ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

BRUNO KAHN

Algebraic tori as Nisnevich sheaves with transfers

Tome XXIII, nº 3 (2014), p. 699-715.

<http://afst.cedram.org/item?id=AFST_2014_6_23_3_699_0>

© Université Paul Sabatier, Toulouse, 2014, tous droits réservés.

L'accès aux articles de la revue « Annales de la faculté des sciences de Toulouse Mathématiques » (http://afst.cedram.org/), implique l'accord avec les conditions générales d'utilisation (http://afst.cedram.org/legal/). Toute reproduction en tout ou partie de cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

Annales de la Faculté des Sciences de Toulouse Vol. XXIII, n° 3, 2014
pp. 699–715

Algebraic tori as Nisnevich sheaves with transfers

Bruno Kahn[(1)]

Abstract. — We relate R-equivalence on tori with Voevodsky's theory of homotopy invariant Nisnevich sheaves with transfers and effective motivic complexes.

Résumé. — On relie la R-équivalence sur les tores aux faisceaux Nisnevich avec transferts invariants par homotopie et aux complexes motiviques effectifs, étudiés par Voevodsky.

Contents

1. Main results

Let k be a field and let T be a k-torus. The R-equivalence classes on T have been extensively studied by several authors, notably by Colliot-Thélène and Sansuc in a series of papers including [4] and [5]: they play a central rôle in many rationality issues. In this note, we show that Voevodsky's triangulated category of motives sheds a new light on this question: see Corollaries 1.3, 1.7 and 1.8 below.

(1) IMJ-PRG, UMR 7586, Case 247, 4 place Jussieu, 75252 Paris Cedex 05, France
bruno.kahn@imj-prg.fr

More generally, let G be a semi-abelian variety over k, which is an extension of an abelian variety A by a torus T. Denote by HI the category of homotopy invariant Nisnevich sheaves with transfers over k in the sense of Voevodsky [19]. Then G has a natural structure of an object of HI ([17, proof of Lemma 3.2], [1, Lemma 1.3.2]). Let L be the group of cocharacters of T.

PROPOSITION 1.1. — *There is a natural isomorphism* $G_{-1} \xrightarrow{\sim} L$ *in* HI.

Here $_{-1}$ is the contraction operation of [18, p. 96], whose definition is recalled in the proof below.

Proof. — Recall that if $\mathcal{F}$ is a presheaf [with transfers] on smooth k-schemes, the presheaf [with transfers] $\mathcal{F}^p_{-1}$ is defined by

$$U \mapsto \operatorname{Coker}(\mathcal{F}(U \times \mathbf{A}^1) \to \mathcal{F}(U \times \mathbb{G}_m)).$$

If $\mathcal{F}$ is homotopy invariant, we may replace $U \times \mathbf{A}^1$ by U and the rational point $1 \in \mathbb{G}_m$ realises $\mathcal{F}^p_{-1}(U)$ as a functorial direct summand of $\mathcal{F}(U \times \mathbb{G}_m)$.

If $\mathcal{F}$ is a Nisnevich sheaf [with transfers], $\mathcal{F}_{-1}$ is defined as the sheaf associated to $\mathcal{F}^p_{-1}$.

Now $A(U \times \mathbf{A}^1) \xrightarrow{\sim} A(U \times \mathbb{G}_m)$ since A is an abelian variety, hence $A^p_{-1} = 0$. We therefore have an isomorphism of presheaves $T^p_{-1} \xrightarrow{\sim} G^p_{-1}$, and *a fortiori* an isomorphism of Nisnevich sheaves $T_{-1} \xrightarrow{\sim} G_{-1}$.

Let $p : \mathbb{G}_m \to \operatorname{Spec} k$ be the structural map. One easily checks that the *étale* sheaf $\operatorname{Coker}(T \xrightarrow{i} p_* p^* T)$ is canonically isomorphic to L. Since i is split, its cokernel is still L if we view it as a morphism of presheaves, hence of Nisnevich sheaves. □

From now on, we assume k perfect. Let $\mathrm{DM}^{\mathrm{eff}}_-$ be the triangulated category of effective motivic complexes introduced in [19]: it has a t-structure with heart HI. It also has a tensor structure and a (partially defined) internal Hom. We then have an isomorphism

$$L[0] = G_{-1}[0] \simeq \underline{\mathrm{Hom}}_{\mathrm{DM}^{\mathrm{eff}}_-}(\mathbb{G}_m[0], G[0])$$

[10, Rk. 4.4], hence by adjunction a morphism in $\mathrm{DM}^{\mathrm{eff}}_-$

$$L[0] \otimes \mathbb{G}_m[0] \to G. \tag{1.1}$$

Let $\nu_{\leqslant 0} G[0]$ denote the cone of (1.1): by [11, Lemma 6.3] or [8, §2], $\nu_{\leqslant 0} G[0]$ is the *birational motivic complex* associated to G. We want to compute its homology sheaves.

