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Arithmetical modular forms and new constructions of p -adic L -functions on classical groups

ALEXEI PANCHISHKIN⁽¹⁾

RÉSUMÉ. – Une nouvelle approche pour construire des fonctions L p -adiques pour les groupes classiques est présentée comme un projet en cours avec Thanh Hung Dang and Anh Tuan Do (Hanoi, Vietnam). Pour un groupe algébrique G sur un corps de nombres K les fonctions L complexes sont certains produits d'Euler $L(s, \pi, r, \chi)$. En particulier, notre construction couvre les fonctions L étudiées par Shimura dans [52] via la méthode de doublement de Piatetski-Shapiro et Rallis. Un avatar p -adique $L(s, \pi, r, \chi)$ est une fonction p -adique analytique $L_p(s, \pi, r, \chi)$ de $s \in \mathbb{Z}_p$, $\chi \bmod p^r$ interpolant les valeurs spéciales normalisées algébriques $L^*(s, \pi, r, \chi)$ de la fonction L complexe analytique attachée. Nous utilisons les formes presque-holomorphes et quasi-modulaires générales pour calculer et pour interpoler les valeurs spéciales normalisées.

ABSTRACT. – An approach to constructions of automorphic L -functions and their p -adic avatars is presented as a work in progress with Thanh Hung Dang and Anh Tuan Do (Hanoi, Vietnam). For an algebraic group G over a number field K these L functions are certain Euler products $L(s, \pi, r, \chi)$. In particular, our constructions cover the L -functions in [52] via the doubling method of Piatetski-Shapiro and Rallis.

A p -adic avatar of $L(s, \pi, r, \chi)$ is a p -adic analytic function $L_p(s, \pi, r, \chi)$ of p -adic arguments $s \in \mathbb{Z}_p$, $\chi \bmod p^r$ which interpolates algebraic numbers defined through the normalized critical values $L^*(s, \pi, r, \chi)$ of the corresponding complex analytic L -function. We present a method using arithmetic nearly-holomorphic forms and general quasi-modular forms, related to algebraic automorphic forms. It gives new technique of constructing p -adic zeta-functions via general quasi-modular forms and their Fourier coefficients.

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Introduction

Traditionally, arithmetical modular forms belong to the world of arithmetic, but also to the world of geometry, algebra and analysis. On the other hand, it became very popular and useful, to attach zeta-functions (or L -functions) to mathematical objects of various nature as certain generating functions or Euler products.

Firstly, such L -functions give a tool to link these objects to each other (expressing a general form of functoriality), and secondly, it allows in favorable cases to obtain answers to fundamental questions about these objects; such answers are often expressed in the form of a number (complex or p -adic).

To a discovery of beautiful mathematical worlds I greatly owe to our joint studies with Vadim Schechtman. It is a great privilege and luck for me, to know him since 1971, entering the same year Mech-mat Faculty of MSU. We studied together with Mikhail Tsfasman, Serguei Vladut, Dima Logachev, Misha Vishik, Igor Skorniyakov, Igor Artamkin, Misha Kiefer, and other wonderful people, see the list of our year in <http://vanteev.narod.ru/Spisok1971.htm>.

Starting from the first semester we often passed examinations together, sometimes in advance. We attended excellent seminars of E.M.LANDIS, E.S.GOLOD, Yu.A. BACHTURIN and A.Yu. OLSHANSKY, and even we already tried the seminars of I.M.GELFAND, Yu.I.MANIN, A.A.KIRILLOV, V.I.ARNOLD and I.R.SHAFAREVICH.

New amazing worlds of numbers, functions and varieties were opened to us in these seminars, and also, in regular meetings of the Moscow Mathematical Society, where crowds of mathematicians from all parts of the city were gathering. Many interesting things were learned in corridors and staircases. These mathematical domains turned out to be closely related to each other by analogies.

Many thanks and deep gratitude to Vadik for his inspiration and stimulation to discovering together the world of "big mathematics", a term often used in those seminars!
Congratulations to Vadim Schechtman on the occasion of his 60th Anniversary! Best wishes of good health, many new achievements! Much enthusiasm in mathematics and life!

It is a pleasure to consider the mathematical topic of this article on new constructions of p -adic L -functions on classical groups, proposed here,

as an illustration of links between different mathematical areas within the Mathematics Realm.

The choice the Chair of Higher Algebra in 1973

An exceptional influence on us in this period was the two-year course "Galois Theory" by I.R.SHAFAREVICH, who explained to us also the representations of finite groups, group cohomology, extensions, crystallographic groups,...

In 1973-75, in a joint seminar of Yu.I.MANIN and A.A.KIRILLOV, p -adic zeta functions were treated, especially p -adic integration, Serre's p -adic modular forms, and Galois representations, with participation of Neal Koblitz, V.Drinfeld, I.I.Piatetski-Shapiro, V.Berkovich, Yu.Zarhin, P.Kurchanov, A.Nassybullin Each time we learned "WHAT, HOW and WHY", and we still often use all these things until now in our research and teaching.

Analogies between numbers and functions

The ideas of using these fruitful analogies and various other tools came to us largely from I. R. SHAFAREVICH and Y. I. MANIN. Let us quote Manin's Introduction to "Periods of cusp forms and p -adic Hecke series" (1973) :

...Elementary questions about congruences and equations have found themselves becoming interwoven in an intricate and rich complex of constructions drawn from abstract harmonic analysis, topology, highly technical ramifications of homological algebra, algebraic geometry, measure theory, logic, and so on (corresponding to the spirit of Gödel's theorem on the incompleteness of the techniques of elementary arithmetic and on our capabilities of recognizing even those truths which we are in a position to "prove"...) A new "synthetic" number theory, taking in the legacy of the "analytic" theory, is possibly taking shape under our very eyes ...

We all were much influenced by this approach, especially by analogies between numbers and functions developed much in this period in the following results:

- (a) Proof of Mordell's Conjecture and Shafarevich conjectures by G. Faltings (1983) developing ideas of I.R.Shafarevich (Stockholm, 1962), A.N.Parshin, Yu.G.Zarhin, ...

- (b) Grothendieck's theory of motives, and L -functions attached to them, giving analogy of a "cell decomposition" of an algebraic variety.
- (c) Proof of Weil Conjectures and Ramanujan Conjecture by Deligne (1974) using the ℓ -adic étale cohomology
- (d) Applications of geometric ideas to the Information Transmission Theory such as Geometric codes, in particular, viewing "code words as functions" on finite sets of points on algebraic varieties over finite fields.
- (e) Fast multiplication (Fast Fourier Transform) in particular, viewing integers as "polynomials over a base like 2^n "
- (f) Drinfeld modules and Drinfeld modular varieties, the notion coming from an analogy with the Chebyshev polynomials, the unique polynomials satisfying $T_n(\cos(\vartheta)) = \cos(n\vartheta)$.

Student years, first mathematical works and publications

There was much joint activity during our student years, including travels to Mozhaisk region, to Yaroslavl' mathematical schools, to several places at the Black Sea coast,...

There were many jokes, much humor, musics, books, graduation diploma work, first mathematical publications,...

Then came postgraduate study, PhD thesis,...

