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Approximate tensorization of entropy at high temperature

PIETRO CAPUTO⁽¹⁾, GEORG MENZ⁽²⁾, PRASAD TETALI⁽³⁾

RÉSUMÉ. — On montre que pour des variables aléatoires faiblement dépendentes l'entropie relative satisfait une version approximée de la propriété de tensorisation associée au cas indépendant. Cela implique une famille d'inégalités de Sobolev logarithmiques indépendentes de la dimension. Pour des systèmes de spin en interaction sur un graphe, la condition de dépendence faible devient une sorte de condition de unicité de Dobrushin. Nos résultats représentent par ailleurs une version discrète d'un travail récent par Katalin Marton [27]. On considère aussi des généralisations naturelles de ces résultats tels que des inégalités de Shearer approximées.

ABSTRACT. — We show that for weakly dependent random variables the relative entropy functional satisfies an approximate version of the standard tensorization property which holds in the independent case. As a corollary we obtain a family of dimensionless logarithmic Sobolev inequalities. In the context of spin systems on a graph, the weak dependence requirements resemble the well known Dobrushin uniqueness conditions. Our results can be considered as a discrete counterpart of a recent work of Katalin Marton [27]. We also discuss some natural generalizations such as approximate Shearer estimates and subadditivity of entropy.

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1. Introduction

Consider a product measurable space (Ω, \mathcal{F}) of the form

$$(\Omega, \mathcal{F}) = (\Omega_1, \mathcal{F}_1) \times \cdots \times (\Omega_N, \mathcal{F}_N), \quad (1.1)$$

where $(\Omega_k, \mathcal{F}_k)$, $k = 1, \dots, N$ are given measurable spaces. Let μ be a probability measure on (Ω, \mathcal{F}) . When μ is a product $\mu = \otimes_{k=1}^N \mu_k$, with μ_k a probability measure on $(\Omega_k, \mathcal{F}_k)$, then it is well known (see e.g. [1]) that the entropy functional satisfies the inequality

$$\text{Ent}_\mu(f) \leq \sum_k \mu [\text{Ent}_{\mu_k}(f)], \quad (1.2)$$

for all bounded measurable functions $f : \Omega \mapsto [0, \infty)$. Here, as usual,

$$\text{Ent}_\mu(f) = \mu [f \log f] - \mu [f] \log \mu [f],$$

which equals $\mu[f]$ times the relative entropy of $\nu = (f/\mu[f])\mu$ with respect to μ . We refer to inequality (1.2) as the tensorization property of entropy. In the general case where μ is not a product measure, we define the probability measure μ_k by conditioning on all variables $x_j \in \Omega_j$, with $j \neq k$. Thus $\mu_k[f]$ denotes the function given by

$$\mu_k[f](x) = \mu [f | x_j, j \neq k], \quad (1.3)$$

and $\text{Ent}_{\mu_k}(f)$ denotes the function $\mu_k[f \log f] - \mu_k[f] \log(\mu_k[f])$. We shall investigate the validity of an approximate tensorization statement of the form

$$\text{Ent}_\mu(f) \leq C \sum_k \mu [\text{Ent}_{\mu_k}(f)], \quad (1.4)$$

for all bounded measurable functions $f : \Omega \mapsto [0, \infty)$, where $C > 0$ is a constant independent of f . We say that μ satisfies $AT(C)$ whenever (1.4) holds. As we discuss below, if μ satisfies such a bound, then one can deduce entropy related functional inequalities such as log-Sobolev or modified log-Sobolev inequalities for the N -component systems as a consequence of the corresponding inequalities for each component.

The idea that a system with weakly dependent components should display some kind of tensorization of entropy is implicitly at the heart of the large body of literature devoted to the proof of logarithmic Sobolev inequalities for spin systems satisfying Dobrushin's uniqueness conditions or more general spatial mixing conditions; see [33, 30, 23, 25, 26, 20, 11, 13]. Perhaps surprisingly, none of these works addresses explicitly the validity of the

statement (1.4). Recently, the inequality (1.4) has been considered by Marton [27] in the case of continuous spins, with $\Omega = \mathbb{R}^N$ and μ an absolutely continuous measure of the form $\mu(dx) = e^{-V(x)}dx$. The author proves (1.4) under suitable weak dependence conditions that are formulated in terms of the Hessian of V . This allows her to derive the logarithmic Sobolev inequality beyond the usual Bakry-Émery criterion [2] or the more recent Otto-Reznikoff criterion [29].

In this paper, we focus on deriving general sufficient conditions on $(\Omega, \mathcal{F}, \mu)$ for inequality (1.4) to be satisfied. In particular, for spin systems with bounded local interactions, we shall establish that approximate tensorization holds as soon as the temperature is high enough, regardless of the single spin space and the underlying spatial structure.

Next, we observe that the tensorization property (1.2) is a member of a much larger family of inequalities, often called Shearer inequalities, satisfied by product measures. In Section 2.4 below we briefly discuss the problem of establishing approximate Shearer estimates for weakly dependent non-product measures.

1.1. Approximate tensorization and the Heat Bath chain

Before describing our results in detail, let us discuss some basic relations between approximate tensorization and functional inequalities for the *Heat Bath* Markov chain (also known as Glauber dynamics or Gibbs sampler). To define the latter, consider the operator \mathcal{L} given by

$$\mathcal{L}f(x) = \sum_k (\mu_k[f](x) - f(x)),$$

where $f : \Omega \mapsto \mathbb{R}$. Then \mathcal{L} defines the infinitesimal generator of a continuous time Markov chain on Ω , such that with rate 1 independently each component Ω_k , $k = 1, \dots, N$ is updated by replacing x_k with a value x'_k sampled from the conditional distribution $\mu[\cdot | x_j, j \neq k]$. The operator \mathcal{L} is a bounded self-adjoint operator in $L^2(\Omega, \mu)$ and the Markov chain is reversible with respect to μ . We denote by $(e^{t\mathcal{L}}, t \geq 0)$ the heat bath semigroup; see e.g. [3]. The Dirichlet form of the process is given by

$$\mathcal{E}(f, g) = \mu[f(-\mathcal{L}g)] = \sum_k \mu[\text{Cov}_{\mu_k}(f, g)], \quad (1.5)$$

where $f, g \in L^2(\Omega, \mu)$ and $\text{Cov}_{\mu_k}(f, g)$ denotes the covariance

$$\text{Cov}_{\mu_k}(f, g) = \mu_k[fg] - \mu_k[f]\mu_k[g].$$

The following inequalities are commonly studied in the Markov chain literature. Say that μ satisfies $P(C)$, or the Poincaré inequality with constant C , if

$$\text{Var}_\mu(g) \leq C \sum_k \mu [\text{Var}_{\mu_k}(g)], \tag{1.6}$$

for any bounded function $g : \Omega \mapsto \mathbb{R}$, where $\text{Var}_\mu(g) = \mu[g^2] - \mu[g]^2$ denotes the variance. Moreover, say that μ satisfies $LS(C)$, or the Log-Sobolev inequality with constant C , if

$$\text{Ent}_\mu(f) \leq C \sum_k \mu [\text{Var}_{\mu_k}(\sqrt{f})],$$

for all bounded measurable functions $f : \Omega \mapsto [0, \infty)$. Finally, say that μ satisfies $MLS(C)$, or the Modified Log-Sobolev inequality with constant C , if

$$\text{Ent}_\mu(f) \leq C \sum_k \mu [\text{Cov}_{\mu_k}(f, \log f)],$$

for all bounded measurable functions $f : \Omega \mapsto [0, \infty)$. Modified log-Sobolev inequalities have received increasing attention in recent years [13, 18, 19, 8, 5], also in view of their connections with mixing times of Markov chains [28].

It is well known that $P(C)$ is equivalent to exponential decay of the variance in the form $\text{Var}_\mu(f_t) \leq e^{-2t/C} \text{Var}_\mu(f)$, for all $f \in L^2(\Omega, \mu)$ and for all $t \geq 0$, where $f_t = e^{t\mathcal{L}}f$. Similarly, $MLS(C)$ is equivalent to exponential decay of the entropy in the form

$$\text{Ent}_\mu(f_t) \leq e^{-t/C} \text{Ent}_\mu(f), \tag{1.7}$$

for all $t \geq 0$, for all functions bounded measurable $f \geq 0$, while $LS(C)$ is equivalent to a hypercontractivity property of the heat bath semigroup; see [15]. The following implications are also well known [15]: for any $C > 0$, $LS(C) \Rightarrow MLS(C/4) \Rightarrow P(C/2)$. The approximate tensorization property $AT(C)$ is naturally linked to the above inequalities as summarized below.