For this, consider a coflasque resolution[1]

$$0 \to Q \to L_0 \to L \to 0 \tag{1.2}$$

of L in the sense of [4, p. 179]. Taking a coflasque resolution of Q and iterating, we get a resolution of L by invertible lattices:

$$\ldots \to L_n \to \ldots \to L_0 \to L \to 0. \tag{1.3}$$

We set

$$Q_n = \begin{cases} Q & \text{for } n = 1 \\ \operatorname{Ker}(L_{n-1} \to L_{n-2}) & \text{for } n > 1. \end{cases}$$

Theorem 1.2. — *a) Let T_n denote the torus with cocharacter group L_n. Then $\nu_{\leqslant 0}G[0]$ is isomorphic to the complex*

$$\ldots \to T_n \to \ldots \to T_0 \to G \to 0.$$

b) Let S_n be the torus with cocharacter group Q_n. For any connected smooth k-scheme X with function field K, we have

$$H_n(\nu_{\leqslant 0}G[0])(X) = \begin{cases} 0 & \text{if } n < 0 \\ G(K)/R & \text{if } n = 0 \\ S_n(K)/R & \text{if } n > 0. \end{cases}$$

The proof is given in Section 3.

Corollary 1.3. — *The assignment $Sm(k) \ni X \mapsto \bigoplus_{x \in X^{(0)}} G(k(x))/R$ provides G/R with the structure of a homotopy invariant Nisnevich sheaf with transfers. In particular, any morphism $\varphi : Y \to X$ of smooth connected k-schemes induces a morphism $\varphi^* : G(k(X))/R \to G(k(Y))/R$.*

This functoriality is essential to formulate Theorem 1.5 below. For φ a closed immersion of codimension 1, it recovers a specialisation map on R-equivalence classes with respect to a discrete valuation of rank 1 which was obtained (for tori) by completely different methods, *e.g.* [5, Th. 3.1 and Cor. 4.2] or [7]. (I am indebted to Colliot-Thélène for pointing out these references.)

Corollary 1.4. — *a) If k is finitely generated, the n-th homology sheaf of $\nu_{\leqslant 0}G[0]$ takes values in finitely generated abelian groups, and even in finite groups if $n > 0$ or G is a torus.*

b) If G is a torus, then $\nu_{\leqslant 0}G[0] = 0$ if G is split by a Galois extension E/k whose Galois group has cyclic Sylow subgroups. This condition is automatic if k is (quasi-)finite.

(1) See Section 2 for this and further terminology.

The proof is also given in Section 3.

Given two semi-abelian varieties G, G', we would now like to understand the maps

$$\mathrm{Hom}_k(G, G') \to \mathrm{Hom}_{\mathrm{DM}_-^{\mathrm{eff}}}(\nu_{\leqslant 0}G[0], \nu_{\leqslant 0}G'[0]) \to \mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R).$$

In Section 4, we succeed in elucidating the nature of their composition to a large extent, at least if G is a torus. Our main result, in the spirit of Yoneda's lemma, is

THEOREM 1.5. — *Let G, G' be two semi-abelian varieties, with G a torus. Suppose given, for every function field K/k, a homomorphism $f_K : G(K)/R \to G'(K)/R$ such that f_K is natural with respect to the functoriality of Corollary 1.3. Then*

a) There exists an extension $\tilde{G}$ of G by a permutation torus, and a homomorphism $f : \tilde{G} \to G'$ inducing (f_K).

b) f_K is surjective for all K if and only if there exist extensions $\tilde{G}, \tilde{G}'$ of G and G' by permutation tori such that f_K is induced by a split surjective homomorphism $\tilde{G} \to \tilde{G}'$.

The proof is given in §4.3. See Proposition 4.7, Corollary 4.9, Remark 4.10 and Proposition 4.11 for complements.