We were much interested mutually in our mathematical results, and we are very proud to belong to the famous mathematical school, founded by our great teachers

Igor Rostislavovich SHAFAREVICH,

Yuri Ivanovich MANIN, Evguenii Solomonovich GOLOD;

the school and going back to

Pafnutij Lvovich CHEBYSHEV,

and it is well presented as the 4th Mathematical Genealogy Tree, see [57], indicated to me by Vadim.

Acknowledgement. — Many thanks to Alexandre Varchenko for soliciting a paper for this collection. The paper grew from several talks: for "Journées Arithmétiques", University Grenoble-1, 2013) Workshop "Explicit methods in the theory of automorphic forms" (Shanghai, Tongji University, March 2014), Rencontres Clermont-Ferrand-Grenoble (Institut Fourier, June 2014), Workshop "Algebraic geometry and number theory", June 2014,

Workshop "Zeta V", (Laboratory J.-V. Poncelet, Moscow, December 2014), Workshop "Non-Archimedean Analysis" of VIASM, Halong Bay, and numerous talks in the Universities of Vinh, Dong Hoi, Thanh Long University, Institut of Technology Hoc Vien Ki Thuat Quan Su in Vietnam, April 2015).

Let all these Institutions be thanked for their kind hospitality.

I am very grateful to Lionel Schwartz, Marcel Morales, and Yolande Jimenez (Format Vietnam) for organizing my travel to Hanoi in April 2015.

My special thanks go to Khoai Ha Huy for inviting me to VIASM, and to Siegfried Boecherer and Vladimir Berkovich for fruitful discussions.

1. Automorphic L -functions and their p -adic avatars

Our main objects in this paper are automorphic L -functions and their p -adic avatars.

For an algebraic group G over a number field K these L functions are defined as certain Euler products. More precisely, we apply our constructions for the L -functions studied in Shimura's book [52].

EXAMPLE 1.1 ($G = \mathrm{GL}(2)$, $K = \mathbb{Q}$, $L_f(s) = \sum_{n \geq 1} a_n n^{-s}$, $s \in \mathbb{C}$). — Here $f(z) = \sum_{n \geq 1} a_n q^n$ is a modular form on the upper-half plane

$$H = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\} = \mathrm{SL}(2)/\mathrm{SO}(2), q = e^{2\pi iz}.$$

An Euler product has the form

$$L_f(s) = \prod_{p \text{ primes}} (1 - a_p p^{-s} + \psi_f(p) p^{k-1-2s})^{-1}$$

where k is the weight and ψ_f the Dirichlet character of f . It is defined iff the automorphic representation π_f attached to f is irreducible.

Recall that π_f is generated by the lift \tilde{f} of f to the group $G(\mathbb{A})$, where \mathbb{A} is the ring of adèles

$$\mathbb{A} = \{x = (x_\infty, x_p)_p \mid x_\infty \in \mathbb{R}, x_p \in \mathbb{Q}_p, \\ \text{such that } x_p \in \mathbb{Z}_p \text{ for all but a finite number of } p\}.$$

A p -adic avatar of $L_f(s)$ (Manin-Mazur)

It is a p -adic analytic function $L_{f,p}(s, \chi)$ of p -adic arguments $s \in \mathbb{Z}_p$, $\chi \bmod p^r$ which interpolates algebraic numbers

$$L_f^*(s, \chi)/\omega^\pm \in \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \hat{\mathbb{Q}}_p \text{ (the Tate field)}$$

for $1 \leq s \leq k-1$, ω^\pm are periods of f where the complex analytic L function of f is defined for all $s \in \mathbb{C}$ so that in the absolutely convergent case $\text{Re}(s) > (k+1)/2$,

$$L_f^*(s, \chi) = (2\pi)^{-s} \Gamma(s) \sum_{n \geq 1} \chi(n) a_n n^{-s}$$

which extends to holomorphic function with a functional equation. According to Manin and Shimura, this number is algebraic if the period ω^\pm is chosen according to the parity $\chi(-1)(-1)^{-s} = \pm 1$.

Constructions of p -adic avatars

In the general case of an irreducible automorphic representation of the adelic group $G(\mathbb{A}_K)$ there is an L -function

$$L(s, \pi, r, \chi) = \prod_{\mathfrak{p}_v \text{ primes in } K} \prod_{j=1}^m (1 - \beta_{j, \mathfrak{p}_v} N_{\mathfrak{p}_v}^{-s})^{-1}$$

where

$$\prod_{j=1}^m (1 - \beta_{j, \mathfrak{p}} X) = \det(1_m - r(\text{diag}(\alpha_{i, \mathfrak{p}})_i X)),$$

$\alpha_{i, \mathfrak{p}}$ are the Satake parameters of $\pi = \bigotimes_v \pi_v$ $v \in \Sigma_K$ (places in K), $\mathfrak{p} = \mathfrak{p}_v$. Here $h_v = \text{diag}(\alpha_{i, \mathfrak{p}})_i$ live in the Langlands group ${}^L G(\mathbb{C})$, $r : {}^L G(\mathbb{C}) \rightarrow \text{GL}_m(\mathbb{C})$ denotes a finite dimensional representation, and $\chi : \mathbb{A}_K^*/K^* \rightarrow \mathbb{C}^*$ is a character of finite order. Constructions admit extension to rather general automorphic representations on Shimura varieties via the following tools:

- Modular symbols and their higher analogues (linear forms on cohomology spaces related to automorphic forms)
- Petersson products with a fixed automorphic form, or
- linear forms coming from the Fourier coefficients (or Whittaker functions), or through the
- CM-values (special points on Shimura varieties),

Accessible cases: symplectic and unitary groups

- $G = \mathrm{GL}_1$ over \mathbb{Q} (Kubota-Leopoldt-Mazur) for the Dirichlet L -function $L(s, \chi)$.
- $G = \mathrm{GL}_1$ over a totally real field F (Deligne-Ribet, using algebraicity result by Klingen).
- $G = \mathrm{GL}_1$ over a CM-field K , i.e. a totally imaginary extension of a totally real field F (N.Katz, Manin-Vishik).
- the Siegel modular case $G = \mathrm{GSp}_n$ (the Siegel modular case, $F = \mathbb{Q}$).
- General symplectic and unitary groups over a CM-field K .

2. Automorphic L -functions attached to symplectic and unitary groups

Let us briefly describe the L -functions attached to symplectic and unitary groups as certain Euler products in Chapter 5 of [52], with critical values computed in Chapter 7, Theorem 28.8 using general nearly holomorphic arithmetical automorphic forms for the group.

$$G = G(\varphi) = \{ \alpha \in \mathrm{GL}_m(K) \mid \alpha \varphi^t \alpha^\rho = \nu(\alpha) \varphi, \nu(\alpha) \in F^* \},$$

where $\varphi = \eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ or $\varphi = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}$, see also Ch.Skinner and E.Urban [53] and Shimura G., [52].

