano decomposition with factor* C'*.*

(ii) *All Mano decompositions of* M *are obtained that way.*

Proof. —

(i). — It comes by a mechanical calculation.

(ii). — Let $M = P'Q'$ a Mano decomposition with factor C'. Set $\Lambda := P'^{-1}P = Q'Q^{-1}$. Since $P'^{-1}P$ has poles only at $[x_1, x_2; q]$ and $Q'Q^{-1}$ at $[x_3, x_4; q]$ and since by (NR) these sets do not meet, $\Lambda \in \mathrm{GL}_2(\mathcal{O}(\mathbf{C}^*))$. From $(\sigma_q P)CP^{-1} = \frac{1}{x}R = (\sigma_q P')C'P'^{-1}$ we draw that $C' = \Lambda[C]$.

Actually, Λ is a Laurent polynomial (with matrix coefficients). Indeed, write $\Lambda = \sum x^n \Lambda_n$. Then relation $C' = \Lambda[C]$ implies that $C'\Lambda_n = q^n \Lambda_n C$. But equation with matricial unknown X $C'X - X(q^n C) = 0$ has non trivial solutions if and only if $\mathrm{Sp}\, C'$ and $\mathrm{Sp}(q^n C)$ intersect, which is possible only for a finite number of values of n. This statement can be made more precise using Proposition 5.10 and its corollary.

5.2.3. Normal forms for C

Proposition 5.10 (Normal forms for C). — *The central factor C can be taken in one and only one of the following forms:*

$$C = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \qquad \textit{(trivial form)},$$

$$C = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad \xi_1 \neq \xi_2 \qquad \textit{(generic form)},$$

$$C = \begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix} \qquad \textit{(logarithmic form)},$$

with ξ_1, ξ_2 or ξ in the fundamental annulus $\mathcal{C}_q : |q| < |z| \leq 1$. Moreover, the form is unique except that in generic form ξ_1, ξ_2 can be permuted.

Proof. — Among gauge transforms are conjugacies (by $\mathrm{GL}_2(\mathbf{C})$) so, by standard reduction theory, C can be taken diagonal or in the form $\begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix}$. Then gauge transformation by so called “shearing matrices” $\mathrm{Diag}(x^\mu, x^\nu)$, $\mu, \nu \in \mathbf{Z}$, allows one to bring ξ_1, ξ_2 or ξ into $\mathcal{C}_q$. The possibility to permute ξ_1, ξ_2 comes from the equality:

$$\begin{pmatrix} \xi_2 & 0 \\ 0 & \xi_1 \end{pmatrix} = J\left(\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}\right), \quad \text{where } J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We now show that this is the only defect of uniqueness. So let C, C' in one of the quoted forms and let $\Lambda = (\lambda_{i,j})_{1 \leq i,j \leq 2} \in \mathrm{GL}_2(\mathcal{O}(\mathbf{C}^*))$ such that $\Lambda[C] = C'$, or, equivalently, $\sigma_q \Lambda = C'\Lambda C^{-1}$.

Case 1, C and C generic or trivial. — Write $C = \mathrm{Diag}(\xi_1, \xi_2)$ and $C = \mathrm{Diag}(\xi_1, \xi_2)$, whence $\sigma_q \lambda_{i,j} = \frac{\xi_i}{\xi_j} \lambda_{i,j}$. By Subsection 2.4 we know that this is possible with $\lambda_{i,j}$ $O(\mathbf{C}\ ) \setminus \{0\}$ only if $\frac{\xi_i}{\xi_j}$ $q^{\mathbf{Z}}$; since ξ_i, ξ_j C_q, this would mean $\xi_i = \xi_j$ and $\lambda_{i,j}$ $\mathbf{C}$. The end of the proof is standard linear algebra.

Case 2, C and C logarithmic. — Write $C = \begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix}$, $C = \begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix}$, whence:

$$\sigma_q \Lambda = \frac{\xi}{\xi} \begin{pmatrix} \lambda_{1,1} + \lambda_{2,1} & \lambda_{1,2} + \lambda_{2,2} - (\lambda_{1,1} + \lambda_{2,1}) \\ \lambda_{2,1} & \lambda_{2,2} - \lambda_{2,1} \end{pmatrix}.$$

If $\xi = \xi$, equation $\sigma_q \lambda_{2,1} = \frac{\xi}{\xi} \lambda_{2,1}$ implies $\lambda_{2,1} = 0$; then equation $\sigma_q \lambda_{1,1} = \frac{\xi}{\xi} \lambda_{1,1}$ implies $\lambda_{1,1} = 0$; but then Λ is not invertible, contradiction. So $\xi = \xi$ and $\lambda_{2,1}$ $\mathbf{C}$. But then we know from Part 2.4.2 that $\sigma_q \lambda_{1,1} = \lambda_{1,1} + \lambda_{2,1}$ is possible with $\lambda_{1,1}$ $O(\mathbf{C}\ )$ only if $\lambda_{2,1} = 0$. The end of the proof along the same lines is easy.

Case 3, mixed case. — Similar calculations left to the reader show this case to be impossible because there is no q-logarithm in $O(\mathbf{C}\ )$.

From the above proof, one can also draw:

Corollary 5.11. — *If C is in normal form, then*

$$\Lambda[C] = C \qquad (\Lambda \quad \mathrm{GL}_2(\mathbf{C}) \text{ and } [\Lambda, C] = 0).$$

5.3. Necessary conditions for the equality $M = PQ$

As a preliminary observation, note that if $M = PQ$ is a Mano decomposition with factor C in normal form, then we can still replace P by any $P = P\Lambda^{-1}$ where Λ $\mathrm{GL}_2(\mathbf{C})$ commutes with C. The apparently more general case Λ $\mathrm{GL}_2(O(\mathbf{C}\ ))$, $\Lambda[C] = C$ boils down to this one by Corollary 5.11.

5.3.1. Possible forms of the factor P

We can (and will) search the factor C in one of the forms shown in Proposition 5.10. We also know from the property of $\det C$ stated in Subsection 5.2 that, according to the case, $\xi_1 \xi_2 \equiv (\rho_1 \rho_2)/(x_1 x_2)$ or $\xi^2 \equiv (\rho_1 \rho_2)/(x_1 x_2)$.

Trivial form. In the case $C = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}$, we must have:

$$P = \begin{pmatrix} \alpha_{1,1}\,\theta_q\left(\frac{\xi}{\rho_1}x\right) & \alpha_{1,2}\,\theta_q\left(\frac{\xi}{\rho_1}x\right) \\ \alpha_{2,1}\,\theta_q\left(\frac{\xi}{\rho_2}x\right) & \alpha_{2,2}\,\theta_q\left(\frac{\xi}{\rho_2}x\right) \end{pmatrix} = \begin{pmatrix} \theta_q\left(\frac{\xi}{\rho_1}x\right) & 0 \\ 0 & \theta_q\left(\frac{\xi}{\rho_2}x\right) \end{pmatrix} \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix},$$

for some $\alpha_{i,j} \in \mathbf{C}$, $i, j = 1, 2$. Thus $\det P = 0 \Leftrightarrow \alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1} = 0$. Actually, the constant right factor $(\alpha_{i,j})$ can be taken rid of by the observation at the beginning of this section, i.e. we can take:

$$P = \begin{pmatrix} \theta_q\left(\frac{\xi}{\rho_1}x\right) & 0 \\ 0 & \theta_q\left(\frac{\xi}{\rho_2}x\right) \end{pmatrix}.$$

Generic form. In the case $C = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$, $\xi_1 \not\equiv \xi_2$, we must have:

$$P = \begin{pmatrix} \alpha_{1,1}\,\theta_q\left(\frac{\xi_1}{\rho_1}x\right) & \alpha_{1,2}\,\theta_q\left(\frac{\xi_2}{\rho_1}x\right) \\ \alpha_{2,1}\,\theta_q\left(\frac{\xi_1}{\rho_2}x\right) & \alpha_{2,2}\,\theta_q\left(\frac{\xi_2}{\rho_2}x\right) \end{pmatrix},$$

for some $\alpha_{i,j} \in \mathbf{C}$, $i, j = 1, 2$. Then:

$$\det P = \alpha_{1,1}\alpha_{2,2}\,\theta_q\left(\frac{\xi_1}{\rho_1}x\right)\theta_q\left(\frac{\xi_2}{\rho_2}x\right) - \alpha_{1,2}\alpha_{2,1}\,\theta_q\left(\frac{\xi_2}{\rho_1}x\right)\theta_q\left(\frac{\xi_1}{\rho_2}x\right),$$

so that:

$$\det P = 0 \Leftrightarrow \alpha_{1,1}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,1} = 0.$$

Indeed, since $\xi_1 \not\equiv \xi_2$, because of (NR) the functions $\theta_q\left(\frac{\xi_1}{\rho_1}x\right)\theta_q\left(\frac{\xi_2}{\rho_2}x\right)$ and $\theta_q\left(\frac{\xi_2}{\rho_1}x\right)\theta_q\left(\frac{\xi_1}{\rho_2}x\right)$ have no common zero.

Logarithmic form. In the case $C = \begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix}$, the coefficients $p_{i,j}$ of P satisfy

$$\begin{cases} \sigma_q p_{i,1} = \frac{\rho_i}{\xi} p_{i,1}, \\ \sigma_q p_{i,2} = \frac{\rho_i}{\xi}(p_{i,2} - p_{i,1}), \end{cases} \quad i = 1, 2.$$

According to Lemma 2.1, setting $\phi_i(x) := \theta_q\left(\frac{\xi}{\rho_i}x\right)$ and $\psi_i(x) := x\phi_i(x) = \frac{\xi}{\rho_i}x\theta_q\left(\frac{\xi}{\rho_i}x\right)$, we must have: $P = \begin{pmatrix} \alpha_{1,1}\varphi_1 & \alpha_{1,1}\psi_1 + \alpha_{1,2}\varphi_1 \\ \alpha_{2,1}\varphi_2 & \alpha_{2,1}\psi_2 + \alpha_{2,2}\varphi_2 \end{pmatrix}$ for some $\alpha_{i,j} \in \mathbf{C}$, $i, j = 1, 2$, so that

$$\begin{aligned} \det P &= (\alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2})\phi_1\phi_2 + \alpha_{1,1}\alpha_{2,1}x(\phi_1\phi_2 - \phi_2\phi_1) \\ &= x\phi_1\phi_2 \times \text{logarithmic derivative of } \left(x^{\alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2}}\left(\frac{\phi_2}{\phi_1}\right)^{\alpha_{1,1}\alpha_{2,1}}\right), \end{aligned}$$

whence

$$\det P = 0 \iff \begin{cases} \alpha_{1,1}\alpha_{2,1} = 0 \text{ and} \\ \alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2} = 0 \end{cases} \iff \begin{cases} (\alpha_{1,1} = \alpha_{2,1} = 0) \text{ or} \\ (\alpha_{1,1} = \alpha_{1,2} = 0) \text{ or} \\ (\alpha_{2,1} = \alpha_{2,2} = 0). \end{cases}$$

5.3.2. Further necessary conditions

From Lemma 5.8, we know that, P being given, $(p_1q_2 : p_2q_1) \in \mathbf{P}^1(\mathbf{C})$ is well defined. We identify this point of the projective line with $\frac{p_1q_2}{p_2q_1} \in \mathbf{S} = \mathbf{C} \cup \{\infty\}$, considered as the target space of elliptic functions, in particular of the function Φ defined in Part 5.1.3.

In what follows, we write ξ_1, ξ_2 the eigenvalues of C, with maybe (in the trivial or in the logarithmic case) $\xi_1 = \xi_2$. Their images $\overline{\xi_1}, \overline{\xi_2} \in \mathbf{E}_q$ are the *exponents* of C.

Lemma 5.12. — *In all cases above, the exponents $\overline{\xi_1}, \overline{\xi_2}$ constitute the fiber $\Phi^{-1}\left(\frac{p_1q_2}{p_2q_1}\right)$, i.e.:*

$$\Phi^{-1}\left(\frac{p_1q_2}{p_2q_1}\right) = \left\{\overline{\xi_1}, \overline{\xi_2}\right\}.$$

Proof. — Since $\xi_1\xi_2 \equiv (\rho_1\rho_2)/(x_1x_2)$, it will be enough to prove that:

$$\Phi(\xi_1) = \frac{p_1q_2}{p_2q_1}.$$

In all cases above, the first column of $P(x)$ is $A(x) := \begin{pmatrix} \alpha_{1,1}\,\theta_q\left(\frac{\xi_1}{\rho_1}x\right) \\ \alpha_{2,1}\,\theta_q\left(\frac{\xi_1}{\rho_2}x\right) \end{pmatrix}$. If $A(x_1)$ and $A(x_2)$ are both non zero, we can take them as column vectors $\binom{p_i}{q_i}$. An immediate calculation then gives $\frac{p_1q_2}{p_2q_1} = \Phi(\xi_1)$ as wanted. Assume that for instance $A(x_1) = 0$ (the case $A(x_2) = 0$ being entirely similar). We have $\alpha_{1,1}\,\theta_q\left(\frac{\xi_1}{\rho_1}x\right) = \alpha_{2,1}\,\theta_q\left(\frac{\xi_1}{\rho_2}x\right) = 0$. We cannot have $\alpha_{1,1} = \alpha_{2,1} = 0$ because then $\det P$ would vanish identically; neither can we have $\theta_q\left(\frac{\xi_1}{\rho_1}x\right) = \theta_q\left(\frac{\xi_1}{\rho_2}x\right) = 0$, because of condition (NR). Therefore, we have two cases to consider:

(1) $\alpha_{1,1} = 0$, $\alpha_{2,1} = 0$ and $\theta_q\left(\frac{\xi_1}{\rho_1}x\right) = 0$. Then $\Phi(\xi_1) = 0$. Also $\theta_q\left(\frac{\xi_1}{\rho_2}x\right) = 0$, so $A(x_2) = 0$, so we can take it as $\binom{p_2}{q_2}$, so $q_2 = \alpha_{2,1}\,\theta_q\left(\frac{\xi_1}{\rho_2}x\right) = 0$, so $\frac{p_1q_2}{p_2q_1} = 0 = \Phi(\xi_1)$ as wanted.

(2) $\alpha_{1,1} = 0$, $\alpha_{2,1} = 0$ and $\theta_q\left(\frac{\xi_1}{\rho_2}x\right) = 0$. A similar calculation yields $\Phi(\xi_1) = \frac{p_1q_2}{p_2q_1} = \infty$.

5.4. Existence of Mano decomposition

Let $R, S, \underline{x}$ the local data described in Subsection 5.1, subject to the assumptions (FR), (NR) and (NS) stated there.

Let $\Phi(\xi) := \dfrac{\theta_q\left(\frac{x_1}{\rho_1}\xi\right)\theta_q\left(\frac{x_2}{\rho_2}\xi\right)}{\theta_q\left(\frac{x_1}{\rho_2}\xi\right)\theta_q\left(\frac{x_2}{\rho_1}\xi\right)}$ the elliptic function defined and studied in Part 5.1.3.

For every matrix $M \in F_{R,S,\underline{x}}$, recall $\Pi(M) := (f_1 g_2 : f_2 g_1) \in \mathbf{P}^1(\mathbf{C})$ the projective invariant defined and studied in Part 5.1.1.

Theorem 5.13 (Existence of Mano decomposition). — *Every matrix $M \in F_{R,S,\underline{x}}$ admits a Mano decomposition $M = PQ$ with (some) factor C. More precisely:*

1. *If $\Pi(M)$ is not a critical value of Φ, the decomposition is generic (i.e. the factor C is in generic form).*
2. *If $\Pi(M)$ is a critical value of Φ and $\Pi(M) \neq 0, \infty$, the decomposition is logarithmic.*
3. *If $\Pi(M)$ is a critical value of Φ and $\Pi(M) = 0$ or ∞, the decomposition is logarithmic, except for the respective degenerate cases: $f_1 = g_2 = 0$ and $f_2 = g_1 = 0$; in these degenerate cases, the decomposition is trivial.*

The rest of this section is devoted to the proof. We shall keep the notations $\begin{pmatrix} f_i \\ g_i \end{pmatrix}$, $i = 1, 2$, for a non zero column of $M(x_i)$ (5.1.1) and similarly $\begin{pmatrix} p_i \\ q_i \end{pmatrix}$, $i = 1, 2$, for a non zero column of the left factor $P(x_i)$ (5.2.1).

We write the fiber of Φ at $\Pi(M) \in \mathbf{P}^1(\mathbf{C})$ as:

$$\Phi^{-1}(\Pi(M)) = \Phi^{-1}\left(\frac{f_1 g_2}{f_2 g_1}\right) = \left\{\overline{\xi_1}, \overline{\xi_2}\right\},$$

where $\xi_1, \xi_2 \in C_q$ (Part 5.1.3). Thus we have:

$$\xi_1 \xi_2 \equiv \frac{\rho_1 \rho_2}{x_1 x_2}. \tag{5.2}$$

5.4.1. Preliminary reductions

Proposition 5.14. — *In order to prove Theorem 5.13, it is enough to find a factor C and $P \in \mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*))$ such that:*

1. $\sigma_q P = RP(Cx)^{-1}$,
2. $\det P \neq 0$ *and* $\det P$ *vanishes at* x_1,

(3) *$P^{-1}M$ is well defined (i.e. has no pole) at x_1, x_2.*

Proof. — By Subsection 2.4, $\det P$ vanishes at $[x_1, x_2; q]$ and $Q := P^{-1}M$ vanishes at $[x_3, x_4; q]$ under the same conditions as usual and immediate calculation shows that we got a Mano decomposition.

Now, from Cramer's rule $P^{-1} = (\det P)^{-1}\tilde{P}$, we see that the last condition can be replaced by: $(\tilde{P}M)(x_i) = 0$, $i = 1, 2$. This in turn is equivalent to: $\begin{pmatrix} f_i \\ g_i \end{pmatrix}$ is a linear combination of the columns of $P(x_i)$ for $i = 1, 2$, i.e. that it is proportional to the selected non zero columns.

Corollary 5.15. — *Condition* (3) *above can be replaced by:*

(3') $\begin{pmatrix} f_i \\ g_i \end{pmatrix} \in \mathbf{C}\binom{p_i}{q_i}$, $i = 1, 2$.

Using Lemmas 5.4 and 5.8, we complete the above as follows:

Proposition 5.16. — *Let P a left factor of M in a Mano decomposition and keep previous notations $\begin{pmatrix} f_i \\ g_i \end{pmatrix}$ and $\binom{p_i}{q_i}$, $i = 1, 2$. Then one cannot have $p_1q_2 = p_2q_1 = 0$, the ratio $(p_1q_2 : p_2q_1) \in \mathbf{P}^1(\mathbf{C})$ is well defined from P and $(p_1q_2 : p_2q_1) = (f_1g_2 : f_2g_1)$.*

5.4.2. Proof of existence, case I: $f_1f_2g_1g_2 = 0$

From the proposition above, $p_1p_2q_1q_2 = 0$. We shall use Lemma 5.4.

Subcase Ia: $\xi_1 = \xi_2$. — For some $\alpha_1, \alpha_2 \in \mathbf{C}$ to be determined (not both zero), we set:

$$P_0 = \begin{pmatrix} \theta_q\left(\frac{\xi_1}{\rho_1}x\right) & \alpha_1\,\theta_q\left(\frac{\xi_2}{\rho_1}x\right) \\ \theta_q\left(\frac{\xi_1}{\rho_2}x\right) & \alpha_2\,\theta_q\left(\frac{\xi_2}{\rho_2}x\right) \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}.$$

Then, writing $T_1(x) := \theta_q\left(\frac{\xi_1}{\rho_1}x\right)\theta_q\left(\frac{\xi_2}{\rho_2}x\right)$ and $T_2(x) := \theta_q\left(\frac{\xi_2}{\rho_1}x\right)\theta_q\left(\frac{\xi_1}{\rho_2}x\right)$, we have:

$$\det P_0 = \alpha_2 T_1 - \alpha_1 T_2,$$

so taking $\alpha_1 := T_2(x_1)$ and $\alpha_2 := T_1(x_1)$ (which, by (NR), cannot both be 0), we see that P_0 satisfies the first two conditions of Proposition 5.14. Writing $\binom{p_i}{q_i}$ the first column of $P_0(x_i)$, $i = 1, 2$, we fall, by choice of ξ_1, under the assumptions of Lemma 5.4. We set $\Lambda := \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with λ, μ as provided by Lemma 5.4 and then $P := \Lambda P_0$. Using the fact that Λ and C commute with each other, we easily conclude that P satisfies all three conditions of Proposition 5.14, so Theorem 5.13 is proved in this case.

Subcase Ib: $\xi_1 = \xi_2 =: \xi$. — From the definition of Φ, we see that $\theta_q\left(\frac{x_j}{\rho_i}\xi\right) = 0$ for $i, j = 1, 2$. Writing $\phi_i(x) := \theta_q\left(\frac{\xi x}{\rho_i}\right)$, $i = 1, 2$, it follows that $(\phi_1\phi_2)(x_1) = 0$. We take $C := \begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix}$ and set $P_0 := \begin{pmatrix} \varphi_1 & x\varphi_1 + \alpha_1\varphi_1 \\ \varphi_2 & x\varphi_2 + \alpha_2\varphi_2 \end{pmatrix}$, so that $\sigma_q P_0 = RP(Cx)^{-1}$. Also $\det P_0 = (\alpha_2 - \alpha_1)\phi_1\phi_2 + x(\phi_1\phi_2 - \phi_1\phi_2)$ is equal to $x\phi_1\phi_2$ times the logarithmic derivative of $x^{\alpha_2 - \alpha_1}\frac{\varphi_2}{\varphi_1}$ and certainly does not vanish identically. Since $(\phi_1\phi_2)(x_1) = 0$, a proper choice of α_1, α_2 yields $\det P_0(x_1) = 0$ and the argument can be completed exactly as in subcase Ia to obtain a left factor $P := \Lambda P_0$.

5.4.3. Proof of existence, case II: $f_1 f_2 g_1 g_2 = 0$

Here we have $(f_1g_2)(f_2g_1) = 0$ while, by Lemma 5.3, f_1g_2 and f_2g_1 cannot both be zero. If $f_1g_2 = 0 = f_2g_1$, we must have $\Phi(\xi_1) = \Phi(\xi_2) = 0$; if $f_1g_2 = 0 = f_2g_1$, we must have $\Phi(\xi_1) = \Phi(\xi_2) = \infty$. We consider the former case only, the latter being entirely similar. Thus we have $\Phi^{-1}(0) = \left\{\overline{\xi_1}, \overline{\xi_2}\right\}$.

Subcase IIa: $\xi_1 = \xi_2$. — If $f_1 = 0$, we take, for some $\alpha \in \mathbf{C}$ to be determined:

$$P = \begin{pmatrix} \theta_q\left(\frac{\xi_1}{\rho_1}x\right) & 0 \\ \alpha\,\theta_q\left(\frac{\xi_1}{\rho_2}x\right) & \theta_q\left(\frac{\xi_2}{\rho_2}x\right) \end{pmatrix}.$$

Then $\det P = \theta_q\left(\frac{\xi_1}{\rho_1}x\right)\theta_q\left(\frac{\xi_2}{\rho_2}x\right)$ does not vanish identically; moreover:

$$P(x_1) = \begin{pmatrix} 0 & 0 \\ \alpha & \star \end{pmatrix} \quad \text{and} \quad P(x_2) = \begin{pmatrix} \star & 0 \\ \alpha & 0 \end{pmatrix},$$

where each $\star$ stands for some non zero complex number. Both determinants vanish, as required. We can take as $\binom{p_1}{q_1}$ the right column of $P(x_1)$, which is indeed non zero and proportional to $\binom{f_1}{g_1} = \binom{0}{\star}$. Likewise, we can take as $\binom{p_2}{q_2}$ the left column of $P(x_2)$, which is indeed non zero and can be made proportional to $\binom{f_2}{g_2} = \binom{\star}{g_2}$ by an appropriate choice of α. This terminates the proof in this case.

If $g_2 = 0$, we take, for some $\alpha \in \mathbf{C}$ to be determined:

$$P = \begin{pmatrix} \theta_q\left(\frac{\xi_1}{\rho_1}x\right) & \alpha\,\theta_q\left(\frac{\xi_2}{\rho_1}x\right) \\ 0 & \theta_q\left(\frac{\xi_2}{\rho_2}x\right) \end{pmatrix}.$$

We leave to the reader to complete the argument in this case.

Subcase IIb: $\xi_1 = \xi_2 =: \xi$. — Then $\xi \equiv -\frac{\rho_1}{\sigma_1} \equiv -\frac{\rho_2}{\sigma_2}$ and 0 is a critical value[44] of Φ. If $f_1 = 0$, one has $M(x_1) = \left(\begin{smallmatrix} 0 & 0 \\ & \end{smallmatrix}\right)$ and we want $p_1 = 0$. If $g_2 = 0$, one has $M(x_2) = \left(\begin{smallmatrix} & \\ 0 & 0 \end{smallmatrix}\right)$ and we want $q_2 = 0$.

IIb (i): $f_1 = g_2 = 0$. — We take $C := \mathrm{Diag}(\xi, \xi)$ and $P := \mathrm{Diag}\left(\theta_q\left(\frac{\xi}{\rho_1}x\right), \theta_q\left(\frac{\xi}{\rho_2}x\right)\right)$. The right column of $P(x_1)$ is $\left(\begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}\right)$ and the left column of $P(x_2)$ is $\left(\begin{smallmatrix} p_2 \\ q_2 \end{smallmatrix}\right)$.

IIb (ii): $f_1 = 0$, $g_2 = 0$. — We take $C := \begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix}$. We take $P := \begin{pmatrix} \alpha_{1,1}\varphi_1 & \alpha_{1,1}\psi_1 + \alpha_{1,2}\varphi_1 \\ \alpha_{2,1}\varphi_2 & \alpha_{2,1}\psi_2 + \alpha_{2,2}\varphi_2 \end{pmatrix}$ for some $\alpha_{i,j} \in \mathbf{C}$, $i, j = 1, 2$ with the usual notations for ϕ_i (see Subsection 5.3.1). We have here $\phi_1(x_1) = \phi_2(x_2) = 0$ and, as a consequence, $\phi_1(x_1), \phi_2(x_2), \phi_2(x_1), \phi_1(x_2) = 0$. Then $P(x_1) = \left(\begin{smallmatrix} 0 & 0 \\ & ? \end{smallmatrix}\right)$, its left column is $\left(\begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}\right)$, indeed colinear with $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$. Also, setting $\alpha_{1,1} := 0$ and $\alpha_{2,1} := 1$, one has $P(x_2) = \begin{pmatrix} 0 & \alpha_{1,2}\varphi_1(x_2) \\ 0 & \end{pmatrix}$, its right column can be chosen as $\left(\begin{smallmatrix} p_2 \\ q_2 \end{smallmatrix}\right)$ and made colinear to $\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$ by an appropriate choice of $\alpha_{1,2}$.