PROPOSITION 1.1. — *The following implications hold for any $C > 0$:*

$$AT(C) \Rightarrow P(C), \quad LS(C) \Rightarrow AT(C) \Rightarrow MLS(C)$$

Proof. — $AT(C) \Rightarrow P(C)$ follows by linearization: (1.6) can be obtained from (1.4) by considering functions f of the form $1 + \varepsilon g$ with g bounded and taking the limit $\varepsilon \rightarrow 0$. In words, approximate tensorization of entropy

implies approximate tensorization of variance, with the same constant C . To prove $AT(C) \Rightarrow MLS(C)$, observe that by Jensen's inequality, for all functions $f \geq 0$ and $k \in [N]$:

$$\text{Ent}_{\mu_k}(f) \leq \text{Cov}_{\mu_k}(f, \log f).$$

Finally, $LS(C) \Rightarrow AT(C)$ follows from the well known fact that $\text{Var}_{\mu}(\sqrt{f}) \leq \text{Ent}_{\mu}(f)$ for any probability measure μ and any bounded function $f \geq 0$. \square

2. Main results

For simplicity of exposition we formulate our results in the case where each of the spaces Ω_k in (1.1) is finite or at most countably infinite, but there is no difficulty in transferring the same proof e.g. to the case $\Omega = \mathbb{R}^N$. However, one should keep in mind that our main assumptions involve L_{∞} norms and therefore they are not ideally suited to deal with unbounded interactions.

The weak dependence assumption is formulated as follows. For each $k \in [N] = \{1, \dots, N\}$, set $\bar{\Omega}_k = \times_{j \in [N]: j \neq k} \Omega_j$ and write $\bar{x}_k \in \bar{\Omega}_k$ for the vector $(x_j, j \neq k)$. Similarly, for $i \neq k$, set $\bar{\Omega}_{k,i} = \times_{j \in [N]: j \neq i, j \neq k} \Omega_j$ and write $\bar{x}_{k,i} \in \bar{\Omega}_{k,i}$ for the vector $(x_j, j \neq k, j \neq i)$. For $x = (x_1, \dots, x_N)$ we write $x = (x_k, \bar{x}_k)$ and let

$$\mu_k^{\bar{x}_k}(x_k) = \mu(x_k | \bar{x}_k) = \frac{\mu(x_k, \bar{x}_k)}{\sum_{y_k \in \Omega_k} \mu(y_k, \bar{x}_k)}$$

denote the conditional probability on Ω_k , so that

$$\mu_k[g](x) = \sum_{y_k \in \Omega_k} \mu_k^{\bar{x}_k}(y_k) g(y_k, \bar{x}_k),$$

for any bounded $g : \Omega \mapsto \mathbb{R}$. For fixed $i \neq k$, consider the function $\varphi_{i,k} : \Omega_i^2 \times \Omega_k \times \bar{\Omega}_{k,i} \mapsto \mathbb{R}$ defined by

$$\varphi_{i,k}(x_i, y_i, x_k, \bar{x}_{i,k}) = \frac{\mu_k^{x_i, \bar{x}_{i,k}}(x_k)}{\mu_k^{y_i, \bar{x}_{i,k}}(x_k)}. \tag{2.1}$$

Our main assumption is formulated in terms of the $\varphi_{i,k}$ as follows. Define the coefficients

$$\alpha_{i,k} = \sup_{x_i, y_i \in \Omega_i, x_k \in \Omega_k, \bar{x}_{i,k} \in \bar{\Omega}_{i,k}} \varphi_{i,k}(x_i, y_i, x_k, \bar{x}_{i,k}), \tag{2.2}$$

$$\delta_{i,k} = \sup_{x_i, y_i \in \Omega_i, x_k, y_k \in \Omega_k, \bar{x}_{i,k} \in \bar{\Omega}_{i,k}} |\varphi_{i,k}(x_i, y_i, x_k, \bar{x}_{i,k}) - \varphi_{i,k}(x_i, y_i, y_k, \bar{x}_{i,k})|.$$

Notice that if μ is a product measure then $\alpha_{i,k} = 1$ and $\delta_{i,k} = 0$ for all $i \neq k$.

THEOREM 2.1. — *Suppose that the measure μ satisfies the ergodicity assumption*

$$\lim_{t \rightarrow \infty} \text{Ent}_\mu(f_t) = 0, \tag{2.3}$$

for every bounded measurable function $f : \Omega \mapsto \mathbb{R}_+$, where $f_t = e^{t\mathcal{L}}f$, and $(e^{t\mathcal{L}}, t \geq 0)$ is the heat bath semigroup. Assume that the coefficients $\{\alpha_{i,k}, \delta_{i,k}\}$ satisfy $\gamma + \kappa < 1$ where

$$\gamma = \max_i \sum_{k: k \neq i} |\alpha_{i,k} \alpha_{k,i} - 1|, \quad \kappa = \frac{1}{4} \max_i \sum_{k: k \neq i} (\delta_{k,i} + \delta_{i,k}) \alpha_{i,k} \alpha_{k,i}.$$

Then the approximate tensorization (1.4) holds with $C = (1 - \gamma - \kappa)^{-1}$.

The proof of Theorem 2.1 is given in Section 3. As in Marton’s paper [27], the proof follows the semigroup approach. An important difference in our argument is that we use the heat bath semigroup where Marton uses the Langevin diffusion. While the overall strategy of the proof is similar, our setting requires a different technique because of the lack of differential calculus. Moreover, in contrast with [27], we do not need to require a one-site log-Sobolev inequality in our assumptions. This allows us to establish the approximate tensorization for the invariant measure of Markov chains without log-Sobolev inequality, or even without Poincaré inequality, see the comments after Corollary 2.3 below.

2.1. Applications

Next, we discuss the implications of Theorem 2.1 in specific examples. For ease of exposition we limit ourselves to probability measures of the following form. Let $\mu_0(x)$ denote a product measure on Ω of the form $\mu_0(x) = \prod_{i=1}^N \mu_{0,i}(x_i)$, where $\mu_{0,i}$ is a probability measure on Ω_i for each $i \in [N]$, and consider the probability measure μ on Ω given by

$$\mu(x) = \frac{\mu_0(x) e^{\beta W(x)}}{Z}, \quad W(x) = \frac{1}{2} \sum_{i,j \in [N]} J_{i,j} w_{i,j}(x_i, x_j), \tag{2.4}$$

where Z is the normalizing factor, the coefficients $J_{i,j} \in \mathbb{R}$ are assumed to satisfy $J_{i,i} = 0$, $J_{i,j} = J_{j,i}$, and we assume that the functions $w_{i,j}$ satisfy

$$\|w_{i,j}\|_\infty = \sup_{x_i, x_j} |w_{i,j}(x_i, x_j)| < \infty. \tag{2.5}$$

Here $\beta > 0$ is a parameter, the inverse temperature, measuring the strength of the interaction. At $\beta = 0$ there is no dependence and the inequality (1.4)

holds with $C = 1$. Notice that the function W in (2.4) is bounded in the sense that

$$\|W\|_\infty \leq \frac{1}{2} \sum_{i,j \in [N]} |J_{i,j}| \|w_{i,j}\|_\infty < \infty,$$

where the bound depends on N in general. We first observe that a simple perturbation argument can be applied to obtain approximate tensorization for any $\beta > 0$, with a constant C depending on $\|W\|_\infty$ and β .

LEMMA 2.2. — *Under the assumption (2.5) the measure μ in (2.4) satisfies the approximate tensorization (1.4) with $C = e^{6\beta\|W\|_\infty}$. In particular, (1.7) holds with the same C .*

Proof. — One has

$$\text{Ent}_\mu(f) \leq e^{2\beta\|W\|_\infty} \text{Ent}_{\mu_0}(f) \leq e^{2\beta\|W\|_\infty} \sum_k \mu_0 [\text{Ent}_{\mu_{0,k}}(f)],$$

where the first bound follows from the Holley-Stroock perturbation argument [21], while the second one is (1.2). One more application of the Holley-Stroock perturbation estimate yields $\text{Ent}_{\mu_{0,k}}(f) \leq e^{2\beta\|W\|_\infty} \text{Ent}_{\mu_k}(f)$, and using $\mu_0(x) \leq e^{2\beta\|W\|_\infty} \mu(x)$ one obtains the claim. \square

The above lemma, using Proposition 1.1 and the estimate in (1.7), shows in particular that the ergodicity assumption (2.3) is always satisfied in this setting. However, it represents a very poor estimate unless $\|W\|_\infty$ does not depend on N . Below, we consider cases where the function W is not bounded uniformly in N , including systems, such as the Ising model, where a phase transition can occur by varying the parameter β . The main corollary of Theorem 2.1 is summarized as follows.