The groups and automorphic forms studied in Shimura's book

Let F be a totally real algebraic number field, K be a totally imaginary quadratic extension of F and ρ be the generator of $\mathrm{Gal}(K/F)$. Take $\eta_n =$

$\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and define

$$G = \mathrm{Sp}(n, F) \quad (\text{Case Sp})$$

$$G = \{ \alpha \in \mathrm{GL}_{2n}(K) \mid \alpha \eta_n \alpha^* = \eta_n \} \quad (\text{Case UT = unitary tube})$$

$$G = \{ \alpha \in \mathrm{GL}_{2n}(K) \mid \alpha T \alpha^* = T \} \quad (\text{Case UB = unitary ball})$$

according to three cases. Assume $F = \mathbb{Q}$ for a while. The group of the real points G_∞ acts on the associated domain

$$\mathcal{H} = \begin{cases} \{ z \in M(n, n, \mathbb{C}) \mid {}^t z = z, \mathrm{Im}(z) > 0 \} & (\text{Case Sp}) \\ \{ z \in M(n, n, \mathbb{C}) \mid i(z^* - z) > 0 \} & (\text{Case UT}) \\ \{ z \in M(p, q, \mathbb{C}) \mid 1_q - z^* z > 0 \} & (\text{Case UB}), \end{cases}$$

$(p, q), p + q = n$ being the signature of iT . Here $z^* = {}^t\bar{z}$ and $>$ means that a hermitian matrix is positive definite. In Case UB, there is the standard automorphic factor $M(g, z)$, $g \in G_\infty$, $z \in \mathcal{H}$ taking values in $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$.

Shimura’s arithmeticity in the theory of automorphic forms [52], p -adic zeta functions and nearly-holomorphic forms on classical groups

Automorphic L -functions via general quasi-modular forms. Automorphic L -functions and their p -adic avatars can be obtained for quite general automorphic representations on Shimura varieties by constructing p -adic distributions out of algebraic numbers attached to automorphic forms. These numbers satisfy certain Kummer-type congruences established in different ways: via

- Normalized Petersson products with a fixed automorphic form, or
- linear forms coming from the Fourier coefficients (or Whittaker functions), or through the
- CM-values (special points on Shimura varieties), see The Iwasawa Main Conjecture for $\mathrm{GL}(2)$ by C. Skinner and E. Urban, [MC], Shimura G., Arithmeticity in the theory of automorphic forms [Sh00].

The combinatorial structure of the Fourier coefficients of the holomorphic forms used in these constructions is quite complicated.

In order to prove the congruences needed for the p -adic constructions, we use a simplification due to nearly-holomorphic and general quasi-modular forms, related to algebraic automorphic forms. In this paper, a new method of constructing p -adic zeta-functions is presented using general quasi-modular forms and their Fourier coefficients.

In order to describe both algebraicity and congruences of the critical values of the zeta functions of automorphic forms on unitary and symplectic groups, we follow the review by H. Yoshida [58] of Shimura’s book ”Arithmeticity in the theory of automorphic forms” [52]. Shimura’s mathematics developed by stages:

- (A) Complex multiplication of abelian varieties =>
- (B) The theory of canonical models = Shimura varieties =>
- (C) Critical values of zeta functions and periods of automorphic forms.

(B) includes (A) as 0-dimensional special case of canonical models. The relation of (B) and (C) is more involved, but (B) provides a solid foundation of the notion of the arithmetic automorphic forms. Also unitary Shimura varieties have recently attracted much interest (in particular by Ch. Skinner and E. Urban), see [53], in relation with the proof of the The Iwasawa Main Conjecture for $GL(2)$.

Integral representations and critical values of the zeta functions

Automorphic forms are assumed scalar valued in this part. For Cases Sp and UT, Eisenstein series $E(z, s)$ associated to the maximal parabolic subgroup of G of Siegel type is introduced. Its analytic behaviour and those values of $\sigma \in 2^{-1}\mathbb{Z}$ at which $E(z, \sigma)$ is nearly holomorphic and arithmetic are studied in [52]. This is achieved by proving a relation giving passage from s to $s-1$ for $E(z, s)$, involving a differential operator, then examining Fourier coefficients of Eisenstein series using the theory of confluent hypergeometric functions on tube domains.

For a Hecke eigenform f on $G_{\mathbb{A}}$ and an algebraic Hecke character χ on the idele group of K (in Case Sp, $K = F$), the zeta function $\mathcal{Z}(s, f, \chi)$ is defined. Viewing it as an Euler product extended over prime ideals of F , the degree of the Euler factor is $2n + 1$ in Case Sp, $4n$ in Case UT, and $2n$ in Case UB, except for finitely many prime ideals, see Chapter 5 of [52].

This zeta function is almost the same as the so called standard L -function attached to f twisted by χ but it turns out to be more general in the unitary case, see also [13].

Main results on critical values of the L -functions studied in Shimura's book [52] is stated in Theorem 28.5, 28.8 (Cases Sp, UT), and in Theorem 29.5 in Case UB.

THEOREM 2.1 (algebraicity of critical values in Cases Sp and UT). — *Let $\mathbf{f} \in \mathcal{V}(\bar{\mathbb{Q}})$ be a non zero arithmetical automorphic form of type Sp or UT. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell} |x_{\mathbf{a}}|^{-\ell}$ with $\ell \in \mathbb{Z}^{\mathbf{a}}$, and let $\sigma_0 \in 2^{-1}\mathbb{Z}$. Assume the following conditions (in the notations of Chapter 7 of [52] for the weight k_v, μ_v, ℓ_v)*

- Case Sp $2n + 1 - k_v - \mu_v - \leq k_v - \mu_v$, where $\mu_v = 0$ if $[k_v] - \ell_v \in 2\mathbb{Z}$
 and $\mu_v = 1$ if $[k_v] - \ell_v \notin 2\mathbb{Z}$; $\sigma_0 - k_v + \mu_v$ for every $v \in \mathbf{a}$
 if $\sigma_0 > n$ and $\sigma_0 - 1 - k_v + \mu_v \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$ if $\sigma_0 \leq n$.
- Case UT $4n - (2k_{v\rho} + \ell_v) \leq 2\sigma_0 \leq m_v - [k_v - k_{v\rho} - \ell_v]$ and $2\sigma_0 - \ell_v \in 2\mathbb{Z}$
 for every $v \in \mathbf{a}$.

Further exclude the following cases

- (A) Case Sp $\sigma_0 = n + 1, F = \mathbb{Q}$ and $\chi^2 = 1$;
- (B) Case Sp $\sigma_0 = n + (3/2), F = \mathbb{Q}; \chi^2 = 1$ and $[k] - \ell \in 2\mathbb{Z}$
- (C) Case Sp $\sigma_0 = 0, \mathfrak{c} = \mathfrak{g}$ and $\chi = 1$;
- (D) Case Sp $0 < \sigma_0 \leq n, \mathfrak{c} = \mathfrak{g}, \chi^2 = 1$ and the conductor of χ is \mathfrak{g} ;
- (E) Case UT $2\sigma_0 = 2n + 1, F = \mathbb{Q}, \chi_1 = \theta$, and $k_v - k_{v\rho} = \ell_v$;
- (F) Case UT $0 \leq 2\sigma_0 < 2n, \mathfrak{c} = \mathfrak{g}, \chi_1 = \theta^{2\sigma_0}$ and the conductor of χ is \mathfrak{r}

Then

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) / \langle \mathbf{f}, \mathbf{f} \rangle \in \bar{\mathbb{Q}},$$

where $d = [F : \mathbb{Q}]$, $|m| = \sum_{v \in \mathfrak{a}} m_v$, and

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n, & \text{Case Sp, } k \in \mathbb{Z}^{\mathfrak{a}}, \text{ and } \sigma_0 > n_0, \\ n\sigma_0 - n^2, & \text{Case Sp, } k \notin \mathbb{Z}^{\mathfrak{a}}, \text{ or } \sigma_0 \leq n_0, \\ 2n\sigma_0 - 2n^2 + n & \text{Case UT} \end{cases}$$

We establish a p -adic analogue of Theorem 28.8 (in Cases Sp and UT) representing algebraic parts of critical values as values of certain p -adic analytic zeta functions.