IIb (iii): $f_1 = 0$, $g_2 = 0$. — We leave to the reader to find the argument in this case (symmetric of the previous one).

This ends the proof of the theorem.

6. The Jimbo–Sakai family (II)

We first recall our general assumptions on the Jimbo–Sakai family, as described in Subsection 4.1 and completed in Subsection 5.1. The local data are:

$$R := \mathrm{Diag}(\rho_1, \rho_2), \quad S := \mathrm{Diag}(\sigma_1, \sigma_2) \quad \text{and} \quad \underline{x} := \{x_1, x_2, x_3, x_4\}.$$

We assume *Fuchs relation* (FR), *strong non resonancy* (NR) and add *non splitting* (NS) with respect to the selected pair $\underline{x}' := \{x_1, x_2\}$. We also write $\underline{x}'' := \{x_3, x_4\}$.

Our goal here is to give a geometric description of the *monodromy data space* $F := F_{R,s,\underline{x}}$ underlying the Jimbo–Sakai approach to the study of the discrete Painlevé equation q-PVI. We gave such a (crude) description in Section 4. In this section, we obtain a more general and more precise description, using for that the Mano decomposition studied in Section 5.

In hope that the reader doesn't get lost in the maze of computations of cases and subcases, here is a brief summary of the process. In short, F will

(44) In the case $f_2 g_1 = 0$ and $\xi_1 = \xi_2$, we would have $\xi \equiv -\frac{\rho_1}{\sigma_2} \equiv -\frac{\rho_2}{\sigma_1}$ and the critical value would be ∞.

be considered as fibered over the base consisting of all possible factors C in the Mano decomposition:

(1) The space X of possible matrices C is that of all those $C \quad \mathrm{GL}_2(\mathbf{C})$ such that $\det C$ is compatible with the prescribed local data: $\det C \equiv \frac{\rho_1\rho_2}{x_1 x_2} \equiv \sigma_1\sigma_2 x_3 x_4$.

(2) Since F_C, the fiber over C, is really determined by the class of C under gauge equivalence, the true base space B of our fibration is a quotient of X. We describe it with the help of normal forms for C; generically they are diagonal and unique up to permutation of the diagonal terms, so we should take in account an involution.

(3) The space F is the quotient of $F := F_{R,s,\underline{x}}$ by the equivalence relation , which was defined in Part 3.1.2. Let F_C the subspace of F made up of those matrices admitting Mano decomposition with factor C and $F_C := \frac{F_C}{}$. We parameterize each F_C by some space of complex matrices[45] and then each F_C can be described with the help of some (multi-)linear algebra.

(4) Of course the description of F_C is not the same in the *generic*, *trivial*[46] and *logarithmic* cases. But moreover in the generic case there are some *special* values for which the fiber is not the same as the *general* fiber (for instance fibers related to those we encountered in Part 5.1.2).

(5) Last we rebuild F as he union of the fibers F_C.

6.1. An assumption on the local data and a preliminary consequence

In Part 6.3.3, we shall be led to introduce two subsets of the fundamental annulus $\mathcal{C}_q$, related to the special fibers mentioned hereabove. For $x \quad \mathbf{C}$, recall (from the q-notations in Subsection 2.2) that we write $R(x)$ the unique representative of x in $\mathcal{C}_q$ (i.e. $R(x) \quad \mathcal{C}_q$ and $R(x) \equiv x$). Let:

$$\Xi \; := \{R(-\rho_1/x_1), R(-\rho_1/x_2), R(-\rho_2/x_1), R(-\rho_2/x_2)\},$$

$$\Xi \; := \{R(-\sigma_1 x_3), R(-\sigma_1 x_4), R(-\sigma_2 x_3), R(-\sigma_2 x_4)\}.$$

Then, in order to simplify the exposition, we shall assume[47] (see equation (6.1)) that:

Assumption $\mathbf{Hyp_8}$: $\quad \Xi := \Xi \quad \Xi$ has eight (pairwise distinct) elements.

[45] As in all the paper, this is possible because the spaces of solutions of “elementary” q-di□erence equations come equipped with explicit finite bases.

[46] Actually the trivial case will be excluded *de facto*, see herebelow Part 6.1.1.

[47] Our analysis can easily be extended to other cases, yielding similar though slightly di□erent geometries.

6.1.1. Our assumption excludes the trivial case

We draw at once a consequence of this assumption: *the trivial case for C (see the third situation described in Theorem 5.13) cannot occur.* Indeed, by 5.3.1, in the case $C = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}$, we must have:

$$P = \begin{pmatrix} \alpha_{1,1}\,\theta_q\left(\frac{\xi}{\rho_1}x\right) & \alpha_{1,2}\,\theta_q\left(\frac{\xi}{\rho_1}x\right) \\ \alpha_{2,1}\,\theta_q\left(\frac{\xi}{\rho_2}x\right) & \alpha_{2,2}\,\theta_q\left(\frac{\xi}{\rho_2}x\right) \end{pmatrix} = \begin{pmatrix} \theta_q\left(\frac{\xi}{\rho_1}x\right) & 0 \\ 0 & \theta_q\left(\frac{\xi}{\rho_2}x\right) \end{pmatrix} A,$$

$$\text{where } A := \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix},$$

In the same way:

$$Q = \begin{pmatrix} \beta_{1,1}\,\theta_q\left(\frac{\sigma_1}{\xi}x\right) & \beta_{1,2}\,\theta_q\left(\frac{\sigma_2}{\xi}x\right) \\ \beta_{2,1}\,\theta_q\left(\frac{\sigma_1}{\xi}x\right) & \beta_{2,2}\,\theta_q\left(\frac{\sigma_2}{\xi}x\right) \end{pmatrix} = B \begin{pmatrix} \theta_q\left(\frac{\sigma_1}{\xi}x\right) & 0 \\ 0 & \theta_q\left(\frac{\sigma_2}{\xi}x\right) \end{pmatrix},$$

$$\text{where } B := \begin{pmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{pmatrix}.$$

Usual conditions on P, Q imply first that $\det A, \det B \neq 0$; and then, since $\det P(x_1) = \det Q(x_3) = 0$, that:

$$\left(\xi \equiv -\rho_1/x_1 \text{ or } \xi \equiv -\rho_2/x_1\right) \quad \text{and} \quad \left(\xi \equiv -\sigma_1 x_3 \text{ or } \xi \equiv -\sigma_2 x_3\right).$$

This would imply $\xi \in \Xi \cap \Xi$, which, by assumption, is impossible.

6.1.2. Under our assumption, 0 and ∞ cannot be critical values of Φ

This means that case (3) in Theorem 5.13 cannot occur at all, so this statement actually subsumes the statement hereabove about the impossibility of the trivial case.

The argument is the following: if for instance 0 is a critical value, then from the definition of Φ (recalled at the beginning of Subsection 5.4, just before Theorem 5.13), $-\rho_1/x_1 \equiv -\rho_2/x_2$. Similarly, if ∞ is a critical value, then $-\rho_1/x_2 \equiv -\rho_2/x_1$. Both congruences are excluded by the fact that all elements of Ξ are distinct.

6.2. Fibration of F

Recall that $F = \mathcal{F}/\sim$ where $\mathcal{F} := \mathcal{F}_{R,s,\underline{x}}$ and the equivalence relation $\sim$ is defined by $M \sim \Gamma M \Delta^{-1}$, where Γ and Δ are diagonal matrices. It is induced by the group action of the product of the group of 2 × 2 invertible diagonal matrices by itself (actually, by the free action of the 3-dimensional torus $\frac{D_2(\mathbf{C}) \times D_2(\mathbf{C})}{\mathbf{C}^* (I_2, I_2)}$, see Subsection 4.2).

6.2.1. Partition of F

Let

$$X := \left\{ C \in \mathrm{GL}_2(\mathbf{C}) \;\middle|\; \det C \equiv \frac{\rho_1 \rho_2}{x_1 x_2} \right\},$$

the set of possible factors C for Mano decompositions. For all $C \in X$, set:

$$F'_C := \left\{ P \in \mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*)) \;\middle|\; \begin{array}{l} \sigma_q P = R P (Cx)^{-1}, \det P \neq 0, \\ \det P \text{ vanishes on } \underline{x} \end{array} \right\}$$

$$F''_C := \left\{ Q \in \mathrm{Mat}_2(\mathcal{O}(\mathbf{C}^*)) \;\middle|\; \begin{array}{l} \sigma_q Q = C Q (Sx)^{-1}, \det Q \neq 0, \\ \det Q \text{ vanishes on } \underline{x} \end{array} \right\}$$

Thus, as usual, all zeroes of $\det P$ and $\det Q$ are simple and located on $[\underline{x}; q]$ and $[\underline{x}; q]$ respectively.

For every $C \in X$, we have a well defined product map:

$$\begin{aligned} F'_C \times F''_C &\longrightarrow F, \\ (P, Q) &\longrightarrow PQ. \end{aligned}$$

We call F_C its image: it is the set of those $M \in F$ that admit a Mano decomposition with factor C. By Theorem 5.13:

$$F = \bigcup_{C \in X} F_C.$$

Also, by Proposition 5.9, $F_{C_1} = F_{C_2}$ if, and only if, C_1 and C_2 are gauge equivalent; otherwise F_{C_1} and F_{C_2} are disjoint. So, writing $\sim$ the restriction of the gauge equivalence relation to X, with a slight abuse of notation (confusing C with its class in $X/\sim$) we have a partition:

$$F = \bigsqcup_{C \in X/\sim} F_C.$$

6.2.2. Partition of $\mathcal{F}$

In order to apply the partition of F to the quotient $\mathcal{F} = F/\sim$, we need to complete Proposition 5.9 (gauge freedom for C).

Proposition 6.1. — *Let $C \in \mathcal{N}$ and let $(P_1,Q_1),(P_2,Q_2) \in F_C \times F_C$. Then:*

$$P_1Q_1 \sim P_2Q_2 \iff (P_2,Q_2) = (\Gamma P_1\Lambda^{-1}, \Lambda Q_1\Delta^{-1})$$
$$\text{for some } \Gamma,\Delta \text{ diagonal and } \Lambda \in \mathrm{GL}_2(\mathbf{C}) \text{ such that } \Lambda C = C\Lambda.$$

Proof. — We have $P_2Q_2 = \Gamma P_1Q_1\Delta^{-1}$ for some Γ,Δ diagonal and $(\Gamma P_1)(Q_1\Delta^{-1})$ is another Mano decomposition with factor C for P_2Q_2, so Proposition 5.9 yields some Λ such that $\Lambda[C] = C$. Since C is normalized, $\Lambda \in \mathrm{GL}_2(\mathbf{C})$ and $\Lambda C = C\Lambda$.

As a consequence, we define on each $F_C \times F_C$, $C \in \mathcal{N}$, an equivalence relation by:

$$(P_1,Q_1) \sim (P_2,Q_2) \underset{\text{def}}{\iff} (P_2,Q_2) = (\Gamma P_1\Lambda^{-1}, \Lambda Q_1\Delta^{-1})$$
$$\text{for some } \Gamma,\Delta \text{ diagonal and } \Lambda \in \mathrm{GL}_2(\mathbf{C}) \text{ such that } \Lambda C = C\Lambda.$$

We then deduce from the discussion in Part 6.2.1 that:

$$F = \bigsqcup_{C \in \frac{X}{\sim}} F_C,$$

where we have a well defined bijection:

$$\frac{F_C \times F_C}{\sim} \longrightarrow F_C.$$

6.2.3. The base space of the fibration

A priori, we should take as base space of our fibration the quotient $\frac{X}{\sim}$ of the set $X \subset \mathrm{GL}_2(\mathbf{C})$ by the gauge equivalence relation. However, under the assumption formulated in Subsection 6.1 we saw in Part 6.1.1 that the *trivial case* is impossible and reduced scalar matrices may be excluded.

For a more precise description, we shall use the normal forms for C found in Part 5.2.3, Proposition 5.10; we define the following subsets of X:

$$N_g := \left\{ \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix} \,\middle|\, \xi_1,\xi_2 \in \mathbf{C}_q \text{ and } \xi_1 \neq \xi_2 \text{ and } \xi_1\xi_2 \equiv \frac{\rho_1\rho_2}{x_1x_2} \right\},$$

$$N_t := \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \,\middle|\, \xi \in \mathbf{C}_q \text{ and } \xi^2 \equiv \frac{\rho_1\rho_2}{x_1x_2} \right\},$$

$$N_u := \left\{ \begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix} \,\middle|\, \xi \in \mathbf{C}_q \text{ and } \xi^2 \equiv \frac{\rho_1\rho_2}{x_1x_2} \right\},$$

$$N := N_g \sqcup N_t \sqcup N_u,$$
$$N' := N_g \sqcup N_u.$$

Every element of X is equivalent to a element of N and the only possible non trivial equivalences within N are relations within N_g of the form $\text{Diag}(\xi_1, \xi_2) \sim \text{Diag}(\xi_2, \xi_1)$. Factors $C \in N_g$, resp. $C \in N_t$, resp. $C \in N_u$ correspond to what we called the generic, resp. the trivial, resp. the logarithmic case. Since any element in the group $\mathbf{E}_q$ has exactly 4 square roots (recall that we use the multiplicative notation), $\text{card}\, N_t = \text{card}\, N_u = 4$.

Our true base space (N_t being excluded by Part 6.1.1) is therefore the quotient:

$$B := \frac{N'}{\sim},$$

where $\sim$ is the relation induced by the involution trivial on N_t and defined on N_g as $\text{Diag}(\xi_1, \xi_2) \to \text{Diag}(\xi_2, \xi_1)$. Sending $\text{Diag}(\xi_1, \xi_2)$ to ξ_1 and $\begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix}$ to ξ defines a bijection of N' to C_q, hence to $\mathbf{E}_q$. The corresponding involution on $\mathbf{E}_q$ is the map:

$$\alpha \longrightarrow \frac{a}{\alpha},$$

where a is the class of $\frac{\rho_1 \rho_2}{x_1 x_2}$. We already met this involution in Part 5.1.4.

Proposition 6.2. — *As a quotient holomorphic curve, the base space B is isomorphic to the projective line $\mathbf{P}^1(\mathbf{C})$. The quotient map is realized by Φ, i.e. it is the mapping:*

$$N' \longrightarrow \mathbf{P}^1(\mathbf{C}), \quad \left[\xi\right] \longrightarrow \Phi(\xi).$$

Let $M \in F$, let ξ_1, ξ_2 the eigenvalues of the factor C in the Mano decomposition of M and P the left factor. The projective invariant $\Pi(M)$ was defined after Lemma 5.3.

In the discussion at the beginning of Subsection 5.4, we used Lemma 5.12, according to which:

$$\Phi^{-1}\left(\frac{p_1 q_2}{p_2 q_1}\right) = \left\{\overline{\xi_1}, \overline{\xi_2}\right\},$$

where the p_i, q_i are related to P as explained there. Then, in Proposition 5.16, we found that:

$$(p_1 q_2 : p_2 q_1) = (f_1 g_2 : f_2 g_1),$$

where $(f_1 g_2 : f_2 g_1) = \Pi(M)$. Combining these facts, we find that:

$$\Pi(M) = \Phi(\xi_1) = \Phi(\xi_2).$$

Using Proposition 6.2 above, we can now recognize the true role of the projective invariant Π:

Theorem 6.3. — *The fibration obtained from $F \to X$ when going to the quotient is (up to natural identifications):*

$$\mathcal{F} \xrightarrow{\Pi} \mathbf{P}^1(\mathbf{C}).$$

Proof. — Recall from Part 6.2.1 the partition $F = \coprod_{C \in X} F_C$, which allows us to define a map $F \to X$. The identification of X with $\mathbf{P}^1(\mathbf{C})$ is provided by the map Φ. We get a commutative diagram:

$$\begin{array}{ccc} F & \longrightarrow & X/ \\ \Big\downarrow & & \Big\downarrow \Phi \\ \mathcal{F} & \xrightarrow[\Pi]{} & \mathbf{P}^1(\mathbf{C}) \end{array}$$

The lower horizontal line is thus identified with the fibration.

6.3. Description of $\mathcal{F}_C$ in the generic case

As can be guessed, we have to distinguish the generic and logarithmic cases for C (the trivial case has been excluded, see Subsection 6.1).

Let $C = \mathrm{Diag}(\xi_1, \xi_2)$, where $\xi_1, \xi_2 \in \mathbf{C}_q$, $\xi_1 \neq \xi_2$ and $\det C = \xi_1\xi_2 \equiv \frac{\rho_1\rho_2}{x_1x_2}$. The matrices $\Lambda \in \mathrm{GL}_2(\mathbf{C})$ commuting with C are the diagonal matrices.

6.3.1. Spaces of matrices

The elements of F_C are the matrices:

$$P = \begin{pmatrix} \alpha_{1,1}\,\theta_q\left(\frac{\xi_1}{\rho_1}x\right) & \alpha_{1,2}\,\theta_q\left(\frac{\xi_2}{\rho_1}x\right) \\ \alpha_{2,1}\,\theta_q\left(\frac{\xi_1}{\rho_2}x\right) & \alpha_{2,2}\,\theta_q\left(\frac{\xi_2}{\rho_2}x\right) \end{pmatrix} \text{ for some } \alpha_{i,j} \in \mathbf{C}, i, j = 1, 2,$$

$$\text{such that } \begin{cases} (\alpha_{1,1}\alpha_{2,2}, \alpha_{1,2}\alpha_{2,1}) \neq (0, 0), \\ \det P(x_1) = \det P(x_2) = 0. \end{cases}$$

Condition $\det P(x_1) = \det P(x_2) = 0$ leads us to introduce the elliptic function:

$$\Phi_C(x) := \frac{\theta_q\left(\frac{\xi_1}{\rho_1}x\right)\theta_q\left(\frac{\xi_2}{\rho_2}x\right)}{\theta_q\left(\frac{\xi_2}{\rho_1}x\right)\theta_q\left(\frac{\xi_1}{\rho_2}x\right)},$$

which is of degree 2 since there is no cancellation of zeroes (this follows from (NR) and the fact that we are in the generic case); also Φ_C is such that $\Phi_C(x_1) = \Phi_C(x_2)$ (because of the condition on $\det C$). Then:

$$(\det P(x_1) = \det P(x_2) = 0) \iff \frac{\alpha_{1,1}\alpha_{2,2}}{\alpha_{1,2}\alpha_{2,1}} = \frac{1}{\Phi_C(x_i)}, i = 1, 2.$$

Said equality is understood to hold in $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$. We are thus led to set:

$$s := \frac{1}{\Phi_C(x_1)} = \frac{1}{\Phi_C(x_2)} \in \mathbf{C} \cup \{\infty\}$$

and to define the following spaces of matrices; first:

$$\mathrm{Mat}_2(\mathbf{C}) := \{A := (\alpha_{i,j}) \in \mathrm{Mat}_2(\mathbf{C}) \mid (\alpha_{1,1}\alpha_{2,2}, \alpha_{1,2}\alpha_{2,1}) \neq (0,0)\},$$

which we endow with a mapping:

$$\begin{aligned} \alpha : \mathrm{Mat}_2(\mathbf{C}) &\longrightarrow \mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}, \\ A := (\alpha_{i,j}) &\longmapsto \frac{\alpha_{1,1}\alpha_{2,2}}{\alpha_{1,2}\alpha_{2,1}}. \end{aligned}$$

Second:

$$\mathrm{Mat}_2(\mathbf{C})_C := \{A := (\alpha_{i,j}) \in \mathrm{Mat}_2(\mathbf{C}) \mid \alpha(A) = s\}.$$

We then have a bijection:

$$\begin{aligned} \mathrm{Mat}_2(\mathbf{C})_C &\longrightarrow F_C, \\ A := (\alpha_{i,j}) &\longmapsto P := \left(\alpha_{i,j}\, \theta_q\left(\frac{\xi_j}{\rho_i}x\right)\right). \end{aligned}$$

We now observe by direct computation (or reasoning on line and columns) that, if $A \mapsto P$ and if $\Gamma, \Lambda \in \mathrm{GL}_2(\mathbf{C})$ are diagonal, then $\Gamma A \Lambda^{-1} \mapsto \Gamma P \Lambda^{-1}$.

A similar study about the right factor $Q \in F_C$ in the Mano decomposition leads us to introduce the elliptic function:

$$\Phi_C(x) := \frac{\theta_q\left(\frac{\sigma_1}{\xi_1}x\right)\theta_q\left(\frac{\sigma_2}{\xi_2}x\right)}{\theta_q\left(\frac{\sigma_2}{\xi_1}x\right)\theta_q\left(\frac{\sigma_1}{\xi_2}x\right)},$$

which is of degree 2 and such that $\Phi_C(x_3) = \Phi_C(x_4)$; and to set:

$$t := \frac{1}{\Phi_C(x_3)} = \frac{1}{\Phi_C(x_4)} \in \mathbf{C} \cup \{\infty\}.$$

We then define the space of matrices:

$$\mathrm{Mat}_2(\mathbf{C})_C := \{B := (\beta_{i,j}) \in \mathrm{Mat}_2(\mathbf{C}) \mid \alpha(B) = t\}.$$

This yields a bijection:

$$\begin{aligned} \mathrm{Mat}_2(\mathbf{C})_C &\longrightarrow F_C, \\ B := (\beta_{i,j}) &\longmapsto Q := \left(\beta_{i,j}\, \theta_q\left(\frac{\sigma_j}{\xi_i}x\right)\right). \end{aligned}$$

Again we observe that, if $B \mapsto Q$ and if $\Lambda, \Delta \in \mathrm{GL}_2(\mathbf{C})$ are diagonal, then $\Lambda B \Delta^{-1} \mapsto \Lambda Q \Delta^{-1}$.

Proposition 6.4. — *We thereby define a bijection:*

$$\underline{\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C} \longrightarrow \underline{F_C \times F_C} \longrightarrow \mathcal{F}_C,$$

where the equivalence relation in the left hand side is defined by (A,B) $(\Gamma A \Lambda^{-1}, \Lambda B \Delta^{-1})$ *for all* (A,B) $\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C$ *and for all invertible diagonal matrices* Γ, Λ, Δ*. The rightmost arrow was previously defined: it is induced by the multiplication map* $F_C \times F_C \to \mathcal{F}_C$*,* $(P,Q) \to PQ$*.*

6.3.2. Keeping track of the involution $C = \mathrm{Diag}(\xi_1, \xi_2) \to \tilde{C} := JCJ = \mathrm{Diag}(\xi_2, \xi_1)$

Recall that $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and that, writing $\tilde{C} := JCJ$, we have $F_C = F_{\tilde{C}}$ by Proposition 5.9 and therefore $\mathcal{F}_C = \mathcal{F}_{\tilde{C}}$. We intend to produce a "coordinate" on $\mathcal{F}_C$ by using the bijection described in the above proposition. This is not the same if we use C or $\tilde{C}$: we have to make explicit this involutive relationship.

More precisely, we have a bijection:

$$\begin{aligned} F_C \times F_C &\longrightarrow F_{\tilde{C}} \times F_{\tilde{C}}, \\ (P,Q) &\longrightarrow (PJ, JQ) \end{aligned}$$

yielding the left vertical map of a commutative diagram:

$$\begin{array}{ccc} F_C \times F_C & \longrightarrow & \mathcal{F}_C \\ \Big| & & \Big\| = \\ F_{\tilde{C}} \times F_{\tilde{C}} & \longrightarrow & \mathcal{F}_{\tilde{C}}. \end{array}$$

Intepreting the multiplications by J in terms of exchanging lines or columns, we get another commutative diagram of bijections:

$$\begin{array}{ccc} \mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C & \longrightarrow & F_C \times F_C \\ \Big| & & \Big| \\ \mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C & \longrightarrow & F_{\tilde{C}} \times F_{\tilde{C}}, \end{array}$$

where the left vertical map is $(A,B) \to (AJ, JB)$.

Write $\tilde{\Lambda} := J \Lambda J$. From the obvious relations:

$$(\Gamma A \Lambda^{-1}) J = \Gamma (AJ) \tilde{\Lambda}^{-1} \quad \text{and} \quad J(\Lambda B \Delta^{-1}) = \tilde{\Lambda}(JB)\Delta^{-1},$$

we deduce that said map $(A,B) \to (AJ, JB)$ is compatible with the equivalence relation on $\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C$. So in the end we get a commutative diagram:

$$\begin{array}{ccccc} \underline{\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C} & \longrightarrow & \underline{F_C \times F_C} & \longrightarrow & F_C \\ \Big| & & \Big| & & \Big\| \\ \underline{\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C} & \longrightarrow & \underline{F_{\tilde{C}} \times F_{\tilde{C}}} & \longrightarrow & F_{\tilde{C}} \end{array}$$

6.3.3. A list of special cases

We shall describe

$$\underline{\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C}$$

by looking for normal forms for pairs $(A,B) \in \mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C$. When all coefficients of A and B are non zero, the answer is simpler, so we first discuss here the possibility that some coefficients vanish. We shall only analyse the case that $\alpha_{1,1} = 0$; the other cases will be expounded dogmatically, the arguments being entirely similar.