COROLLARY 2.3. — *Define $\varepsilon_{i,k} = 4\beta|J_{k,i}|\|w_{i,k}\|_\infty$ and assume*

$$q := \max_i \sum_{k: k \neq i} e^{\varepsilon_{i,k}} (e^{2\varepsilon_{i,k}} - 1) < \frac{2}{3}. \tag{2.6}$$

Then, the measure μ in (2.4) has the approximate tensorization (1.4) with $C = (1 - \frac{3}{2}q)^{-1}$.

The proof of Corollary 2.3 is given in Section 4. It is interesting to note that the estimate of Corollary 2.3 is uniform in the choice of the single probability distributions $\mu_{0,i}$ in (2.4), since the smallness condition (2.6) does not involve the single measures $\mu_{0,i}$. In particular, the single measures $\mu_{0,i}$ are not required to satisfy a Poincaré inequality or any other condition. Below, we discuss some specific applications of Corollary 2.3. For simplicity we limit ourselves to Glauber dynamics for discrete spin systems and interacting birth and death chains.

2.2. Spin systems

Consider the Ising model defined as follows. Let $\Omega = \{-1, +1\}^N$, and set

$$\mu(x) = \frac{1}{Z} \exp\left(\frac{1}{2}\beta \sum_{i,j \in [N]} J_{i,j} x_i x_j + \sum_{i \in [N]} h_i x_i\right), \quad (2.7)$$

where Z is the normalizing factor, $\beta > 0$ is the inverse temperature, the couplings $J_{i,j} \in \mathbb{R}$ are assumed to satisfy $J_{i,j} = J_{j,i}$, and the $h_i \in \mathbb{R}$ are the so-called external fields. Since the external fields can be absorbed in the single measures $\mu_{0,i}$, it is immediate to check that the above μ has the form (2.4) with $w_{i,j}(x_i, x_j) = x_i x_j$. Therefore, uniformly in the external fields, the Ising model defined in (2.7) has the approximate tensorization (1.4) with $C = (1 - \frac{3}{2}q)^{-1}$ as soon as (2.6) holds.

A special case is the *ferromagnetic Ising model on a graph* $G = (V, E)$, $|V| = N$, which corresponds to the choice $J_{i,j} = \mathbf{1}(\{i, j\} \in E)$. In this case one can obtain the following explicit criterion. Let $\Delta = \max_i \sum_{k: k \neq i} |J_{ki}|$ denote the maximal degree of the graph. Using e.g. $e^t(e^{2t} - 1) < 3t$ for $t < 1/5$ one finds that if $4\beta < 1/5$, then $e^{\varepsilon_{i,k}}(e^{2\varepsilon_{i,k}} - 1) < 12\beta |J_{k,i}|$, so that (2.6) is satisfied as soon as $\beta \leq \beta_0 \Delta^{-1}$ with e.g. $\beta_0 = (18)^{-1}$. Another example is the mean field model or *Curie-Weiss model*, which corresponds to $J_{i,j} = \frac{1}{N}$ for all $i, j \in [N]$. In this case, reasoning as above one obtains that (2.6) is satisfied as soon as e.g. $\beta \leq \beta_0 = 0.1$. The critical point of the Curie-Weiss model is at $\beta = 1$, and therefore it is well known that approximate tensorization cannot hold for $\beta \geq 1$ since already the Poincaré inequality (1.6) fails beyond this point; see [16]. The above numbers β_0 can be improved slightly by a more accurate analysis of the values of β which allow the estimate (2.6), but it is clear that they will generally be far from the optimal values.

The result of Corollary 2.3 can actually be extended to a much larger class of spin systems, where the spin takes a finite number $s \geq 2$ of values. For example, letting $\Omega = \{1, \dots, s\}^N$ one may define the *Potts model* Gibbs measure

$$\mu(x) = \frac{1}{Z} \exp\left(\frac{1}{2}\beta \sum_{i,j \in [N]} J_{i,j} \mathbf{1}(x_i = x_j) + \sum_{i \in [N]} h_i x_i\right).$$

With the same arguments of Corollary 2.3, one obtains, for example, for the Potts model on a graph G with maximal degree Δ , that there exists $\beta_0(s) > 0$ such that the approximate tensorization (1.4) holds as soon as $\beta \leq \beta_0(s) \Delta^{-1}$ uniformly in the external fields.

Remark 2.4. — We point out that in the case of spin systems on the lattice \mathbb{Z}^d , it is known that the Log-Sobolev inequality holds for the heat

bath dynamics under so-called “strong spatial mixing” conditions; see [30, 25, 11, 13]. Moreover, it is known that these spatial mixing conditions can cover a larger region of the phase diagram than our Dobrushin condition (2.6); see [25]. Our assumption (2.6) compares well with Zegarlin’s earlier result [32]. Since $LS(C)$ implies $AT(C)$ by Proposition 1.1, our results are weaker than already known estimates in these cases. The only interest here could be the very different nature of the proof.

2.3. Interacting birth and death chains

Here we investigate some special cases of the general model (2.4) with unbounded variables. To fix ideas, consider the case where $\Omega = \mathbb{Z}_+^N$, where $\mathbb{Z}_+ = \{0, 1, \dots\}$. Consider the probability measure μ on Ω given by (2.4), where the $\mu_{0,i}$ are fixed reference probability measures on $\Omega_i = \mathbb{Z}_+$ defined as follows. Let $\nu(n)$, $n \in \mathbb{Z}_+$ denote a probability measure such that

$$\nu(n)^2 \geq \frac{n+1}{n} \nu(n+1)\nu(n-1), \quad n \geq 1.$$

Such a measure is called ultra log-concave; see e.g. [22]. The basic example is the Poisson distribution with parameter $\lambda > 0$, with $\nu(n) = e^{-\lambda} \lambda^n / n!$. Let $F_i : \mathbb{Z}_+ \mapsto \mathbb{R}$ denote arbitrary functions such that $F_\infty := \max_i \|F_i\|_\infty < \infty$ and define the probability

$$\mu_{0,i}(x_i) = \frac{\nu(x_i) e^{F_i(x_i)}}{Z},$$

where Z denotes the normalization. From [7, Theorem 3.1] we know that for each i , $\mu_{0,i}$ satisfies the following modified log-Sobolev inequality in \mathbb{Z}_+ :

$$\text{Ent}_{\mu_{0,i}}(f) \leq C_0 \mu_{0,i} [\partial_i f \partial_i \log f], \tag{2.8}$$

with $C_0 = e^{4F_\infty} \nu(1) / \nu(0)$, where $f : \mathbb{Z}_+ \mapsto \mathbb{R}_+$ and $\partial_i f(x_i) := f(x_i + 1) - f(x_i)$, $x_i \in \mathbb{Z}_+$. The inequality (2.8) expresses the exponential decay of entropy for the birth and death process with birth rate $b(n) = 1$ and with death rate $d(n) = \mathbf{1}(n \geq 1) \mu_{0,i}(n-1) / \mu_{0,i}(n)$, see [7, 12]. The gradient operator ∂_i is extended to functions f on Ω by setting $\partial_i f(x) = f(x + e_i) - f(x)$, e_i denoting the unit vector in the i -th direction.

COROLLARY 2.5. — *Consider the measure μ given by (2.4) with $\mu_{0,i}$ as above. Suppose the interaction term βW satisfies the assumption (2.6). Then μ satisfies (1.4) with $C = (1 - \frac{3}{2}q)^{-1}$. Moreover, one has the modified log-Sobolev inequality*

$$\text{Ent}_\mu(f) \leq K \sum_i \mu [\partial_i f \partial_i \log f], \tag{2.9}$$

with constant $K = CC_0 e^{1/3}$, for all $f : \Omega \mapsto \mathbb{R}_+$.

The inequality (2.9), which can be interpreted as the exponential decay of entropy for interacting birth and death processes, could have been established also by an extension of the discrete Γ_2 approach of [7], see the recent paper [14]; see also [13] for an alternative approach. Let us remark that in contrast with the case of bounded spin systems it is essential here to consider the modified log-Sobolev inequality rather than the log-Sobolev inequality itself, since even the one-dimensional bound (2.8) could fail if we replace the energy $\mu_{0,i} [\partial_i f \partial_i \log f]$ by $\mu_{0,i} [(\partial_i \sqrt{f})^2]$, as it is seen for example when ν is the Poisson distribution. Finally, we point out that it would be desirable to prove a modified log-Sobolev inequality as in (2.9) under weaker assumptions than (2.5), in order to include unbounded interactions of log-concave type, see [8] for some examples. This would be natural from a discrete Γ_2 perspective; see [14, Section 3.2] where some progress in this direction was recently made in the case $N = 2$. For continuous unbounded spins, the log-Sobolev inequality at high temperature, beyond the Bakry-Émery criterion, has been established in [34, 31, 6, 29, 27].