3. Constructing p -adic zeta-functions via quasi-modular forms

We present here a new method of constructing p -adic zeta-functions based on the use of general quasi-modular forms on classical groups.

The combinatorial structure of the Fourier coefficients of the holomorphic forms used in these constructions is quite complicated. We present a method of simplification using nearly-holomorphic and general quasi-modular forms, related to algebraic automorphic forms. It gives a new method of constructing p -adic zeta-functions using general quasi-modular forms and their Fourier coefficients. The symmetric space

$$\mathcal{H} = G(\mathbb{R}) / ((\text{maximal-compact subgroup})\mathcal{K} \times \text{Center})$$

parametrizes certain families of abelian varieties A_z ($z \in \mathcal{H}$) so that $F \subset \text{End}(A_z) \otimes \mathbb{Q}$. The CM-points z correspond to a maximal multiplication ring $\text{End}(A_z)$.

For the group $GL(2)$, N.Katz used arithmetical elements (real-analytic and p -adic)

Instead of holomorphic forms in these representation spaces. These elements correspond also to quasi-modular forms coming from derivatives which can be defined in general using Shimura's arithmeticity and the Maass-Shimura operators. A relation real-analytic \leftrightarrow p -adic modular forms comes from the notion of p -adic modular forms invented by J.-P.Serre [Se73] as p -adic limits of q -expansions of modular forms with rational coefficients for $\Gamma = SL_2(\mathbb{Z})$. The present method of constructing p -adic automorphic L -functions uses general quasi-modular forms, and their link to algebraic p -adic modular forms.

Real-analytic and p -adic modular forms

In Serre's case for $\Gamma = SL_2(\mathbb{Z})$, the ring M_p of p -adic modular forms contains the subring $\mathcal{M} = \bigoplus_{k \geq 0} \mathcal{M}_k(\Gamma, \mathbb{Z}) = \mathbb{Z}[E_4, E_6]$, and it contains as element with q -expansion $E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n$. On the other hand,

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2, \text{ where } S = \frac{1}{4\pi y},$$

is a nearly holomorphic modular form (its coefficients are polynomials of S over \mathbb{Q}). Let \mathcal{N} be the ring of such forms. Then $\tilde{E}_2|_{S=0} = E_2$ and it was proved by J.-P. Serre that E_2 is a p -adic modular form. Elements of the ring $QM = \mathcal{N}|_{S=0} = 0$ will be called general quasi-modular forms. These phenomena are quite general and can be used in computations and proofs. In June 2014 in a talk in Grenoble, S.Boecherer extended these results to the Siegel modular case.

Using algebraic and p -adic modular forms

There are several methods to compute various L -values starting from the constant term of the Eisenstein series in [Se73],

$$G_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n = \frac{\Gamma(k)}{(2\pi i)^k} \sum_{(c,d)} '(cz+d)^{-k} \text{ (for } k \geq 4);$$

and using Petersson products of nearly-holomorphic Siegel modular forms and arithmetical automorphic forms as in [52]:

the Rankin-Selberg method,

the doubling method (pull-back method).

A known example is the standard zeta function $D(s, f, \chi)$ of a Siegel cusp eigenform $f \in \mathcal{S}_n^k(\Gamma)$ of genus n (with local factors of degree $2n + 1$) and χ a Dirichlet character.

Theorem (the case of even genus n ([11]), via the Rankin-Selberg method) gives a p -adic interpolation of the normalized critical values $D^*(s, f, \chi)$ using Andrianov-Kalinin integral representation of these values $1 + n - k \leq s \leq k - n$ through the Petersson product $\langle f, \theta_{T_0} \delta^r E \rangle$ where δ^r is a certain composition of Maass-Shimura differential operators, θ_{T_0} a theta-series of weight $n/2$, attached to a fixed $n \times n$ matrix T_0 .

THEOREM 3.1 (p -adic interpolation of $D(s, f, \chi)$). — (1) (the case of odd genus (Boecherer-Schmidt, [8])

Assume that n is arbitrary genus, and a prime p ordinary then there exists a p -adic interpolation of $D(s, f, \chi)$

(2) (Anh-Tuan Do (non-ordinary case, PhD Thesis of March 2014)), via the doubling method)

Assume that n is arbitrary genus, and p an arbitrary prime not dividing level of f then there exists a p -adic interpolation of $D(s, f, \chi)$.

Proof uses the following Boecherer-Garrett-Shimura identity (a pull-back formula) which allows to compute the critical values through certain double Petersson product by integrating over $z \in \mathbb{H}_n$ the identity:

$$\Lambda(l + 2s, \chi) D(l + 2s - n, f, \chi) f = \langle f(w), E_{l, \nu, \chi, s}^{2n}(\text{diag}[z, w])_w \rangle,$$

Here $k = l + \nu, \nu \geq 0, \Lambda(l + 2s, \chi)$ is a product of special values of Dirichlet L -functions and Γ -functions, $E_{l, \nu, \chi, s}^{2n}$ a higher twist of a Siegel-Eisenstein series on $(z, w) \in \mathbb{H}_n \times \mathbb{H}_n$ (see [Boe85], [Boe-Schm]).

A p -adic construction uses congruences for the L -values, expressed through the Fourier coefficients of the Siegel modular forms and nearly-modular forms. In the present approach of computing the Petersson products and L -values, an injection of algebraic nearly holomorphic modular forms into p -adic modular forms is used.

Injecting nearly-holomorphic forms into p -adic modular forms

A recent discovery by Takashi Ichikawa (Saga University), [Ich12], J. reine angew. Math., [Ich13] allows to inject nearly-holomorphic arithmetical (vector valued) Siegel modular forms into p -adic modular forms. Via the Fourier expansions, the image of this injection is represented by certain quasi-modular holomorphic forms like $E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$, with algebraic Fourier expansions. This description provides many advantages, both

computational and theoretical, in the study of algebraic parts of Petersson products and L -values, which we would like to develop here. In fact, the realization of nearly holomorphic forms as p -adic modular forms has been studied by Eric Urban, who calls them "Nearly overconvergent modular forms" [55], Chapter 10.

Urban only treats the elliptic modular case in that paper, but I believe he and Skinner are working on applications of a more general theory. This work is related to a recent preprint [4] by S. Boecherer and Shoyu Nagaoka where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are p -adic modular forms. Moreover they show that derivatives of such Siegel modular forms are p -adic. Parts of these results are also valid for vector-valued modular forms.