If $\alpha_{1,1} = 0$, then $\alpha_{1,2}\alpha_{2,1} = 0$. Let $P \in F_C$ the corresponding matrix; then

$$\det P = -\alpha_{1,2}\alpha_{2,1}\,\theta_q\left(\frac{\xi_2}{\rho_1}x\right)\theta_q\left(\frac{\xi_1}{\rho_2}x\right)$$

must vanish at x_1 and x_2, which means that one of its theta factors vanishes at x_1 and the other at x_2. This implies that either $\xi_1 \equiv -\frac{\rho_2}{x_1}$ and $\xi_2 \equiv -\frac{\rho_1}{x_2}$ or $\xi_1 \equiv -\frac{\rho_1}{x_2}$ and $\xi_2 \equiv -\frac{\rho_2}{x_1}$. (We could alternatively use the fact that $\frac{\alpha_{1,1}\alpha_{2,2}}{\alpha_{1,2}\alpha_{2,1}} = \frac{1}{\Phi_C(x_1)} = \frac{1}{\Phi_C(x_2)}$.) With the notations of Subsections 5.1 and 5.2, we also see that either $f_2 = 0$ and $\Phi(\xi_1) = \Phi(\xi_2) = \infty$, or $f_1 = 0$ and $\Phi(\xi_1) = \Phi(\xi_2) = 0$.

Conversely, if for instance $\xi_1 \equiv -\frac{\rho_2}{x_1}$ (and therefore $\xi_2 \equiv -\frac{\rho_1}{x_2}$), we have $\theta_q\left(\frac{\xi_1}{\rho_2}x_1\right) = 0$, whence $\det P(x_1) = \alpha_{1,1}\alpha_{2,2}\,\theta_q\left(\frac{\xi_1}{\rho_1}x_1\right)\theta_q\left(\frac{\xi_2}{\rho_2}x_1\right) = 0$. Under the assumption (NR) and due to the fact that we are in the generic case, $\theta_q\left(\frac{\xi_1}{\rho_1}x\right)\theta_q\left(\frac{\xi_2}{\rho_2}x\right)$ has no common zero with $\theta_q\left(\frac{\xi_1}{\rho_2}x\right)$. So the above in turn implies that $\alpha_{1,1}\alpha_{2,2} = 0$.

We summarize in the following table the list of all possible cases of vanishing of some $\alpha_{i,j}$:

vanishing coefficient	class of ξ_1 (mod $q^{\mathbf{Z}}$)	class of ξ_2 (mod $q^{\mathbf{Z}}$)	value of one of the f_i, g_i	$\Phi(\xi_1) =$ $\Phi(\xi_2) =$
$\alpha_{1,1} = 0$	$\xi_1 \equiv -\rho_2/x_1$	$\xi_2 \equiv -\rho_1/x_2$	$f_2 = 0$	∞
$\alpha_{1,1} = 0$	$\xi_1 \equiv -\rho_2/x_2$	$\xi_2 \equiv -\rho_1/x_1$	$f_1 = 0$	0
$\alpha_{2,2} = 0$	$\xi_1 \equiv -\rho_2/x_1$	$\xi_2 \equiv -\rho_1/x_2$	$g_1 = 0$	∞
$\alpha_{2,2} = 0$	$\xi_1 \equiv -\rho_2/x_2$	$\xi_2 \equiv -\rho_1/x_1$	$g_2 = 0$	0
$\alpha_{1,2} = 0$	$\xi_1 \equiv -\rho_1/x_1$	$\xi_2 \equiv -\rho_2/x_2$	$f_1 = 0$	0
$\alpha_{1,2} = 0$	$\xi_1 \equiv -\rho_1/x_2$	$\xi_2 \equiv -\rho_2/x_1$	$f_2 = 0$	∞
$\alpha_{2,1} = 0$	$\xi_1 \equiv -\rho_1/x_1$	$\xi_2 \equiv -\rho_2/x_2$	$g_2 = 0$	0
$\alpha_{2,1} = 0$	$\xi_1 \equiv -\rho_1/x_2$	$\xi_2 \equiv -\rho_2/x_1$	$g_1 = 0$	∞

Just for the few following definitions, we shall, for every a $\mathbf{C}$, write $R(a)$ its unique representative in C_q, i.e. $\{R(a)\} = [a;q] \cap C_q$. We introduce the sets of special values:

$$\Xi_1 := \{R(-\rho_1/x_1), R(-\rho_1/x_2)\},$$
$$\Xi_2 := \{R(-\rho_2/x_1), R(-\rho_2/x_2)\},$$
$$\Xi := \Xi_1 \quad \Xi_2 \quad \mathbf{C},$$

Then we can also express our conditions as:

$$\xi_1 \quad \Xi_1 \qquad \xi_2 \quad \Xi_2 \qquad \alpha_{1,1}\alpha_{2,2} = 0,$$
$$\xi_1 \quad \Xi_2 \qquad \xi_2 \quad \Xi_1 \qquad \alpha_{1,2}\alpha_{2,1} = 0.$$

There are similar results for the vanishing of the $\beta_{i,j}$, but we do not tabulate them (although they will be used when necessary). In short, all the $\beta_{i,j}$ are non zero except maybe if ξ_1 is congruent modulo $q^{\mathbf{Z}}$ to one of the $-\sigma_j x_i$, $i = 3, 4$, $j = 1, 2$. (Here, one must take in account vanishing of such expressions as $\theta_q \; \frac{\sigma_j}{\xi_1} x_i$.) So we define new sets of special values:

$$\Xi_1 := \{R(-\sigma_1 x_3), R(-\sigma_1 x_4)\},$$
$$\Xi_2 := \{R(-\sigma_2 x_3), R(-\sigma_2 x_4)\},$$
$$\Xi := \Xi_1 \quad \Xi_2 \quad \mathbf{C}.$$

We have:

$$\xi_1 \quad \Xi_2 \qquad \xi_2 \quad \Xi_1 \qquad \beta_{1,1}\beta_{2,2} = 0,$$
$$\xi_1 \quad \Xi_1 \qquad \xi_2 \quad \Xi_2 \qquad \beta_{1,2}\beta_{2,1} = 0.$$

Altogether we get the following set of special values:

$$\Xi := \Xi \quad \Xi.$$

From now on, we shall assume for simplicity that these eight special values (i.e. their classes in $\mathbf{E}_q$) are pairwise distinct:

$$\text{Assumption } \mathbf{Hyp_8}: \quad \operatorname{card}\Xi = 8. \tag{6.1}$$

Besides, this implies that $\xi_1 \equiv \xi_2$ for all $\xi_1 \quad \Xi$, so all the "general values" in $\mathbf{C}\ \setminus \Xi$ actually fall under the generic case being presently studied.

6.3.4. General fiber

We assume here that $\xi_1, \xi_2 \quad \Xi$. Then, for all pairs $(A, B) \quad \mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C$, all the coefficients $\alpha_{i,j}, \beta_{i,j}$, $i, j = 1, 2$ are non zero.

Straightforward use of the action of Γ, Λ allows one to bring A to the form $\left(\begin{smallmatrix} 1 & 1 \\ 1 & ? \end{smallmatrix}\right)$ and the missing coefficient is necessarily s. Then the only possible actions of Γ, Λ preserving that form are those such that $\gamma_1/\lambda_1 = \gamma_1/\lambda_2 = \gamma_2/\lambda_1 = 1$, whence $\Gamma = \Lambda = \lambda I_2$ for some $\lambda \quad \mathbf{C}$ .This means that we can only act on B, while preserving the normal form for A, by maps $B \to \lambda B \Delta^{-1}$. This allows one to bring B to the form $\left(\begin{smallmatrix} 1 & 1 \\ \eta & y \end{smallmatrix}\right)$ with the relation $y = t\eta$. In the end, we obtain a normal form for (A, B):

$$(A, B) \quad (A_0, D_\eta B_0),$$
$$\text{where} \quad A_0 := \begin{pmatrix} 1 & 1 \\ 1 & s \end{pmatrix}, \quad B_0 := \begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}, \quad D_\eta := \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}.$$

Here s, t are fixed by the value of the base point $(\xi_1, \xi_2) \quad X$ but $x \quad \mathbf{C}$ is the one free parameter (indeed, a coordinate) characterizing the class of (A, B).

The action of the involution permutes the columns of A and the lines of B. It sends $(A_0, D_\eta B_0)$ to:

$$(A_0 J, J D_\eta B_0) = \left(\begin{pmatrix} 1 & 1 \\ s & 1 \end{pmatrix}, \begin{pmatrix} \eta & \eta t \\ 1 & 1 \end{pmatrix}\right) \quad \left(\begin{pmatrix} 1 & 1 \\ 1 & s^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ \eta^{-1} & \eta^{-1} t^{-1} \end{pmatrix}\right),$$

the equivalence being induced by the action of:

$$(\Gamma, \Lambda, \Delta) := \left(\mathrm{Diag}(1, s^{-1}), I_2, \mathrm{Diag}(\eta, \eta t)\right).$$

Note that here we consider the fiber over $\Theta = \mathrm{Diag}(\xi_2, \xi_1)$ and it is readily checked that s and t must respectively be replaced by s^{-1} and t^{-1}. We conclude that the coordinate η must correspondingly be replaced by η^{-1}.

6.3.5. Special fibers

Assume for instance that $\xi_1 \equiv -\rho_1/x_1$ (the other possibilities are entirely similar). Let $(A, B) \in \mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C$. Then all the coefficients $\beta_{i,j}$, $i, j = 1, 2$ as well as $\alpha_{1,1}$ and $\alpha_{2,2}$ are non zero; but $\alpha_{1,2}\alpha_{2,1} = 0$ (see the table in Part 6.3.3).

If $\alpha_{1,2} = 0 = \alpha_{2,1}$, use of Γ, Λ allows one to reduce A to the form $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. This pattern can only be preserved by further transformations such that $\Gamma = \Lambda = \lambda I_2$, so possible transformations of B must have the form $B \to \lambda B \Delta^{-1}$. This allows one to bring B to the form $D_\eta B_0$ as before. So pairs (A, B) of this type give rise to one line in F_C, parameterized by $\mathbf{C}$. The case $\alpha_{1,2} = 0 = \alpha_{2,1}$ is obviously similar and leads to the same conclusion.

Now, if $\alpha_{1,2} = \alpha_{2,1} = 0$, A can be brought to the form I_2 but all pairs (Γ, Λ) such that $\Gamma = \Lambda$ preserve that form, so that all actions $B \to \Lambda B \Delta^{-1}$ are allowed and B can be brought to the form B_0. This means that all these pairs define a unique point in F_C. Since this point is a degeneracy of each of the two punctured $\mathbf{C}$ lines found above, the resulting figure (for the special fiber over $\xi_1 \equiv -\rho_1/x_1$ or for any of the other possibilities) is: two $\mathbf{C}$ lines intersecting at a point.

Now we discuss the involution. It replaces $\xi_1 \equiv -\rho_1/x_1$ by $\xi_1 \equiv -\rho_2/x_2$ and the reduced pair $\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, D_\eta B_0\right)$ by $\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, J D_\eta B_0\right)$, which is equivalent to $\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ \eta^{-1} & \eta^{-1}t^{-1} \end{pmatrix}\right)$. So the two $\mathbf{C}$-lines above $-\rho_1/x_1$ go isomorphically to the two special lines above $-\rho_2/x_2$; and obviously, the degenerate point to the degenerate point. Therefore, after going to the quotient by the involution, each special fiber still consists in two $\mathbf{C}$ lines intersecting at a point.

6.4. Description of F_C in the logarithmic case

Let $C := \begin{pmatrix} \xi & \xi \\ 0 & \xi \end{pmatrix}$, where $\xi \in \mathbf{C}_q$ and $\xi^2 \equiv \frac{\rho_1\rho_2}{x_1x_2}$. Matrices $\Lambda \in \mathrm{GL}_2(\mathbf{C})$ commuting with C are those of the form $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$, where $\lambda \in \mathbf{C}$ and $\mu \in \mathbf{C}$.

From 5.3.1, we know that corresponding left factors in Mano decomposition have the form $P = \begin{pmatrix} \alpha_{1,1}\varphi_1 & \alpha_{1,1}\psi_1 + \alpha_{1,2}\varphi_1 \\ \alpha_{2,1}\varphi_2 & \alpha_{2,1}\psi_2 + \alpha_{2,2}\varphi_2 \end{pmatrix}$ for some $\alpha_{i,j} \in \mathbf{C}$, $i, j = 1, 2$, where $\phi_i(x) := \theta_q\left(\frac{\xi}{\rho_i}x\right)$ and $\psi_i(x) := x\phi_i(x) = \frac{\xi}{\rho_i}x\theta_q\left(\frac{\xi}{\rho_i}x\right)$, so that:

$$\det P = (\alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2})\phi_1\phi_2 + \alpha_{1,1}\alpha_{2,1}x(\phi_1\phi_2 - \phi_2\phi_1).$$

Also condition $\det P = 0$ is equivalent to $(\alpha_{1,1}\alpha_{2,1}, \alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2}) = (0, 0)$.

We note that $\phi_1\phi_2(x_1) = 0$. Indeed, if for instance $\phi_1(x_1) = 0$, then $\xi \equiv -\rho_1/x_1$, so $\xi \in \Xi$ which we saw was impossible. (Direct argument: the congruence property on ξ^2 would imply $\xi \equiv -\rho_2/x_2$, whence $\rho_1/x_1 \equiv \rho_2/x_2$, contrary to the assumption that Ξ has eight pairwise distinct elements.)

We also deduce that $\alpha_{1,1}\alpha_{2,1} = 0$ because otherwise the condition $\det P(x_1) = 0$ would imply $\alpha_{1,1}\alpha_{2,2} - \alpha_{2,1} = 0$, then $(\alpha_{1,1}\alpha_{2,1}, \alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2}) = (0, 0)$. So in fact a small calculation yields:

$$\det P(x_1) = 0 \iff \frac{\alpha_{2,2}}{\alpha_{2,1}} - \frac{\alpha_{1,2}}{\alpha_{1,1}} = u, \text{ where } u := x_1 \left(\frac{\phi_1'}{\phi_1} - \frac{\phi_2'}{\phi_2}\right)(x_1).$$

So here we introduce the space of matrices:

$$\mathrm{Mat}_2(\mathbf{C})_C := \left\{ A := (\alpha_{i,j}) \in \mathrm{Mat}_2(\mathbf{C}) \,\middle|\, \begin{matrix} \alpha_{1,1}, \alpha_{2,2} = 0, \\ \frac{\alpha_{2,2}}{\alpha_{2,1}} - \frac{\alpha_{1,2}}{\alpha_{1,1}} = u. \end{matrix} \right\}.$$

An easy calculation shows that action $P \mapsto \Gamma P \Lambda^{-1}$, with diagonal Γ and Λ as shown above, translates to the similar action on A.

We now go into the corresponding calculations for the right factors Q in Mano decomposition; they have the form $Q = \begin{pmatrix} \beta_{1,1}\bar\varphi_1 - \beta_{2,1}\bar\psi_1 & \beta_{1,2}\bar\varphi_2 - \beta_{2,2}\bar\psi_2 \\ \beta_{2,1}\bar\varphi_1 & \beta_{2,2}\bar\varphi_2 \end{pmatrix}$ for some $\beta_{i,j} \in \mathbf{C}$, $i, j = 1, 2$, where $\bar\phi_j(x) := \theta_q\left(\frac{\sigma_j}{\xi} x\right)$ and $\bar\psi_j(x) := x\bar\phi_j'(x)$, so that:

$$\det Q = (\beta_{1,1}\beta_{2,2} - \beta_{2,1}\beta_{1,2})\bar\phi_1\bar\phi_2 + \beta_{2,1}\beta_{2,2}\, x(\bar\phi_1\bar\phi_2' - \bar\phi_2\bar\phi_1').$$

In the same way as before, we are led to set:

$$v := x_3 \left(\frac{\bar\phi_1'}{\bar\phi_1} - \frac{\bar\phi_2'}{\bar\phi_2}\right)(x_3)$$

and to define:

$$\mathrm{Mat}_2(\mathbf{C})_C := \left\{ B := (\beta_{i,j}) \in \mathrm{Mat}_2(\mathbf{C}) \,\middle|\, \begin{matrix} \beta_{1,1}, \beta_{2,2} = 0, \\ \frac{\beta_{2,2}}{\beta_{1,2}} - \frac{\beta_{2,1}}{\beta_{1,1}} = v. \end{matrix} \right\}.$$

Again we find that action $Q \mapsto \Lambda Q \Delta^{-1}$, with diagonal Δ and Λ as shown above, translates to the similar action on B. So we get a bijection:

$$\frac{\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C}{\sim} \longrightarrow F_C,$$

where relation $\sim$ on $\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C$ is defined by the action $(A, B) \mapsto (\Gamma A \Lambda^{-1}, \Lambda B \Delta^{-1})$ of triples $(\Gamma, \Lambda, \Delta)$, where Γ, Δ are diagonal invertible and $\Lambda := \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$, for some $\lambda \in \mathbf{C}^*$ and $\mu \in \mathbf{C}$. It is easily checked that this action indeed sends $\mathrm{Mat}_2(\mathbf{C})_C \times \mathrm{Mat}_2(\mathbf{C})_C$ to itself.

Action of Γ can be used to reduce A to the form $\begin{pmatrix} 1 & ? \\ 1 & ? \end{pmatrix}$, then action of Λ to the form $\left(\begin{smallmatrix} 1 & 0 \\ 1 & u \end{smallmatrix}\right)$, where the down right coefficient u is forced upon us by the condition defining $\mathrm{Mat}_2(\mathbf{C})_C$. Then the only possibility to preserve this form is to have $\Gamma = \Lambda$, a scalar matrix. So the remaining possible actions on B are $B \to \lambda B \Delta^{-1}$, $\lambda \in \mathbf{C}^*$ and Δ diagonal invertible. This can be used to reduce B to the form $\begin{pmatrix} 1 & 1 \\ x & y \end{pmatrix}$ with condition $y - x = v$. So the mapping:

$$x \longmapsto \text{the class of } \begin{pmatrix} 1 & 1 \\ x & x+v \end{pmatrix}$$

induces a bijective parameterisation of F_C by $\mathbf{C}$.

6.5. Putting it all together

Recall our assumptions from the beginning of Section 6: Fuchs relation (FR), strong non resonancy (NR) and non splitting (NS); to which we added in Part 6.3.3 assumption $\mathbf{Hyp}_8$.

There are three components in F, two of which project to finite subsets of the base:

(1) The logarithmic part $\bigcup_{C \in N_u} F_C$ is in bijection with $\mathrm{Y} \times \mathbf{C}$, where Y is the set of square roots of $\pi\left(\frac{\rho_1\rho_2}{x_1x_2}\right) = \pi(\sigma_1\sigma_2x_3x_4)$ in $\mathbf{E}_q$ (so $\operatorname{card} \mathrm{Y} = 4$). We simplify the formulation by saying that "the logarithmic part is $\mathrm{Y} \times \mathbf{C}$", and similarly for the following.

(2) Putting together the special fibers in the generic part and identifying Ξ by its image in $\mathbf{E}_q$, we have (set theoretically) the quotient of $\Xi \times (\mathbf{C} \sqcup \mathbf{C} \sqcup \{\cdot\})$ by the involution. Choosing a representative subset Ξ_0 of Ξ for the involution $\xi_1 \leftrightarrow \xi_2$ (thus $\operatorname{card} \Xi_0 = 4$), we see that this quotient can be identified with $\Xi_0 \times (\mathbf{C} \sqcup \mathbf{C} \sqcup \{\cdot\})$.

(3) Putting together the general fibers, we have (again set theoretically) the quotient of $(\mathbf{E}_q \setminus (\Xi \cup \mathrm{Y})) \times \mathbf{C}$ by the involution. We shall give a closer look at this component in Part 6.5.3.

6.5.1. The fibering: generic part

We must justify our contention that $\Pi : F \to \mathbf{P}^1(\mathbf{C})$ is a fibration and that its general (non logarithmic) part has exactly four special fibers. So we set $\mathbf{E}_q^\bullet := \mathbf{E}_q \setminus \mathrm{Y}$ and $\mathbf{P}^1(\mathbf{C})^\bullet$ its image under Φ, a projective line minus four points (actually the critical values of Φ), so that the restriction $\Phi : \mathbf{E}_q^\bullet \to \mathbf{P}^1(\mathbf{C})^\bullet$ is an *unramified* degree 2 covering. We write $C_q^\bullet$ the subset of C_q corresponding to $\mathbf{E}_q^\bullet$, so that π induces a bijection $C_q^\bullet \to \mathbf{E}_q^\bullet$.

For $\xi \in C_q^\bullet$, we write $\tilde{\xi} \in C_q^\bullet$ its image under the involution, i.e. the unique element of $C_q^\bullet$ such that $\xi\tilde{\xi} \equiv \rho_1\rho_2/(x_1x_2) = \sigma_1\sigma_2x_3x_4$; and $C(\xi) := \text{Diag}\left(\xi, \tilde{\xi}\right)$. Accordingly, we write $F_\xi := F_{C(\xi)}$ and $F_{\tilde{\xi}} := F_{C(\tilde{\xi})}$.

Last, in order to have a unified picture, we set:

$$\Phi(\xi) := \Phi_{C(\xi)}(x_1) = \Phi_{C(\xi)}(x_2)$$
$$= \frac{\theta_q\left(\frac{\xi}{\rho_1}x_1\right)\theta_q\left(\frac{\tilde{\xi}}{\rho_2}x_1\right)}{\theta_q\left(\frac{\tilde{\xi}}{\rho_1}x_1\right)\theta_q\left(\frac{\xi}{\rho_2}x_1\right)} = \frac{\theta_q\left(\frac{\xi}{\rho_1}x_1\right)\theta_q\left(\frac{\rho_1}{x_2\xi}\right)}{\theta_q\left(\frac{\xi}{\rho_2}x_1\right)\theta_q\left(\frac{\rho_2}{x_2\xi}\right)} = \frac{\rho_1}{\rho_2}\,\frac{\theta_q\left(\frac{x_1}{\rho_1}\xi\right)\theta_q\left(\frac{x_2}{\rho_1}\xi\right)}{\theta_q\left(\frac{x_1}{\rho_2}\xi\right)\theta_q\left(\frac{x_2}{\rho_2}\xi\right)}$$

and

$$\Phi(\xi) := \Phi_{C(\xi)}(x_3) = \Phi_{C(\xi)}(x_4)$$
$$= \frac{\theta_q\left(\frac{\sigma_1}{\xi}x_3\right)\theta_q\left(\frac{\sigma_2}{\tilde{\xi}}x_3\right)}{\theta_q\left(\frac{\sigma_2}{\xi}x_3\right)\theta_q\left(\frac{\sigma_1}{\tilde{\xi}}x_3\right)} = \frac{\theta_q\left(\frac{\sigma_1}{\xi}x_3\right)\theta_q\left(\frac{\xi}{\sigma_1x_4}\right)}{\theta_q\left(\frac{\sigma_2}{\xi}x_3\right)\theta_q\left(\frac{\xi}{\sigma_2x_4}\right)} = \frac{\sigma_1}{\sigma_2}\,\frac{\theta_q\left(\frac{\xi}{\sigma_1x_3}\right)\theta_q\left(\frac{\xi}{\sigma_1x_4}\right)}{\theta_q\left(\frac{\xi}{\sigma_2x_3}\right)\theta_q\left(\frac{\xi}{\sigma_2x_4}\right)}.$$

(We used the Fuchs relation and the functional equation $\theta_q(1/x) = (1/x)\,\theta_q(x)$, see Subsection 2.3)

After Subsection 6.3, we have a cartesian square:

$$\begin{array}{ccc}
\text{Mat}_2(\mathbf{C}) \times \text{Mat}_2(\mathbf{C}) & \longrightarrow & \coprod_{\xi \in C_q^\bullet} F_\xi \\
\downarrow{\scriptstyle \alpha\times\alpha} & & \downarrow \\
\mathbf{P}^1(\mathbf{C})^\bullet \times \mathbf{P}^1(\mathbf{C})^\bullet & \xrightarrow{(1/\Phi,\,1/\Phi)} & C_q^\bullet
\end{array}$$

Taking in account the $(\Gamma, \Lambda, \Delta)$ action, this gives rise to a bigger commutative diagram:

$$\begin{array}{ccccc}
\frac{\text{Mat}_2(\mathbf{C}) \times \text{Mat}_2(\mathbf{C})}{(\Gamma,\Lambda,\Delta)-\text{action}} & \longrightarrow & \coprod_{\xi \in C_q^\bullet} F_\xi & \xrightarrow{\text{quotient by involution}} & F \\
\downarrow{\scriptstyle \alpha\times\alpha} & & \downarrow & & \downarrow{\scriptstyle \Pi} \\
\mathbf{P}^1(\mathbf{C})^\bullet \times \mathbf{P}^1(\mathbf{C})^\bullet & \xrightarrow{(1/\Phi,\,1/\Phi)} & C_q^\bullet & \xrightarrow{\Phi} & \mathbf{P}^1(\mathbf{C})
\end{array}$$

The left hand square is cartesian, the right hand square is only commutative.

Now it follows that the fibers in the generic part have the following form:

$$\Pi^{-1}(\Phi(\xi)) = \frac{F_\xi \quad F_{\tilde{\xi}}}{\text{involution}},$$

and the case-by-case computation in Subsection 6.3 says that the fiber is degenerate ("special case") if, and only if $\Phi(\xi) \in \{0, \infty\}$ or $\Phi(\xi) \in \{0, \infty\}$, that is, after the above computations, if $\xi \in \Xi$ or if $\xi \in \Xi$ respectively.

Taking in account the involution $\xi \leftrightarrow \bar{\xi}$, for which each of Ξ , Ξ is invariant, this means that there are *four* critical values in $\mathbf{P}^1(\mathbf{C})^\bullet$ giving rise to special fibers $\Pi^{-1}(-)$ of the form $\mathbf{C} \quad \mathbf{C} \quad \{\cdot\}$:

$$\Phi(-\rho_1/x_1) = \Phi(-\rho_2/x_2) = 0,$$

$$\Phi(-\rho_1/x_2) = \Phi(-\rho_2/x_1) = \infty,$$

$$\Phi(-\sigma_1 x_3) = \Phi(-\sigma_2 x_4) = \frac{\theta_q\left(\frac{\sigma_1}{\rho_1} x_1 x_3\right) \theta_q\left(\frac{\sigma_1}{\rho_2} x_2 x_3\right)}{\theta_q\left(\frac{\sigma_1}{\rho_1} x_2 x_3\right) \theta_q\left(\frac{\sigma_1}{\rho_2} x_1 x_3\right)},$$

$$\Phi\left(-\sigma_1 x_4\right) = \Phi\left(-\sigma_2 x_3\right) = \frac{\theta_q\left(\frac{\sigma_2}{\rho_1} x_1 x_3\right) \theta_q\left(\frac{\sigma_2}{\rho_2} x_2 x_3\right)}{\theta_q\left(\frac{\sigma_2}{\rho_1} x_2 x_3\right) \theta_q\left(\frac{\sigma_2}{\rho_2} x_1 x_3\right)}.$$

Other expressions are possible for the last two critical values, but we found no simple ones.

