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YANN BRENIER

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Geometric diffusions of 1-currents

YANN BRENIER ⁽¹⁾

ABSTRACT. — We get diffusion equations of geometric nature for 1-currents through two different approaches. Partial existence and uniqueness results are discussed.

RÉSUMÉ. — Nous obtenons, par deux approches différentes, des équations de diffusion de nature géométrique pour les 1-courants. Nous discutons quelques résultats d'existence et d'unicité.

Introduction

Our main goal is to construct geometric diffusion equations for 1-currents, or, in equivalent terms, for divergence-free vector fields. Loosely speaking, a 1-current B can be thought (at least approximately) as a superposition of loops

$$(s \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, a \in \mathcal{A}) \rightarrow X(s, a) \in \mathbb{R}^d$$

labelled by a and defined by duality on test functions ϕ as follows

$$\langle B, \phi \rangle = \int_{\mathcal{A}} \int_{\mathbb{T}} \phi(X(s, a)) \partial_s X(s, a) \, ds \, d\lambda(a)$$

where (\mathcal{A}, λ) is a suitable probability space for the label a . We immediately see that such a current can also be interpreted as a divergence-free vector field on \mathbb{R}^d : $x \in \mathbb{R}^d \rightarrow B(x) = (B^i(x))_{i=1, \dots, d} \in \mathbb{R}^d$. Indeed, in the sense of distribution and written in coordinates with implicit summation on repeated indices:

$$\begin{aligned} \langle \partial_i B^i, \phi \rangle &= -\langle B^i, \partial_i \phi \rangle = - \int_{\mathcal{A}} \int_{\mathbb{T}} \partial_i \phi(X(s, a)) \partial_s X^i(s, a) \, ds \, d\lambda(a) \\ &= - \int_{\mathcal{A}} \left\{ \int_{\mathbb{T}} \frac{d}{ds} [\phi(X(s))] \, ds \right\} d\lambda(a) = 0. \end{aligned}$$

⁽¹⁾ CNRS UMR 7640, Ecole Polytechnique, Palaiseau, France

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Conversely, at least approximately and locally, every divergence-free vector field can be written as a superposition of loops. (A precise statement can be found in a well-known paper by Stanislav Smirnov [12].)

By geometric diffusion, we mean, loosely speaking, a diffusion process which is compatible with the loop structure of 1-currents. To get such a diffusion equation, we will rely on two approaches:

- (1) a “bottom-up” strategy in which the diffusion equation is just an emergent system based on approximations of loops by point particles and a suitable concept of diffusion for point particles;
- (2) a “top-down” strategy in which the diffusion equation is directly obtained, by a simple quadratic time rescaling, from a very pure geometric equation designed in the 30s by physicists Max Born and Leopold Infeld.

Through these two approaches we do not exactly get the same equations. However, at least in the case of the three dimensional Euclidean space ($d = 3$), they are both of the following form (written in coordinates with implicit summation on repeated indices)

$$\partial_t \rho + \partial_i (\rho v^i) = 0, \quad \rho v^i = \partial_k (\eta B^i B^k) - \partial^i p, \quad (0.1)$$

$$\partial_t B^i + \partial_j (B^i v^j - B^j v^i + \mu \partial^i (\nu B^j) - \mu \partial^j (\nu B^i)) = 0. \quad (0.2)$$

Here $(\rho, p, v, B) \in \mathbb{R}^{1+1+3+3}$ can be interpreted as the density, pressure, velocity and magnetic fields of some electrically charged fluid, with “constitutive laws” giving (μ, ν, η, p) as functions of ρ . The current itself is encoded in the vector field B and its scalar companion ρ . We can rewrite these equations in a more compact form as

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \rho v = \nabla \cdot (\eta B \otimes B) - \nabla p, \quad (0.3)$$

$$\partial_t B + \nabla \times (B \times v) + \nabla \times (\mu \nabla \times (\nu B)) = 0. \quad (0.4)$$

In terms of “constitutive laws”, the first, “bottom-up”, approach corresponds to the choice $\mu = 0$, $\eta = \rho^{-1}$, $p = \rho$ (ν being irrelevant since $\mu = 0$), while the second, “top-down”, approach rather corresponds to $\mu = \nu = \eta = \rho^{-1} = -p$ (which involves the “Chaplygin pressure law”, sometimes used in Cosmology, with sound speed $(\frac{dp}{d\rho})^{1/2} = \rho^{-1}$).