Arithmetical nearly-holomorphic Siegel modular forms

Nearly-holomorphic Siegel modular forms over a subfield k of \mathbb{C} are certain \mathbb{C}^d -valued smooth functions f of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$ given by the following expression $f(Z) = \sum_T P_T(S)q^T$ where T runs through the set B_n of all half-integral semi-positive matrices, $S = (4\pi Y)^{-1}$ a symmetric matrix, $q^T = \exp(2\pi\sqrt{-1}\text{tr}(TZ))$, $P_T(S)$ are vectors of degree d whose entries are polynomials over k of the entries of S .

Review of the algebraic theory

Following [17], consider the columns Z_1, Z_2, \dots, Z_n of $Z \in \mathbb{H}_n$ and the \mathbb{Z} -lattice L_Z in \mathbb{C}^n generated by $\{E_1, E_2, \dots, E_n, Z_1, Z_2, \dots, Z_n\}$, where E_1, E_2, \dots, E_n , are the columns of the identity matrix E . The torus $A_Z = \mathbb{C}^n/L_Z$ is an abelian variety, and there is an analytic family $\mathcal{A} \rightarrow \mathbb{H}_n$ whose fiber over the point Z is $A_Z \in \mathbb{H}_n$. Let us consider the quotient space $\mathbb{H}_n/\Gamma(N)$ of the Siegel upper half space \mathbb{H}_n of degree n by the integral symplectic group

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \mid \begin{array}{l} A_\gamma \equiv D_\gamma \equiv 1_n \\ B_\gamma \equiv C_\gamma \equiv 0_n \end{array} \right\}$$

If $N > 3$, $\Gamma(N)$ acts without fixed points on $\mathcal{A} = \mathcal{A}_n$ and the quotient is a smooth algebraic family $\mathcal{A}_{n,N}$ of abelian varieties with level N structure over the quasi-projective variety $\mathcal{H}_{n,N}(\mathbb{C}) = \mathbb{H}_n/\Gamma(N)$ defined over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N -th root of 1. For positive integers n and N , $\mathcal{H}_{n,N}$ is the moduli space classifying principally polarized abelian schemes of relative dimension n with a symplectic level N structure.

De Rham and Hodge vector bundles

The fiber varieties \mathcal{A} and $\mathcal{A}_{n,N}$ give rise to a series of vector bundles over \mathcal{H}_n and $\mathcal{H}_{n,N}$. Notations

- $\mathcal{H}_{DR}^1(\mathcal{A}/\mathbb{H}_n)$ and $\mathcal{H}_{DR}^1(\mathcal{A}_{n,N}/\mathcal{H}_{n,N})$ the relative algebraic De Rham cohomology bundles of dimension $2n$ over \mathbb{H}_n and $\mathcal{H}_{n,N}$ respectively. Their fibers at $Z \in \mathbb{H}_n$ are $H^1 := \text{Hom}_{\mathbb{C}}(L_Z \otimes \mathbb{C}, \mathbb{C})$ generated by α_i, β_i :

$$\alpha_i \left(\sum_j a_j E_j + b_j Z_j \right) = a_i, \quad \beta_i \left(\sum_j a_j E_j + b_j Z_j \right) = b_i, \quad (i = 1, \dots, n),$$

- \mathcal{H}_{∞}^1 is the \mathcal{C}^{∞} vector bundle associated to \mathcal{H}_{DR}^1 (over \mathbb{H}_n and $\mathcal{H}_{n,N}$). It splits as a direct sum

$$\mathcal{H}_{\infty}^1 = \mathcal{H}_{\infty}^{1,0} \oplus \mathcal{H}_{\infty}^{0,1}$$

and induces the Hodge decomposition on the De Rham cohomology of each fiber.

- The summand $\omega = \mathcal{H}_{\infty}^{1,0}$ is the bundle of relative 1-forms for either $\mathcal{A}/\mathcal{H}_n$ or $\mathcal{A}_{n,N}/\mathcal{H}_{n,N}$. Let us denote by $\pi : \mathcal{A}_{n,N} \rightarrow \mathcal{H}_{n,N}$ the universal abelian scheme with 0-section s , and by the Hodge bundle of rank n defined as

$$\mathbb{E} = \pi_* (\Omega_{\mathcal{A}_{n,N}/\mathcal{H}_{n,N}}^1) = s^* (\Omega_{\mathcal{A}_{n,N}/\mathcal{H}_{n,N}}^1)$$

- The bundle of holomorphic 1-forms on the base \mathbb{H}_n or on $\mathcal{H}_{n,N}$, is denoted Ω .

Algebraic Siegel modular forms

are defined as global sections of \mathbb{E}_{ρ} , the locally free sheaf on $\mathcal{H}_{n,N} \otimes R$ obtained from twisting the Hodge bundle \mathbb{E} by ρ .

DEFINITION 3.2. — *Let R be a $\mathbb{Z}[1/N, \zeta_N]$ -algebra. For an algebra homomorphism $\rho : \text{GL}_n \rightarrow \text{GL}_d$ over R , define algebraic Siegel modular forms over R as elements of $\mathcal{M}_{\rho}(R) = H_0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_{\rho})$, called of weight ρ , degree n , level N . If $\rho = \det^{\otimes k} : \text{GL}_n \rightarrow \mathbb{G}_m$, then elements of $\mathcal{M}_k(R) = \mathcal{M}_{\det^k}(R)$ are called of weight k .*

For $R = \mathbb{C}$, each $Z \in \mathbb{H}_n$, let $\mathcal{A}_Z = \mathbb{C}^n / (\mathbb{Z}^n + \mathbb{Z}^n \cdot Z)$ be the corresponding abelian variety over \mathbb{C} , and (u_1, \dots, u_n) be the natural coordinates on the universal cover \mathbb{C}^n of \mathcal{A}_Z . Then \mathbb{E} is trivialized over \mathbb{H}_n by du_1, \dots, du_n , and $f \in \mathcal{M}_{\rho}(\mathbb{C})$ is a complex analytic section of \mathbb{E}_{ρ} on $\mathcal{H}_{n,N}(C) = \mathbb{H}_n/\Gamma(N)$.

Hence, an element $f \in \mathcal{M}_\rho(\mathbb{C})$ is a \mathbb{C}^d -valued holomorphic function on \mathbb{H}_n satisfying the ρ -automorphic condition:

$$f(Z) = \rho(C_\gamma Z + D_\gamma)^{-1} \cdot f(\gamma(Z)) \left(Z \in \mathbb{H}_n, \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \right)$$

because of the identification $\mathcal{A}_Z \xrightarrow{\sim} \mathcal{A}_{\gamma Z} : {}^t(u_1, \dots, u_n) \mapsto (CZ + D)^{-1t}(u_1, \dots, u_n)$ and γ acts equivariantly on the trivialization of \mathbb{E} over \mathbb{H}_n as the left multiplication by $(CZ + D)^{-1}$.