6.5.2. More about special fibers

So special fibers correspond to the vanishing of one of the $\alpha_{i,j}$ or one of the $\beta_{i,j}$; with for instance the conditions $\alpha_{1,1} = 0$ and $\alpha_{2,2} = 0$ each providing a line of the same special fiber (and of course these two lines intersect where $\alpha_{1,1} = \alpha_{2,2} = 0$).

The first two of the above four special fibers were already met when analyzing the behaviour of Π in Part 5.1.2. Indeed, when encoding the factors P and Q of the Mano decomposition $M = PQ$ by matrices $A := (\alpha_{i,j})$ and $B := (\beta_{i,j})$, we find that one of the $\alpha_{i,j}$ is zero if, and only if one coefficient of P vanishes and this, by already explained arguments, is equivalent to: $M(x_1)$ or $M(x_2)$ has a null line. We already saw in Part 5.1.1 that this is equivalent to one (at least) of f_1, f_2, g_1, g_2 vanishes, i.e. to $\Pi(M) \quad \{0, \infty\}$. And actually each of the four lines found in Part 6.3.5 corresponds to one of those conditions.

This means that M belonging to one of the lines of one of the first two special fibers can be read either on $M(x_1)$ or on $M(x_2)$, independently of each other. So if we refine our notation and write $\Pi_{1,2}$ for the above Π and more generally $\Pi_{i,j}$ for the one obtaining by using x_i, x_j instead of x_1, x_2, we see that each of these four lines is a line of one of the first two special fibers of $\Pi_{1,3}$ or $\Pi_{1,4}$ or $\Pi_{2,3}$ or $\Pi_{2,4}$.

Now we are going to see that the last two of the four special fibers can be read on $M(x_3)$ and $M(x_4)$, although not on $\Pi_{3,4}$. Actually, the same line of argument as above leads to: one of the $\beta_{i,j}$ is zero if, and only if $M(x_3)$ or

$M(x_4)$ has a zero *column*. This leads us to introduce for each of them (both have rank 1) one non zero *line*, say (u_3, v_3) and (u_4, v_4); and to define:

$$\Pi_{3,4}(M) := \frac{u_3 v_4}{u_4 v_3} \in \mathbf{P}^1(\mathbf{C}).$$

Then we conclude from the above discussion that M is in one of the last two special fibers if, and only if $\Pi_{3,4}(M) \in \{0, \infty\}$.

Remark 6.5. — The relation of the $\Pi_{i,j}$ and $\Pi_{i,j}$ projective invariants is subtle and interesting in its own right. Matrices $M(x_i)$ have rank one, thus can be written $C_i \times L_i$, a product of a column by a line matrix, both non zero and defined up to a non zero scalar factor. We saw in Lemma 5.4 that generically $\Pi_{i,j}$ is a complete invariant for the left action of diagonal matrices on pairs (C_i, C_j). Clearly, $\Pi_{i,j}$ is a complete invariant for the right action of diagonal matrices on pairs (L_i, L_j).

6.5.3. Algebro-geometric description of the general component

The space of interest is the quotient of $(\mathbf{E}_q \setminus (\Xi \cup \Upsilon)) \times \mathbf{C}^*$ by the involution $(\overline{\xi_1}, \eta) \to (\overline{\xi_2}, \eta^{-1})$, where $\overline{\xi_1 \xi_2} = a$, the particular class written above (recall that we write multiplicatively the group law in $\mathbf{E}_q$). We extend this involution to $\mathbf{E}_q \times \mathbf{C}^*$. Also after choosing a particular square root α of a, we can instead use a parameter $t \in \mathbf{E}_q$ such that $\overline{\xi_1} = \alpha t$ and $\overline{\xi_2} = \alpha t^{-1}$. So we must find the quotient of $\mathbf{E}_q \times \mathbf{C}^*$ by the involution $\tau : (t, \eta) \to (t^{-1}, \eta^{-1})$.

In the usual projective model of $\mathbf{E}_q$, the point at infinity is its own inverse. So it makes sense to restrict the involution to the affine algebraic set $\mathbf{E}_q^* \times \mathbf{C}^*$, where $\mathbf{E}_q^* := \mathbf{E}_q \setminus \{\infty\}$. For the latter we have an algebro-geometric model:

$$\mathbf{E}_q^* = \operatorname{Spec} \mathbf{C}[x, y] \quad \text{where} \quad \mathbf{C}[x, y] := \frac{\mathbf{C}[X, Y]}{Y^2 - f(X)},$$

for some separable cubic polynomial $f(X)$. The inversion map on $\mathbf{E}_q^*$ is dual to the automorphism of $\mathbf{C}[x, y]$ defined by $y \to -y$. In this model (the indeterminate z denoting a "coordinate" for η):

$$\mathbf{E}_q^* \times \mathbf{C}^* = \operatorname{Spec} \mathbf{C}[x, y][z, 1/z]$$

and the involution is dual to the automorphism of $\mathbf{C}[x, y][z, 1/z]$ defined by $y \to -y$, $z \to 1/z$.

The quotient of a complex affine algebraic variety by a finite group (here the group generated by the involution) is obtained by computing its affine algebra as the subalgebra fixed by the dual action of the group. So:

$$\frac{\mathbf{E}_q^* \times \mathbf{C}^*}{\text{involution } \tau} = \operatorname{Spec} \mathbf{C}[x, y][z, 1/z]^{\tau}.$$

We proceed to compute the invariant subalgebra $\mathbf{C}[x, y][z, 1/z]^\tau$. An element of $\mathbf{C}[x, y][z, 1/z]$ can be uniquely written as

$$g = \sum_{n \in \mathbf{Z}} a_n(x) z^n + y \sum_{n \in \mathbf{Z}} b_n(x) z^n, \text{ where all the } a_n, b_n \in \mathbf{C}[X].$$

Invariance by τ translates into:

$$g = a_0(x) + \sum_{n>1} a_n(x)(z^n + z^{-n}) + y \sum_{n>1} b_n(x)(z^n - z^{-n}).$$

Setting $w := \frac{z+z^{-1}}{2}$ and $v := \frac{z-z^{-1}}{2} y$, we get the form:

$$g = A(x, w) + vB(x, w).$$

Clearly, x and w are algebraically independent and $v \notin \mathbf{C}[x, w]$. Also:

$$v^2 = \frac{(z - z^{-1})^2}{4} y^2 = (w^2 - 1) f(x).$$

This describes an algebraic surface, a degree 2 covering of the plane Spec $\mathbf{C}[x, w]$ ramified over the set of equation $(w^2 - 1) f(x) = 0$ (a union of six lines).

Since for any fixed $v_0 = \pm 1$ the equation $w^2 = f(x)(v_0^2 - 1)$ defines an affine elliptic curve, our surface is also a pencil of elliptic curves parameterized by $\mathbf{C} \setminus \{+1, -1\}$.

To recover the *projective* picture from the above *affine* one, just note that each fixed point α of the involution, i.e. each $\alpha = \bar{\xi}$ with $\xi^2 \equiv \frac{\rho_1\rho_2}{x_1 x_2}$, gives rise to such an affine chart. This will be done in some detail in Section 7. Herebelow we attempt at a geometric description of the projective surface.

6.6. Geometric description of the whole of F

6.6.1. Definitions

We recall the classical definitions of an *elliptic surface*. The first one [3] is the following.

Definition 6.6. — *Let S be a complex projective surface. We will say that it is an elliptic surface if there exists a smooth curve B and a surjective morphism $p : S \to B$ whose generic fiber is an elliptic curve.*

In fact a variant of this definition is better for our purposes [76, Def. 3.1, p. 7].

Definition 6.7. — *An elliptic surface S over B is a smooth projective surface S with an elliptic fibration over B, i.e. a surjective morphism p : $S \to B$, such that:*

(i) *almost all fibers are smooth curves of genus* 1*;*
(ii) *no fiber contains an exceptional curve of the first kind.*

It is better to say "smooth curves of genus 1" than "elliptic curve". An elliptic curve is a smooth curve of genus 1 with a marked point and Definition 6.6 could suggest that there exists a section.

We recall that an exceptional curve of the first kind is a smooth rational curve of self-intersection -1 (also called (-1)-curve). Naturally, (-1)-curves occur as exceptional divisors of blow-ups of surfaces at smooth points. One can always successively blow-down (-1)-curves to reduce to a smooth minimal model. Therefore an elliptic fibration in the sense of the first definition 6.6 can be transformed by a succession of blow-downs into an elliptic fibration in the sense of the second definition 6.7.

A *section* of an elliptic surface $p : S \to B$ is a morphism $s : B \to S$ such that $p \circ s = \mathrm{Id}_B$. An elliptic surface does not necessarily admit a section.

6.6.2. An elliptic fibration. Algebraic charts on F

We set $Y := \frac{\mathbf{E}_q \times \mathbf{C}}{\text{involution } \tau}$. (recall that the involution τ was defined at the beginning of Part 6.5.3). We have two maps $p : Y \to \mathbf{C}$ and $\Psi : Y \to \mathbf{P}^1(\mathbf{C})$: the maps induced respectively by

$$p : (\xi, \eta) \to w := \frac{1}{2}(\eta + 1/\eta) \quad \text{and} \quad \Psi : (\xi, \eta) \to \Phi(\xi).$$

- The map p gives an elliptic fibration of Y with two exceptionnal fibers above ± 1.
- For $\eta = \pm 1$, the canonical map $\mathbf{E}_q \times \{\eta\} \to Y$ induces an isomorphism of $\mathbf{E}_q$ onto the generic fiber $p^{-1}(w)$ ($w = \frac{1}{2}(\eta + \eta^{-1})$).
- The map Ψ induces an isomorphism between the exceptional fiber $p^{-1}(1)$ (resp. $p^{-1}(-1)$) and $\mathbf{P}^1(\mathbf{C})$.
- For $u \in \Phi(Y) \subset \mathbf{P}^1(\mathbf{C})$, the fiber $\Psi^{-1}(u)$ is parameterized bijectively by $w \in \mathbf{C}$. We get 4 sections of the elliptic fibration p.
- For $u \in \mathbf{P}^1(\mathbf{C}) \setminus \Phi(Y)$ we can describe the fiber $\Psi^{-1}(u)$ as two copies of $\mathbf{C}$ glued at the points 1 and -1 on each copy.

The surface $\mathbf{E}_q \times \mathbf{C}$ is smooth, therefore if $a \in Y$ is not the image of a *fixed point* of the involution τ, then Y is smooth at a. There are 8 fixed points: Y × $\{\pm 1\}$. Their images belong to the union of the 4 logarithmic fibers and to the union of the two exceptional fibers $p^{-1}(\pm 1)$. One verifies that these images are isolated singular points[48] of Y.

We can extend τ into an involution $\tilde{\tau}$ on $\mathbf{E}_q \times \mathbf{P}^1(\mathbf{C})$. We set $\tilde{Y} := \frac{\mathbf{E}_q \times \mathbf{P}^1(\mathbf{C})}{\text{involution } \tilde{\tau}}$ and we extend the elliptic fibration p into an elliptic fibration $\tilde{p} : \tilde{Y} \to \mathbf{P}^1(\mathbf{C})$. We extend Ψ into $\tilde{\Psi}$.

If we remove from $\mathbf{E}_q$ the set Y, that is the 4 fixed points of the involution $\tau : \xi \to \rho_1\rho_2/x_1x_2\xi$, we get $\mathbf{E}_q^\bullet = \mathbf{E}_q \setminus$ Y. If we remove from $\mathbf{E}_q$ the 8 points of Ξ, we get $\mathbf{E}_q{}^\dagger := \mathbf{E}_q \setminus \Xi$. We set $\mathbf{E}_q^{\bullet\dagger} := \mathbf{E}_q^\bullet \cap \mathbf{E}_q{}^\dagger$. We recall that the points of Ξ are not fixed by τ, therefore we have removed 12 points. The surfaces $\mathbf{E}_q^\bullet \times \mathbf{C}$, $\mathbf{E}_q{}^\dagger \times \mathbf{C}$ and $\mathbf{E}_q{}^{\bullet\dagger} \times \mathbf{C}$ are invariant by τ.

We set:

$$Y^\bullet := \frac{\mathbf{E}_q^\bullet \times \mathbf{C}}{\text{involution } \tau}, \quad Y^\dagger := \frac{\mathbf{E}_q{}^\dagger \times \mathbf{C}}{\text{involution } \tau}, \quad Y^{\bullet\dagger} := \frac{\mathbf{E}_q^{\bullet\dagger} \times \mathbf{C}}{\text{involution } \tau}.$$

We have some “punctured elliptic fibrations”:

$$p^\bullet : Y^\bullet \to \mathbf{C}, \quad p^\dagger : Y^\dagger \to \mathbf{C}, \quad p^{\bullet\dagger} : Y^{\bullet\dagger} \to \mathbf{C}$$

induced (by restriction) by the elliptic fibration $p : Y \to \mathbf{C}$.

We recall that we have an injective analytic map $\psi : Y^{\bullet\dagger} \to F$. Considering the algebraic structure on $\mathbf{E}_q$ and the 3 affine charts described in the preceding section, we can interpret this map (in 3 different ways) as an *algebraic chart* of the surface F. The image of this chart misses 12 lines of F.

6.6.3. Description of some fibers of Π

If we remove from $\mathbf{P}^1(\mathbf{C})$ the set Π(Y) (that is the images of the 4 “logarithmic fibers”) we get $\mathbf{P}^1(\mathbf{C})^\bullet = \mathbf{P}^1(\mathbf{C}) \setminus \Pi(\mathrm{Y})$. If we remove from $\mathbf{P}^1(\mathbf{C})$ the 4 points of $\Pi(\Xi)$ (that is 0, ∞ and two other points), we get $\mathbf{P}^1(\mathbf{C})^\dagger := \mathbf{P}^1(\mathbf{C}) \setminus \Pi(\Xi)$. We set $\mathbf{P}^1(\mathbf{C})^{\bullet\dagger} := \mathbf{P}^1(\mathbf{C})^\bullet \cap \mathbf{P}^1(\mathbf{C})^\dagger$: we have removed 8 points.

Using the elliptic fibration p we can describe the fibers of Π above $\mathbf{P}^1(\mathbf{C})^{\bullet\dagger}$ (the generic fibers).

Let $u \in \mathbf{P}^1(\mathbf{C})^{\bullet\dagger}$. Then ψ is defined and we have $\Pi \circ \psi = \Psi$, hence ψ induces an isomorphism of $\Psi^{-1}(u)$ onto $\Pi^{-1}(u)$. Therefore we can describe

(48) They are rational double points.

the algebraic curve $\Pi^{-1}(u)$ as two copies of $\mathbf{C}$ glued at the points 1 and -1 on each copy[49].

The open set $\Psi^{-1}\left(\mathbf{P}^1(\mathbf{C})^{\bullet}\right) = Y^{\bullet}$ is smooth (it does not contain singular points). A fortiori $\Psi^{-1}\left(\mathbf{P}^1(\mathbf{C})^{\bullet\dagger}\right) = Y^{\bullet\dagger}$ is smooth and $\psi(Y^{\bullet\dagger}) = \Pi^{-1}\left(\mathbf{P}^1(\mathbf{C})^{\bullet\dagger}\right) F$ is also smooth. We conjecture that F is also smooth in a convenient neighborhood of each logarithmic fiber. (We will return to this question later, cf. Part 7.2.3).

Conjecture 6.8. — *The inverse image* $\Pi^{-1}\left(\mathbf{P}^1(\mathbf{C})^{\dagger}\right) F$ *is smooth.*

Otherwise speaking we can add to $\psi(Y^{\bullet\dagger})$ the 4 logarithmic fibers and we get a smooth open subset of F.

When $u \in \mathbf{P}^1(\mathbf{C})^{\bullet\dagger}$ tends to $u_0 \in \Pi((Y))$, the generic fiber tends is some sense towards a logarithmic fiber (cf. Part 7.2.3). In order to prove the conjecture it seems necessary to understand more precisely what happens. Blow ups of the 8 singular points of Y could be useful (cf. Part 7.2.3).

The fibers above $u \in \Pi(\Xi)$ are made of two affine lines intersecting at one point (for $u = 0$ and $u = \infty$ it is the Theorem 5.5, for the two other values, see Part 6.3.5).

It is difficult to describe what happens to the generic fibers $\Pi^{-1}(u)$ when $u \in \mathbf{P}^1(\mathbf{C})^{\bullet\dagger}$ tends to $u_1 \in \Pi(\Xi)$. It seems that the two points $\Pi^{-1}(u) \cap p^{-1}(1)$ and the two points $\Pi^{-1}(u) \cap p^{-1}(-1)$ glue into an unique point: the special point of $\Pi^{-1}(u_1)$ which is the intersection of the two affine lines.

A possible approach for a description is to reparameterize $\mathbf{C} \setminus \{\pm 1\}$ using an affine transform of $\mathbf{C}$ sending ± 1 to $\pm\epsilon$. We get a family of elliptic fibrations above the family of punctured lines $\{\mathbf{C} \setminus \{\pm\epsilon\}\}$. Then we can try to describe what happens when $\epsilon \to 0$. A similar method works perfectly for a description of the fibration of the cubic surface S_{VI} analog to the fibration by Π: cf. Part 7.1.3 below *"A simple model"*.

6.6.4. An heuristic description of the fibration by Π

For a generic $u \in \mathbf{P}^1(\mathbf{C})$, we can describe the algebraic curve $\widetilde{\Psi}^{-1}(u)$ as two copies of the projective line $\mathbf{P}^1(\mathbf{C})$ glued at the points 1 and -1 on each copy. We will give another description of the abstract algebraic curve $\widetilde{\Psi}^{-1}(u)$ (with its fibration induced by p) as an algebraic curve of $(\mathbf{P}^1(\mathbf{C}))^2$ (with the fibration induced by the projection on the first factor). Using this

(49) A model is given by two parabolas in $\mathbf{C}^2$ in general position.

picture we will give herebelow a simple *heuristic* description of the fibration of F by Π as a family of curves.

Let $\overline{W}$ be a $(2,2)$ curve of $(\mathbf{P}^1(\mathbf{C}))^2$ decomposed into two $(1,1)$ curves bitangent at two distinct points A^+ and A^-. We denote $\overline{\mathbf{p}}$ the projection of $\overline{W}$ on $\mathbf{P}^1(\mathbf{C})$ induced by restriction of the projection on the first factor $(\mathbf{P}^1(\mathbf{C}))^2 \to \mathbf{P}^1(\mathbf{C})$. Up to a Möbius transform on the first factor, we can suppose that $\overline{\mathbf{p}}(A^\pm) = \pm\epsilon$ (ϵ $\mathbf{C}$). The fibers of $\overline{\mathbf{p}}$ above a generic point $\mathbf{w}$ $\mathbf{P}^1(\mathbf{C}) \setminus \{\pm\epsilon\}$ are sets made of two points. The special fibers above $\pm\epsilon$ are one point sets.

For u $\mathbf{P}^1(\mathbf{C})^{\bullet\dagger}$, the two fibrations of abstract algebraic curves $\overline{p}$: $\overline{\Pi^{-1}(u)} \to \mathbf{P}^1(\mathbf{C})$ and $\overline{\mathbf{p}} : \overline{W} \to \mathbf{P}^1(\mathbf{C})$ are isomorphic: $\mathbf{C} \setminus \{\pm 1\}$ is send bijectively to $\mathbf{P}^1(\mathbf{C}) \setminus \{\pm\epsilon\}$. (We leave the verification to the reader.)

We set $W := \overline{W} \setminus \overline{\mathbf{p}}^{-1}(\infty)$. Then the two algebraic fibrations $p : \Pi^{-1}(u) \to \mathbf{C}$ and $\mathbf{p} : W \to \mathbf{C}$ are isomorphic.

We consider an algebraic family of fibered curves $(\overline{W}_\lambda, \overline{\mathbf{p}}_\lambda, \mathbf{P}^1(\mathbf{C}))_{\lambda \quad \mathbf{P}^1(\mathbf{C})}$ such that for a generic value of λ the pair $(\overline{W}_\lambda, \overline{\mathbf{p}}_\lambda)$ is of the type $(\overline{W}, p)$. We allow for the curve $\overline{W}_\lambda$, as an algebraic curve of $(\mathbf{P}^1(\mathbf{C}))^2$, some degeneracies of the two following types.

- The two $(1,1)$ curves of $\overline{W}_\lambda$ degenerate into a double $(1,1)$ curve. Then the projection $\overline{\mathbf{p}}_\lambda$ becomes an isomorphism.
- The two $(1,1)$ curves of $\overline{W}_\lambda$ degenerate into two double lines intersecting at only one point B ($\epsilon \to 0$, A^+ and A^- glue together into the point B).

7. Geometry, surgery and pants

We will freely use some notations introduced in Section 1.2.

7.1. The classical geometry of a smooth cubic complex surface and the representations of a free group of rank 3

There are strong relations between the classical geometry of a smooth complex cubic surface (27 lines, 45 tritangent planes ..., cf. [13]) and some properties of the representations into $\mathrm{SL}_2(\mathbf{C})$ of a free group of rank 3. As far as we know these (simple but important) relations remained unnoticed until recently[50] .

7.1.1. The geometry of the cubic surfaces $S(a)$

By the classical theory, if Σ is a smooth projective cubic surface:

- Σ admits 27 lines and each line is a (-1)-line;
- Σ admits 45 tritangent planes;
- the intersection of each tri-tangent plane with Σ is the union of 3 lines forming a triangle;
- each line of Σ belongs exactly to 5 tri-tangent planes.

The equation of the projective surface $\overline{S(a)} \subset \mathbf{P}^3(\mathbf{C})$ in projective coordinates $(\bar{X}_0, \bar{X}_t, \bar{X}_1, \bar{T})$ is:

$$\bar{X}_0\bar{X}_t\bar{X}_1 + \bar{X}_0^2\bar{T} + \bar{X}_t^2\bar{T} + \bar{X}_1^2\bar{T} - A_0\bar{X}_0\bar{T}^2 - A_t\bar{X}_t\bar{T}^2 - A_1\bar{X}_1\bar{T}^2 + A_\infty\bar{T}^3 = 0.$$

The plane at infinity $\bar{T} = 0$ is a tri-tangent plane and its intersection with the surface is the triangle $\bar{X}_0\bar{X}_t\bar{X}_1 = 0$. Therefore the affine cubic surface $S(a)$ contains exactly 24 lines. We have the following description of these lines.

Each line at infinity is contained in 4 tri-tangent planes different from the plane at infinity. The intersection of such a tri-tangent plane and $\overline{S(a)}$ is a triangle, therefore the intersection with $S(a)$ is the union of 2 affine lines with a common point. Therefore for each line at infinity we get 8 affine lines on $S(a)$. Using the coordinates X_0, X_t, X_1 we see that for each $l = 0, t, 1$ there exists 4 exceptional values of X_l such that $\{X_l = 0\} \cap S(a)$ is the union of 2 affine lines.

Below we will interpret the 24 lines on $S(a)$ in terms of representations.

Proposition 7.1. — *Let* $a_0, a_t, a_1, a_\infty \in \mathbf{C}$ *be* arbitrary*. The* 24 *lines* distinct or not *defined in* $\mathbf{C}^3$ *by the following equations are contained in the cubic surface* $S(a) \subset \mathbf{C}^3$*:*

$$\begin{aligned}
X_k &= e_ie_j^{-1} + e_je_i^{-1}, & e_iX_i + e_jX_j &= a_\infty + e_ie_ja_k, \qquad (7.1)\\
X_k &= e_ie_j^{-1} + e_je_i^{-1}, & e_iX_j + e_jX_i &= a_k + e_ie_ja_\infty,\\
X_k &= e_ie_j + e_i^{-1}e_j^{-1}, & X_i + e_ie_jX_j &= e_ja_k + e_ia_\infty,\\
X_k &= e_ie_j + e_i^{-1}e_j^{-1}, & X_j + e_ie_jX_i &= e_ja_\infty + e_ia_k,
\end{aligned}$$

[50] They are due to Martin Klimes, Emmanuel Paul and the second author, and they are studied in a work in progress on the confluence of the Painlevé equations [43, 44].

$$
\begin{aligned}
X_k &= e_k e_\infty^{-1} + e_\infty e_k^{-1}, & e_\infty X_i + e_k X_j &= a_i + e_k e_\infty a_j, \\
X_k &= e_k e_\infty^{-1} + e_\infty e_k^{-1}, & e_k X_i + e_\infty X_j &= a_j + e_k e_\infty a_i, \\
X_k &= e_k e_\infty + e_k^{-1} e_\infty^{-1}, & X_i + e_k e_\infty X_j &= e_k a_j + e_\infty a_i, \\
X_k &= e_k e_\infty + e_k^{-1} e_\infty^{-1}, & X_j + e_k e_\infty X_i &= e_k a_i + e_\infty a_j.
\end{aligned}
$$

Proof. — The result follows immediately from some decompositions of $F(X, a)$ (cf. [43, Prop. 4.5]). We give only one of these decompositions:

$$
\begin{aligned}
F(X : a) = (X_k - e_i e_j^{-1} - e_j e_i^{-1})&(F_{X_k} - X_k + e_i e_j^{-1} + e_j e_i^{-1}) \\
&- e_i^{-1} e_j^{-1}(e_i X_i + e_j X_j - a_\infty - e_i e_j a_k) \\
&\qquad \times (e_i X_j + e_j X_i - a_k - e_i e_j a_\infty).
\end{aligned}
$$

7.1.2. Reducibility of representations of the free group of rank 2

We recall the well known conditions of reducibility for the representations of a free group of rank 2 into $\mathrm{SL}_2(\mathbf{C})$ and some classical results (cf. [35]). We denote $\Gamma_2 := \ u, v\ $ the free group of rank 2 generated by the letters $u,\ v$.

Definition 7.2. — *A pair of matrices* $(M\ , M\ )\quad (\mathrm{SL}_2(\mathbf{C}))^2$ *is said* reducible *if there exists a common (non trivial, non total) invariant subspace.*

It is equivalent to say that the corresponding representation $\omega : \Gamma_2 \to \mathrm{SL}_2(\mathbf{C})$ defined by $\omega(u) := M$ and $\omega(v) := M$ is reducible.