The most striking difference is the absence of the “diffusion term” $\nabla \times (\mu \nabla \times (\nu B))$ in the first approach. This is why we keep a slight preference for the “top-down” approach which indeed provides a richer model with an additional term. However, we are able to work out the “top-down” approach only in the case of the three dimensional Euclidean space. Moreover, the “bottom-up” approach may present the advantage that the topology of the loops is, at least formally, preserved during the time evolution, which is

not the case of the “top-down” equation, precisely because of the additional term we have just discussed. Of course, by mixing the two approaches it is possible, although not very elegant, to get more general “constitutive laws” for (μ, ν, η, p) as given functions of p .

Finally, we will provide some elements of existence and uniqueness theory not exactly for the diffusion equations we have obtained but rather for their “incompressible” versions when the density field ρ is forced to be uniform (and, as consequence, p becomes a Lagrange multiplier of the incompressibility constraint).

1. A first approach based on diffusions of point particles

In this section we derive a diffusion equations for 1-currents in the following way.

- (1) We first approximate the usual linear heat equation in \mathbb{R}^d by an N -body first order dynamical system, involving a mollification of the delta distribution with length ε . The convergence of such an approximation as $1/N$ and ε go to zero in an appropriate order (typically $1/N \ll \varepsilon^d$) is well established in numerical analysis (see [4] and, also [7] for *nonlinear* diffusion equations). Keeping ε fixed and letting N go to $+\infty$ leads to an integro-differential equation.
- (2) We use the previous integro-differential equation with fixed mollification length ε in the case of a loop parameterized by abscissa $s \in \mathbb{T}$. This way we describe the diffusion of the loop in the “target space” \mathbb{R}^d . Crucially, we add an extra diffusion process with respect to the abscissa s (i.e. in the “source space” $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) and get a new integro-differential equation for the loop.
- (3) We write the integro-differential equations in “Eulerian coordinates”, as we would do for a fluid, which requires two fields v and B , playing the role of velocity and magnetic fields, as in Magnetohydrodynamics.
- (4) We formally let the mollification length ε go to zero and obtain the desired equation that describes the diffusion of loops in both source and target spaces.

1.1. Step 1: a mollified heath equation for particles

The heat equation for a smooth positive density field

$$(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \rho(t, x) > 0$$

can be seen as a transport equation with velocity field $v = v(t, x) = -\nabla \log \rho(t, x)$. Indeed,

$$(\partial_t - \Delta)\rho = \partial_t \rho + \nabla \cdot (\rho v), \quad v = -\nabla \log \rho.$$

This system can be “mollified” as follows. We first mollify ρ

$$\rho_\varepsilon(t, x) = \int_{\mathbb{R}^d} \rho(t, x + \varepsilon \xi) \exp(-\pi |\xi|^2) d\xi \tag{1.1}$$

and, then, transport ρ by $v_\varepsilon = -\nabla \log \rho_\varepsilon$:

$$\partial_t \rho + \nabla \cdot (\rho v_\varepsilon) = 0.$$

In sharp contrast with the regular heat equation, this approximate heat equation is a transport equation with a velocity field $v_\varepsilon = -\nabla \log \rho_\varepsilon$ which is uniformly smooth down to time $t = 0$, even when the initial ρ is not smooth or not strictly positive, such as, for instance, a finite sum of Dirac masses. In particular, the “mollified heat equation” admits *exact* solutions made of Dirac masses

$$\rho(t, x) = \frac{1}{N} \sum_{k=1}^N \delta(x - X_k(t)), \quad \frac{dX_k}{dt}(t) = -(\nabla \log \rho_\varepsilon)(t, X_k(t)),$$

for which it is possible to get a self-consistent set of ODEs. Indeed, we get for each $x = X_k(t)$,

$$\frac{dX_i}{dt}(t) = \frac{2\pi \sum_{k=1}^N (X_i(t) - X_k(t)) g_{ik}(t)}{\varepsilon^2 \sum_{k=1}^N g_{ik}(t)}, \quad i = 1, \dots, N \tag{1.2}$$

where

$$g_{ik}(t) = \exp\left(-\pi \frac{|X_i(t) - X_k(t)|^2}{\varepsilon^2}\right). \tag{1.3}$$

Thus, this “N-body” ordinary differential system describes, at least approximately, the diffusion of point particles according to the linear heat equation as $1/N$ and ε go to zero, according to an appropriate ordering (typically $1/N \ll \varepsilon^d$). (See [4] and also [7] for non linear diffusions.) Observe that these particles do not follow any Brownian motion. On the contrary their trajectories are smooth and fully deterministic.