Algebraic Fourier expansion

can be defined algebraically using an algebraic test object over the ring $\mathcal{R}_n = \mathbb{Z}[[q_{11}, \dots, q_{nn}]][[q_{ij}, q_{ij}^{\pm 1}]_{i,j=1,\dots,n}]$; where q_{ij} ($1 \leq i, j \leq n$) are variables with symmetry $q_{ij} = q_{ji}$. Mumford constructs in [Mu72] an object represented over n as

$$\begin{aligned} & (\mathcal{R}_n \otimes \mathbb{G}_m)^n / \langle (q_{ij})_{i=1,\dots,n} \mid 1 \leq j \leq n \rangle, (\mathbb{G}_m)^n \\ & = \text{Spec}(\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]), \end{aligned}$$

For the level N , at each 0-dimensional cusp c on $\mathcal{H}_{n,N}^*$ (Satake's minimal compactification of $\mathcal{H}_{n,N}$), this construction gives an abelian variety over the formal power series ring

$$\mathcal{R}_{n,N} = \mathbb{Z}[1/N, \zeta_N][[q_{11}^{1/N}, \dots, q_{nn}^{1/N}]][[q_{ij}^{\pm 1}]_{i,j=1,\dots,n}]$$

with a symplectic level N structure, and $\omega_i = dx_i/x_i$ ($1 \leq i \leq n$) form a basis of regular 1-forms. We may view algebraically Siegel modular forms as certain sections of vector bundles over $\mathcal{H}_{n,N}$. Using the morphism $\text{Spec}(\mathcal{R}_{n,N}) \rightarrow \mathcal{H}_{n,N}$, \mathbb{E} becomes $(\mathcal{R}_{n,N} \otimes R)^n$ in the basis $\omega_i = dx_i/x_i$, ($1 \leq i \leq n$) of regular 1-forms.

Fourier expansion map and q -expansion principle

For an algebraic representation $\rho : \text{GL}_n \rightarrow \text{GL}_d$, \mathbb{E}_ρ becomes in the above basis ω_i

$$\mathbb{E}_\rho \times_{\mathcal{H}_{n,N} \otimes R} \text{Spec}(\mathcal{R}_{n,N} \otimes R) = (\mathcal{R}_{n,N} \otimes R)^d.$$

For an R -module M , the space of Siegel modular forms with coefficients in M of weight ρ is defined as

$$\mathcal{M}_\rho(M) = H^0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_\rho \otimes_R M).$$

Then the evaluation on Mumford's abelian scheme gives a homomorphism

$$F_c : \mathcal{M}_\rho(M) \rightarrow (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N, \zeta_N]} M)^d$$

which is called the Fourier expansion map associated with c . According to [24], Theorem 2, F_c satisfies the following **q -expansion principle**:

If M' is a sub R -module of M and $f \in \mathcal{M}_\rho(M)$ satisfies that

$$F_c(f) \in (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N, \zeta_N]} M')^d,$$

then $f \in \mathcal{M}_\rho(M')$.

For q -expansion principle in the unitary case, see [12], [13].

Algebraic nearly holomorphic forms as formal Fourier expansions over a commutative ring A

Algebraically we use the notation

$$q^T = \prod_{i=1}^n q_{ii}^{T_{ii}} \prod_{i < j} q_{ij}^{2T_{ij}} \in A[[q_{11}, \dots, q_{nn}]][[q_{ij}, q_{ij}^{\pm 1}]_{i,j=1, \dots, n}$$

(with $q^T = \exp(2\pi i \text{tr}(TZ))$, $q_{ij} = \exp(2\pi(\sqrt{-1}Z_{ij}))$ for $A = \mathbb{C}$). The elements q^T form a multiplicative semi-group so that $q^{T_1 \cdot T_2} = q^{T_1 + T_2}$, and one may consider f as a formal q -expansion over an arbitrary ring A via elements of the semi - group algebra $A[[q^{B_n}]]$.

Algebraic definition of arithmetical nearly holomorphic forms, see [52] $f \in S_e(\text{Sym}^2(A^n), A[[q^{B_n}]]^d)$, where S_e denotes the A -polynomial mappings of degree e on symmetric matrices $S \in \text{Sym}^2(A^n)$ of order n with vector values in $A[[q^{B_n}]]^d$.

Notation: $f = \sum_T a_T(S)q^T \in \mathcal{N}(A)$.

General quasi-modular forms. For all $f = \sum_T a_T(S)q^T \in \mathcal{N}(A)$ define general quasi-modular forms as elements of the form

$$\kappa(f) = \sum_T a_T(0)q^T = f|_{S=0}.$$

Notation: $\kappa(f) \in \mathcal{QM}(A)$.

Computing the Petersson products

The Petersson product of a given modular form $f(Z) = \sum_T a_T q^T \in \mathcal{M}_\rho(\bar{\mathbb{Q}})$ by another modular form $h(Z) = \sum_T a_T q^T \in \mathcal{M}_{\rho^*}(\bar{\mathbb{Q}})$ produces a linear form

$$\ell_f : \mapsto \frac{\langle f, h \rangle}{\langle f, f \rangle}$$

defined over a subfield $k \subset \mathbb{Q}$. Thus ℓ_f can be expressed through the Fourier coefficients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coefficients: $\ell_{T_i} : h \mapsto b_{T_i} (i = 1, \dots, n)$. It follows that $\ell_f(h) = \sum_i \gamma_i b_{T_i}$, where $\gamma_i \in k$.

Applications to constructions of p -adic L -functions

We present here a survey of some methods of construction of p -adic L -functions. Two important ideas that are not as well known as they should be are developed briefly in this section.

There exist two kinds of L -functions

- Complex L -functions (Euler products) on $\mathbb{C} = \text{Hom}(\mathbb{R}_+^*; \mathbb{C}^*)$,
- p -adic L -functions on the \mathbb{C}_p -analytic group $\text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ (Mellin transforms \mathcal{L}_μ of p -adic measures μ on \mathbb{Z}_p^*).

Both are used in order to obtain a number (L -value) from an automorphic form. Such a number can be algebraic (after normalization) via the embeddings,

$$\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}, \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$$

and we may compare the complex and p -adic L -values at many points.

How to define and to compute p -adic L -functions? The Mellin transform of a p -adic distribution μ on \mathbb{Z}_p^* gives an analytic function on the group of p -adic characters

$$x \mapsto \mathcal{L}_\mu(x) = \int_{\mathbb{Z}_p^*} x(y) d\mu(y), \quad x \in X_{\mathbb{Z}_p^*} = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$$

A general idea is to construct p -adic measures directly from Fourier coefficients of modular forms proving Kummer-type congruences for L -values. Here we present a new method to construct p -adic L -functions via quasi-modulat forms:

How to prove Kummer-type congruences using the Fourier coefficients?