Let $\omega : \Gamma_2 \to \mathrm{SL}_2(\mathbf{C})$ be a linear representation. We set $M\ := \omega(u)$ and $M\ := \omega(v)$. We denote e and $(e\ )^{-1}$ (resp. e and $(e\ )^{-1}$) the eigenvalues of M (resp. M). We denote e and e^{-1} the eigenvalues of $M := M\ M$.

Proposition 7.3. — *The following assertions are equivalent*

(i) *The representation* ω *is reducible.*
(ii) *The pair* $(M\ , M\ )$ *is reducible.*
(iii) *We have:* $e = e\ e$ *or* $e = e\ (e\ )^{-1}$ *or* $e = (e\ )^{-1} e$ *or* $e = (e\ )^{-1}(e\ )^{-1}$*.*
(iv) *We have* $\mathrm{Tr}\, M = e\ e\ + (e\ e\ )^{-1}$ *or* $\mathrm{Tr}\, M = e\ (e\ )^{-1} + (e\ )^{-1} e$ *.*

Proof. —

- The assertions (i) and (ii) are evidently equivalent.
- If the pair $(M\ , M\ )$ is reducible, then there exists a common eigenvector $v\quad \mathbf{C}^2$. Then $M\ v = \lambda\ v$, $M\ v = \lambda\ v$ and $Mv = \lambda v$, where $\lambda\ , \lambda$ and λ are respectively eigenvalues of M , M and M. Therefore one of the conditions of (iii) is satisfied.
- The assertion (iii) implies clearly the assertion (iv).

- If ω is irreducible, then one can prove that:

  Tr $M = e\,e + (e\,e)^{-1}$ and Tr $M = e\,(e)^{-1} + (e)^{-1}e$

  (cf. [35, (4.2.9), p. 83]). Therefore if (iv) is satisfied, then ω is necessarily reducible. This proves the assertion (i).

Let ω be a representation and M , M as above. We suppose that Tr M = ±2 and Tr M = ±2. Then M and M are diagonalisable. There allways exists a *mixed basis* $\{v, v\}$ of $\mathbf{C}^2$ formed by an eigenvector of M and an eigenvector of M . In general there are (up to rescaling of the eigenvectors) 4 ways one can form such a basis. The 4 cases of reducibility of ω correspond to the degeneracy of one of these 4 basis.

We recall that if ω is irreducible, then it is determined, up to equivalence, by the traces of M , M and M (cf. [35, Thm. 4.2.1, p. 80]).

7.1.3. Partial reducibility and lines on $S(a)$

We describe a relation between a notion of partial reducibility of a representation and the lines on the cubic surface $S(a)$. This relation is apparently new(51), it has been found recently by M. Klimes, E. Paul and the second author [43, 44].

As in 1.2.1, we denote $\Gamma_3 := u_0, u_t, u_1$ the free group of rank 3 generated by the letters $u_0,\ u_t,\ u_1$ and we set $u_\infty = u_1^{-1}u_t^{-1}u_0^{-1}$. Let $\rho : \Gamma_3 \to \mathrm{SL}_2(\mathbf{C})$ be a linear representation. We set $M_l := \rho(u_l)$ ($l = 0, t, 1, \infty$). We denote e_l and e_l^{-1} ($l = 0, t, 1, \infty$) the eigenvalues of M_l.

We have the following characterizations of smoothness (some are classical and some are apparently new).

Theorem 7.4. — *Let a $\mathbf{C}^4$. We suppose $a_l = \pm 2$ ($l = 0, t, 1, \infty$) (non resonance). The following conditions are equivalent:*

(i) *The affine cubic surface $S(a)$ is smooth.*
(ii) *The projective cubic surface $\overline{S(a)}$ is smooth.*
(iii) *The 24 lines (7.1) are pairwise distinct.*
(iv) *The 3 following conditions are satisfied:*
  - *the 4 numbers built from the e_l ($l = 0, t, 1, \infty$)*

$$e_te_1^{-1} + e_1e_t^{-1},\ \ e_te_1 + e_t^{-1}e_1^{-1},\ \ e_0e_\infty^{-1} + e_\infty e_0^{-1},\ \ e_0e_\infty + e_0^{-1}e_\infty^{-1} \tag{7.2}$$

  are pairwise distinct;

(51) If we replace representations by wild representations, then this relation can be extended to all the Painlevé equations.

- *the* 4 *numbers:*

$$e_1e_\infty^{-1} + e_\infty e_1^{-1},\ \ e_1e_\infty + e_1^{-1}e_\infty^{-1},\ \ e_te_0^{-1} + e_0e_t^{-1},\ \ e_te_0 + e_t^{-1}e_0^{-1} \qquad (7.3)$$

are pairwise distinct,

- *the* 4 *numbers:*

$$e_1e_\infty^{-1} + e_\infty e_1^{-1},\ \ e_1e_\infty + e_1^{-1}e_\infty^{-1},\ \ e_te_0^{-1} + e_0e_t^{-1},\ \ e_te_0 + e_t^{-1}e_0^{-1} \qquad (7.4)$$

are pairwise distinct.

(v) *We have the* 8 *conditions:* $e_0\, e_t^{\pm 1}e_1^{\pm 1}e_\infty^{\pm 1} = 1$ *(the* 3 *signs are chosen independantly).*

(vi) *If* ρ *is a representation such that* $\mathrm{Tr}\,\rho(u_l) = a_l$ *for all* $l = 0, t, 1, \infty$*, then it is irreducible.*

Proof. —

- A singular point of $\overline{S(a)}$ is allways contained into $S(a)$, therefore (i) ⇔ (ii).
- The conditions (iv) and (v) are clearly equivalent.
- We have (iii) ⇔ (i). If the 24 lines are distinct, then $\overline{S(a)}$ contains 27 distinct lines and therefore it is smooth.
- We have (iv)   (iii). If we suppose (iv), then we get 3 sets of 8 two by two distinct lines. Each set corresponds to 4 distinct values of X_0 or X_t or X_1. It is easy to check that a line cannot belong to 2 different such sets.
- We have (i)   (v)   (iii). If $S(a)$ is smooth, then (1.4) is impossible, therefore (v) is true; (iv) and (iii) follows.
- We have (ii) ⇔ (vi). Easy.

If we use the parameters θ_l, then the conditions (v) are translated into:

$$\theta_0 \pm \theta_l \pm \theta_1 \pm \theta_\infty \quad \mathbf{Z}$$

(cf. [48, Thm. 4.1]).

We suppose that the surface $S(a)$ is smooth. For each pair (l, m) of elements of $\{0, t, 1, \infty\}$, each of the 2 planes:

$$\begin{cases} X_n = e_le_m + e_l^{-1}e_m^{-1} \\ X_n = e_le_m^{-1} + e_l^{-1}e_m \end{cases} \quad \text{with } (l, m, n) = \begin{cases} (i, j, k) \\ (k, \infty, k) \end{cases} \qquad (7.5)$$

intersects $S(a)$ at 2 lines. The resulting 4 lines correspond to the reducibility of the pair of matrices (M_l, M_m).

More precisely if two matrices M_l and M_m are diagonalizable, for each of them there exists a pair of invariant subspaces giving rise to a basis. Then there are in general 4 possibilities of pairing of invariant subspaces out of which one can form a *mixed* basis. The cases of reducibility of the pair

(M_l, M_m) corresponds to the degeneracy of (at least) one of these mixed bases. Each of the 4 lines corresponds to such a case of degeneracy.

Definition 7.5. — *Let* $\rho : \Gamma_3 \to \mathrm{SL}_2(\mathbf{C})$*. We will say that* ρ *is* partially reducible *if there exists* $i, j \in \{0, t, 1, \infty\}$*,* $i \neq j$*, such that the pair of matrices* $(\rho(u_i), \rho(u_j))$ *is reducible.*

Proposition 7.6. — *Let* $a \in \mathbf{C}^4$ *arbitrary and* $\rho : \Gamma_3 \to \mathrm{SL}_2(\mathbf{C})$ *a representation such that* $\mathrm{Tr}\,\rho(u_l) = a_l$ *(*$l = 0, t, 1, \infty$*).*

(i) *If the representation* ρ *is* partially reducible*, then its equivalence class belongs to one of the* 24 *lines (distinct or not) defined in Proposition 7.1.*

(ii) *We suppose that* $S(a)$ *is smooth. Then* ρ *is partially reducible if and only if its equivalence class belongs to one of the* 24 *lines of* $S(a)$*.*

Proof. — The representation ρ is partially reducible if and only if there exists $k = 1, 2, 3$ such that:

$$X_k = e_i e_j^{-1} + e_i e_j^{-1} \text{ or } X_k = e_i e_j + e_i^{-1} e_j^{-1}$$
$$\text{or } X_k = e_k e_\infty^{-1} + e_\infty e_k^{-1} \text{ or } X_k = e_k e_\infty + e_k^{-1} e_\infty^{-1}. \quad (7.6)$$

Then assertion (i) follows from Proposition 7.1 and assertion (ii) follows from Theorem 7.4.

The condition (7.6) appear in various papers in "$theta$ notation":

$$\sigma_k \pm \theta_i \pm \theta_j \in 2\mathbf{Z}, \quad \sigma_k \pm \theta_k \pm \theta_\infty \in 2\mathbf{Z},$$

where $X_k = e^{i\pi\sigma_k} + e^{-i\pi\sigma_k}$. cf. [37, condition $(A.3)_{PVI}$, p. 1141] and [27, footnote 9, p. 85].

A fibration. We suppose that we are in the "generic case" (i. e. $S_{VI}(a)$ is smooth).

Let $\Pi_0 : S_{VI}(a) \to \mathbf{C}$, $\Pi_0 : (X_0, X_t, X_\infty) \to X_0$. We recall:

$$S_{VI}(a) = \left\{ (X_0, X_t, X_1) \in \mathbf{C}^3 \;\middle|\; \begin{array}{l} X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 \\ - A_0 X_0 - A_t X_t - A_1 X_1 + A_\infty = 0 \end{array} \right\}.$$

For $c \in \mathbf{C}$, $\Pi_0^{-1}(c)$ is interpreted as an affine conic in the (X_t, X_1)-plane:

$$X_t^2 + X_1^2 + c X_t X_1 - A_t X_t - A_1 X_1 - c A_0 + A_\infty = 0.$$

The generic fiber is isomorphic to $\mathbf{C}$. The exceptional fibers are of two types:

- either $X_0 = \pm 2$, then $X_t^2 + X_1^2 \pm 2 X_t X_1 = (X_t \pm X_1)^2$, the fiber is a *parabola* and it is isomorphic to $\mathbf{C}$;

- either we are in a partially reductible case, that is in one of the 4 cases:

$$X_0 = e_t e_1^{-1} + e_1 e_t^{-1} \text{ or } X_0 = e_t e_1 + e_t^{-1} e_1^{-1}$$
$$\text{or } X_0 = e_0 e_\infty^{-1} + e_\infty e_0^{-1} \text{ or } X_0 = e_0 e_\infty + e_0^{-1} e_\infty^{-1}, \quad (7.7)$$

then the fiber is *degenerated into two lines*. The intersection of these two lines is a critical point of Π_0. Its image is a critical value of Π_0.

If we remove the 6 exceptional fibers (that is 8 lines and two curves) from $S_{VI}(a)$ then we can parameterize the remaining set by a Zariski open set of $\mathbf{C} \times \mathbf{C}$. Such parameterizations appear in many papers [33], [37] ... We will return to this question below (cf. Part 7.2.2). In the q-case, we will deduce later from the Mano decomposition a similar parameterization (cf. Part 7.2.3).

If c_0 is a critical value of Π_0, then we can describe $\Pi_0^{-1}(\Delta)$ for a small open disc $\Delta \subset \mathbf{C}$ centered at c_0. We blow down one of the lines of $\Pi^{-1}(c_0)$ into the smooth surface $\Pi_0^{-1}(\Delta)$ (the closure of such a line in $\overline{S(a)}$ is a (-1)-line). We get a surface analytically diffeomorphic to $\Delta \times \mathbf{C}$. Therefore $\Pi_0^{-1}(\Delta)$ is analytically diffeomorphic to a blow-up of $\Delta \times \mathbf{C}$.

The projective version of the above description is the following. We choose one of the lines Δ of the triangle at infinity of the surface $\overline{S_{VI}(a)}$. The family of planes passing by this line cut the surface along a linear family of cubic curves. Each one is decomposed into the union of Δ and a conic curve. There are 5 conic curves degenerated into 2 lines: they correspond to the 5 tritangent planes, the plane at infinity and 4 other planes. There are two other exceptional curves: when the conic is tangent to Δ.

A simple model. We will give two descriptions of a hyperbolic paraboloid of $\mathbf{C}^3$ (an affine quadric surface):

– by a blow-up of a point in $\mathbf{C}^2$;
– by a singular fibration.

Let $Q \subset \mathbf{C}^3$ defined by the equation $Y = XZ$ (hyperbolic paraboloid). Let $f : \mathbf{C}^3 \to \mathbf{C}$ defined by $f : (X, Y, Z) \to Y$ and $g : \mathbf{C}^3 \to \mathbf{C}^2$ (the (X,Y) plane) defined by $g : (X, Y, Z) \to (X, f(X, Y, Z) = XZ)$. We denote Φ (resp. ϕ) the restriction of f (resp. g) to Q.

– The map ϕ is a bijection of $Q \setminus \phi^{-1}(0,0)$ onto $\mathbf{C}^2 \setminus \{(0,0)\}$ and $\phi^{-1}(0,0)$ is the line of Q defined by $\{(X, Y, Z) \in C^3 | X = Y = 0\}$. If we parameterize Q by (X, Z) then ϕ is expressed by $(X, T) \to (X, Y = XT)$. The quadric Q is a blow-up of $\mathbf{C}^2$ at the point $(0, 0)$. If we blow down into Q the line $(X = Z = 0)$ by g we get $\mathbf{C}^2$.

- We can describe Q by the fibers $\Phi^{-1}(c)$ of Φ. If $c \neq 0$, then the fiber is the affine conic $\{(X, c, Z) \in \mathbf{C}^3 \mid X \neq 0,\ Z = c/X\}$. If $c = 0$, then the fiber is the union of the two lines $(X = 0)$ and $(Z = 0)$.

7.1.4. Dynamics on $S_{VI}(a)$

The cubic surface of P_{VI} admits 3 polynomial involutions $s_0,\ s_t,\ s_1$. These involutions are *anti-symplectic* and they generate a subgroup of the group of algebraic automorphisms of S_{VI} which is isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ [11, Thm. 3.1, p. 2948]. The products $g_{i,j} := s_i \circ s_j$ $(i \neq j)$ are *symplectic polynomial automorphisms*. The corresponding automorphisms of the Okamoto variety of initial conditions obtained by conjugation by RH are the *non-linear monodromies* of PVI around the singular points $0, 1, \infty$ [11].

7.2. Pant decompositions, surgery, parametrizations

7.2.1. Pant decompositions of representations of a free group of rank 3.

Let $\rho : \Gamma_3 \to \mathrm{SL}_2(\mathbf{C})$ be a linear representation. We use the same notations as before. We suppose a fixed such that $a_l = \operatorname{Tr} M_l \neq \pm 2$, $l = 0, t, 1, \infty$. We associate to ρ the two representations of Γ_2:

$$\omega_{0,t} : \Gamma_2 \to \mathrm{SL}_2(\mathbf{C}) \quad \text{and} \quad \omega_{1,\infty} : \Gamma_2 \to \mathrm{SL}_2(\mathbf{C})$$

We will say that $(\omega_{0,t}, \omega_{1,\infty})$ is the *pant decomposition* of ρ associated to the partition $\{0, t, 1, \infty\} = \{0, t\} \cup \{1, \infty\}$. There are two others pant decompositions associated to the two other partitions.

We denote $\tilde{\rho}$ and $\tilde{\omega}$ the equivalence classes of representations. The pair $(\tilde{\omega}_{0,t}, \tilde{\omega}_{1,\infty})$ depends only on $\tilde{\rho}$.

We suppose that $\omega_{0,t}$ and $\omega_{1,\infty}$ are irreducible, equivalently $\tilde{\rho}$ belongs to $S(a)$ minus the 8 critical lines. Then the knowledge of $(\tilde{\omega}_{0,t}, \tilde{\omega}_{1,\infty})$ is equivalent to the knowledge of $X_0 = \operatorname{Tr} M_0 M_t = \operatorname{Tr} M_1 M_\infty$ (a is fixed).

In order to recover $\tilde{\rho}$ from X_0, we need another parameter. We set $X_0 = e + e^{-1}$. There are two different cases.

- We suppose $X_0 = \pm 2$ (equivalently $e = e^{-1}$). We can choose representations ω and ω of Γ_2 such that:

$$\mathrm{Tr}\,\omega\,(u) = a_0, \quad \mathrm{Tr}\,\omega\,(v) = a_t, \quad \omega\,(uv) = \mathrm{Diag}(e, e^{-1}),$$

$$\mathrm{Tr}\,\omega\,(u) = a_\infty, \quad \mathrm{Tr}\,\omega\,(v) = a_1, \quad \omega\,(uv) = \mathrm{Diag}(e^{-1}, e).$$

  There is a freeness in the choice: we can replace ω (resp. ω) by an overall conjugate by an arbitrary matrix commuting with $\mathrm{Diag}(e, e^{-1})$, that is of the form $\mathrm{Diag}(t, t^{-1})$ with t $\mathbf{C}$

- We suppose $X_0 = \pm 2$. We verify that the "trivial case" $\omega\,(uv) = \pm I_2$ is impossible. Then we can choose representations ω and ω of Γ_2 such that:

$$\mathrm{Tr}\,\omega\,(u) = a_0, \quad \mathrm{Tr}\,\omega\,(v) = a_t, \quad \omega\,(uv) = \begin{pmatrix} e & 1 \\ 0 & e \end{pmatrix},$$

$$\mathrm{Tr}\,\omega\,(u) = a_\infty, \quad \mathrm{Tr}\,\omega\,(v) = a_1, \quad \omega\,(uv) = \begin{pmatrix} e^{-1} & -1 \\ 0 & e^{-1} \end{pmatrix}.$$

  There is a freeness in the choice: we can replace ω (resp. ω) by an overall conjugate by an arbitrary matrix commuting with $\left(\begin{smallmatrix} e & 1 \\ 0 & e \end{smallmatrix}\right)$, that is of the form $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ with t $\mathbf{C}$.

7.2.2. Pant decompositions and pant parametrizations

Pant decompositions of a n-punctured sphere. Trace coordinates. Let n $\mathbf{N}$, $n > 4$. We denote S^2_n the n-punctured sphere. We replace the n punctures by little holes obtained by cutting along non-intersecting simple closed curves surrounding the punctures, we get a *n-holed sphere* that we also denote S^2_n.

Definition 7.7. — *A* pant decomposition *is defined by cutting S^2_n along $n-3$ simple closed curves γ_r, $r = 1, \ldots, n-3$, on S^2_n in such a way that this will decompose S^2_n into a disjoint union of $n-2$ three-holed spheres $S^2_{3,t}$, $t = 1, \ldots, n-2$. The collection $\{\gamma_1, \ldots, \gamma_{n-3}\}$ of curves is called the* cut system*.*

The origin of the terminology is clear: a pair of pants is homeomorphic to a 3 holed sphere.

If $n = 4$, then the cut system if a set of 3 curves. Each curve separates the set of punctures $\{0, t, 1, \infty\}$ into two unordered pairs of unordered sets of two elements:

$$((0,t),(1,\infty)), \quad ((0,1),(t,\infty)), \quad ((0,\infty),(t,1)).$$

We will also call (abusively...) such a pair *a pant decomposition* of the 4-punctured sphere. In that sense we get 3 pant decompositions.

We can interpret the character variety S associated to the free group Γ_{n-1} (of rank $n-1$) as the set of representations ρ of $\pi_1(S^2_n)$ into $\mathrm{SL}_2(\mathbf{C})$ modulo equivalence of representations. Then useful sets of coordinates on S are given by the trace functions $\operatorname{Tr}\rho(\gamma)$ associated to any simple closed curve γ on S^2_4. It is a classical fact that minimal sets of trace functions that can be used to parameterize S can be identified using *pant decompositions*. We will give only the basic idea. For more details see for example [33] (that we follow in our description). In the next part we will detail the case $n = 4$. To a pant decomposition we can associate decompositions of a representation ρ. More abstractly we get a notion of pant decomposition of a representation of a free group of rank n. This generalizes the definition introduced in the preceding paragraph.

To each curve γ_r we can associate the union of the two 3-holed spheres which have γ_r in their boundary (one of the 3 components). We get a 4-holed sphere $S^2_{n,r}$. We choose an orientation on each curve γ_r. This allows us to introduce a numbering of the 4 boundary components of $S^2_{n,r}$. Then we can consider the curves γ^r_s and γ^r_t which encircle respectively the pair of component $(1,2)$ and $(2,3)$. The collection of pairs of trace functions $(\operatorname{Tr}\rho(\gamma^r_s), \operatorname{Tr}\rho(\gamma^r_t))$, $r = 1,\dots,n-3$, can be used to parameterize the character variety.

Pant parametrization of a 4 holed sphere. Jimbo formulae. Let S^2_4 be the four punctured sphere. Its fundamental group $\pi_1(S^2_4)$ is isomorphic to a free group of rank 3: we can choose as generators the homotopy classes of three simple loops turning around 3 punctures. We choose simple loops γ_i, $i = 1,2,3,4$ turning respectively around the 4 punctures and based at a point z_0 of the punctured sphere.

As above, up to a Möbius transformation, we can choose as punctures $0, t, 1, \infty$ for some value of $t \quad \mathbf{C} \setminus \{0,1\}$. Then we denote γ_i, $i = 0, t, 1, \infty$, the simple loops and $M_i = \rho(\gamma_i)$. We suppose the M_i semi-simple with eigenvalues e_i, e_i^{-1} and we set, as above, $a_i := \operatorname{Tr} M_i = e_i + e_i^{-1}$, $i = 0, t, 1, \infty$, and

$$X_0 = \operatorname{Tr} M_1 M_t, \quad X_t = \operatorname{Tr} M_1 M_0, \quad X_1 = \operatorname{Tr} M_t M_0.$$

We can apply this to the monodromy representation of a system 1.9, then we have local monodromy exponents $\theta_i \quad \mathbf{C}$ and $a_i = 2\cos\theta_i$. We suppose that the non resonance conditions are satisfied: $a_i = \pm 2$ or equivalently $\theta_i / \mathbf{Z}$.

We can interpret $M_t M_0$ as the monodromy associated to an oriented curve separating the singularities in two packs $(0, t)$ and $(1, \infty)$, and therefore the 4 punctured sphere S^2_4 into two 3 punctured sphere S^2_3. The corresponding monodromy exponent is denoted σ_1: $X_1 = 2\cos 2\pi\sigma_1$. We define similarly σ_0 and σ_t: $X_0 = 2\cos 2\pi\sigma_0$ and $X_t = 2\cos 2\pi\sigma_t$.

We will recall some Jimbo formulae [37] and interpret it in relation with (a variant of[52]) the fibration of the cubic surface $S_{VI}(a)$ described page 1217. We use the presentation of [33, 6.1, p. 19], with a change of notations.

We fix a. If we further fix X_1, then the equation

$$X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - A_0 X_0 - A_t X_t - A_1 X_1 + A_\infty = 0$$

(where the A_l, which depends only on A are fixed) defines a *conic* in the variables X_0, X_t. This conic admits a *rational parameterization* [37], [33, cf. (6.67a), (6.67b), (6.68a), (6.68b), (6.68c), (6.67d)]:

$$\begin{aligned} (X_1^2 - 4)X_0 &= D_{0,+}s + D_{0,-}s^{-1} + D_{0,0} \\ (X_1^2 - 4)X_t &= D_{t,+}s + D_{t,-}s^{-1} + D_{t,0}, \end{aligned} \tag{7.8}$$

with coefficients given by:

$$\begin{aligned} D_{0,0} &:= X_1 A_t - 2A_0, \quad D_{t,0} = X_1 A_0 - 2A_t, \\ D_{0,\pm} &:= 16 \sum_{\epsilon = \pm 1} \sin\pi(\theta_t \mp \sigma_1 + \epsilon\theta_0) + \sin\pi(\theta_1 \mp \sigma_1 + \epsilon\theta_\infty), \\ D_{t,\pm} &:= -D_{0,\pm}\, e^{\mp 2i\pi\sigma_1}. \end{aligned} \tag{7.9}$$

More precisely the above formulae give a rational parametrization of the conic if we suppose that he following conditions are satisfied:

(1) $X_1 \neq \pm 2$, or equivalently $\sigma_1 \notin \mathbf{Z}$;
(2) the 4 conditions (7.6) for X_1 are excluded, or equivalently:

$$\sigma_1 \pm \theta_i \pm \theta_j \notin 2\mathbf{Z}, \quad \sigma_1 \pm \theta_k \pm \theta_\infty \notin 2\mathbf{Z}.$$

The first case correspond to the 2 parabolic fibers. The second case to the 4 cases of decomposition of the conic into two lines.

We can compare with the fibration of F by Π described in 6. The first case correspond to the logarithmic fibers. The second to the exceptional non logarithmic fibers.

Formulae (7.8) define a parametrization of the surface $S_{VI}(a)$ by the (X_1, s) (resp. (σ_1, s)) satisfying the above restrictions. It is a *pants parametrization*. The image misses 8 lines and 2 parabolas. There are two others similar pants parametrizations (we replace X_1 by X_0 or X_t).

(52) We will fix X_1 in place of X_0.

7.2.3. q-pants parametrizations

q-pants decompositions. At the beginning of Subsection 6.1 we introduced special values associated to the decomposition $\{1,2,3,4\} = \{1,2)\}\ \{3,4\}$. We recall these values (adding indices):

$$\Xi_{1,2} := \{R(-\rho_1/x_1), R(-\rho_1/x_2), R(-\rho_2/x_1), R(-\rho_2/x_2)\},$$
$$\Xi_{1,2} := \{R(-\sigma_1 x_3), R(-\sigma_1 x_4), R(-\sigma_2 x_3), R(-\sigma_2 x_4)\}.$$

We assumed (this was $\mathbf{Hyp}_8$, see equation (6.1)) that $\Xi_{1,2} := \Xi_{1,2}\quad \Xi_{1,2}$ has eight (pairwise distinct) elements. We can consider similar conditions for the five other decompositions. We can assume the six conditions, then we will say that $\mathbf{Hyp}_{48}$ is satisfied.