1.2. Step 2: ε -diffusion of a single loop

Generalizing the case of point particles to a single loop $s \in \mathbb{R}/\mathbb{Z} \rightarrow x = X(s) \in \mathbb{R}^d$, we suggest the equation

$$\partial_t X(t, s) - \partial_{ss}^2 X(t, s) = \frac{2\pi \int (X(t, s) - X(t, s'))g(t, s, s') ds'}{\varepsilon^2 \int g(t, s, s') ds'} \quad (1.4)$$

$$g(t, s, s') = \exp\left(-\pi \frac{|X(t, s) - X(t, s')|^2}{\varepsilon^2}\right) \quad (1.5)$$

as a *double* diffusion process, first in the “source” space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, treated in a standard way, with the usual one-dimensional heat operator $\partial_t - \partial_{ss}^2$ on \mathbb{T} , second in the “target” space \mathbb{R}^d , at least approximately, with the integro-differential equation (1.2), (1.3) we used for point particles in the previous subsection.

Remark 1.1 (superposition of loops). — We may define a similar integro-differential equation not only for a single loop but also for a finite superposition of loops, which is good enough to approximate generic 1-currents. But we may also approximate such a 1-current by a (presumably widely convoluted) single loop, just by concatenation of the former loops. Thus, at this stage, there is no need to consider more than a single loop.

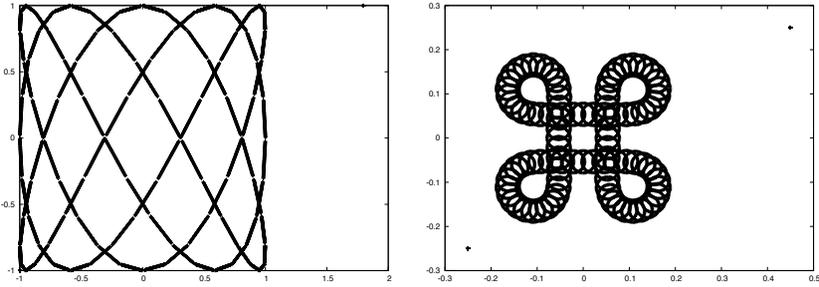
Remark 1.2 (a fully discrete scheme). — A simple-minded fully discrete explicit scheme just reads

$$X_i^{n+1} = \frac{X_{i+1}^n + X_{i-1}^n}{2} + \frac{2\pi\delta t \sum_{k=1}^N (X_i^n - X_k^n)g_{ik}^n}{\varepsilon^2 \sum_{k=1}^N g_{ik}^n}$$

(for $i = 1, \dots, N$ with $N + 1 \sim 1$ by periodicity)

$$g_{ik}^n = \exp\left(-\pi \frac{|X_i^n - X_k^n|^2}{\varepsilon^2}\right)$$

where δt is the time step chosen so that $2\delta t N^2 = 1$. Let us show the result of a numerical simulation when the initial loop is discretized with $N = 100$ particles, as shown on the first figure. On the second picture, we plot the location at time 1 of the ε -blobs surrounding the hundred particles generated by the scheme. Notice that this simulation is just an illustration. Indeed, the parameters N and ε are very far from the right regime to get the diffusion equation we want to design.



1.3. Step 3: Eulerian formulation

We define the eulerian “density” and “magnetic” fields

$$\rho(t, x) = \int \delta(x - X(t, s)) ds, \quad B(t, x) = \int \partial_s X(t, s) \delta(x - X(t, s)) ds. \quad (1.6)$$

Let us assume X to be a smooth function of both t and s and let us postulate the existence of a smooth “velocity field” $v = v(t, x) \in \mathbb{R}^d$ that transports the loop, so that

$$\partial_t X(t, s) = v(t, X(t, s)) \quad (1.7)$$

(which is reasonable as long as there is no self-crossing of the loop, i.e. as $X(t, s) = X(t, \tilde{s})$ implies $s = \tilde{s}$). Then, we rigorously get, in the sense of distributions,

$$\partial_t B + \nabla \cdot (B \otimes v - v \otimes B) = 0. \quad (1.8)$$

Indeed, using implicit summations on repeated indices,

$$\begin{aligned} \langle \partial_t B^i, \phi_i \rangle &= - \int \partial_t \phi_i(t, X(t, s)) \partial_s X^i(t, s) dt ds \\ &= - \int \partial_t [\phi_i(t, X(t, s))] \partial_s X^i(t, s) dt ds \\ &\quad + \int \partial_k \phi_i(t, X(t, s)) \partial_t X^k(t, s) \partial_s X^i(t, s) dt ds \\ &= - \int \partial_s [\phi_i(t, X(t, s))] \partial_t X^i(t, s) dt ds \\ &\quad + \int \partial_k \phi_i(t, X(t, s)) \partial_t X^k(t, s) \partial_s X^i(t, s) dt ds \end{aligned}$$