Suppose that we are given some L -function $L_f^*(s, \chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" $(s_j; \chi_j)$ one has an integral representation $\boxed{L_f^*(s, \chi) = \langle f, h_j \rangle}$ with all $h_j = \sum_T b_{j,T} q^T \in \mathcal{M}$ in a certain finite-dimensional space \mathcal{M} containing f and defined over

$\bar{\mathbb{Q}}$. We want to prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^*, \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \pmod{p^N} \implies \sum_j \beta_j \frac{L_f^*(s, \chi)}{\langle f, f \rangle} \equiv 0 \pmod{p^N}$$

for any choice of

$$\beta_j \in \bar{\mathbb{Q}}, k_j = \begin{cases} s_j - s_0 & \text{if } s_0 = \min_j s_j \text{ or} \\ k_j = s_0 - s_j & \text{if } s_0 = \max_j s_j. \end{cases}$$

Using the above expression for $\ell_f(h_j) = \sum_i \gamma_i, j b_{j, T_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j} \beta_j b_{j, T_i} \equiv 0 \pmod{p^N}.$$

Reduction to a finite dimensional case

In order to prove the congruences

$$\sum_{i,j} \gamma_{i,j} \beta_j b_{j, T_i} \equiv 0 \pmod{p^N}.$$

in general we use the functions h_j which belong only to a certain infinite dimensional $\bar{\mathbb{Q}}$ -vector space $\mathcal{M} = \mathcal{M}(\bar{\mathbb{Q}})$

$$\mathcal{M}(\bar{\mathbb{Q}}) := \bigcup_{m \geq 0} \mathcal{M}_{\rho^*}(\bar{\mathbb{Q}})(Np^m, \bar{\mathbb{Q}}).$$

Starting from the functions h_j , we use their characteristic projection $\pi = \pi^\alpha$ on the characteristic subspace \mathcal{M}^α (of eigenvectors) associated to a non-zero eigenvalue α of Atkin's U -operator on f which turns out to be of fixed finite dimension so that for all j , $\pi^\alpha(h_j) \in \mathcal{M}^\alpha$.

From holomorphic to nearly holomorphic and p -adic modular forms

Next we explain, how to treat the functions h_j which belong to a certain infinite dimensional $\bar{\mathbb{Q}}$ -vector space $\mathcal{N} \subset \mathcal{N}_\rho(\bar{\mathbb{Q}})$ (of nearly holomorphic modular forms).

Usually, h_j can be expressed through the functions of the form $\delta_{k_j}(\varphi_0(\chi_j))$ for a certain non-negative power k_j of the Maass-Shimura-type differential operator applied to a holomorphic form $\varphi_0(\chi_j)$. Then the idea is to proceed in two steps:

1) to pass from the infinite dimensional $\bar{\mathbb{Q}}$ -vector space $\mathcal{N} = \mathcal{N}(\bar{\mathbb{Q}})$ of nearly holomorphic modular forms,

$$\mathcal{N}(\bar{\mathbb{Q}}) := \bigcup_{m \geq 0} \mathcal{N}_{k,r}(\bar{\mathbb{Q}})(Np^m, \bar{\mathbb{Q}})$$

(of the depth r) to a fixed finite dimensional characteristic subspace $\mathcal{N}^\alpha \subset \mathcal{N}(Np)$ of U_p in the same way as for the holomorphic forms. This step controls Petersson products using the conjugate f^0 of an eigenfunction f_0 of $U(p)$:

$$\langle f^0, h \rangle = \alpha^{-m} \langle f^0, h|U(p)^m \rangle = \langle f^0, \pi^\alpha(h) \rangle.$$

From holomorphic to nearly holomorphic and p -adic modular forms: Ichikawa's mapping

2) Let us apply Ichikawa's mapping $\iota_p : \mathcal{N}(Np) \rightarrow \mathcal{M}_p(Np)$ to a certain space $\mathcal{M}_p(Np)$ of p -adic Siegel modular forms. Notice also that the realization of nearly holomorphic forms as p -adic modular forms has been studied by Eric Urban, who calls them "Nearly overconvergent modular forms" [55], Chapter 10.

Let us assume algebraically,

$$h_j = \sum_T b_{j,T}(S)q^T \mapsto (\kappa_j) = \sum_T b_{j,T}(0)q^T$$

which is also a certain Siegel quasi-modular form. Under this mapping, computation become much easier, as the action of δ^{k_j} becomes simply a k_j -power of the Ramanujan Θ -operator

$$\Theta : \sum_T b_T q^T \mapsto \sum_T \det(T) b_T(0) q^T$$

in the scalar-valued case. In the vector-valued case such operators were studied in [BoeNa13].

After this step, proving the Kummer-type congruences reduces to those for the Fourier coefficients the quasi-modular forms $\kappa(h_j(\chi_j))$ which can be explicitly evaluated using the Θ -operator.

Computing with Siegel modular forms over a ring A

There are several types of Siegel modular forms (vector-valued, nearly-holomorphic, quasi-modular, p -adic). Consider modular forms over a ring $A = \mathbb{C}, \mathbb{C}_p, \Lambda = \mathbb{Z}_p[[T]], \dots$ as certain formal Fourier expansions over A . Let us fix the congruence subgroup Γ of a nearly holomorphic modular form

$f \in \mathcal{N}_\rho$ and its depth r as the maximal S -degree of the polynomial Fourier Fourier coefficients $a_T(S)$ of a nearly holomorphic form

$$h_j = \sum_T b_{j,T}(S)q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0)q^T$$

which is also a certain Siegel quasi-modular form. Under this mapping, computations become much easier, as the action of δ^{k_j} becomes simply a k_j -power of the Ramanujan Θ -operator

$$\Theta : \sum_T a_T(S)q^T \in \mathcal{N}(A)$$

over R , and denote by $\mathcal{N}_{\rho,r}(\Gamma, A)$ the A -module of all such forms. This module is often locally-free of finite rank, that is, it becomes a finite-dimensional F -vector space over the fraction field $F = \text{Frac}(A)$.

Types of modular forms

- \mathcal{M}_ρ (holomorphic vector-valued Siegel modular forms attached to an algebraic representation $\rho : \text{GL}_n \rightarrow \text{GL}_d$)
- $QM = \mathcal{N}|_{S=0}$ (quasi-modular vector-valued forms attached to ρ)
- \mathcal{N}_ρ (holomorphic vector-valued Siegel modular forms, algebraic p -adic vector-valued forms attached to ρ over a number field $k \subset \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$)

Definitions and interrelations:

- $QM_{\rho,r} = \kappa(\mathcal{N}_{\rho,r})$, where $\kappa : f \mapsto f|_{S=0} = \sum_T P_T(0)q^T$, with the notation $\mathcal{R}_{n,\infty} = \mathbb{C}[[q_{11}, \dots, q_{nn}]] [q_{ij}, q_{ij}^{-1}]_{ij=1, \dots, n}$
- $\mathcal{M}_{\rho,r}^b(R, \Gamma) = F_c(\iota_p(\mathcal{N}_{\rho,r}(R, \Gamma))) \subset \mathcal{R}_{n,p}^d$ where

$$\mathcal{R}_{n,p} = \mathbb{C}_p[[q_{11}, \dots, q_{nn}]] [q_{ij}, q_{ij}^{-1}]_{ij=1, \dots, n}$$

Let us fix the level Γ , the depth r , and a subring R of $\overline{\mathbb{Q}}$, then all the R -modules $\mathcal{M}_{\rho,r}(R, \Gamma)$, $\mathcal{N}_{\rho,r}(R, \Gamma)$, $QM_{\rho,r}(R, \Gamma)$, $\mathcal{M}_{\rho,r}^b(R, \Gamma)$ are then locally free of finite rank.

In interesting cases, there is an inclusion

$$QM_{\rho,r}(R, \Gamma) \hookrightarrow \mathcal{M}_{\rho,r}^b(R, \Gamma).$$

If $\Gamma = \text{SL}_2(\mathbb{Z})$, $k = 2$, $P = E_2$ is a p -adic modular form, see [Se73], p.211.