In all this part we suppose that (FR), (NR), (NS) and $\mathbf{Hyp}_{48}$ are satisfied.

We consider the six decompositions of the set $\{1,2,3,4\}$ (indexing the intermediate singularities x_1, x_2, x_3, x_4) into two *ordered packs* of *unordered* elements:

$$((1,2),(3,4))\,,\ \ ((1,3),(2,4))\,,((1,4),(2,3))\,,\ \ ((2,3),(1,4))\,,$$
$$((2,4),(1,3))\,,\ \ ((3,4),(1,2))\,;$$

$$((i,j),(k,l)) = ((j,i),(k,l)) = ((i,j),(l,k)) = ((j,i),(l,k))\,.$$

A decomposition $((i,j),(k,l))$ is indexed[53] by (i,j) such that $i < j$. The corresponding decomposition of the set of intermediate singularities $((x_i,x_j),(x_k,x_l))$ is called a *q-pants decomposition*. As we explained before, the heuristic idea is to select a particular pair of singularities x_i, x_j among x_1, x_2, x_3, x_4, with the idea of "localize" the "q-monodromy" around that pair.

Be careful, in the classical case of representations of the free group Γ_3 generated by $u_0,\ u_t,\ u_1,\ u_\infty$ up to the relation $u_0 u_t u_1 u_\infty = 1$ (or equivalently of the fundamental group of the 4-punctured sphere $\mathbf{P}^1(\mathbf{C}) \setminus \{0,t,1,\infty\}$), the pant-decompositions are indexed by the 3 decompositions of $\{0,t,1,\infty\}$ into two *unordered packs* of unordered elements:

$$((0,t),(1,\infty))\,,\quad ((0,1),(t,\infty))\,,\quad ((0,\infty),(t,1))\,.$$

This is an important difference between the representations and the "q-representations". For the Fricke coordinates we have $\operatorname{Tr} M_0 M_t = \operatorname{Tr} M_1 M_\infty$ but for the q-analogs we have $\Pi_{1,2} = \Pi_{3,4}$.

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(53) It can be convenient to allow also indexation by (j,i), with $j > i$, when it simplifies some notations.

To each indices decomposition is associated a Mano decomposition. If necessary we will index the objects appearing in the study of this Mano decomposition by the corresponding (i,j): $\Pi_{i,j}$, $\Xi_{1,2},\ldots$.

q-pants parameterizations and q-pants charts. We remove from $\mathbf{E}_q$ the 4 fixed points of the involution $\xi \to \rho_1\rho_2/x_1x_2\xi$ and the 8 points in $\Xi_{1,2}$. We get a punctured elliptic curve denoted by $\mathbf{E}_q{}^{\bullet,\dagger;1,2}$. We denote $U^{\bullet,\dagger;1,2}_{\mathbf{C}^*}$ the inverse image of the punctured elliptic curve by the canonical map $\mathbf{C}^* \to \mathbf{E}_q$. We set[54] :

$$s(\xi_1,\xi_2) := \frac{\theta_q\left(\frac{\xi_2}{\rho_1}x_1\right)\theta_q\left(\frac{\xi_1}{\rho_2}x_1\right)}{\theta_q\left(\frac{\xi_1}{\rho_1}x_1\right)\theta_q\left(\frac{\xi_2}{\rho_2}x_1\right)} \quad \text{and} \quad s(\xi) := s(\xi,\rho_1\rho_2/x_1x_2\xi),$$
$$t(\xi_1,\xi_2) := \frac{\theta_q\left(\frac{\sigma_2}{\xi_1}x\right)\theta_q\left(\frac{\sigma_1}{\xi_2}x\right)}{\theta_q\left(\frac{\sigma_1}{\xi_1}x\right)\theta_q\left(\frac{\sigma_2}{\xi_2}x\right)}, \quad \text{and} \quad t(\xi) := t(\xi,\rho_1\rho_2/x_1x_2\xi), \tag{7.10}$$

$$\underline{P}(\xi_1,\xi_2,x;\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22}) := P\left(\left(\alpha_{ij}\,\theta_q\left(\frac{\xi_j}{\rho_i}x\right)\right)\right)$$
$$\text{and} \quad P_{1,2}(\xi,x) := \underline{P}\left(\xi,\rho_1\rho_2/x_1x_2\xi,x;1,1,1,s(\xi)\right),$$
$$\underline{Q}(\xi_1,\xi_2,x;\beta_{11},\beta_{12},\beta_{21},\beta_{22}) := Q\left(\left(\beta_{ij}\,\theta_q\left(\frac{\sigma_j}{\xi_i}x\right)\right)\right) \tag{7.11}$$
$$\text{and} \quad Q_{1,2}(\xi,x) := \underline{P}\left(\xi,\rho_1\rho_2/x_1x_2\xi,x;1,1,1,t(\xi)\right).$$

Lemma 7.8. — *We suppose the ρ_i and σ_j $(i,j=1,2)$ fixed (satisfying the “good conditions”). We have:*

$$\theta_q\left(\frac{\xi_j}{\rho_i}x\right) \in V_{1,\frac{\rho_i}{\xi_j}}, \quad \theta_q\left(\frac{\sigma_j}{\xi_i}x\right) \in V_{1,\frac{\xi_i}{\sigma_j}}, \quad \theta_q\left(\frac{\xi_h}{\rho_i}x\right)\theta_q\left(\frac{\sigma_j}{\xi_h}x\right) \in V_{2,\frac{\rho_i}{\sigma_j}}.$$

The maps $\zeta_{i,j,h} : \mathbf{C}^* \to V_{2,\frac{\rho_i}{\sigma_j}}$ $(i,j,h=1,2)$ defined by $\zeta_{i,j,h} : \xi_h \to \theta_q\left(\frac{\xi_h}{\rho_i}x\right)\theta_q\left(\frac{\sigma_j}{\xi_h}x\right)$ are analytic on $\mathbf{C}^*$.

Proof. — We fix (i,j). Let (e_1,e_2) be a basis of $V_{2,\frac{\rho_i}{\sigma_j}}$. There exist two functions $C_1(\xi_h)$ and $C_2(\xi_h)$ of ξ_h, uniquely determined, such that:

$$\zeta_{i,j,h}(\xi_h) = C_1(\xi_h)e_1 + C_1(\xi_h)e_2.$$

We have:

$$\zeta_{i,j,h}(\xi_h)(qx) = C_1(\xi_h)e_1(qx) + C_1(\xi_h)e_2(qx).$$

For $x \in \mathbf{C}^*$ fixed, the two functions: $\xi_h \to \zeta_{i,j,h}(\xi_h)(x)$ and $\xi_h \to \zeta_{i,j,h}(\xi_h)(qx)$ are analytic on $\mathbf{C}^*$, therefore C_1 and C_2 are analytic on $\mathbf{C}^*$.

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[54] If necessary one can precise $s_{1,2}$ and $t_{1,2}$.

The functions s and t are meromorphic on $\mathbf{C}$, they are analytic on the inverse image $U^{\bullet,\dagger;1,2}$ of ${\mathbf{E}_q}^{\bullet,\dagger;1,2}$. Identifying $V := V_{2,\frac{\rho_1}{\sigma_1}} \times V_{2,\frac{\rho_1}{\sigma_2}} \times V_{2,\frac{\rho_2}{\sigma_1}} \times V_{2,\frac{\rho_2}{\sigma_2}}$ with a set of matrices, we define a map $\underline{M}_{1,2} : U^{\bullet,\dagger;1,2} \times \mathbf{C} \quad \to V$ by:

$$\underline{M}_{1,2}(\xi,\eta) := P_{1,2}(\xi,x)\ \mathrm{Diag}(1,\eta)\, Q_{1,2}(\xi,x). \tag{7.12}$$

If we fix ξ, then $M_{1,2}(\xi,w)$ is *linear* in η (in the trivial sense). The map $\underline{M}_{1,2}$ is analytic in the variable ξ and it extends uniquely in a map meromorphic on $\mathbf{C} \quad \times \mathbf{C}$. We have: $M_{1,2}(q\xi,w) = M_{1,2}(\xi,w)$. Therefore $M_{1,2}$ induces an analytic map $M_{1,2} : {\mathbf{E}_q}^{\bullet,\dagger;1,2} \times \mathbf{C} \quad \to V$ and this map extends uniquely into a map meromorphic on $\mathbf{E}_q \times \mathbf{C}$. Therefore it can be interpreted as a *rational map* from $\mathbf{E}_q \times \mathbf{C}$ to the linear space V. This map is *regular* on ${\mathbf{E}_q}^{\bullet,\dagger;1,2} \times \mathbf{C}$. The image of $M_{1,2}$ is contained in F, therefore we get by corestriction a map (abuse of notations . . .) $M_{1,2} : {\mathbf{E}_q}^{\bullet,\dagger;1,2} \times \mathbf{C} \quad \to F$.

If we compose by the quotient map $F \to \mathcal{F}$, then we get $\overline{M}_{1,2} : {\mathbf{E}_q}^{\bullet,\dagger;1,2} \times \mathbf{C} \quad \to \mathcal{F}$. This map is called the *q-pant parameterization* associated to the q-pant decomposition $(1,2)$.

We have[55]

$$\begin{aligned} \Pi_{12} \circ M_{1,2}(\xi,\eta) = \Phi_{1,2}(\xi), \quad M_{1,2}(q\xi,\eta) \quad M_{1,2}(\xi,\eta), \\ M_{1,2}(\xi,\eta) = M_{1,2}(\rho_1\rho_2/x_1x_2\xi,\eta^{-1}). \end{aligned} \tag{7.13}$$

The map $\overline{M}_{1,2}$ is not injective: the fiber is $((\xi,\eta),(\rho_1\rho_2/x_1x_2\xi,\eta^{-1}))$. Therefore $\overline{M}_{1,2}$ induces an *injective* map:

$$\psi_{1,2} : Y^{\bullet,\dagger,1,2} \to \mathcal{F},$$

where

$$Y^{\bullet\dagger;1,2} := \frac{{\mathbf{E}_q}^{\bullet\dagger;1,2} \times \mathbf{C}}{\text{involution } \tau_{12}}.$$

This map is called the *q-pant chart* associated to the q-pant decomposition $(1,2)$. We can interpret $Y^{\bullet\dagger;1,2}$ as an algebraic variety. We denote it $Y^{\bullet\dagger;1,2}_{alg}$ and we denote:

$$\psi_{alg;1,2} : Y^{\bullet\dagger;1,2}_{alg} \to \mathcal{F}$$

the corresponding *regular* map. It is called the algebraic q-pant chart associated to the q-pant decomposition $(1,2)$. There are respectively 6 similar q-pant parameterizations, q-pant charts, algebraic q-pant charts.

We can consider the maps:

$$\Pi_{i\ ,j} \quad \circ \psi_{alg;i,j} : Y^{\bullet\dagger;i,j} \to \mathbf{P}^1(\mathbf{C}).$$

(55) We denote abusively $\Pi \circ M \ = \Pi \circ \overline{M}$.

They are *regular* maps and can be computed *explicitly*. At the level of q-pant parameterization we can compute explicitly $\Pi_{i,j} \circ M_{i,j}$ using $M_{i,j}(x_i)$ and $M_{i,j}(x_j)$.

If we interpret the six $\Pi_{i,j}$ as "coordinates", then we get *q-coordinate charts from the pant decomposition* (i, j), the q-analogs of the coordinate charts from a pant decomposition of the classical case given by (7.8).

Remark 7.9. — In Part 6.6.2 we extended the involution τ into an involution $\tilde{\tau}$ on $\mathbf{E}_q \times \mathbf{P}^1(\mathbf{C})$ and set $\tilde{Y} := \frac{\mathbf{E}_q \times \mathbf{P}^1(\mathbf{C})}{\text{involution } \tilde{\tau}}$. Afterwards we extended the elliptic fibration p into an elliptic fibration $\tilde{p} : \tilde{Y} \to \mathbf{P}^1(\mathbf{C})$ and the application Ψ into an application $\tilde{\Psi}$. At the end of Part 6.5.1 we remarked that the surface $\{v^2 = (w^2 - 1)f(x)\}$ is a double covering of the affine plane $\mathbf{C}^2$ (the (x, w) plane) ramified above 6 lines. We consider the map $\pi := (\tilde{p}, \tilde{\Psi}) : \tilde{Y} \to (\mathbf{P}^1(\mathbf{C}))^2$. It is a double covering of $(\mathbf{P}^1(\mathbf{C}))^2$ ramified along 6 lines a $(2, 3)$-sextic of $(\mathbf{P}^1(\mathbf{C}))^2$. The surface $\tilde{Y}$ has 8 singular points (rational double points), above the pairwise intersections of the 6 lines. Blowing up these 8 points, we get a smooth surface X (one can compare with Example 7.25). It is possible to compute explicitely a system of algebraic charts for X. Then the surface X minus the 12 lines above $\tilde{\Psi}^{-1}(\mathbf{P}^1(\mathbf{C})^{\bullet\dagger})$, that we denote $X^\dagger$, could perhaps be used for a parameterization of F by explicit Zariski open sets. More precisely, we conjecture that it is possible to extend the q-pant chart ψ into a regular injective map $X \to F$ (cf. Remark 7.9) such that its image is a smooth open set of F containing the 4 logarithmic fibers. Conjecture 6.8 would follow.

A smoothness conjecture. We end this subsection with a conjecture[(56)]. This conjecture is strongly related to the configuration of the lines on the surface F. We will return to this question in the next paragraphs.

We set $U_{i,j} = F \setminus (\Pi_{i,j}^{-1}(0) \quad \Pi_{i,j}^{-1}(\infty) \quad (\Pi\ )_{i,j}^{-1}(0) \quad (\Pi\ )_{i,j}^{-1}(\infty))$; it is F minus the 4 exceptional non logarithmic fibers. The image of $\psi_{i,j}$ is F minus all the exceptional fibers, therefore it is $U_{i,j}$ minus the logarithmic fibers.

Conjecture 7.10. — *We suppose that* (FR), (NR), (NS) *and* $\mathbf{Hyp}_{48}$ *are satisfied. Then F is* smooth.

We will prove below (cf. Proposition 7.11) that F minus all the logarithmic fibers is covered by the union of the images[(57)] of the six q-pant charts $\psi_{i,j}$ $(i, j \quad \{1, 2, 3, 4\}, i < j)$. Then the above conjecture will follow immediately from Conjecture 6.8

(56) It is a q-analog of Theorem 7.4.

(57) The image of each q-pants chart is F minus 8 lines (depending on the chart).

The rational functions $\Pi_{i,j}$ on the surface F can be interpreted as q-analogs of the Fricke coordinates. Then there are natural questions:

- What are the algebraic relations between the $\Pi_{i,j}$? How to compute them ?
- Is it possible to use the $\Pi_{i,j}$ to build an embedding of F into some $\left(\mathbf{P}^1(\mathbf{C})\right)^m$ ($m \in \mathbf{N}$, $m > 3$) ?
- If we denote X the closure of the image of F by such an embedding, what can we say of the surface X ?

We will return later to these questions, cf. Part 7.2.7.

7.2.4. q-pants decompositions and partial reducibility

We consider the quotient of $\mathbf{P}^1(\mathbf{C})$ by the action of $q^{\mathbf{Z}}$. We write it:

$$\left[\mathbf{P}^1(\mathbf{C}); q\right] := \{[0;q]\} \cup \mathbf{E}_q \cup \{[\infty;q]\}.$$

We have a "cotangent bundle" (defined by the first projection):

$$\left[\mathbf{P}^1(\mathbf{C}); q\right] \times \mathbf{C}^* \to \left[\mathbf{P}^1(\mathbf{C}); q\right].$$

It is the q-analog of the cotangent bundle of $\mathbf{P}^1(\mathbf{C})$ (of more generally a Riemann surface) in the differential case; we will call it the *q-cotangent bundle*.

The fiber above $[a;q]$ is $[a;q] \times \mathbf{C}^*$. When the "point" $[a;q]$ is a singularity, we will consider $\mathbf{C}^*$ as the space of *possible monodromy exponents*. More precisely above $[a;q] = [0;q]$ or $[a;q] = [\infty;q]$ we can choose arbitrarily a monodromy exponent into $\mathbf{C}^*$ but if $[a;q] \in \mathbf{E}_q$, the only possible choice above $[a;q]$ is $\zeta \in \mathbf{C}^*$ such that $[\zeta : q] = [-a^{-1};q]$.

In the context of our description of q-PVI we consider the following list of pairs of points of the total space of the q-cotangent bundle[58] :

$$\big(([0;q];\rho_1),([0;q];\rho_2)\big), \quad \big(([\infty;q];\sigma_1),([\infty;q];\sigma_2)\big), \\ \left(\left([x_i;q];-x_i^{-1}\right),\left([x_j;q];-x_j^{-1}\right)\right). \quad (7.14)$$

($i = 1,2,3,4$, $i \neq j$). Be careful:

$$\left(\left([x_i;q];-x_i^{-1}\right),\left([x_j;q];-x_j^{-1}\right)\right)$$

is written in the chart of the q-cotangent bundle coming from $\mathbf{C}$. In the chart coming from $\mathbf{P}^1(\mathbf{C}) \setminus \{0\}$, we write the same element

$$\left(\left([x_i^{-1};q];-x_i\right),\left([x_j^{-1};q];-x_j\right)\right).$$

(58) We skip the problem of ordering or not of such pair.

The Mano decomposition allows us to decompose the *global* monodromy around 0, ∞ and the four intermediate singularities into a pair of *local* monodromies: around 0 and one pair of singularities on one side and one pair of singularities and ∞ on the other side. As we will see it is better to interpret 0, ∞ and the pair (x_i, x_j) as the corresponding elements in the list (7.14).

We have a criterion of reducibility for each local monodromy. Using the above list (7.14), we will see that it is a perfect q-analog of the criterium in the differential case (cf. Proposition 7.6).

Looking on the left hand side of RH, the Mano decomposition can be interpreted as a decomposion of the system into two hypergeometric systems. We recall that it is the beginning of our story: in [46] Mano gave a *direct* method (based on isomonodromy and a q-analogy with Jimbo decompoition [37]) in order to decompose the original system.

The reducibility of a local monodromy on the right hand side of RH is equivalent to the reducibility of the corresponding hypergeometric system on the left hand side (cf. Subsection 3.3).

The reducibility of the local monodromy around 0 and the pair (x_i, x_j) is coded by:

$$(([0;q];\rho_1), ([0;q];\rho_2)) \quad \text{and} \quad \left(\left([x_i;q]; -x_i^{-1}\right), \left([x_j;q]; -x_j^{-1}\right)\right).$$

The 4 conditions of reducibility are:

$$\xi_1 = -\rho_1/x_i, \quad \xi_1 = -\rho_1/x_j, \quad \xi_1 = -\rho_2/x_i, \quad \xi_1 = -\rho_2/x_j.$$

They correspond to the following pairings:

$$\begin{aligned} ([0;q];\rho_1) &\longleftrightarrow \left([x_i;q]; -x_i^{-1}\right), \quad & ([0;q];\rho_1) &\longleftrightarrow \left([x_j;q]; -x_j^{-1}\right), \\ ([0;q];\rho_2) &\longleftrightarrow \left([x_i;q]; -x_i^{-1}\right), \quad & ([0;q];\rho_2) &\longleftrightarrow \left([x_j;q]; -x_j^{-1}\right). \end{aligned}$$

For simplicity we will denote these pairings $(\rho_h \leftrightarrow x_i)$ $(h = 1, 2;\ i = 1, 2, 3, 4)$.

The reducibility of the local monodromy around the pair (x_k, x_l) and ∞ is coded by:

$$(([\infty;q];\sigma_1), ([\infty;q];\sigma_2)) \quad \text{and} \quad \left(\left([x_k^{-1};q]; -x_k\right), \left([x_l^{-1};q]; -x_l\right)\right).$$

The 4 conditions of reducibility are:

$$\xi_1 = -\sigma_1 x_k, \quad \xi_1 = -\sigma_1 x_l, \quad \xi_1 = -\sigma_2 x_k, \quad \xi_1 = -\sigma_2 x_l.$$

They correspond to the following pairings:

$$\begin{aligned} ([\infty;q];\sigma_1) &\longleftrightarrow \left([x_k^{-1};q]; -x_k\right), \quad & ([\infty;q];\sigma_1) &\longleftrightarrow \left([x_l^{-1};q]; -x_l\right), \\ ([\infty;q];\sigma_2) &\longleftrightarrow \left([x_k^{-1};q]; -x_k\right), \quad & ([\infty;q];\sigma_2) &\longleftrightarrow \left([x_l^{-1};q]; -x_l\right). \end{aligned}$$

For simplicity we will denote these pairings $(\sigma_h \leftrightarrow x_i)$ $(h = 1, 2; i = 1, 2, 3, 4)$.

We will see in the next subsection that each pairing $\longleftrightarrow$ corresponds to a *line* on the surface F. We have 8 pairings and therefore 8 lines.

The pairing $(\rho_1 \leftrightarrow x_i)$ (resp. $(\rho_2 \leftrightarrow x_i)$) appears in the 3 Mano decompositions (i, j), (i, k) and (i, l) (where (i, j, k, l) is a permutation of $(1, 2, 3, 4)$). In the next paragraph we will prove that the corresponding 3 lines *coincide*.

Similarly the pairing $(\sigma_1 \leftrightarrow x_i)$ (resp. $(\sigma_2 \leftrightarrow x_i)$) appears in the 3 Mano decompositions (i, j), (i, k) and (i, l) (where (i, j, k, l) is a permutation of $(1, 2, 3, 4)$) We will also prove that the corresponding 3 lines *coincide*.

7.2.5. Description of some lines on the surface F

In all this part we suppose that (FR), (NR), (NS) and $\mathbf{Hyp}_{48}$ are satisfied.

We will give a global description of the special non-logarithmic lines on the surface F and observe that this description is a translation by q-analogies of the dictionnary between the set of lines on the cubic surface on one side and the partial reducibility of representations of the other side that we described in the classical cases (cf. Part 7.1.3).

We will derive from this description a notion of *reducibility of the local monodromy around* 0 *and* x_i (resp. ∞ *and* x_i).

We recall:

$$\Phi_{1,2}(-\rho_1/x_1) = \Phi_{1,2}(-\rho_2/x_2) = 0,$$
$$\Phi_{1,2}(-\rho_1/x_2) = \Phi_{1,2}(-\rho_2/x_1) = \infty,$$

and we set:

$$\mathbf{e}_q^{1;1,2;3}(\underline{\rho}, \underline{\sigma}, \underline{x}) := \Phi_{1,2}(-\sigma_1 x_3) = \Phi_{1,2}(-\sigma_2 x_4) = \frac{\theta_q\left(\frac{\sigma_1}{\rho_1} x_1 x_3\right) \theta_q\left(\frac{\sigma_1}{\rho_2} x_2 x_3\right)}{\theta_q\left(\frac{\sigma_1}{\rho_1} x_2 x_3\right) \theta_q\left(\frac{\sigma_1}{\rho_2} x_1 x_3\right)},$$

$$\mathbf{e}_q^{2;1,2;3}(\underline{\rho}, \underline{\sigma}, \underline{x}) := \Phi_{1,2}\left(-\sigma_1 x_4\right) = \Phi_{1,2}\left(-\sigma_2 x_3\right) = \frac{\theta_q\left(\frac{\sigma_2}{\rho_1} x_1 x_3\right) \theta_q\left(\frac{\sigma_2}{\rho_2} x_2 x_3\right)}{\theta_q\left(\frac{\sigma_2}{\rho_1} x_2 x_3\right) \theta_q\left(\frac{\sigma_2}{\rho_2} x_1 x_3\right)}.$$

We verify:

$$\mathbf{e}_q^{1;1,2;3}(\underline{\rho}, \underline{\sigma}, \underline{x}) = \mathbf{e}_q^{2;1,2;4}(\underline{\rho}, \underline{\sigma}, \underline{x}) \quad \text{and} \quad \mathbf{e}_q^{2;1,2;3}(\underline{\rho}, \underline{\sigma}, \underline{x}) = \mathbf{e}_q^{1;1,2;4}(\underline{\rho}, \underline{\sigma}, \underline{x}).$$

Using the other decompositions we define similarly $\mathbf{e}_q^{h;i,j;k}(\underline{\rho}, \underline{\sigma}, \underline{x})$ (where $h = 1, 2$ and (i, j, k) is a set of 3 distinct elements of $\{1, 2, 3, 4\}$). We have: $\mathbf{e}_q^{h;i,j;k} = \mathbf{e}_q^{h;j,i;k}$ and we verify:

$$\mathbf{e}_q^{1;i,j;k}(\underline{\rho}, \underline{\sigma}, \underline{x}) = \mathbf{e}_q^{2;i,j;l}(\underline{\rho}, \underline{\sigma}, \underline{x}) \text{ and } \mathbf{e}_q^{2;i,j;k}(\underline{\rho}, \underline{\sigma}, \underline{x}) = \mathbf{e}_q^{1;i,j;l}(\underline{\rho}, \underline{\sigma}, \underline{x}).$$

The function $\mathbf{e}_q^{h;i,j;k}$ is elliptic in σ_h. We interpret the 12 functions as q-analogs of the $a_l = 2\cos\theta_l$ ($l = 0, t, 1, \infty$) of the di□erential case (the traces of the local monodromies). A big di□erence is that the $\mathbf{e}_q^{h;i,j;k}$ involve *all* the local data.

We will call the $\mathbf{e}_q^{h;i,j;k}$ the *q-local monodromy invariants.* We conjecture that when the monodromy exponents $\underline{\rho},\ \underline{\sigma},\ \underline{x}$ move these q-local monodromy invariants "parameterize algebraically" the variation of $F(\underline{\rho}, \underline{\sigma}, \underline{x})$.