2.4. Approximate Shearer inequalities and subadditivity

We conclude this introduction with some notes on possible extensions of the previous results. Let \mathcal{B} be a cover of $[N]$, that is a family of subsets $B \subset [N]$ whose union equals $[N]$. Let $\deg_k(\mathcal{B})$ denote the degree of k in \mathcal{B} , that is the number of subsets $B \in \mathcal{B}$ such that $B \ni k$, and set

$$n_-(\mathcal{B}) = \min\{\deg_k(\mathcal{B}), k \in [N]\}, \quad n_+(\mathcal{B}) = \max\{\deg_k(\mathcal{B}), k \in [N]\}.$$

for the minimal and maximal degree, respectively. For any $B \subset [N]$, we write

$$\mu_B = \mu(\cdot | x_j, j \in B^c),$$

for the conditional probability measure on Ω_i , $i \in B$, obtained by conditioning μ on the value of all $x_j \in \Omega_j$, $j \notin B$. When $B = \{k\}$ for some k , then μ_B coincides with μ_k defined in (1.3). Also, for any function f , we write $f_B = \mu[f | x_B]$, where $x_B = \{x_i, i \in B\}$. Note that, when f is a probability density with respect to μ , then f_B is the density of the marginal of $f\mu$ on x_B with respect to the marginal of μ on x_B . When $B = \{k\}$ we simply write f_k for $f_{\{k\}}$. We recall that any probability measure μ satisfies the decomposition

$$\text{Ent}_\mu(f) = \text{Ent}_\mu(f_B) + \mu[\text{Ent}_{\mu_{B^c}}(f)]. \quad (2.10)$$

We formulate the following version of Shearer-type estimates.

PROPOSITION 2.6. — *For any product measure $\mu = \otimes_{k=1}^N \mu_k$, for any cover \mathcal{B} , for any bounded measurable $f \geq 0$,*

$$\text{Ent}_\mu(f) \leq \frac{1}{n_-(\mathcal{B})} \sum_{B \in \mathcal{B}} \mu[\text{Ent}_{\mu_B}(f)]. \quad (2.11)$$

This bound can be derived from the classical Shearer estimate for Shannon entropy [24, 4]. For the reader's convenience we give a proof of Proposition 2.6 along these lines in Section 5. Note that using (2.10) one has that (2.11) is equivalent to the inequality

$$\sum_{A \in \bar{\mathcal{B}}} \text{Ent}_\mu(f_A) \leq n_+(\bar{\mathcal{B}}) \text{Ent}_\mu(f), \quad (2.12)$$

where $\bar{\mathcal{B}}$ denotes the complementary cover $\{A = [N] \setminus B, B \in \mathcal{B}\}$.

It is immediate to check that the tensorization statement (1.2) is the special case of (2.11) corresponding to the singleton cover $\mathcal{B} = \mathcal{B}_1 := \{\{k\}, k \in [N]\}$. Similarly, for the complementary cover $\mathcal{B} = \mathcal{B}_{N-1} := \bar{\mathcal{B}}_1$, (2.12) reduces to the well known subadditivity property of entropy for product measures:

$$\sum_k \text{Ent}_\mu(f_k) \leq \text{Ent}_\mu(f).$$

In the case of non-product measures with weak dependences, it is natural to investigate the validity of an approximate form of Proposition 2.6 such as

$$\text{Ent}_\mu(f) \leq \frac{C(\mathcal{B})}{n_-(\mathcal{B})} \sum_{B \in \mathcal{B}} \mu[\text{Ent}_{\mu_B}(f)],$$

where $C(\mathcal{B}) > 0$ is a suitable constant. Note that, in analogy with Proposition 1.1, approximate Shearer estimates are naturally linked to Log-Sobolev inequalities and exponential decay of entropy for the block version of the heat bath chain with infinitesimal generator given by

$$\mathcal{L}^{\mathcal{B}} = \sum_{B \in \mathcal{B}} (\mu_B - 1).$$

The following is an immediate corollary of our main result Theorem 2.1.

COROLLARY 2.7. — *Suppose μ satisfies the assumptions of Theorem 2.1 and let $C > 0$ be the constant appearing in that theorem. Then for any cover \mathcal{B} , setting $\Delta(\mathcal{B}) := \max\{|B|, B \in \mathcal{B}\}$,*

$$\text{Ent}_\mu(f) \leq C \frac{\Delta(\mathcal{B})}{n_-(\mathcal{B})} \sum_{B \in \mathcal{B}} \mu[\text{Ent}_{\mu_B}(f)]. \quad (2.13)$$

We remark that (2.13) is far from optimal if $\Delta(\mathcal{B})$ is large, and it becomes useless if the maximal cardinality of \mathcal{B} grows linearly in N . In particular, it cannot be used to prove an approximate subadditivity statement (corresponding to $\mathcal{B} = \mathcal{B}_{N-1}$, $\Delta(\mathcal{B}) = N - 1$) of the form

$$\sum_k \text{Ent}_\mu(f_k) \leq C \text{Ent}_\mu(f), \tag{2.14}$$

with a nontrivial constant $C > 0$. The approximate subadditivity estimate (2.14) has been obtained with the constant $C = 2$ in [9, 10] when μ is the uniform measure on the N -dimensional sphere or the uniform measure on the symmetric group of permutations S_N . While the value $C = 2$ is sharp for the sphere [9], it remains open to find the optimal value of C for the symmetric group. We are not aware of any result of that kind for e.g. high-temperature Ising systems.

3. Proof of Theorem 2.1

Let $\mathcal{E}(\cdot, \cdot)$ denote the Dirichlet form of the heat-bath chain discussed in Section 1.1. Let $(e^{t\mathcal{L}}, t \geq 0)$ denote the heat-bath semigroup. For any bounded nonnegative function f on Ω , we write $f_t = e^{t\mathcal{L}}f$ for its evolution at time t . We need the following lemma.

LEMMA 3.1. — *For every bounded $f \geq 0$, $k \in [N]$ one has*

$$\text{Ent}_\mu(f) = \int_0^\infty \mathcal{E}(f_t, \log f_t) dt, \tag{3.1}$$

$$\mu[\text{Ent}_{\mu_k}(f)] = \int_0^\infty \mathcal{E}(f_t, \log(f_t/\mu_k[f_t])) dt. \tag{3.2}$$

Proof. — From the assumption (2.3) one has $\text{Ent}_\mu(f_t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, to prove (3.1) it is sufficient to observe that

$$\frac{d}{dt} \text{Ent}_\mu(f_t) = \mu[\mathcal{L}f_t \log f_t] = -\mathcal{E}(f_t, \log f_t),$$

where we use $\frac{d}{dt}f_t = \mathcal{L}f_t$ and $\mu[f_t] = \mu[f]$ for all $t \geq 0$.

From Jensen’s inequality one has $0 \leq \mu[\text{Ent}_{\mu_k}(f_t)] \leq \text{Ent}_\mu(f_t)$ and the latter tends to zero as $t \rightarrow \infty$ by (2.3). Therefore, (3.2) follows from

$$\frac{d}{dt} \mu[\text{Ent}_{\mu_k}(f_t)] = -\mathcal{E}(f_t, \log(f_t/\mu_k[f_t])). \tag{3.3}$$

To prove (3.3), notice that

$$\frac{d}{dt} \text{Ent}_{\mu_k}(f_t) = \mu_k[\mathcal{L}f_t \log f_t] - \mu_k[\mathcal{L}f_t] \log \mu_k[f_t].$$

Taking expectation with respect to μ one has

$$\frac{d}{dt} \mu [\text{Ent}_{\mu_k}(f_t)] = \mu[\mathcal{L}f_t \log f_t] - \mu[\mathcal{L}f_t \log \mu_k[f_t]] = -\mathcal{E}(f_t, \log(f_t/\mu_k[f_t])).$$

□

Thanks to Lemma 3.1, in order to prove (1.4), it is sufficient to prove that for all $f \geq 0$:

$$\sum_k \mathcal{E}(f, \log(f/\mu_k[f])) \geq \varrho \mathcal{E}(f, \log f), \quad (3.4)$$

with $\varrho = C^{-1} > 0$. Writing (1.5) explicitly one has

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{i=1}^N \sum_{x \in \Omega} \sum_{y_i \in \Omega_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i g(x; y_i),$$

using the notation

$$\nabla_i f(x; y_i) := f(\bar{x}_i, y_i) - f(x).$$

Then (3.4) becomes

$$\begin{aligned} & \sum_{k,i=1}^N \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i (\log f - \log \mu_k[f])(x; y_i) \\ & \geq \varrho \sum_{i=1}^N \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i \log f(x; y_i). \end{aligned} \quad (3.5)$$

Noting that $\nabla_k \log \mu_k[f](x; y_i) = 0$ for all k , the left hand side in (3.5) satisfies

$$\begin{aligned} & \sum_{k,i} \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i (\log f - \log \mu_k[f])(x; y_i) \\ & = \sum_i \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i \log f(x; y_i) \\ & \quad + \sum_{k,i: k \neq i} \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i \log f(x; y_i) \\ & \quad - \sum_{k,i: k \neq i} \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i \log \mu_k[f](x; y_i). \end{aligned} \quad (3.6)$$

Let us consider the three terms appearing in the right hand side of (3.6). The first term is exactly what we have in the right hand side of (3.5) apart from

the constant ϱ . The essence of the argument below is to show that the last term is approximately compensated by the term preceding it, the correction being proportional to the first term with a proportionality constant that can be made tiny in the regime of weak interaction, so that (3.5) holds with some positive ϱ . This program starts with a pointwise estimate of the term $\nabla_i \log \mu_k[f]$.