(after a double integration by part, both in t and s)

$$\begin{aligned}
 &= \int \partial_k \phi_i(t, X(t, s)) \{ -\partial_s X^k(t, s) \partial_t X^i(t, s) \\
 &\qquad\qquad\qquad + \partial_t X^k(t, s) \partial_s X^i(t, s) \} dt ds \\
 &= \int \partial_k \phi_i(t, X(t, s)) \{ -\partial_s X^k(t, s) v^i(t, X(t, s)) \\
 &\qquad\qquad\qquad + v^k(t, X(t, s)) \partial_s X^i(t, s) \} dt ds \\
 &= -\langle B^k, \partial_k \phi_i v^i \rangle + \langle B^i, \partial_k \phi_i v^k \rangle
 \end{aligned}$$

which is exactly (1.8) written in the sense of distributions. In a similar but simpler way, we also get, still in the sense of distributions,

$$\partial_t \rho + \nabla \cdot (\rho v) = 0. \tag{1.9}$$

In order to express the integro-differential equation (1.4), (1.5) in terms of v and B , we need further assumptions. We assume there is a second smooth vector field $b = b(t, x) \in \mathbb{R}^d$ that supports the loop in the sense

$$\partial_s X(t, s) = b(t, X(t, s)) \tag{1.10}$$

(which is consistent with the assumption we already made that the loop is smooth without self-intersection). Notice that, in the sense of distribution, B is just ρb . Indeed, for any test function ϕ , we can write, by definition (1.6) of ρ and B ,

$$\begin{aligned}
 \langle \rho, b\phi \rangle &= \int b(t, X(t, s)) \phi(t, X(t, s)) dt ds \\
 &= \int \partial_s X(t, s) \phi(t, X(t, s)) dt ds = \langle B, \phi \rangle.
 \end{aligned}$$

Let us now write the left-hand side of our integro-differential equation for a loop (1.4), multiplied by $\phi_i(t, X(t, s))$ and integrated in (t, s) , where ϕ is a (vector-valued) test-function. We find

$$\begin{aligned}
 &\int \phi_i(t, X(t, s)) (\partial_t X^i(t, s) - \partial_{ss}^2 X^i(t, s)) dt ds \\
 &\qquad\qquad\qquad = \langle \rho, v^i \phi_i \rangle \quad (\text{by (1.7)}) \\
 &+ \int \partial_k \phi_i(t, X(t, s)) \partial_s X^k(t, s) \partial_s X^i(t, s) dt ds \\
 &\qquad\qquad\qquad (\text{integrating by part in } s \text{ on } \mathbb{T}) \\
 &= \langle \rho, v^i \phi_i \rangle + \int \partial_k \phi_i(t, X(t, s)) b^k(t, X(t, s)) b^i(t, X(t, s)) dt ds
 \end{aligned}$$

(by (1.10))

$$= \langle \rho, v^i \phi_i + \partial_k \phi_i b^k b^i \rangle$$

(by definition (1.6) of ρ).

Finally, written in differential form, the left-hand side of (1.4) reads:

$$\rho v - \nabla \cdot (\rho b \otimes b). \quad (1.11)$$

Let us now move to the right-hand side of (1.4). we first get

$$\int g(t, s, s') ds' = \int \exp \left(-\pi \frac{|X(t, s) - X(t, s')|^2}{\varepsilon^2} \right) ds ds' \quad (\text{by definition (1.5)})$$

$$= \int \exp \left(-\pi \frac{|X(t, s) - y|^2}{\varepsilon^2} \right) \rho(t, dy) \quad (\text{by definition (1.6)})$$

$$= \rho_\varepsilon(t, X(t, s)) \quad (\text{by definition (1.1)}).$$

Similarly,

$$\begin{aligned} & \int (X(t, s) - X(t, s')) g(t, s, s') ds' \\ &= \int (X(t, s) - y) \exp \left(-\pi \frac{|X(t, s) - y|^2}{\varepsilon^2} \right) \rho(t, dy) \\ &= \frac{-\varepsilon^2}{2\pi} (\nabla \rho_\varepsilon)(t, X(t, s)). \end{aligned}$$

Thus the right-hand side of (1.4) just reads

$$-\frac{\nabla \rho_\varepsilon}{\rho_\varepsilon}(t, X(t, s)).$$

Multiplying by $\phi(t, X(t, s))$ and integrating in (t, s) , we find

$$-\langle \rho, \phi \cdot \frac{\nabla \rho_\varepsilon}{\rho_\varepsilon} \rangle.$$

Combined with (1.11), we have finally expressed equation (1.4) as

$$\rho v = \nabla \cdot (\rho b \otimes b) - \rho \frac{\nabla \rho_\varepsilon}{\rho_\varepsilon}. \quad (1.12)$$