We have:

$$(\Pi\ )^{-1}_{1,2}(0) = \Pi^{-1}_{1,2}\left(\mathbf{e}_q^{1;1,2;3}\right), \quad (\Pi\ )^{-1}_{1,2}(\infty) = \Pi^{-1}_{1,2}\left(\mathbf{e}_q^{2;1,2;3}\right).$$

To the pairing $(\rho_1 \leftrightarrow x_i)$ (resp. $(\rho_2 \leftrightarrow x_i)$) we associate a line L_{ρ_1,σ_i} (resp. L_{ρ_2,σ_i}) of F: it is the set of the classes of the matrices M such that the first (resp. second) *line* of $M(x_i)$ is null. Similarly, to the pairing $(\sigma_1 \leftrightarrow x_i)$ (resp. $(\sigma_2 \leftrightarrow x_i)$) we associate a line L_{σ_1,x_i} (resp. L_{σ_2,x_i}) of F: it is the set defined by the classes of the matrices M such that the first (resp. second) *column* of $M(x_i)$ is null. Using these lines we can describe the exceptional non logarithmic fibers of $\Pi_{i,j}$ (and $\Pi_{i,j}$). We detail the case $(i, j) = (1, 2)$; the others are similar.

We denote $\overline{M}\quad F$ the equivalence class of $M\quad F$. We have:

$$f_i = 0 \qquad \overline{M} \quad L_{\rho_1,x_i} \qquad M(x_i) = \begin{pmatrix} 0 & 0 \\ & \end{pmatrix}$$

$$\text{and} \quad g_i = 0 \qquad \overline{M} \quad L_{\rho_2,x_i} \qquad M(x_i) = \begin{pmatrix} & \\ 0 & 0 \end{pmatrix}.$$

Then (cf. Theorem 5.5):

$$\Pi^{-1}_{1,2}(0) = \{f_1 = 0\} \quad \{g_2 = 0\} = L_{\rho_1,x_1} \quad L_{\rho_2,x_2}$$

$$\text{and} \quad \Pi^{-1}_{1,2}(\infty) = \{f_2 = 0\} \quad \{g_1 = 0\} = L_{\rho_1,x_2} \quad L_{\rho_2,x_1}.$$

$$f_i = 0 \qquad \overline{M} \quad L_{\rho_1,x_i} \qquad M(x_i) = \begin{pmatrix} 0 & 0 \\ & \end{pmatrix}$$

$$\text{and} \quad g_i = 0 \qquad \overline{M} \quad L_{\rho_2,x_i} \qquad M(x_i) = \begin{pmatrix} & \\ 0 & 0 \end{pmatrix}.$$

Similarly we have:

$$f_i = 0 \qquad \overline{M} \quad L_{\sigma_1,x_i} \qquad M(x_i) = \begin{pmatrix} 0 & \\ 0 & \end{pmatrix}$$

$$\text{and} \quad g_i = 0 \qquad \overline{M} \quad L_{\sigma_2,x_i} \qquad M(x_i) = \begin{pmatrix} & 0 \\ & 0 \end{pmatrix}.$$

Then:

$$(\Pi\ )^{-1}_{1,2}(0) = \{f_3 = 0\} \quad \{g_4 = 0\} = L_{\sigma_1,x_3} \quad L_{\sigma_2,x_4}$$
$$\text{and} \quad (\Pi\ )^{-1}_{1,2}(\infty) = \{f_4 = 0\} \quad \{g_3 = 0\} = L_{\sigma_1,x_4} \quad L_{\sigma_2,x_3}.$$

The two by two intersections of the 4 special fibers are empty, therefore the 8 lines are distinct and we have for these lines the following incidence relations:

$$\begin{gathered}
L_{\rho_1,x_1} \cap L_{\rho_1,x_2} = ?, \quad L_{\rho_2,x_1} \cap L_{\rho_2,x_2} = ?, \\
L_{\sigma_1,x_3} \cap L_{\sigma_1,x_4} = ?, \quad L_{\sigma_2,x_3} \cap L_{\sigma_2,x_4} = ?, \\
h, h\ = 1,2, \qquad i = 1,2, \qquad j = 3,4, \quad L_{\rho_h,x_i} \cap L_{\sigma_{h\ },x_j} = ? \qquad (7.15) \\
L_{\rho_1,x_1} \cap L_{\rho_2,x_2} = \{\text{one point}\}, \quad L_{\rho_1,x_2} \cap L_{\rho_2,x_1} = \{\text{one point}\}, \\
L_{\sigma_1,x_3} \cap L_{\sigma_2,x_4} = \{\text{one point}\}, \quad L_{\sigma_2,x_3} \cap L_{\sigma_1,x_4} = \{\text{one point}\}.
\end{gathered}$$

Replacing the decomposition $(1,2)$ by another decomposition, we get similar results.

We cannot have $M(x_i) = \begin{pmatrix} 0 & 0 \\ & 0 \end{pmatrix}$, $M(x_i) = \begin{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}$, $M(x_i) = \begin{pmatrix} 0 & \\ 0 & 0 \end{pmatrix}$, $M(x_i) = \begin{pmatrix} 0 & 0 \\ 0 & \end{pmatrix}$, therefore:

$$h, h\ = 1,2,\ i = 1,2,3,4, \quad L_{\rho_h,x_i} \cap L_{\sigma_{h\ },x_i} = ?.$$

For $h, h\ = 1,2$ and i, j fixed, $i = j$, the lines L_{ρ_h,x_i} and L_{σ_{h},x_j} appear in two di□erent exceptionnal fibers $\Pi^{-1}_{i,k}$ ($k = i$ and $k = j$), therefore $L_{\rho_h,x_i} \cap L_{\sigma_{h},x_j} = ?$.

Putting things together, we verify that we have 16 *di erent* lines with the following incidence relations:

- for $h, h\ = 1,2$, $i, j = 1,2,3,4$, $L_{\rho_h,x_i} \cap L_{\sigma_{h\ },x_j} = ?$.
- for $h = 1,2$, if $i = j$ then $L_{\rho_h,x_i} \cap L_{\rho_h,x_j} = ?$;
- for $h = 1,2$, if $i = j$ then $L_{\sigma_h,x_i} \cap L_{\sigma_h,x_j} = ?$;
- if (i,j,k,l) is a permutation of $(1,2,3,4)$, then:
  - L_{ρ_1,x_i} meets L_{ρ_2,x_j}, L_{ρ_2,x_k}, L_{ρ_2,x_l} and the 3 intersection points are *distinct*;
  - L_{ρ_2,x_i} meets L_{ρ_1,x_j}, L_{ρ_1,x_k}, L_{ρ_1,x_l} and the 3 intersection points are *distinct*
- if (i,j,k,l) is a permutation of $(1,2,3,4)$, then:
  - L_{σ_1,x_i} meets L_{σ_2,x_j}, L_{σ_2,x_k}, L_{σ_2,x_l} and the 3 intersection points are *distinct*;
  - L_{σ_2,x_i} meets L_{σ_1,x_j}, L_{σ_1,x_k}, L_{σ_1,x_l} and the 3 intersection points are *distinct*.

Each line is contained into exactly 3 special fibers:

$$
\begin{array}{llllll}
L_{\rho_1,x_i} & \Pi_{i,j}^{-1}(0), & L_{\rho_1,x_i} & \Pi_{i,k}^{-1}(0), & L_{\rho_1,x_i} & \Pi_{i,l}^{-1}(0)\\
L_{\rho_2,x_i} & \Pi_{i,j}^{-1}(\infty), & L_{\rho_2,x_i} & \Pi_{i,k}^{-1}(\infty), & L_{\rho_2,x_i} & \Pi_{i,l}^{-1}(\infty).\\
L_{\sigma_1,x_i} & \Pi_{j,k}^{-1}(\mathbf{e}_q^{1;j,k;i}), & L_{\sigma_1,x_i} & \Pi_{k,l}^{-1}(\mathbf{e}_q^{1;k,l;i}), & L_{\sigma_1,x_i} & \Pi_{l,j}^{-1}(\mathbf{e}_q^{1;l,j;i})\\
L_{\sigma_2,x_i} & \Pi_{j,k}^{-1}(\mathbf{e}_q^{2;j,k;i}), & L_{\sigma_2,x_i} & \Pi_{k,l}^{-1}(\mathbf{e}_q^{2;k,l;i}), & L_{\sigma_2,x_i} & \Pi_{l,j}^{-1}(\mathbf{e}_q^{2;l,j;i}).
\end{array}
\tag{7.16}
$$

Moreover this line is equal to each pairwise intersection of the 3 special fibers.

If (i,j,k,l) is a permutation of $(1,2,3,4)$, then we have:

$$
\begin{aligned}
L_{\rho_1,x_i} &= \Pi_{i,j}^{-1}(0)\cap\Pi_{i,k}^{-1}(0) = \Pi_{i,k}^{-1}(0)\cap\Pi_{i,l}^{-1}(0) = \Pi_{i,l}^{-1}(0)\cap\Pi_{i,j}^{-1}(0)\\
L_{\rho_2,x_i} &= \Pi_{i,j}^{-1}(\infty)\cap\Pi_{i,k}^{-1}(\infty) = \Pi_{i,k}^{-1}(\infty)\cap\Pi_{i,l}^{-1}(\infty)\\
&= \Pi_{i,l}^{-1}(\infty)\cap\Pi_{i,j}^{-1}(\infty)\\
L_{\sigma_1,x_i} &= \Pi_{j,k}^{-1}(\mathbf{e}_q^{1;j,k;i})\cap\Pi_{k,l}^{-1}(\mathbf{e}_q^{1;k,l;i}) = \Pi_{k,l}^{-1}(\mathbf{e}_q^{1;k,l;i})\cap\Pi_{l,j}^{-1}(\mathbf{e}_q^{1;l,j;i})\\
&= \Pi_{l,j}^{-1}(\mathbf{e}_q^{1;l,j;i})\cap\Pi_{j,k}^{-1}(\mathbf{e}_q^{1;j,k;i})\\
L_{\sigma_2,x_i} &= \Pi_{j,k}^{-1}(\mathbf{e}_q^{2;j,k;i})\cap\Pi_{k,l}^{-1}(\mathbf{e}_q^{2;k,l;i}) = \Pi_{k,l}^{-1}(\mathbf{e}_q^{2;k,l;i})\cap\Pi_{l,j}^{-1}(\mathbf{e}_q^{2;l,j;i})\\
&= \Pi_{l,j}^{-1}(\mathbf{e}_q^{2;l,j;i})\cap\Pi_{j,k}^{-1}(\mathbf{e}_q^{2;j,k;i})
\end{aligned}
\tag{7.17}
$$

We recall the notation:

$$U_{i,j} = F \setminus \Big(\Pi_{i,j}^{-1}(0) \quad \Pi_{i,j}^{-1}(\infty) \quad (\Pi\ )_{i,j}^{-1}(0) \quad (\Pi\ )_{i,j}^{-1}(\infty)\Big).$$

Proposition 7.11. —

(i) *Each line intersects* 3 *other lines at* 3 *di erent points.*

(ii) *If* a F, *there are at most two di erent lines passing by* a.

(iii) *The set* $U_{i,j}$ *contains* F *less all the logarithmic fibers.*

(iv) *The set of* 16 *lines has two connected components: the set* $\{L_{\rho_h,x_i}\}_{h=1,2,;i=1,2,3,4}$ *and the set* $\{L_{\sigma_h,x_i}\}_{h=1,2,;i=1,2,3,4}$.

Proof. — (i) and (ii) follows easily from the above relations

(iii). — Let a F which does not belong to a logarithmic fiber. We suppose that $a \,/\, U_{1,2}\quad U_{1,3}\quad U_{1,4}$. Then we have in particular $a \,/\, U_{1,2}$ and a must belong to one of the 4 exceptional fibers and therefore to one of the eight lines. We have a similar result for $(1,3)$ and $(1,4)$, therefore a belongs to one of the four lines L_{ρ_h,x_1}, $L_{\sigma_h\ ,x_1}$. We prove similarly that a belongs to one of the four lines L_{ρ_h,x_2}, L_{σ_h,x_2}, to one of the four lines L_{ρ_h,x_3}, $L_{\sigma_h\ ,x_3}$ and to one of the four lines L_{ρ_h,x_4}, $L_{\sigma_h\ ,x_4}$. Then a belongs to 4 distinct lines. This contradicts (ii).

(iv). — The intersection of the two sets is empty and, using (i), we can verify that each set is connected.

If Conjecture 6.8 is true, then the above proposition implies that F is smooth, that is Conjecture 7.10.

7.2.6. An image of F in $(\mathbf{P}^1(\mathbf{C}))^3$

The image and what we know about it. We consider the map $T_{1,2} : F \to (\mathbf{P}^1(\mathbf{C}))^3$ defined by:

$$T_{1,2} := (\Pi_{1,2}, \Pi_{2,3}, \Pi_{3,4}) : \overline{M} \to \left(u(\overline{M}) = \Pi_{1,2}(\overline{M}), v(\overline{M}) = \Pi_{2,3}(\overline{M}), u\,(\overline{M}) = \Pi_{3,4}(\overline{M}) \right).$$

A better notation would be $T_{1,2,3,4}$: note that $T_{1,2,3,4} = T_{1,2,4,3}$ because there is an exchange between v and u . We will use (carefully ...) $T_{1,2}$ for simplicity.

We would like to describe the Zariski closure[59] Y in $(\mathbf{P}^1(\mathbf{C}))^3$ of the image $T_{1,2}(F)$ and in particular the closures of the images of the 16 lines on F.

Definition 7.12. — *The* $(1,2)$-skeleton *of F is the closure in* $(\mathbf{P}^1(\mathbf{C}))^3$ *of the image by $T_{1,2}$ of the set of the* 16 *lines. We denote it*

$$Sk_{1,2}(F) \subset T_{1,2}(F) \subset (\mathbf{P}^1(\mathbf{C}))^3.$$

If (i,j,k,l) is a *circular* permutation of $(1,2,3,4)$ we can define similarly the (i,j)-skeleton $\mathrm{Sk}_{(i,j)}(F)$.

It is important to understand the skeleton structure and to describe the inclusion $\mathrm{Sk}_{1,2}(F) \subset T_{1,2}(F)$. In the next paragraph we will explain how (under some "reasonable" conjectures) it is possible to "sew the surface onto the skeleton bones".

We have $L_{\rho_1,x_2} = \Pi_{1,2}^{-1}(0) \cap \Pi_{2,3}^{-1}(0)$, therefore the image of this line in $(\mathbf{P}^1(\mathbf{C}))^3$ is contained in the line $\{u = v = 0\}$, the intersection of the planes $\{u = 0\}$ and $\{v = 0\}$. As u is not constant on the image, this image is an open connected affine subset of the projective line $\{u = v = 0\}$ and therefore it is equal to $\{u = v = 0\}$ punctured at a point. Its closure is the projective line.

We have similar results for the lines:

$$L_{\rho_2,x_2},\ L_{\rho_1,x_3},\ L_{\rho_2,x_3},\ L_{\sigma_1,x_1},\ L_{\sigma_2,x_1}, L_{\sigma_1,x_4},\ L_{\sigma_2,x_4}. \tag{7.18}$$

We will call *half-skeleton* the set $\mathrm{Sk}_{1,2}(F)$ formed by the closures of the images of the 8 lines by $T_{1,2}$. The half-skeleton is the closure of the union

[59] It is possible that $T_{1,2}(F)$ is closed ...

of the images of the 4 special fibers of $\Pi_{2,3}$. The half-skeleton is made of 4 pairs of projective lines respectively contained in some planes $\{v = \beta\}$. Each line is defined by $\{u = \alpha, v = \beta\}$ or $\{v = \beta, u = \gamma\}$.

An important point is that it is possible to *describe explicitly this half-skeleton* using *only some q-local monodromy invariants*.

We use $u, v, u \quad \mathbf{C} \quad \{\infty\}$ as coordinates on $(\mathbf{P}^1(\mathbf{C}))^3$. We can describe the closed half-skeleton using some pairs of plane equations taken from two different lines of the following table:

$$\begin{array}{llll} \{u = 0\}, & \{u = \infty\}, & \{u = \mathbf{e}_q^{1;1,2;3}\}, & \{u = \mathbf{e}_q^{2;1,2;3}\} \\ \{v = 0\}, & \{v = \infty\}, & \{v = \mathbf{e}_q^{1;2,3;4}\}, & \{v = \mathbf{e}_q^{2;2,3;4}\} \\ \{u = 0\}, & \{u = \infty\}, & \{u = \mathbf{e}_q^{1;3,4;1}\}, & \{u = \mathbf{e}_q^{2;3,4;1}\} \end{array} \tag{7.19}$$

Proposition 7.13. — *The half-skeleton $Sk_{1,2}(F)$ is the union the* 8 *lines:*

$$\begin{gathered} \{u = v = 0\},\ \{v = u = 0\},\ \{u = v = \infty\},\ \{v = u = \infty\}, \\ \{u = \mathbf{e}_q^{1;1,2;3}, v = \mathbf{e}_q^{2;2,3;4}\},\ \{u = \mathbf{e}_q^{2;1,2;3}, v = \mathbf{e}_q^{1;2,3;4}\}, \\ \{v = \mathbf{e}_q^{2;2,3;4}, u = \mathbf{e}_q^{1;3,4;1}\},\ \{v = \mathbf{e}_q^{1;2,3;4}, u = \mathbf{e}_q^{2;3,4;1}\}. \end{gathered} \tag{7.20}$$

The index $2, 3$ and $1, 4$ do not appear symmetrically in the definition of $T_{1,2}$. It is more difficult to understand the images of the 8 lines:

$$L_{\rho_1,x_1},\ L_{\rho_2,x_1},\ L_{\rho_1,x_4},\ L_{\rho_2,x_4},\ L_{\sigma_1,x_2},\ L_{\sigma_2,x_2}, L_{\sigma_1,x_3},\ L_{\sigma_2,x_3}. \tag{7.21}$$

For example, we have $L_{\sigma_1,x_1} \quad \Pi_{1,2}^{-1}(0)$, therefore its image by $T_{1,2}$ is contained into the plane $\{u = 0\} = \{0\} \times (\mathbf{P}^1(\mathbf{C}))^2$. But we have only one exceptional fiber in the picture and therefore we know a priori *only one plane* into $(\mathbf{P}^1(\mathbf{C}))^3$ containing the images $T_{1,2}(L_{\sigma_1,x_1})$, ...

A first heuristic description of the image. It seems difficult to say more about the image $Y = T_{1,2}(F)$ and in particular about the skeleton $\mathrm{Sk}_{1,2}(F)$ without heavy computations involving the q-pants charts. We plan to return to the question in a future work. Here we will only *try some guesses*.

Conjecture 7.14. — *The lines of the half-skeleton are double curves.*

More precisely, in a neighborhood of a smooth point of the double curve, we have in local coordinates $Y = \{xy = 0\}$ (mild singularity). There could also exist pinch points (in a neighborhood of a pinch point, $Y = \{x^2 - yz^2 = 0\}$) and a finite number of more complicated singular points.

The image of the half-skeleton $\mathrm{Sk}_{1,2}(F)$ by the projection $(\mathbf{P}^1(\mathbf{C}))^3 \to (\mathbf{P}^1(\mathbf{C}))^2$ defined by $(u, v, u') \to (u, u')$ is a union of 8 lines:

$$\begin{array}{llll} \{u = 0\}, & \{u = \infty\}, & \{u = \mathbf{e}_q^{1;3,4;1}\}, & \{u = \mathbf{e}_q^{2;1,3;4}\} \\ \{u' = 0\}, & \{u' = \infty\}, & \{u' = \mathbf{e}_q^{2;3,4;1}\}, & \{u' = \mathbf{e}_q^{1;1,3;4}\} \end{array} \tag{7.22}$$

This image depends only on the four complex numbers $\mathbf{e}_q^{1;3,4;1}$, $\mathbf{e}_q^{2;1,3;4}$, $\mathbf{e}_q^{2;3,4;1}$ and $\mathbf{e}_q^{1;1,3;4}$. It is also clearly the image of the skeleton by the projection (the lines (7.21) are contained in special planes $\{u = \alpha\}$ or $\{u' = \beta\}$).

We recall the following definition.

Definition 7.15. — *Let V, W two complex algebraic varieties. A morphism $f : V \to W$ is a* branched covering *if the two dimensions are the same and if the typical fiber of f is of dimension* 0.

There is a Zariski dense open set $W' \subset W$ such that f is unramified above W' (a classical covering space). The complement of the largest possible W' is called the *branching locus*. If W' is connected then the cardinal of the fiber is constant, it is the *degree* of the branched covering.

Be careful, the classical notion of ramified covering is more restrictive: *all the fibers* are finite sets.

Conjecture 7.16. — *The map π_2 induced by the restriction to Y of the projection $(\mathbf{P}^1(\mathbf{C}))^3 \to (\mathbf{P}^1(\mathbf{C}))^2$ defined by $(u, v, u') \to (u, u')$ is a branched double covering of $(\mathbf{P}^1(\mathbf{C}))^2$. The branching set is the set of eight lines defined by* (7.22).

We will see later (cf. page 1243) that there exists a K3 surface (of Kummer type) X which is a double covering of $(\mathbf{P}^1(\mathbf{C}))^2$ branched along the same set of 8 lines. This suggest that X could be, up an isomorphism, a projective completion of F.

Another heuristic description of the image. There is another, more precise, heuristic description of the image. It is based on the following conjecture.

Conjecture 7.17. —

(i) *The surface $Y = \overline{T_{1,2}(F)}$ is a* $(2, 2, 2)$ *surface of* $(\mathbf{P}^1(\mathbf{C}))^3$.
(ii) *The restriction of $T_{1,2}$ to F minus the* 16 *lines is* injective.

We return to the 12 plane equations of (7.19). Each plane $\{u = \alpha\}$, resp. $\{v = \beta\}$, resp. $\{u' = \gamma\}$ cuts Y along a $(2, 2)$ curve.

When β moves there are four values such that the $(2,2)$ curve is decomposed into two lines (with a common point) and these lines are necessary *double* lines. The four values are: $\beta = 0,\ \infty\ ,\ \mathbf{e}_q^{1;2,3;4},\ \mathbf{e}_q^{2;2,3;4}$. Each pair of lines is picked up in the list formed by the line L_{ρ_1,x_2} and the lines (7.18). More precisely the union of the 4 pairs of lines is the half-skeleton.

We have a similar situation when α and γ move but we do not know *a priori* the four exceptional values of decomposition. When the $(2,2)$-curve is decomposed into two lines, it is decomposed into a double line of the closed half-skeleton and *another double line*. According to Proposition 7.11, the only possibilities for such a double line seem to be:

$$\{u = 0, u = \infty\},\ \ \{u = \infty, u = 0\},\ \ \{u = \mathbf{e}_q^{1;3,4;1}, u = \mathbf{e}_q^{1;1,3;4}\},$$
$$\{u = \mathbf{e}_q^{2;3,4;1}, u = \mathbf{e}_q^{2;1,3;4}\}, \quad (7.23)$$

We are thus led to the following conjecture.

Conjecture 7.18. —

(i) *The skeleton* $Sk_{1,2}(F)$ *is a union of* 12 *lines: the* 8 *lines of the half-skeleton* $Sk_{1,2}(F)$ *and the* 4 *lines* (7.23).
(ii) *The surface* Y *is mildly singular along each line of the skeleton.*

In order to put the set of the 16 lines into $(\mathbf{P}^1(\mathbf{C}))^3$, it is necessay to "fold it". The skeleton has two connected components, it can be described as a "split parallellepipedal structure". Each connected component is a deformed hexagonal structure.

Moreover we can conjecture that there are 4 pinch points on each line and that there exist 12 exceptional planes $\{u = \alpha\}$, $\{v = \beta\}$, $\{u = \gamma\}$, such that each one contains 4 pinch points. This configuration seems to be related to the logarithmic fibers.

We have a description of the fibrations of the surface by the coordinates. We detail it for the coordinate v. When β moves we have 3 types of fiber:

- the generic fiber is a $(2,2)$ curve with two nodes; it is decomposed into two $(1,1)$ curves;
- there are 4 fibers decomposed into two double lines;
- there are 4 fibers corresponding to the planes containing pinch points, they are double $(1,1)$ curves.

According to the above description, we can verify that the images of the skeleton by the projections $(u,v,u) \to (u,u)$, $(u,v,u) \to (u,v)$ and $(u,v,u) \to (v,u)$ are sets of 8 lines that we can explicit using only the q-local monodromy invariants.

Then it is easy to prove that the restriction of each projection to Y is a double covering of $(\mathbf{P}^1(\mathbf{C}))^2$ branched *exactly* along 8 lines, a degenerated $(4,4)$-curve (cf. Conjecture[60] 7.16). We consider for example the first projection, the ramification set is a $(4,4)$ curve of $(\mathbf{P}^1(\mathbf{C}))^2$ and it contains the 8 lines defined by (7.22). The union of these 8 lines is also a $(4,4)$ curve, therefore we have equality.

7.2.7. Conjectural embeddings of F into $(\mathbf{P}^1(\mathbf{C}))^6$ and into $(\mathbf{P}^1(\mathbf{C}))^4$

It seems difficult to get an *embedding* of F into $(\mathbf{P}^1(\mathbf{C}))^3$. We can try to do better[61] with maps which involve more symmetrically the $\Pi_{i,j}$.

Conjecture 7.19. — *We suppose that* (FR)*,* (NR)*,* (NS) *and* $\mathbf{Hyp}_{48}$ *are satisfied.*

(i) *The regular map:*

$$T := (\Pi_{1,2}, \Pi_{2,3}, \Pi_{3,4}, \Pi_{1,4}, \Pi_{1,3}, \Pi_{2,4}) : F \to (\mathbf{P}^1(\mathbf{C}))^6$$

defined by:

$$u := \Pi_{1,2}(\overline{M}), \quad v := \Pi_{2,3}(\overline{M}), \quad w := \Pi_{1,3}(\overline{M}),$$
$$u \ := \Pi_{3,4}(\overline{M}), \quad v \ := \Pi_{1,4}(\overline{M}), \quad w \ := \Pi_{2,4}(\overline{M})$$

is a regular embedding[62]*.*

(ii) *The regular map:*

$$T_{1,2} := (T_{1,2}, \Pi_{1,4}) = (\Pi_{1,2}, \Pi_{2,3}, \Pi_{3,4}, \Pi_{1,4}) : F \to (\mathbf{P}^1(\mathbf{C}))^4$$

defined by:

$$u := \Pi_{1,2}(\overline{M}), \quad v := \Pi_{2,3}(\overline{M}), \quad u \ := \Pi_{3,4}(\overline{M}), \quad v \ := \Pi_{1,4}(\overline{M})$$

is a regular embedding. Similarly the maps $(\Pi_{1,2}, \Pi_{1,3}, \Pi_{3,4}, \Pi_{2,4})$ *and* $\Pi_{2,3}, \Pi_{1,4}, \Pi_{1,3}, \Pi_{2,4}$ *are regular embeddings.*

We end with a conjectural picture.