3.1. Estimate of $\nabla_i \log \mu_k[f]$

As in several related works (see e.g. [33, 30, 23, 25]), estimating gradients of functions of $\mu_k[f]$ yields a principal term (which will be responsible for the compensation in (3.6) alluded to above) plus a covariance term that needs to be suitably small. A new ingredient here with respect to these works is the use of the logarithmic mean $\Lambda(a, b)$, defined as usual by

$$\Lambda(a, b) = \frac{a - b}{\log a - \log b}, \quad a, b \geq 0, a \neq b, \quad (3.7)$$

and $\Lambda(a, a) = a$, for all $a \geq 0$; see however [17] for a recent extensive use of the logarithmic mean in the analysis of entropy decay.

Fix $i \neq k$ and $x_i, y_i \in \Omega_i, \bar{x}_{k,i} \in \bar{\Omega}_{k,i}$. Introduce the probability measure $\nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}$ on Ω_k defined by

$$\nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k) = \frac{\Lambda(f(y_i, x_k, \bar{x}_{k,i}), f(x_i, x_k, \bar{x}_{k,i})) \mu_k^{x_i, \bar{x}_{k,i}}(x_k)}{\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}}}, \quad (3.8)$$

where the normalization $\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}}$ is given by

$$\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} = \sum_{x_k} \Lambda(f(y_i, x_k, \bar{x}_{k,i}), f(x_i, x_k, \bar{x}_{k,i})) \mu_k^{x_i, \bar{x}_{k,i}}(x_k).$$

For simplicity, we omit the dependence on f in the notation (3.8). Recall the definition (2.1) of $\varphi_{i,k}$. The goal of this subsection is to establish the following estimate.

PROPOSITION 3.2. — *For every $k \neq i$, for all $x \in \Omega, y_i \in \Omega_i$:*

$$\begin{aligned} |\nabla_i \log \mu_k[f](x; y_i)| &\leq \alpha_{i,k} \sum_{x'_k} |\nabla_i \log f(\bar{x}_k, x'_k, y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x'_k) + \\ &+ \alpha_{i,k} \frac{\left| \text{Cov}_{\mu_k^{y_i, \bar{x}_{k,i}}} (f(y_i, \cdot, \bar{x}_{k,i}), \varphi_{i,k}(x_i, y_i, \cdot, \bar{x}_{k,i})) \right|}{\left(\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} \sum_{x'_k} f(y_i, x'_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x'_k) \right)^{\frac{1}{2}}}. \end{aligned} \quad (3.9)$$

Proof. We start with simple manipulations: for $k \neq i$,

$$\begin{aligned} \nabla_i \log \mu_k[f](x; y_i) &= \log \mu_k[f](y_i, \bar{x}_{k,i}) - \log \mu_k[f](x_i, \bar{x}_{k,i}) \\ &= \log \sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k) - \log \sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k) \\ &\quad + \log \sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k) - \log \sum_{x_k} f(x_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k). \end{aligned}$$

Using the function Λ in (3.7), we have

$$\begin{aligned} \nabla_i \log \mu_k[f](x; y_i) &= \frac{\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) (\mu_k^{y_i, \bar{x}_{k,i}}(x_k) - \mu_k^{x_i, \bar{x}_{k,i}}(x_k))}{\Lambda(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k), \sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k))} \\ &\quad + \frac{\sum_{x_k} \nabla_i f(x; y_i) \mu_k^{x_i, \bar{x}_{k,i}}(x_k)}{\Lambda(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k), \sum_{x_k} f(x_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k))} \end{aligned}$$

Note that

$$\Lambda(a, b) \geq q \Lambda(a', b'), \quad \text{if } a \geq qa' \text{ and } b \geq qb', \quad (3.10)$$

for $a', b', q \geq 0$. This follows e.g. from the representation $\Lambda(a, b) = \int_0^1 a^{1-t} b^t dt$ of the logarithmic mean (3.7). Since $\Lambda(a, a) = a$, one has

$$\begin{aligned} \Lambda(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k), \sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k)) \\ \geq [\sup_{x'_k} \varphi_{i,k}(y_i, x_i, x'_k, \bar{x}_{k,i})]^{-1} \sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k) \\ \geq (\alpha_{i,k})^{-1} \sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k). \end{aligned}$$

Note that

$$\begin{aligned} \sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) (\mu_k^{y_i, \bar{x}_{k,i}}(x_k) - \mu_k^{x_i, \bar{x}_{k,i}}(x_k)) \\ = - \text{Cov}_{\mu_k^{y_i, \bar{x}_{k,i}}} (f(y_i, \cdot, \bar{x}_{k,i}), \varphi_{i,k}(x_i, y_i, \cdot, \bar{x}_{k,i})). \end{aligned}$$

Moreover, the concavity of $(a, b) \mapsto \Lambda(a, b)$ implies that

$$\begin{aligned} \Lambda(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k), \sum_{x_k} f(x_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k)) \\ \geq \sum_{x_k} \Lambda(f(y_i, x_k, \bar{x}_{k,i}), f(x_i, x_k, \bar{x}_{k,i})) \mu_k^{x_i, \bar{x}_{k,i}}(x_k) = \bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}}. \quad (3.11) \end{aligned}$$

By definition (3.8) we can write:

$$\frac{\sum_{x_k} |\nabla_i f(x; y_i)| \mu_k^{x_i, \bar{x}_{k,i}}(x_k)}{\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}}} = \sum_{x_k} |\nabla_i \log f(x; y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k).$$

Combining the above bounds we have obtained

$$\begin{aligned}
 |\nabla_i \log \mu_k[f](x; y_i)| &\leq \sum_{x_k} |\nabla_i \log f(x; y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k) \\
 &+ \alpha_{i,k} \frac{\left| \text{Cov}_{\mu_k^{y_i, \bar{x}_{k,i}}} \left(f(y_i, \cdot, \bar{x}_{k,i}), \varphi_{i,k}(x_i, y_i, \cdot, \bar{x}_{k,i}) \right) \right|}{\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k)}. \quad (3.12)
 \end{aligned}$$

We now derive a slightly different bound on $\nabla_i \log \mu_k[f](x; y_i)$. Namely,

$$\begin{aligned}
 &\nabla_i \log \mu_k[f](x; y_i) \\
 &= \frac{\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k) - \sum_{x_k} f(x_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k)}{\Lambda \left(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k), \sum_{x_k} f(x_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k) \right)} \\
 &\quad - \frac{\text{Cov}_{\mu_k^{y_i, \bar{x}_{k,i}}} \left(f(y_i, \cdot, \bar{x}_{k,i}), \varphi_{i,k}(x_i, y_i, \cdot, \bar{x}_{k,i}) \right)}{\Lambda \left(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k), \sum_{x_k} f(x_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k) \right)} \\
 &+ \frac{\sum_{x_k} \nabla_i f(x; y_i) \mu_k^{x_i, \bar{x}_{k,i}}(x_k)}{\Lambda \left(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k), \sum_{x_k} f(x_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k) \right)}.
 \end{aligned}$$

Using (3.10) and (3.11), we have

$$\Lambda \left(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k), \sum_{x_k} f(x_i, x_k, \bar{x}_{k,i}) \mu_k^{x_i, \bar{x}_{k,i}}(x_k) \right) \geq \frac{\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}}}{\alpha_{i,k}}.$$

Therefore the first term in the expression of $\nabla_i \log \mu_k[f](x; y_i)$ above is bounded in absolute value by

$$\frac{\alpha_{i,k}}{\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}}} \left| \text{Cov}_{\mu_k^{y_i, \bar{x}_{k,i}}} \left(f(y_i, \cdot, \bar{x}_{k,i}), \varphi_{i,k}(x_i, y_i, \cdot, \bar{x}_{k,i}) \right) \right|.$$