(Notice that this equation makes perfect sense even when ρ is a singular measure provided v and b are smooth enough (at least continuous), just as (1.8)

makes sense even when B is a (vector-valued) measure.) This equation should be combined with (1.8) and (1.9), which reads, since $B = \rho b$,

$$\partial_t(\rho b) + \nabla \cdot (\rho b \otimes v - \rho v \otimes b) = 0, \quad \partial_t \rho + \nabla \cdot (\rho v) = 0. \quad (1.13)$$

So, under assumptions (1.7), (1.10), the integro-differential equation (1.4), (1.5) has been entirely expressed as a consistent system of evolution PDEs for the fields (v, b) , namely (1.12), (1.13). The formal limit as $\varepsilon \rightarrow 0$ of (1.12) is obvious:

$$\rho v = \nabla \cdot (\rho b \otimes b) - \nabla \rho. \quad (1.14)$$

Thus, remembering that the 1-current B is given by $B = \rho b$, we have finally obtained a consistent set of PDEs for (ρ, v, b) , namely (1.13), (1.14), which can be seen as the ‘‘Eulerian version’’ of our diffusion equation for loops (1.4,1.5).

Notice, without surprise, that, as $B = 0$, we consistently recover the scalar heat equation.

Remark 1.3 (a physical interpretation). — Physically speaking, our diffusion equation (1.13), (1.14), is just a friction dominated, ‘‘Darcy’’ version of ideal MHD, with density ρ , velocity v , and magnetic field $B = \rho b$:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t(\rho b) = \nabla \cdot (\rho v \otimes b - \rho b \otimes v), \quad \rho v = \nabla \cdot (\rho b \otimes b) - \nabla \rho.$$

The second equation says that the magnetic lines are transported by the velocity field, which formally implies that their topology is preserved during the diffusion process.

2. Direct derivation from some geometric PDEs

In order to explain our second approach, we first recover, as a prototype, the scalar heat equation out of the Euler equation through a nonlinear (quadratic) rescaling of the time variable. Next, we apply the same quadratic time rescaling to the more involved Born–Infeld (BI) equations [1], or, more precisely, to the *augmented* BI (ABI) equations, which were introduced in [2]. Eventually, we get our diffusion equation, which, as in our first approach, involves three fields (ρ, v, b) and reads

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \rho v = \nabla \cdot (\rho b \otimes b) + \nabla(\rho^{-1}), \quad (2.1)$$

$$\partial_t(\rho b) + \nabla \times (\rho b \times v + \rho^{-1} \nabla \times b) = 0. \quad (2.2)$$

Alternately, we may use a variational formulation of the ABI equations to get the same equations, and we show how this can be generalized to more general equations than the ABI equations in a strict sense. In particular we may recover the diffusion equations (1.13), (1.14) we have found in the

first section. However these generalized ABI equations lack the geometric “purity” of the original BI equations!

2.1. Example of the heat equation recovered from the Euler equations

The heat equation can be recovered in an (unusual) way from Euler’s equations

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = -\nabla p$$

(where $(\rho, p, v) \in \mathbb{R}^{1+1+3}$ are the density, pressure, and velocity fields of a fluid, p being a given function of ρ), through the quadratic time rescaling

$$t \rightarrow \theta = t^2/2, \quad \rho(t, x) \rightarrow \rho(\theta, x), \quad v(t, x) dt \rightarrow v(\theta, x) d\theta,$$

leading to $\partial_\theta \rho + \nabla \cdot (\rho v) = 0$, $\rho v + 2\theta[\partial_\theta(\rho v) + \nabla \cdot (\rho v \otimes v)] = -\nabla p(\rho)$.

After dropping the lowest order term, as $\theta \ll 1$, we get the “Darcy law” $\rho v = -\nabla p(\rho)$ and the porous medium equation $\partial_\theta \rho = \Delta(p(\rho))$.

In particular, for an isotherm gas, for which the pressure is proportional to the density, say $p = \rho$, we get the usual linear heat equation $\partial_\theta \rho = \Delta \rho$.

2.2. The Born–Infeld equations

These very geometric equations were designed by Max Born and Leopold Infeld in 1934 [1] as a nonlinear substitute to the Maxwell equation and they are still for use in String Theory [11].

In general, the Born–Infeld theory involves a $d+1$ dimensional Lorentzian space-time manifold of metric $g_{ij} dx^i dx^j$ and vector potentials $A = A_i dx^i$ that are critical points of the (fully covariant) “action” $\int \sqrt{-\det(g + dA)}$. Here, we concentrate on the 3+1 Minkowski space of special relativity (as Max Born and Leopold Infeld did in 1934).