Question: Prove it in general! (after discussions with S.Boecherer and T.Ichikawa)¹

¹In June 2014, an affirmative answer is given by S.Boecherer for the Siegel modular group

4. Applications to families of arithmetical automorphic forms

We treat only the Siegel modular case here but the results can be extended to the general Sp- and unitary cases (UT in Shimura's terminology).

Computing with families of Siegel modular forms

Let $\Lambda = \mathbb{Z}_p[[T]]$ be the Iwasawa algebra, and consider Serre's ring

$$\mathcal{R}_{n,\Lambda} = \Lambda[[q_{11}, \dots, q_{nn}]] [q_{ij}, q_{ij}^{-1}]_{ij=1, \dots, n}$$

For any pair (k, χ) as above consider the homomorphisms:

$$\kappa_{(k,\chi)} : \Lambda \rightarrow \mathbb{C}_p, \quad \mathcal{R}_{n,\Lambda} \rightarrow \mathcal{R}_{n,\mathbb{C}_p},$$

where $T \mapsto \chi(1+p)(1+p)^k - 1$.

DEFINITION 4.1 (Families of Siegel modular forms). *Let $\mathbf{f} \in \mathcal{R}_{n,\Lambda}$ such that for infinitely many pairs (k, χ) as above,*

$$\kappa_{k,\chi}(\mathbf{f}) \in \mathcal{M}_{\rho_k}((i_p(\bar{\mathbb{Q}}))) \xrightarrow{F_c} \mathcal{R}_{n,\mathbb{C}_p}^d$$

is the Fourier expansion at c of a Siegel modular form over $\bar{\mathbb{Q}}$. All such \mathbf{f} generate the Λ -submodule $\mathcal{M}_{\rho_k}(\Lambda) \subset \mathcal{R}_{n,\Lambda}^d$ of Λ -adic Siegel modular forms of weight ρ .

In the same way, the Λ -submodule

$$QM_{\rho_k}(\Lambda) \subset \mathcal{R}_{n,\Lambda}^d$$

of Λ -adic Siegel quasi-modular forms is defined.

Examples of families of Siegel modular forms

can be constructed via differential operators of Maass

$$\Delta = \det \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial z_{ij}} \right),$$

so that $\Delta q^T = \det(T) q^T$. Shimura's operator

$$\delta_k f(Z) = (-4\pi)^{-n} \det(Z - \bar{Z})^{\frac{1+n}{2} - k} \Delta(\det(Z - \bar{Z})^{k - \frac{1+n}{2} + 1} f)(Z)$$

acts on q^T using $\rho_r : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{GL}(\wedge^r \mathbb{C}^n)$ and its adjoint ρ_r^* :

$$\delta_k(q^T) = \sum_{l=0}^n c_{n-l} \left(k + 1 - \frac{1+n}{2} \right) \mathrm{tr}({}^t \rho_{n-l}(S) \rho_l^*(T)) q^T,$$

where $c_{n-l}(s) = s(s - \frac{1}{2}) \dots (s - \frac{n-l-1}{2})$, $S = (2\pi i(\bar{z} - z))^{-1}$.

- Nearly holomorphic Λ -adic Siegel-Eisenstein series as in [PaSE] can be produced from the pairs $(-s, \chi)$: if s is a nonpositive integer such that $k + 2s > n + 1$

$$E_k(Z, s, \chi) = \prod_{i=0}^{-s-1} c_n(k + 2s + 2i)^{-1} \delta_{k+2s}^{(-s)}(E_{k+2s}(Z, 0, \chi)).$$

- **Ichikawa's construction:** quasi-holomorphic (and p -adic) Siegel-Eisenstein series obtained in [24] using the injection ι_p :

$$\iota_p(\pi^{ns} E_k(Z, s, \chi)) = \prod_{i=0}^{-s-1} c_n(k + 2s + 2i)^{-1} \sum_T \det(T)^{-s} b_{k+2s}(T) q^T,$$

where $E_{k+2s}(Z, 0, \chi) = \sum_T b_{k+2s}(T) q^T, k + 2s > n + 1, s \in \mathbb{Z}$.

- A two-variable family is for the parameters $(k + 2s, s), k + 2s > n + 1, s \in \mathbb{Z}$ will be now constructed. Normalized Siegel-Eisenstein series of two variables Let us start with an explicit family described in [25], [42], [40] as follows $\mathcal{E}_k^n = E_k^n(z) 2^{n/2} \zeta(1 - k) \prod_{i=1}^{[n/2]} \zeta(1 - 2k + 2i) = \sum_T a_T(\mathcal{E}_k^n) q^T$, where for any non-degenerate matrix T of quadratic character ψ_T .

PROPOSITION 4.2. — *Let $k > m + 1$.*

- (1) *For any non-degenerate matrix $h \in C_m$ the following equality holds*

$$a_h(\mathcal{E}_k^m) = 2^{-\frac{m}{2}} \det h^{k - \frac{m+1}{2}} M_h(k) \tag{4.1}$$

$$\times \begin{cases} L(1 - k + \frac{m}{2}, \psi_h) C_h^{\frac{m}{2} - k + (1/2)}, & m \text{ even,} \\ 1, & m \text{ odd,} \end{cases}$$

where C_h is the conductor of ψ_h .

- (2) *for any prime $p > 2$, and $\det(2h)$ not divisible by p , define the p -regular part $a_h(\mathcal{E}_k^m)^{(p)}$ of the coefficient $a_h(\mathcal{E}_k^m)$ of \mathcal{E}_k^m by introducing the factor $\begin{cases} (1 - \psi_h(p) p^{k - \frac{m}{2} - 1}) C_h^{\frac{m}{2} - k + (1/2)}, & m \text{ even,} \\ 1, & m \text{ odd.} \end{cases}$*

Then $a_h(\mathcal{E}_k^m)^{(p)}$ is a p -adic analytic Iwasawa function of $t = (1+p)^k - 1$ for all k with ω^k fixed, and divided by the elementary factor $1 - \psi_h(c_h) c_h^{k - \frac{m}{2}}$.

- Then Ichikawa's construction is applicable and it provides a two-variable family.

Further examples of families of Siegel modular forms

- Ikeda-type families of cusp forms of even genus [47]. Start from a p -adic family

$$\varphi = \{\varphi_{2k}\} : 2k \mapsto \varphi_{2k} = \sum_{n=1}^{\infty} a_n(2k)q^n \in \mathbb{Q}[[q]] \subset \mathbb{C}_p[[q]]$$

where the Fourier coefficients $a_n(2k)$ of the normalized cusp Hecke eigenform φ_{2k} and one of the Satake p -parameters $\alpha(2k) := \alpha_p(2k)$ are given by certain p -adic analytic functions $k \mapsto a_n(2k)$ for all $(n, p) = 1$. The Fourier expansions of the modular forms $F = F_{2n}(\varphi_{2k})$ can be explicitly evaluated where

$$L(F_{2n}(\varphi), St, s) = \zeta(s) \prod_{i=1}^{2n} L(\varphi, s + k + n - i).$$

This sequence provide an example of a p -adic family of Siegel modular forms.

- Ikeda-Myawaki-type families of cusp forms of $n = 3$, [47].
- Families of Klingen-Eisenstein series extended from $n = 2$ to a general case (reported in Journées Arithmétiques, Grenoble, July 2013).

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