Similarly, the second term is bounded by

$$\begin{aligned}
 &\frac{\alpha_{i,k}}{\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}}} \sum_{x_k} |\nabla_i f(x; y_i)| \mu_k^{x_i, \bar{x}_{k,i}}(x_k) \\
 &= \alpha_{i,k} \sum_{x_k} |\nabla_i \log f(x; y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k).
 \end{aligned}$$

Thus, we have obtained the following estimate:

$$\begin{aligned}
 |\nabla_i \log \mu_k[f](x; y_i)| &\leq \frac{\alpha_{i,k}}{\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}}} \left| \text{Cov}_{\mu_k^{y_i, \bar{x}_{k,i}}} \left(f(y_i, \cdot, \bar{x}_{k,i}), \varphi_{i,k}(x_i, y_i, \cdot, \bar{x}_{k,i}) \right) \right| \\
 &+ \alpha_{i,k} \sum_{x_k} |\nabla_i \log f(x; y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k). \quad (3.13)
 \end{aligned}$$

Finally, using $\alpha_{i,k} \geq 1$ and putting together (3.12) and (3.13) it is immediate to obtain the desired bound (3.9). \square

The next task is to plug the bound of Proposition 3.2 into the last term of (3.6). This produces the two terms

$$\begin{aligned}
 A &:= \sum_{k,i: k \neq i} \alpha_{i,k} \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) |\nabla_i f(x; y_i)| \times \\
 &\quad \times \sum_{x'_k} |\nabla_i \log f(\bar{x}_k, x'_k; y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x'_k), \\
 B &:= \sum_{k,i: k \neq i} \alpha_{i,k} \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) |\nabla_i f(x; y_i)| \times \\
 &\quad \times \frac{|\text{Cov}_{\mu_k^{y_i, \bar{x}_{k,i}}} (f(y_i, \cdot, \bar{x}_{k,i}), \varphi_{i,k}(x_i, y_i, \cdot, \bar{x}_{k,i}))|}{(\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} \sum_{x'_k} f(y_i, x'_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x'_k))^{\frac{1}{2}}}. \tag{3.14}
 \end{aligned}$$

Below, we analyze the two terms separately. We start with term A which allows for the main cancellation in (3.6).

3.2. The main cancellation

Let us rewrite $A = \sum_{k,i: k \neq i} A_{k,i}$, with

$$\begin{aligned}
 A_{k,i} &= \alpha_{i,k} \sum_{\bar{x}_{k,i}} \sum_{x_i} \sum_{y_i} \mu(\bar{x}_{k,i}, x_i) \left(\sum_{x_k} \mu_k^{x_i, \bar{x}_{k,i}}(x_k) \mu_i^{x_k, \bar{x}_{k,i}}(y_i) |\nabla_i f(x; y_i)| \right) \times \\
 &\quad \times \left(\sum_{x'_k} |\nabla_i \log f(\bar{x}_k, x'_k; y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x'_k) \right),
 \end{aligned}$$

where we use

$$\mu(x) = \mu(\bar{x}_{k,i}, x_i) \mu_k^{x_i, \bar{x}_{k,i}}(x_k), \quad \mu(\bar{x}_{k,i}, x_i) = \mu(\bar{x}_k) = \sum_{x_k} \mu(x).$$

Since

$$\mu_k^{x_i, \bar{x}_{k,i}}(x_k) |\nabla_i f(x; y_i)| = \bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} |\nabla_i \log f(\bar{x}_k, x_k, y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k), \tag{3.15}$$

we obtain

$$\begin{aligned}
 A_{k,i} &\leq \alpha_{i,k} \sum_{\bar{x}_{k,i}} \sum_{x_i} \mu(\bar{x}_{k,i}, x_i) \sum_{y_i} (\sup_{y_k} \mu_i^{y_k, \bar{x}_{k,i}}(y_i)) \\
 &\quad \times \bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} \left(\sum_{x_k} |\nabla_i \log f(\bar{x}_k, x_k, y_i)| \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k) \right)^2.
 \end{aligned}$$

Using Schwarz' inequality one has

$$\begin{aligned}
 A_{k,i} &\leq \alpha_{i,k} \sum_{\bar{x}_{k,i}} \sum_{x_i} \mu(\bar{x}_{k,i}, x_i) \sum_{y_i} (\sup_{y_k} \mu_i^{y_k, \bar{x}_{k,i}}(y_i)) \\
 &\quad \times \bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} \sum_{x_k} |\nabla_i \log f(\bar{x}_k, x_k, y_i)|^2 \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k) \\
 &\leq \alpha_{i,k} \sum_{\bar{x}_{k,i}} \sum_{x_i} \mu(\bar{x}_{k,i}, x_i) \sum_{y_i} (\sup_{y_k, y'_k} \varphi_{k,i}(y_k, y'_k, y_i, \bar{x}_{k,i})) \\
 &\quad \times \bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} \sum_{x_k} |\nabla_i \log f(\bar{x}_k, x_k, y_i)|^2 \mu_i^{x_k, \bar{x}_{k,i}}(y_i) \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k),
 \end{aligned}$$

where we use

$$\frac{\sup_{y_k} \mu_i^{y_k, \bar{x}_{k,i}}(y_i)}{\inf_{y'_k} \mu_i^{y'_k, \bar{x}_{k,i}}(y_i)} = \sup_{y_k, y'_k} \varphi_{k,i}(y_k, y'_k, y_i, \bar{x}_{k,i}).$$

Therefore, using (3.15) and rearranging summations one arrives at

$$A_{k,i} \leq \alpha_{i,k} \alpha_{k,i} \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i \log f(x; y_i). \quad (3.16)$$

From (3.16) it follows that (3.6) can be bounded from below as follows:

$$\begin{aligned}
 &\sum_{k,i} \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i (\log f - \log \mu_k[f])(x; y_i) \\
 &\quad \geq -B + (1 - \gamma) \sum_i \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i \log f(x; y_i),
 \end{aligned}$$

where B is given in (3.14) and

$$\gamma = \max_i \sum_{k: k \neq i} (\alpha_{i,k} \alpha_{k,i} - 1).$$

3.3. Covariance estimate

The next step is an estimate of the form

$$B \leq \kappa \sum_i \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) \nabla_i f(x; y_i) \nabla_i \log f(x; y_i), \quad (3.17)$$

for a suitable constant $\kappa > 0$. We start with the following statement.

LEMMA 3.3. — For any fixed k and for all configurations $\bar{x}_k \in \bar{\Omega}_k$, for any pair of functions $g, \psi : \Omega_k \mapsto \mathbb{R}$, with $g \geq 0$:

$$\begin{aligned} |\text{Cov}_{\mu_k^{\bar{x}_k}}(g, \psi)| &\leq \frac{1}{2} \left(\sup_{z_k, z'_k} |\psi(z_k) - \psi(z'_k)| \right) \left(\sum_{x_k} g(x_k) \mu_k^{\bar{x}_k}(x_k) \right)^{\frac{1}{2}} \times \\ &\times \left(\sum_{x_k} \sum_{y_k} \mu_k^{\bar{x}_k}(x_k) \mu_k^{\bar{x}_k}(y_k) [g(y_k) - g(x_k)] [\log g(y_k) - \log g(x_k)] \right)^{\frac{1}{2}}. \end{aligned}$$

Proof. — Set $\gamma(x_k, y_k) := \mu_k^{\bar{x}_k}(x_k) \mu_k^{\bar{x}_k}(y_k)$, and write

$$\text{Cov}_{\mu_k^{\bar{x}_k}}(g, \psi) = \frac{1}{2} \sum_{x_k, y_k} \gamma(x_k, y_k) (g(y_k) - g(x_k)) (\psi(y_k) - \psi(x_k)).$$

Therefore,

$$|\text{Cov}_{\mu_k^{\bar{x}_k}}(g, \psi)| \leq \frac{1}{2} \left(\sup_{z_k, z'_k} |\psi(z_k) - \psi(z'_k)| \right) \sum_{x_k, y_k} \gamma(x_k, y_k) |g(y_k) - g(x_k)|.$$

Schwarz' inequality yields

$$\begin{aligned} \sum_{x_k, y_k} \gamma(x_k, y_k) |g(y_k) - g(x_k)| &\leq \left(\sum_{x_k, y_k} \gamma(x_k, y_k) (g(x_k) + g(y_k)) \right)^{\frac{1}{2}} \times \\ &\times \left(\sum_{x_k, y_k} \gamma(x_k, y_k) \frac{(g(x_k) - g(y_k))^2}{g(x_k) + g(y_k)} \right)^{\frac{1}{2}} \\ &= \left(2 \sum_{x_k} g(x_k) \mu_k^{\bar{x}_k}(x_k) \right)^{\frac{1}{2}} \left(\sum_{x_k, y_k} \gamma(x_k, y_k) \frac{(g(x_k) - g(y_k))^2}{g(x_k) + g(y_k)} \right)^{\frac{1}{2}}. \end{aligned}$$