Then, the Born–Infeld equations read, using classical electromagnetic notation,

$$\begin{aligned} \partial_t B + \nabla \times \left(\frac{B \times (D \times B) + D}{\sqrt{1 + D^2 + B^2 + (D \times B)^2}} \right) &= 0, & \nabla \cdot B &= 0, \\ \partial_t D + \nabla \times \left(\frac{D \times (D \times B) - B}{\sqrt{1 + D^2 + B^2 + (D \times B)^2}} \right) &= 0, & \nabla \cdot D &= 0. \end{aligned}$$

Using Noether's theorem, we get 4 extra conservation laws

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t (\rho v) + \nabla \cdot \left(\rho v \otimes v - \frac{B \otimes B - D \otimes D}{\rho} \right) = \nabla (\rho^{-1}),$$

where

$$v = \frac{D \times B}{\rho}, \quad \rho = \sqrt{1 + D^2 + B^2 + (D \times B)^2}. \quad (2.3)$$

Following [2], we observe that it is consistent (and much simpler) to consider $(B, D, \rho, P = \rho v)$ as *independent* variables solving the 10×10 *augmented* system which includes the 4 extra conservation laws

$$\partial_t \rho = -\nabla \cdot P, \quad \partial_t B = -\nabla \times \left(\frac{B \times P + D}{\rho} \right), \quad (2.4)$$

$$\partial_t D = -\nabla \times \left(\frac{D \times P - B}{\rho} \right), \quad (2.5)$$

$$\partial_t P = \nabla \cdot \left(\frac{-P \otimes P + B \otimes B + D \otimes D + I}{\rho} \right),$$

(where I denotes the identity matrix) while ignoring the algebraic constraints (2.3). (Indeed, it is established in [2] that these algebraic constraints are preserved by smooth solutions of the ABI system.)

2.3. Variational interpretation of the ABI equations

The ABI equations (2.4), (2.5) can be derived from a least action principle. Indeed, (2.5) can be seen as the optimality condition for (B, D, ρ, P) to be a critical point of

$$\iint \frac{|P|^2 + |D|^2 - |B|^2 - 1}{2\rho} \, dx \, dt$$

with respect to compactly supported space-time perturbations, under constraint (2.4). So we can define generalizations of the ABI equations just by considering more general actions, in particular of form

$$\iint \left(\frac{|P|^2 + |D|^2}{2\rho} - K(\rho, B) \right) \, dx \, dt, \quad (2.6)$$

where K can be any suitable convex function on $\mathbb{R}_+ \times \mathbb{R}^3$.

2.4. Quadratic time rescaling of the (augmented) Born–Infeld equations

We now perform the following rescaling of the (augmented) BI equations (2.4), (2.6):

$$t \rightarrow \theta = t^2/2, \quad (\rho, B, P, D) \rightarrow \left(\rho, B, P \frac{d\theta}{dt}, D \frac{d\theta}{dt} \right).$$

We obtain, after routine calculations,

$$\begin{aligned} \partial_\theta \rho + \nabla \cdot P &= 0, & \partial_t B &= -\nabla \times \left(\frac{B \times P + D}{\rho} \right), \\ D + 2\theta \left[\partial_\theta D + \nabla \times \left(\frac{D \times P}{\rho} \right) \right] &= \nabla \times (\rho^{-1} B), \\ P + 2\theta \left[\partial_\theta(\rho v) + \nabla \cdot \left(\rho v \otimes v - \frac{D \otimes D}{\rho} \right) \right] &= \nabla \cdot \left(\frac{B \otimes B}{\rho} \right) + \nabla(\rho^{-1}). \end{aligned}$$

Neglecting the higher order terms as $\theta \ll 1$, we get

$$\begin{aligned} \partial_\theta \rho + \nabla \cdot P &= 0, & \partial_\theta B + \nabla \times \left(\frac{B \times P + D}{\rho} \right) &= 0, \\ D &= \nabla \times (\rho^{-1} B), & \rho v &= \nabla \cdot \left(\frac{B \otimes B}{\rho} \right) + \nabla(\rho^{-1}), \end{aligned}$$

which are exactly the desired diffusion equation (2.1), (2.2), once we have written B as $B = \rho b$, P as $P = \rho v$, eliminated D , and moved back to notation t instead of θ for the time variable.