Conjecture 7.20. — *Let* $(\underline{\rho}, \underline{\sigma}, \underline{x})$ *such that* (FR)*,* (NR)*,* (NS) *and* $\mathbf{Hyp}_{48}$ *are satisfied. We denote* $u, v, u\ , v\ \quad \mathbf{C} \quad \infty$ *coordinates on* $(\mathbf{P}^1(\mathbf{C}))^4$*. Then there exist three polynomials* $f_1,\ f_2,\ f_3$ *"on"* $(\mathbf{P}^1(\mathbf{C}))^3$*, of tri-degree* $(2,2,2)$*, such that:*

---

[60] Now this conjecture follows from Conjecture 7.18

[61] We already know that there exists a embedding of F in $(\mathbf{P}^1(\mathbf{C}))^4$, cf. Subsection 4.5.

[62] That is the image is an affine surface into $(\mathbf{P}^1(\mathbf{C}))^6$ and if we endow it with the induced Zariski topology, then it is isomorphic to F.

(i) *the* 3 *equations:*

$$f_1(u, v, u) = 0, \quad f_2(u, v, v) = 0, \quad f_3(u, v, v) = 0$$

define a smooth *surface* X *of* $(\mathbf{P}^1(\mathbf{C}))^4$*;*

(ii) *$F(\underline{\rho}, \underline{\sigma}, \underline{x})$ is isomorphic to a Zariski open subset of X.*

(iii) *We can choose the coefficients of the polynomials f_i as functions of $(\underline{\rho}, \underline{\sigma}, \underline{x})$ in such a way that they depend only on the q-local monodromy invariants, this dependance being rational.*

Each equation $f_i = 0$ defines a $(2, 2, 2)$ surface of $(\mathbf{P}^1(\mathbf{C}))^3$. As explained before, we think that this surface is *singular*. We will suggest below a method of computation of f_i (cf. page 1241).

7.3. K3 surfaces and conjectural description of F

This part is a stub and it contains mainly heuristics. However it could open some pathes towards a clear synthesis of all the rigorous (but complicated ...) informations that we got on the surface F. We plan to return to these questions in a future work.

7.3.1. Definitions and exemples

We recall the following definitions [28].

Definition 7.21. —

(i) *A complex smooth projective surface X is called* K3 surface *if X is simply connected with trivial canonical bundle $\omega_X \approx O_X$.*

(ii) *An Enriques surface is a quotient of a K3 surface X by a fixed point free involution ι (called an Enriques involution).*

There exists a symplectic 2-form on the K3 surface X (unique up to a multiplicative constant). We have[(63)] $H^1(X; O_X) = 0$, $H^1(X; \mathbf{Z}) = 0$ and the rank of $H^2(X; \mathbf{Z})$ is 22. We recall that the *Betti numbers* $b_r(X)$ of a surface X are the integers defined by $b_r(X) := \dim_{\mathbf{Q}} H^r(X; \mathbf{Q})$ (they are topological invariants). A surface X is K3 if and only if its canonical bundle is trivial and if $b_1(X) = 0$. If X is K3, then we have $\dim H^2(X; \mathbf{Z}) = b_2(X) = 22$.

(63) Another equivalent definition of a K3 surface, due to André Weil, around 1948, is $\omega_X \approx O_X$ and $H^1(X; O_X) = 0$.

Enriques surfaces and K3 surface have a null Kodaira dimension. They are not rational surfaces.

We recall the adjunction formula.

Proposition 7.22 (Adjunction formula). — *If $Y \subset X$ is an hypersurface, with X and Y smooth, then we have the two equivalent equalities:*

$$K_Y = (K_X + Y)_{|Y} \quad \text{and} \quad \omega_Y = (\mathcal{O}(Y) \otimes \omega_X)_{|Y} \tag{7.24}$$

We recall the following result.

Proposition 7.23. — *Let L be a smooth curve of genus g contained in a K3 surface, then $(L, L) = 2g - 2$; L is rational if and only if $(L, L) = -2$.*

Proof. — This result follows from the adjunction formula: $\omega_L = \mathcal{O}_L(L)$, therefore $2g - 2 = (L, L)$.

One can prove that if L is a (-2) curve in a K3 surface, then L is smooth.

It follows from the above proposition that there does not exist (-1) smooth curves on a K3 surface.

Example 7.24. — Every smooth quartic surface in $\mathbf{P}^3(\mathbf{C})$ is a K3 surface. Let $m \in \mathbf{N}^*$. We have $\omega_{\mathbf{P}^m(\mathbf{C})} = \mathcal{O}_{\mathbf{P}^m(\mathbf{C})}(-m-1)$. Let $X \subset \mathbf{P}^m(\mathbf{C})$ be a smooth hypersurface defined by an homogeneous polynomial of degree d. By the adjunction formula:

$$\omega_X = \left(\omega_{\mathbf{P}^m(\mathbf{C})} \otimes \mathcal{O}_{\mathbf{P}^m(\mathbf{C})}(d)\right)_{|X} = \mathcal{O}_X(-m-1+d).$$

Therefore, if $m = 3$ and $d = 4$, then $\omega_X = \mathcal{O}_X$. The projective space $\mathbf{P}^3(\mathbf{C})$ is simply connected, therefore, by Lefschetz theorem, X is also simply connected. We can also prove that $H^1(X; \mathcal{O}_X) = 0$ using the short exact sequence:

$$0 \to \mathcal{O}_{\mathbf{P}^3(\mathbf{C})}(-4) \to \mathcal{O}_{\mathbf{P}^3(\mathbf{C})} \to \mathcal{O}_X \to 0$$

and $H^1(\mathbf{P}^3(\mathbf{C}); \mathcal{O}_{\mathbf{P}^3(\mathbf{C})}) = H^2(\mathbf{P}^3(\mathbf{C}); \mathcal{O}_{\mathbf{P}^3(\mathbf{C})}(-4)) = 0$. An interesting example of a smooth quartic hypersurface in $\mathbf{P}^3(\mathbf{C})$ is the Fermat quartic: $X^4 + Y^4 + Z^4 + T^4 = 0$

Example 7.25 (Double plane). — For this example cf. [29]. Consider a double covering $\pi : X \to P^2(\mathbf{C})$ branched along a sextic curve $C \subset P^2(\mathbf{C})$. Then $\pi_*(\mathcal{O}_X) \approx \mathcal{O}_{P^2(\mathbf{C})} \oplus \mathcal{O}(-3)$ and therefore $H^1(X; \mathcal{O}_X) = 0$. We suppose that the branching curve C is non-singular, then X is non singular and the canonical bundle formula for branched coverings shows that $\omega_X = \pi^*(\omega_{P^2(\mathbf{C})} \otimes \mathcal{O}(3)) \approx \mathcal{O}_X$. Therefore X is a K3 surface, called a *double plane*. If the sextic C is the union of 6 generic lines in $P^2(\mathbf{C})$, the double cover X has 15 rational double points. These 15 points correspond to the pairwise intersections of the 6 lines. Blowing-up these 15 singular points produces a K3 surface.

Proposition 7.26. — *The smooth* $(2,2,2)$ *surfaces are the K3 surfaces embedded in* $(\mathbf{P}^1(\mathbf{C}))^3$.

Proof. — Let $X \quad (\mathbf{P}^1(\mathbf{C}))^3$ be a smooth irreducible hypersurface of tri-degree (a,b,c). Using the adjunction formula, we get $K_X = O_X(a-2, b-2, c-2)$ and K_X is trivial if and only if $(a,b,c) = (2,2,2)$. We suppose $(a,b,c) = (2,2,2)$. The fiber bundle $[X]$ is positive. Using the Lefschetz theorem on hyperplane sections we get an isomorphism $H^1((\mathbf{P}^1(\mathbf{C}))^3;\mathbf{Q}) \to H^1(X;\mathbf{Q})$ induced by the canonical injection $X \to (\mathbf{P}^1(\mathbf{C}))^3$. As $H^1((\mathbf{P}^1(\mathbf{C}))^3;\mathbf{Q}) = 0$, we have also $H^1(X;\mathbf{Q}) = 0$ and $b_1(X) = 0$.

7.3.2. The Enriques surface and some surfaces in the same style

The Enriques surface. Around 1895, after many discussions with G. Castelnuovo under the arcades of the city of Bologna, F. Enriques discovered a very interesting surface [12, 17, 18]. We quote [18]:

Nel 1896 mi si è presentata la superficie del 6° *ordine passante doppiamente per gli spigoli di un tetraedro come primo esempio di superficie di genere* $p_g = p_a = 0$*, non razionale.*

The example of Enriques [15] is a smooth normalization of a non-normal sextic surface Y in $\mathbf{P}^3(\mathbf{C})$ that passes with multiplicity 2 through the edges of the coordinate tetrahedron. Its equation (in projective coordinates) is:

$$F := x_1^2x_2^2x_3^2 + x_0^2x_2^2x_3^2 + x_0^2x_1^2x_3^2 + x_0^2x_1^2x_2^2 + x_0x_1x_2x_3\, q(x_0,x_1,x_2,x_3) = 0, \quad (7.25)$$

where q is a non-degenerate quadratic form.

The surface Y has the following singularities: a double curve Γ with ordinary triple points which are also triple points of the surface and some pinch points (4 on each edge of the tetrahedron).

We choose an edge Δ of the tetrahedron. The family of planes passing by Δ cut the surface along a sextic curve. This sextic curve is decomposed into the double line Δ and a quartic curve. There are the following exceptional cases:

1. the plane is a face of the tetrahedron, then the quartic is decomposed into two double lines (there are two such cases);
2. the quartic pass by a pinch point on Δ, then there appears a cusp (there are four such cases).

The quadric curves form a pencil. A base locus of this pencil is the union of 4 edges, excluding the union of 2 opposite edges.

It is interesting to compare the above fibrations with the following fibrations:

- the fibration of the cubic surface $\overline{S_{VI}(a)}$ described in paragraph *A fibration*, page 1217;
- the fibration of the image $Y = T_{1,2}(F)$ of F into $(\mathbf{P}^1(\mathbf{C}))^3$ by the planes $\{v = \text{cste}\}$;
- the fibration of F by each Π_{ij}.

More generally we can consider *all* the sextic surfaces Z in $\mathbf{P}^3(\mathbf{C})$ mildly singular along the 6 edges of the tetrahedron $x_0x_1x_2x_3 = 0$ (cf. [50]). An equation of such a surface is:

$$F_{a,b,c,d;q} := ax_1^2x_2^2x_3^2 + bx_0^2x_2^2x_3^2 + cx_0^2x_1^2x_3^2 \\ + x_0^2x_1^2x_2^2 + x_0x_1x_2x_3\, q(x_0, x_1, x_2, x_3) = 0, \quad (7.26)$$

where q is a non-degenerate quadratic form. There are four sextic monomials and the 10 quadratic monomials of q. Quotienting by an action of $(\mathbf{C}^{*})^4$, we get a 10 parameters family.

Let S be the normalization of Z. We have the following result (cf. [50, Prop. 4.1, p. 5]).

Proposition 7.27. — *The surface S is an Enriques and the covering K3 surface X is a $(2,2,2)$ surface in $(\mathbf{P}^1(\mathbf{C}))^3$ which is invariant by the involution $(u, v, w) \to (-u, -v, -w)$.*

More precisely an equation of the surface X is:

$$au^2v^2w^2 + bu^2 = cv^2 + dw^2 + uvw\, q(1, vw, uw, uv).$$

Some $(2,2,2)$-surfaces in Enriques style. We consider a generalized version Sk of the half-skeleton, the set of 8 lines in $\left(\mathbf{P}^1(\mathbf{C})\right)^3$:

$$\begin{aligned} &\{u = \alpha_1, v = \beta_1\}, \quad \{v = \beta_1, u = \gamma_1\}, \quad \{u = \alpha_2, v = \beta_2\}, \\ &\{v = \beta_2, u = \gamma_2\}, \quad \{u = \alpha_3, v = \beta_4\}, \quad \{v = \beta_4, u = \gamma_3\}, \\ &\{u = \alpha_4, v = \beta_3\}, \quad \{v = \beta_3, u = \gamma_4\}. \end{aligned} \quad (7.27)$$

parameterized by the 12 “numbers” $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$: each of $\underline{\alpha}$ (resp. $\underline{\beta}$, resp. $\underline{\gamma}$) is an arbitrary triple of *distinct* elements of $\mathbf{P}^1(\mathbf{C})$. Up to Möbius transformations on each factor of $\left(\mathbf{P}^1(\mathbf{C})\right)^3$ it is sufficient to consider the case: $\underline{\alpha} := (\alpha_1, 1, 0, \infty)$, $\underline{\beta} := (\beta_1, 1, 0, \infty)$, $\underline{\gamma} := (\gamma_1, 1, 0, \infty)$. Then it remain only 3 parameters. We will write $\text{Sk}_{\underline{\alpha},\, \underline{\beta},\, \underline{\gamma}}$ if we want to precise the parameters.

The half-skeleton corresponds to

$$\begin{aligned}\underline{\alpha} &= (0, \infty, \mathbf{e}_q^{1;3,4;1}, \mathbf{e}_q^{2;1,3;4}),\\ \underline{\beta} &= (0, \infty, \mathbf{e}_q^{1;2,3;4}, \mathbf{e}_q^{2;2,3;4}),\\ \underline{\gamma} &= (0, \infty, \mathbf{e}_q^{2;3,4;1}, \mathbf{e}_q^{1;1,3;4}).\end{aligned} \tag{7.28}$$

The projection of $\mathrm{Sk}_{\underline{\alpha},\ \underline{\beta},\ \underline{\gamma}}$ on the $(u, u\ )$-plane is the following set of 8 coordinates lines:

$$\begin{array}{llll}\{u = \alpha_1\}, & \{u = \alpha_2\}, & \{u = \alpha_3\}, & \{u = \alpha_4\}\\ \{u\ \ = \gamma_1\}, & \{u\ \ = \gamma_2\}, & \{u\ \ = \gamma_3\}, & \{u\ \ = \gamma_4\}.\end{array} \tag{7.29}$$

There is exactly one line of $\mathrm{Sk}_{\underline{\alpha},\underline{\beta},\underline{\gamma}}$ above each line of (7.29).

We can also generalise the skeleton: it is an union of 12 lines denoted $\mathrm{Sk}_{\underline{\alpha},\underline{\beta},\underline{\gamma}}$. It is easy to write the lines equations using $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ (cf. (7.23)).

Enriques considered the family of sextic surfaces of $\mathbf{P}^3(\mathbf{C})$ mildly singular along the six edges of the tetrahedron $xyzt = 0$. Similarly we will consider the (possibly empty) family $\{S_{\underline{\alpha},\underline{\beta},\underline{\gamma}}\}_{\underline{\alpha},\underline{\beta},\underline{\gamma}}$ of $(2,2,2)$ surfaces of $(\mathbf{P}^1(\mathbf{C}))^3$ mildly singular along the 12 lines of $\mathrm{Sk}_{\underline{\alpha},\underline{\beta},\underline{\gamma}}$.

Let $h_{\underline{\alpha},\underline{\beta},\underline{\gamma}}$ be a polynomial of tri-degree $(2,2,2)$ such that $S_{\underline{\alpha},\underline{\beta},\underline{\gamma}} = V(h_{\underline{\alpha},\underline{\beta},\underline{\gamma}})$.

We can write some *necessary conditions* on $h_{\underline{\alpha},\underline{\beta},\underline{\gamma}}$.

$$\begin{aligned}h_{\underline{\alpha},\underline{\beta},\underline{\gamma}} &= a_1(u-\alpha_1)^2(u\ \ -\gamma_1)^2 + (v-\beta_1)^2(\cdots)\\ &= a_2(u-\alpha_2)^2(u\ \ -\gamma_2)^2 + (v-\beta_2)^2(\cdots)\\ &= a_3(u-\alpha_3)^2(u\ \ -\gamma_3)^2 + (v-\beta_3)^2(\cdots),\end{aligned} \tag{7.30}$$

where each $(\cdots)$ is a $(2,2)$ polynomial in $(u, u\ )$ (6 monomials). There are similar conditions replacing $(u, u\ )$ by (u, v) or $(v, u\ )$.

Using the above conditions we can write a linear system where the unknown are the coe□cients of $h_{\underline{\alpha},\underline{\beta},\underline{\gamma}}$. Solving this system one will obtain the family[64] $\{HS_{\underline{\alpha},\underline{\beta},\underline{\gamma}}\}$. We can use Möbius transformations in order to simplify the system.

Using resultants, we can write another linear system. We denote $R_t(P,Q)$ the resultant of two polynomials in t. We write for simplicity $h = h_{\underline{\alpha},\underline{\beta},\underline{\gamma},\underline{\alpha}\ ,\underline{\gamma}}$

[64] It could be empty for generic values of $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$. For the values associated to the q-monodromy invariants, we can conjecture that there exists an unique solution up to scaling.

Then we can consider the three resultants:

$$R_2 := R_v\left(h, \frac{\partial h}{\partial v}\right), \quad R_1 := R_u\left(h, \frac{\partial h}{\partial u}\right) = 0, \quad R_3 := R_u\left(h, \frac{\partial h}{\partial u}\right) = 0$$

and write that they vanish respectively on three systems of 8 lines. We get a linear system: the unknown are the coefficients of h. A solution of the first system is clearly a solution of the second. We do not know if they are equivalent.

We return to Conjecture 7.20. If Conjecture 7.20 is true, then Y is a $(2,2,2)$-surface mildly singular along the 12 lines of the skeleton and one can use the above method to get an equation of Y into $(\mathbf{P}^1(\mathbf{C}))^3$, or equivalently an algebraic relation between $\Pi_{1,2}$, $\Pi_{2,3}$, and $\Pi_{3,4}$.

The surface F and the Kummer surfaces. The $(2,2,2)$ surface Y is a double covering of $(\mathbf{P}^1(\mathbf{C}))^2$ *branched* along the 8 lines. We can conjecture that there exists a smooth projective completion X of F which is also a double covering of $(\mathbf{P}^1(\mathbf{C}))^2$ *branched* along the 8 lines.

We will explain how to *compute* a double covering of $(\mathbf{P}^1(\mathbf{C}))^2$ ramified along the 8 lines. It is a K3 surface, more precisely a K3 surface of Kummer type. It is a good candidate for a projective completion of F (up to an isomorphism).

For simplicity we denote:

$$\begin{array}{llll} \{u = \alpha_1\}, & \{u = \alpha_2\}, & \{u = \alpha_3\}, & \{u = \alpha_4\} \\ \{u' = \gamma_1\}, & \{u' = \gamma_2\}, & \{u' = \gamma_3\}, & \{u' = \gamma_4\} \end{array} \tag{7.31}$$

the equations of the 8 lines.

Let $p : Z \to (\mathbf{P}^1(\mathbf{C}))^2$ be a double *ramified* covering, ramified on the 8 lines. The 16 double points of the ramification locus are (α_i, γ_j). We consider double ramified coverings $A \to \mathbf{P}^1(\mathbf{C})$ and $B \to \mathbf{P}^1(\mathbf{C})$ respectively ramified above $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, A and B being *elliptic curves*. Then Z is isomorphic to the quotient of $A \times B$ by the canonical involution $x \to -x$. It is a *Kummer surface*. It is isomorphic to a nodal quartic surface into $\mathbf{P}^3(\mathbf{C})$ with 16 nodal points. Blowing up at the 16 nodes, we get a K3 surface X'. We conjecture that X' is isomorphic to a projective completion of F.

The K3 surface X' is *uniquely determined*, up to an isomorphism, by the two cross-ratios $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and a fortiori by the 4 complex numbers:

$$\mathbf{e}_q^{1;3,4;1}(\underline{\rho}, \underline{\sigma}, \underline{x}), \quad \mathbf{e}_q^{2;1,3;4}(\underline{\rho}, \underline{\sigma}, \underline{x}), \quad \mathbf{e}_q^{2;3,4;1}(\underline{\rho}, \underline{\sigma}, \underline{x}) \quad \text{and} \quad \mathbf{e}_q^{1;1,3;4}(\underline{\rho}, \underline{\sigma}, \underline{x}).$$

This model could be a q-analog of the “algebraic dependence” of the smooth projective cubic surface $\tilde{S}_{A_0,A_t,A_1,A_\infty}$ on $(A_0, A_t, A_1, A_\infty) \in \mathbf{C}^4$.

8. Conclusion: open questions and perspectives

We have nearly achieved our initial aim. We built the character variety of q-PVI and gave a quite precise description of this variety. There remain some open problems. More generally there is a lot of related questions and possible generalizations. We will give a (non exhaustive[(65)] ...) list.

8.1. Generalized versions of Riemann–Hilbert map

In the differential case the Riemann–Hilbert map is a complex analytic morphism $RH : M \to \tilde{Rep}$ from a moduli space M of connections to a (categorical) moduli space of (generalized) monodromy data. More precisely from a *family* of moduli space of connections to a *family* of moduli space of monodromy data. In the fuchsian case (i. e. PVI) the parameters on the left hand side are (t, θ) and (t, a) on the right hand side ($a_l = 2\cos 2\pi\theta_l$).

In the irregular case it is necessary to add some generalized exponents into the parameters and Stokes multipliers into the monodromy data [47, 59].

Painlevé equations are derived from holomorphic flows on M. The flows are transversal to the parameter fibration. The fibers are the Okamoto spaces of initial conditions.

The above picture works perfectly in the PVI case for *generic values* of the parameters and we have generalized it to the q-PVI case for *fixed generic values of the parameters*. In the differential PVI case if one wants to allow the exceptional values of the parameters, then it is necessary to replace connections by *parabolic connections*[(66)] [30, 31]. There are possible generalizations of our work (or of part of our work).

- The case of a fixed exceptional parameter. It will be necessary to use the *parabolic q-difference modules* of Mochizuki [49].
- The case of q-PVI *with parameters*. We remark that even the simpler hypergeometric case with parameters is not known.
- The case of the equations of the Murata’s list with or without parameters.

(65) In particular we will not discuss the important problems of symplectic structures.
(66) Intuitively one “adds a line”.

We conjecture that it is possible to extend *part of our results* to all the equations of Murata's list (taking account of q-Stokes phenomena). In sharp contrast, there is no hope to extend the Mano decompositions for *all* the equations.

In [37] Jimbo give a splitting of PIII and PV respectively into:

- two confluent hypergeometric equations,
- an hypergeometric equation and a confluent hypergeometric equation.

For a more detailed description of splittings of Painlevé equations, cf. [24, Figure 3: CMR confluence diagram for Painlevé equations].

It is possible to extend Mano result for PVI=P(A_3) to the equations P(A_4), P(A_5) , P(A_6) . One gets respective splittings into(67) :

- a q-Kummer and a Heine q-hypergeometric equation,
- two q-Kummer equations,
- a q-Kummer and a Hahn Exton q-Bessel equation.

We conjecture that it is possible to extend our Mano decomposition for these three cases.

8.2. Relations with q-difference Galois groups

As we said above, defining local monodromies and local Galois groups at intermediate singularities for q-di□erence equations is one of the most important open problems in modern q-di□erence theory. In some sense it would close the problem of "localisation" of Galois groups: the problem of "localisation" of the Galois groups a 0 and ∞ (i.e. the description of the corresponding q-wild groups) was solved in full generality in a series of papers of the two last authors [62, 71].

This will require a more general version of Mano decomposition. Extension to higher degrees should be easy along the same lines, but extension to higher orders (polynomial matrices with coe□cients in $\mathrm{Mat}_n(\mathbf{C})$) seems more di□cult. Moreover the "basic bricks" are not clear.

(67) cf. [52] for basics on irregular q-hypergeometric equations.

8.3. Exceptional lines and points on q-character varieties and exceptional solutions of q-Painlevé equations

In the differential case there is a fundamental heuristic principle: there is a dictionary between the asymptotics of a solution of a Painlevé equation at the singular points and some "natural coordinates" of the corresponding point on the character variety.

This principle is illustrated for PVI by Jimbo formula [37], [6, Appendix B]. For the others Painlevé equations there is a lot of precise results in this direction into the book [20].

In [44], M. Klimes, E. Paul and the second author propose another principle in the same direction: the lines on the character variety (an affine cubic surface) correspond to one parameter families of "special" solutions of the Painlevé equation, an intersection of two such lines corresponds to a "very special" solution. A good illustration is PII: there are 9 lines, they correspond to Boutroux tronquées solutions. The intersections of two lines correspond to tritronquées solutions or to bitronquées solutions(68) .

It could be interesting to look at q-analogs, in particular about the 16 exceptional lines that we exhibited on F.

8.4. q-deformations of CFT

During the last years appeared some papers about possible q-deformations of Conformal Field Theory (CFT). In this context ordinary differential equations are replaced by q-difference equations. We think that q-characters varieties, Mano-decompositions and q-pants parametrizations could be useful (cf. in particular [40]). The irregular case (q-Stokes phenomena and q-sommations) seems interesting in such approaches [1].

We quote [24, p. 9].

Perhaps the most intriguing perspective is to extend our setup to q-isomonodromy problems, in particular q-di erence Painlevé equations, presumably related to the deformed Virasoro algebra [78] *and 5D gauge theories. Among the results pointing in this direction, let us mention a study of the connection problem for q-Painlevé VI* [46] *based on asymptotic factorization of the associated linear problem into two systems solved by the Heine basic hypergeometric series ${}_2\phi_1$, and critical expansions for solutions of q-P(A_1) equation recently obtained in* [42].

(68) Hahn–Exton style solutions.

We quote [79].

Localization techniques for supersymmetric quantum field theories allow one to produce non-perturbative results such as computing partition functions exactly, in stark contrast to general field theories. In many two-dimensional examples of supersymmetric theories, the path integral or partition function is related to geometric invariants and appears as a solution to certain differential equations with geometric and physical interpretation. Recently a program has been initiated to lift these constructions from two- to three-dimensional theories. Beem, Dimofte and Pasquetti argued that the natural 3D analogue of the differential equations whose solutions determine the partition function in two-dimensions are q-difference equations, . . .

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