Since $\Lambda(a, b) \leq (a + b)/2$ one has

$$(a - b)^2 / (a + b) \leq \frac{1}{2} (a - b)^2 / \Lambda(a, b) = \frac{1}{2} (a - b) (\log a - \log b).$$

This shows that

$$\begin{aligned} \sum_{x_k, y_k} \gamma(x_k, y_k) |g(y_k) - g(x_k)| &\leq \left(\sum_{x_k} g(x_k) \mu_k^{\bar{x}_k}(x_k) \right)^{\frac{1}{2}} \times \\ &\times \left(\sum_{x_k, y_k} \gamma(x_k, y_k) [g(y_k) - g(x_k)] [\log g(y_k) - \log g(x_k)] \right)^{\frac{1}{2}}. \end{aligned}$$

□

For all fixed $x_i, y_i, \bar{x}_{k,i}$, we apply Lemma 3.3 with $g(x_k) = f(y_i, x_k, \bar{x}_{k,i})$ and $\psi(x_k) = \varphi_{i,k}(x_i, y_i, x_k, \bar{x}_{k,i})$. One finds

$$\left| \text{Cov}_{\mu_k^{y_i, \bar{x}_{k,i}}} \left(f(y_i, \cdot, \bar{x}_{k,i}), \varphi_{i,k}(x_i, y_i, \cdot, \bar{x}_{k,i}) \right) \right| \leq \frac{1}{2} \delta_{i,k} \left(\sum_{x_k} f(y_i, x_k, \bar{x}_{k,i}) \mu_k^{y_i, \bar{x}_{k,i}}(x_k) \right)^{\frac{1}{2}} (\mathcal{B}_k^{y_i, \bar{x}_{k,i}}(f, \log f))^{\frac{1}{2}},$$

where $\delta_{i,k}$ is defined in (2.2) and we use the notation

$$\mathcal{B}_k^{y_i, \bar{x}_{k,i}}(f, g) = \sum_{x_k} \sum_{y_k} \mu_k^{y_i, \bar{x}_{k,i}}(x_k) \mu_k^{y_i, \bar{x}_{k,i}}(y_k) \nabla_k f(\bar{x}_i, y_i; y_k) \nabla_k g(\bar{x}_i, y_i; y_k).$$

Plugging this into (3.14) one has

$$B \leq \frac{1}{2} \sum_{k,i: k \neq i} \alpha_{i,k} \delta_{i,k} \sum_x \sum_{y_i} \mu(x) \mu_i^{\bar{x}_i}(y_i) |\nabla_i f(x; y_i)| (\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}})^{-1/2} (\mathcal{B}_k^{y_i, \bar{x}_{k,i}}(f, \log f))^{\frac{1}{2}}.$$

Using (3.15) and reasoning as in Section 3.2 one has

$$\begin{aligned} B &\leq \frac{1}{2} \sum_{k,i: k \neq i} \alpha_{i,k} \delta_{i,k} \sum_{\bar{x}_{k,i}} \sum_{x_i} \mu(\bar{x}_{k,i}, x_i) \sum_{y_i} (\sup_{z_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i)) \times \\ &\quad \times \sum_{x_k} \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k) |\nabla_i \log f(x; y_i)| (\bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}})^{1/2} (\mathcal{B}_k^{y_i, \bar{x}_{k,i}}(f, \log f))^{\frac{1}{2}} \\ &\leq \frac{1}{2} \sum_{k,i: k \neq i} \alpha_{i,k} \delta_{i,k} \sum_{\bar{x}_{k,i}} \sum_{x_i} \mu(\bar{x}_{k,i}, x_i) \sum_{y_i} (\sup_{z_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i)) \times \\ &\quad \times \left(\sum_{x_k} \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k) |\nabla_i \log f(x; y_i)|^2 \bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} \right)^{1/2} (\mathcal{B}_k^{y_i, \bar{x}_{k,i}}(f, \log f))^{\frac{1}{2}}. \end{aligned}$$

Using $ab \leq \frac{1}{2} a^2 + \frac{1}{2} b^2$ one has

$$\begin{aligned} B &\leq \frac{1}{4} \sum_{k,i: k \neq i} \alpha_{i,k} \delta_{i,k} \sum_{\bar{x}_{k,i}} \sum_{x_i} \mu(\bar{x}_{k,i}, x_i) \sum_{y_i} (\sup_{z_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i)) \times \tag{3.18} \\ &\quad \times \sum_{x_k} \nu_{k,i}^{x_i, y_i, \bar{x}_{k,i}}(x_k) |\nabla_i \log f(x; y_i)|^2 \bar{\nu}_{k,i}^{x_i, y_i, \bar{x}_{k,i}} + \\ &\quad + \frac{1}{4} \sum_{k,i: k \neq i} \alpha_{i,k} \delta_{i,k} \sum_{\bar{x}_{k,i}} \sum_{x_i} \mu(\bar{x}_{k,i}, x_i) \sum_{y_i} (\sup_{z_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i)) \mathcal{B}_k^{y_i, \bar{x}_{k,i}}(f, \log f). \end{aligned}$$

Using again (3.15) and the coefficients $\alpha_{k,i}$ as in Section 3.2, the first term in (3.18) is bounded by

$$\frac{1}{4} \sum_{k,i: k \neq i} \alpha_{i,k} \alpha_{k,i} \delta_{i,k} \sum_x \mu(x) \sum_{y_i} \mu_i^{x_k, \bar{x}_{k,i}}(y_i) \nabla_i f(x; y_i) \nabla_i \log f(x; y_i). \tag{3.19}$$

Next, we estimate the second term in (3.18). Notice that $\mathcal{B}_k^{y_i, \bar{x}_{k,i}}(f, \log f)$ depends on y_i and not on x_i . We are going to show that

$$\sum_{x_i} \mu(\bar{x}_{k,i}, x_i) (\sup_{z_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i)) \leq \alpha_{k,i} \mu(\bar{x}_{k,i}, y_i). \quad (3.20)$$

Write $\mu(\bar{x}_{k,i}) = \sum_{x_i} \mu(\bar{x}_{k,i}, x_i)$ for the marginal on $\bar{\Omega}_{k,i}$. Let $\mu_{i,k}^{\bar{x}_{k,i}}(y_i, x_k)$ denote the joint law at (Ω_i, Ω_k) conditioned on $\bar{x}_{k,i} \in \bar{\Omega}_{k,i}$, and observe that

$$\sum_{x_k} \mu_{i,k}^{\bar{x}_{k,i}}(y_i, x_k) = \sum_{x_k} \mu_i^{x_k, \bar{x}_{k,i}}(y_i) \mu(x_k | \bar{x}_{k,i}) \geq \inf_{z_k \in \Omega_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i).$$

Therefore,

$$\begin{aligned} \sum_{x_i} \mu(\bar{x}_{k,i}, x_i) (\sup_{z_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i)) &= \mu(\bar{x}_{k,i}) (\sup_{z_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i)) \\ &\leq \alpha_{k,i} \mu(\bar{x}_{k,i}) (\inf_{z_k} \mu_i^{z_k, \bar{x}_{k,i}}(y_i)) \leq \alpha_{k,i} \mu(\bar{x}_{k,i}) \sum_{x_k} \mu_{i,k}^{\bar{x}_{k,i}}(y_i, x_k) \\ &= \alpha_{k,i} \sum_{x_k} \mu(\bar{x}_{k,i}, y_i, x_k) = \alpha_{k,i} \mu(\bar{x}_{k,i}, y_i). \end{aligned}$$

This proves (3.20). Thanks to this estimate, the second term in (3.18) is estimated with

$$\frac{1}{4} \sum_{k,i: k \neq i} \alpha_{i,k} \alpha_{k,i} \delta_{i,k} \sum_x \mu(x) \sum_{y_k} \mu_k^{x_i, \bar{x}_{k,i}}(y_k) \nabla_k f(x; y_k) \nabla_k \log f(x; y_k). \quad (3.21)$$

Thus, summing (3.19) and (3.21), the final estimate on B is given by (3.17) with the coefficient κ defined by:

$$\kappa = \frac{1}{4} \max_i \sum_{k: k \neq i} \delta_{k,i} \alpha_{i,k} \alpha_{k,i} + \frac{1}{4} \max_k \sum_{i: i \neq k} \delta_{k,i} \alpha_{k,i} \alpha_{i,k}.$$

This concludes the proof of Theorem 2.1.