2.5. Variational derivation

The diffusion equation (2.1), (2.2) can be equivalently derived from the least action principle (2.4), (2.6) in the following way (in the spirit of [3]). We first differentiate, with respect to time, the potential part of action (2.6), using equation (2.4), and get

$$\begin{aligned} & \frac{d}{dt} \int K(\rho, B) \, dx \\ &= - \int \partial_\rho K(\rho, B) \nabla \cdot P \, dx - \int \partial_B K(\rho, B) \cdot \nabla \times \left(\frac{B \times P + D}{\rho} \right) \, dx \\ &= - \int \frac{P}{\rho} \cdot \left(-\rho \nabla \partial_\rho K + (\nabla \times \partial_B K) \times B \right) \, dx - \int \frac{D}{\rho} \cdot \nabla \times (\partial_B K) \, dx \end{aligned}$$

$$\begin{aligned}
 &= - \int \frac{|P|^2 + |D|^2 + |\rho \nabla \partial_\rho K - (\nabla \times \partial_B K) \times B|^2 + |\nabla \times (\partial_B K)|^2}{2\rho} dx \\
 &\quad + \int \frac{|P + \rho \nabla \partial_\rho K - (\nabla \times \partial_B K) \times B|^2}{\rho} dx + \int \frac{|D - \nabla \times (\partial_B K)|^2}{2\rho} dx.
 \end{aligned}$$

Then we maximize the dissipation of the potential energy by canceling the two last terms, which means

$$P = -\rho \nabla \partial_\rho K + (\nabla \times \partial_B K) \times B, \quad D = \nabla \times (\partial_B K) \quad (2.7)$$

and finally leads to the energy balance:

$$\frac{d}{dt} \int K(\rho, B) dx = - \int \frac{|P|^2 + |D|^2}{\rho} dx. \quad (2.8)$$

Let us apply this method to the ABI equations (2.4), (2.5). We find:

$$\begin{aligned}
 K &= \frac{1 + |B|^2}{2\rho}, & D &= \nabla \times \frac{B}{\rho}, \\
 P &= \rho \nabla \left(\frac{1}{2\rho^2} \right) + \rho \nabla \frac{|B|^2}{2\rho^2} + \left(\nabla \times \frac{B}{\rho} \right) \times B \\
 &= \rho \nabla \left(\frac{1}{\rho} \right) \frac{1}{\rho} + \rho \nabla \frac{|B|^2}{2\rho^2} + (B \cdot \nabla) \frac{B}{\rho} - \rho \nabla \frac{|B|^2}{2\rho^2} \\
 &= \nabla \left(\frac{1}{\rho} \right) + \nabla \cdot \left(\frac{B \otimes B}{\rho} \right)
 \end{aligned}$$

and we exactly recover (2.1), (2.2). This variational viewpoint is quite powerful. For instance, in the case $K = |B|^2/(2\rho) + \rho \log \rho$, we immediately get

$$P = -\nabla \rho + \nabla \cdot \frac{B \otimes B}{\rho}, \quad D = \nabla \times \frac{B}{\rho},$$

which differs from the equations (1.13), (1.14) we have obtained, through our “bottom-up” approach in the first section, just by the additional diffusion term due to D . As a matter of fact, (1.13), (1.14) can be directly deduced from the reduced action

$$\iint \left(\frac{|P|^2}{2\rho} - \frac{|B|^2}{2\rho} - \rho \log \rho \right) dx dt,$$

combined with the reduced version of (2.4)

$$\partial_t \rho + \nabla \cdot P = 0, \quad \partial_t B + \nabla \times \frac{B \times P}{\rho} = 0,$$

where there is no variable D involved.

3. The incompressible versions of the diffusion equations

Our diffusion equations obtained by two different approaches, respectively (1.13), (1.14) and (2.1), (2.2), are very nonlinear and degenerate; even the existence of local smooth solutions is questionable for (1.13), (1.14)! We will postpone their analysis to a further publication. In this paper we will just briefly discuss the simpler, “incompressible”, versions of (1.13), (1.14) and (2.1), (2.2), when the density field is constrained to be uniform, say $\rho = 1$, which leads, for the first equation (1.13), (1.14), to

$$\nabla \cdot v = 0, \quad v = \nabla \cdot (b \otimes b) + \nabla p, \quad (3.1)$$

$$\partial_t b + \nabla \times (b \times v) = 0, \quad (3.2)$$

where $p = p(t, x) \in \mathbb{R}$ is a “Lagrange multiplier” for the incompressibility constraint $\nabla \cdot v = 0$. This incompressible version of (1.13), (1.14) coincides with one of the “magnetic relaxation” models discussed by Moffatt in the framework of incompressible hydrodynamics [8, 9, 10].

The incompressible version of (2.1), (2.2) is almost identical, combining (3.1) with

$$\partial_t b + \nabla \times (b \times v + \nabla \times b) = 0, \quad (3.3)$$

differing from (3.2) just by the additional diffusion term $\nabla \times \nabla \times b$ which makes its analysis much simpler.