4. Proof of corollaries

Proof of Corollary 2.3. — From Lemma 2.2 it follows that (2.3) is satisfied. Define the function

$$\hat{W}_k(x) = \sum_{j: j \neq k} J_{jk} w_{jk}(x_j, x_k).$$

Then, the measure in (2.4) satisfies

$$\mu_k^{x_i, \bar{x}_{k,i}}(x_k) = \frac{\mu_{0,k}(x_k) e^{\beta \hat{W}_k(x)}}{\sum_{x'_k} \mu_{0,k}(x'_k) e^{\beta \hat{W}_k(x'_k, \bar{x}_k)}}. \quad (4.1)$$

Notice that for $i \neq k$:

$$\hat{W}_k(\bar{x}_{k,i}, y_i, x_k) = \hat{W}_k(\bar{x}_{k,i}, x_i, x_k) + J_{k,i}(w_{i,k}(y_i, x_k) - w_{i,k}(x_i, x_k)).$$

Setting $\varepsilon_{i,k} = 4\beta \|J_{k,i}\| \|w_{i,k}\|_\infty$ it follows that

$$e^{-\varepsilon_{i,k}} \leq \varphi_{i,k}(x_i, y_i, x_k, \bar{x}_{k,i}) \leq e^{\varepsilon_{i,k}}. \quad (4.2)$$

Moreover, from (4.2) one has

$$|\varphi_{i,k}(x_i, y_i, x_k, \bar{x}_{i,k}) - \varphi_{i,k}(x_i, y_i, y_k, \bar{x}_{i,k})| \leq e^{\varepsilon_{i,k}} - e^{-\varepsilon_{i,k}}.$$

Therefore, the coefficients $\alpha_{i,k}$ and $\delta_{i,k}$ satisfy

$$1 \leq \alpha_{i,k} \leq e^{\varepsilon_{i,k}}, \quad 0 \leq \delta_{i,k} \leq e^{\varepsilon_{i,k}} - e^{-\varepsilon_{i,k}}.$$

The numbers γ, κ in Theorem 2.1 can then be bounded by

$$\gamma \leq \max_i \sum_{k \neq i} (e^{2\varepsilon_{i,k}} - 1), \quad \kappa \leq \frac{1}{2} \max_i \sum_{k \neq i} e^{\varepsilon_{i,k}} (e^{2\varepsilon_{i,k}} - 1) = \frac{1}{2} q.$$

Under the assumptions of Corollary 2.3 one has $\kappa < \frac{1}{3}, \gamma \leq q < \frac{2}{3}$, and therefore one may apply Theorem 2.1 to obtain (1.4) with $C = (1 - \gamma - \kappa)^{-1} \leq (1 - \frac{3}{2}q)^{-1}$. \square

Proof of Corollary 2.5. — From Corollary 2.3 we know that (1.4) holds:

$$\text{Ent}_\mu(f) \leq C \sum_k \mu [\text{Ent}_{\mu_k}(f)].$$

From (4.1) we also have:

$$e^{-2\beta \|\hat{W}_k\|_\infty} \leq \frac{\mu_k^{x_i, \bar{x}_{k,i}}(x_k)}{\mu_{0,k}(x_k)} \leq e^{2\beta \|\hat{W}_k\|_\infty}. \quad (4.3)$$

Thus, the perturbation argument from Lemma 2.2 shows that

$$\text{Ent}_\mu(f) \leq C \sum_k e^{2\beta \|\hat{W}_k\|_\infty} \mu [\text{Ent}_{\mu_{0,k}}(f)].$$

At this point we can apply the bound (2.8). Therefore

$$\text{Ent}_\mu(f) \leq C' \sum_k e^{2\beta \|\hat{W}_k\|_\infty} \mu [\mu_{0,k} [\partial_k f \partial_k \log f]],$$

where $C' = CC_0$. Using again (4.3):

$$\text{Ent}_\mu(f) \leq C' \sum_k e^{4\beta \|\hat{W}_k\|_\infty} \mu [\partial_k f \partial_k \log f].$$

Finally, observe that

$$4\beta \|\hat{W}_k\|_\infty \leq 4\beta \sum_{j:j \neq k} |J_{j,k}| \|w_{j,k}\|_\infty = \sum_{j:j \neq k} \varepsilon_{j,k} \leq \frac{q}{2},$$

where we use $\varepsilon_{j,k} \leq \frac{1}{2}(e^{2\varepsilon_{j,k}} - 1)$ and q is defined in Corollary 2.3. Therefore, $e^{4\beta \|\hat{W}_k\|_\infty} \leq e^{q/2} \leq e^{1/3}$ and the conclusion (2.9) follows with $K = C' e^{1/3}$. \square

5. Proof of Shearer-type estimates

Proof of Proposition 2.6. — As usual, we restrict to the discrete setting. Suppose that \mathcal{A} is a uniform cover of $[N]$, namely a cover with constant degree, i.e. $n(\mathcal{A}) := \deg_k(\mathcal{A})$ is independent of k . Let us start by showing that a product measure $\mu = \otimes_{i=1}^N \mu_i$ satisfies

$$\sum_{A \in \mathcal{A}} \text{Ent}_\mu(f_A) \leq n(\mathcal{A}) \text{Ent}_\mu(f). \quad (5.1)$$

By homogeneity, we may assume f to be a density w.r.t. μ , i.e. $\mu[f] = 1$. Call $X = (X_1, \dots, X_N)$ the random vector with values in the discrete space Ω whose probability distribution is $f\mu$. Then $f_A \mu_A$, where $\mu_A := \otimes_{i \in A} \mu_i$, is the law of the marginal $X_A = (X_i, i \in A)$. The Shannon entropy $H(X_A)$ of X_A , for any $A \subset [N]$ satisfies:

$$\begin{aligned} H(X_A) &= - \sum_{x_A} f_A(x_A) \mu_A(x_A) \log(f_A(x_A) \mu_A(x_A)) \\ &= -\text{Ent}_\mu(f_A) - \sum_{x_A} \sum_{i \in A} f_A(x_A) \mu_A(x_A) \log(\mu_i(x_i)) \\ &= -\text{Ent}_\mu(f_A) - \sum_{i \in A} \sum_{x_i} f_i(x_i) \mu_i(x_i) \log(\mu_i(x_i)) \\ &= -\text{Ent}_\mu(f_A) + \sum_{i \in A} H(X_i) + \sum_{i \in A} \mu[f_i \log f_i]. \end{aligned}$$

In other words,

$$\sum_{i \in A} H(X_i) - H(X_A) = \text{Ent}_\mu(f_A) - \sum_{i \in A} \text{Ent}_\mu(f_i). \quad (5.2)$$

The standard Shearer estimate for Shannon entropy (see e.g. [24]) states that

$$n(\mathcal{A}) H(X) \leq \sum_{A \in \mathcal{A}} H(X_A). \quad (5.3)$$

Therefore, summing over $A \in \mathcal{A}$ in (5.2) and using (5.3)

$$\sum_{A \in \mathcal{A}} \text{Ent}_\mu(f_A) - n(\mathcal{A}) \sum_{i \in [N]} \text{Ent}_\mu(f_i) \leq n(\mathcal{A}) \sum_{i \in [N]} H(X_i) - n(\mathcal{A}) H(X).$$

Using (5.2) with $A = [N]$ one obtains (5.1).

Suppose now that \mathcal{A} is a non-uniform cover, i.e. $n_-(\mathcal{A}) < n_+(\mathcal{A})$. We can add singleton sets to \mathcal{A} until we obtain a uniform cover \mathcal{A}' such that $n_+(\mathcal{A}) = n(\mathcal{A}')$. It follows that

$$\sum_{A \in \mathcal{A}} \text{Ent}_\mu(f_A) \leq \sum_{A \in \mathcal{A}'} \text{Ent}_\mu(f_A) \leq n(\mathcal{A}') \text{Ent}_\mu(f) = n_+(\mathcal{A}) \text{Ent}_\mu(f).$$

This proves (2.12), which is equivalent to (2.11). \square

Proof of Corollary 2.7. — From Theorem 2.1 one has

$$\begin{aligned} \text{Ent}_\mu(f) &\leq C \sum_k \mu [\text{Ent}_{\mu_k}(f)] \\ &= C \sum_k \sum_{B \in \mathcal{B}: B \ni k} \frac{1}{\deg_k(\mathcal{B})} \mu [\text{Ent}_{\mu_k}(f)] \\ &\leq \frac{C}{n_-(\mathcal{B})} \sum_{B \in \mathcal{B}} \sum_{k \in B} \mu [\text{Ent}_{\mu_k}(f)]. \end{aligned}$$

It remains to show that for any $B \subset [N]$:

$$\sum_{k \in B} \mu [\text{Ent}_{\mu_k}(f)] \leq |B| \mu [\text{Ent}_{\mu_B}(f)].$$

However, this is immediate since $\mu [\text{Ent}_{\mu_k}(f)] \leq \mu [\text{Ent}_{\mu_B}(f)]$, for any $k \in B$. \square

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