For these incompressible versions, we easily get the counterpart of the “energy balance” (2.7), (2.8) found in Subsection 2.5, respectively

$$\frac{d}{dt} \int |b|^2 dx = - \int (|\nabla \cdot (b \otimes b)|^2 + |\nabla \times b|^2) dx \quad (3.4)$$

for (3.3) and, for (3.2),

$$\frac{d}{dt} \int |b|^2 dx = - \int |\nabla \cdot (b \otimes b)|^2 dx.$$

Analysis of the incompressible versions

For the “incompressible” version of the second diffusion equation, namely (3.1), (3.3), we can use the standard concept of weak solutions, in the sense of distribution. Indeed, we get from (3.4) the *formal* a priori estimate

$$\frac{d}{dt} \int |b(t, x)|^2 dx + \int |v(t, x)|^2 dx + \int |\nabla \times b(t, x)|^2 dx = 0.$$

Working, for simplicity, on the unit periodic cube $\mathbb{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ rather than on the whole space \mathbf{R}^3 , we can get, by standard methods relying on

Sobolev’s inequalities, the global existence of a weak solution b with $\nabla \times b \in L^2([0, +\infty[\times \mathbb{T}^3)^3$ and $b \in C_w^0([0, +\infty[, L^2(\mathbb{T}^3)^3)$ (where the subscript w refers to the weak topology of L^2 ; in other words, for every test function $\phi \in L^2(\mathbb{T}^3)$, the function $t \rightarrow \int_{\mathbb{T}^3} \phi(x)b(t, x) dx$ belongs to $C^0([0, +\infty[)^3$).

The “incompressible” version (3.1), (3.2) is much more challenging and the weak formulation does not seem to be relevant. This is why we have introduced in [3] a concept of “dissipative solutions” (between P.-L. Lions’ concept of dissipative solutions for the Euler equations of incompressible fluids [6] and the formulation of the heat equation by Gigli [5]) for which we have obtained global existence in two space dimension and uniqueness of smooth dissipative solutions for any space dimension. In order to get global existence in space dimension $d = 3$, we use here an *even weaker* definition of dissipative solution, actually closer to Lions’ original definition, namely:

DEFINITION 3.1. — *Let us fix $T > 0$ and $d = 3$. We say that a pair of time dependent zero-mean divergence-free vector fields*

$$(t, x) \in [0, T] \times \mathbb{T}^d \rightarrow (b(t, x), v(t, x)) \in \mathbb{R}^{2d},$$

is a dissipative solution of equations (3.1), (3.2), on the time interval $[0, T]$, if:

$$b \in C_w^0([0, T], L^2(\mathbb{T}^d)^d), \quad v \in L^2([0, T] \times \mathbb{T}^d)^d,$$

and, for any pair of smooth, time-dependent zero-mean divergence-free vector fields

$$(t, x) \in [0, T] \times \mathbb{T}^d \rightarrow (\beta(t, x), w(t, x)) \in \mathbb{R}^{2d},$$

$$\|b_t - \beta_t\|^2 + \int_0^t e^{(t-s)C} \left[\frac{1}{2} \|v_s - w_s\|^2 + J_s \right] ds \leq \|b_0 - \beta_0\|^2 e^{tC}, \quad (3.5)$$

$$J_t = -2((v_t - w_t, \nabla \cdot (\beta_t \otimes \beta_t) - w_t)) + 2((b_t - \beta_t, \partial_t \beta_t + \nabla \times (\beta_t \times w_t)),$$

for all $t \geq 0$, for some constant C bounding from above (up to a numerical constant) the Lipschitz constants (with respect to the space variable x) of the test vector fields (β, w) up to time T .

Nota Bene. — In this expression, we have denoted by $\|\cdot\|$ and $((\cdot, \cdot))$ the L^2 norm and the L^2 inner product on \mathbb{T}^d . We have also used notation b_t , etc. for $b(t, \cdot)$.

Existence and partial uniqueness

We first observe that, once b_0 is fixed, formulation (3.5) is convex in b and v and therefore stable under weak convergence. This makes quite easy the global existence of dissipative solutions, just by passing to the limit in

suitable approximate equations enjoying global smooth solutions such as, for example,

$$\partial_t b + \nabla \times (b \times v) + (\varepsilon \nabla \times)^{2m} b = 0,$$

combined with (3.1), with $m \in \mathbb{N}$ large enough, as $\varepsilon \rightarrow 0$.

Next, whenever (β, w) is a smooth solution (say with bounded Lipschitz constants in x) of (3.1), (3.2), we see that J (defined in (3.5)) vanishes. Then (β, w) is the unique dissipative solution with initial condition β_0 . This follows directly from (3.5).

Therefore, without entering into details, we can obtain for such “dissipative solutions” the same kind of results obtained by Lions [6] for Euler’s equations of incompressible fluids: “such solutions exist; as long as a smooth solution exists with the same initial condition, any such dissipative solution coincides with it.”

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