This relates to questions of stable birationality studied by Colliot-Thélène and Sansuc in [4] and [5], providing alternate proofs and strengthening of some of their results (at least over a perfect field). More precisely, let us introduce the following terminology:

Definition 1.6. — a) A torus is quasi-invertible *if it is a quotient of a invertible torus by an permutation torus.*

b) An extension $0 \to T' \to T \to T'' \to 0$ of tori is Nisnevich-exact *if $T(K) \to T''(K)$ is surjective for any function field K/k.*

(a) was suggested by Xun Jiang; see also [2]. See §2 for "permutation torus" and "invertible torus".)

Thanks to [18, Cor. 4.18], Nisnevich-exact sequences of tori are exact in the Nisnevich topology and even in the Zariski topology. It is easy to see that an extension as in b) is Nisnevich-exact if T' is invertible, but not necessarily if T' is only quasi-invertible. Using [4, Th. 2], one sees that quasi-invertible tori are universally R-trivial. Conversely:

Corollary 1.7. — *a) Let G' be a semi-abelian k-variety such that $G'(K)/R = 0$ for any function field K/k. Then G' is a quasi-invertible torus.*

b) In Theorem 1.5 b), assume that f_K is bijective for all K/k. Then there exists an extension $\tilde{G}$ of G by a permutation torus and a Nisnevich-exact extension $\tilde{G}'$ of G' by a quasi-invertible torus such that f_K is induced by an isomorphism $\tilde{G} \xrightarrow{\sim} \tilde{G}'$.

Proof. — a) This is the special case $G = 0$ of Theorem 1.5 b).

b) By Theorem 1.5 b), we may replace G and G' by extensions by permutation tori such that f_K is induced by a split surjection $f : G \to G'$. Let $T = \mathrm{Ker} f$. Then $T/R = 0$ universally. By a), T is quasi-invertible. Replacing G' by $G' \times T$, we get the desired statement. □

Corollary 1.7 a) is a version of [5, Prop. 7.4] (taking [4, p. 199, Th. 2] into account). Theorem 1.5 was inspired by the desire to understand this result from a different viewpoint. Another characterisation of quasi-invertible tori in loc. cit. is that they are the retract-rational tori.

Corollary 1.8. — *Let $f : G \dashrightarrow G'$ be a rational map of semi-abelian varieties, with G a torus. Then the following conditions are equivalent:*

(i) $f_* : \nu_{\leqslant 0}G[0] \to \nu_{\leqslant 0}G'[0]$ *is an isomorphism (see Proposition 4.7).*

(ii) $f_* : G(K)/R \to G'(K)/R$ *is bijective for any function field K/k.*

(iii) *f is an isomorphism, up to Nisnevich-exact extensions of G and G' by quasi-invertible tori and up to a translation. (See Lemma 4.4.)*

Acknowledgements. — Part of Theorem 1.2 was obtained in the course of discussions with Takao Yamazaki during his stay at the IMJ in October 2010: I would like to thank him for inspiring exchanges. I also thank Daniel Bertrand for a helpful discussion, Xun Jiang for pointing out some errors and the referee for suggesting expository improvements. Finally, I wish to acknowledge inspiration from the work of Colliot-Thélène and Sansuc, which will be obvious throughout this paper.

2. Review of terminology for tori

We take this terminology from [4] and [5].

Definition 2.1 Let G be a profinite group.

a) A lattice *is a G-module which is finitely generated and free over* $\mathbf{Z}$.

b) A lattice L is

- permutation *if it affords a G-invariant $\mathbf{Z}$-basis.*
- invertible *if it is isomorphic to a direct summand of a permutation lattice.*
- coflasque *if $H^1(H,L)=0$ for any open (hence closed) subgroup $H\subseteq G$.*
- flasque *if the dual lattice L^* is coflasque.*

c) A coflasque resolution *of a lattice L is a short exact sequence of lattices*

$$0\to Q\to P\to L\to 0$$

where P is permutation and Q is coflasque. Dually, we have flasque [co]resolutions

$$0\to L\to P\to F\to 0$$

with P permutation and F flasque.

Proposition 2.2 ([4, p. 181, lemme 3]). — *Any lattice has a flasque and a coflasque resolution.*

In [5, Lemma 0.6], the first statement of c) is extended to G-modules which are finitely generated over $\mathbf{Z}$ but not necessatily free.

Let k_s be a separable closure of the field k and take $G=Gal(k_s/k)$. Let T be a k-torus: we shall say that it is *permutation, invertible, flasque, coflasque*, if its character group is (Colliot-Thélène and Sansuc use *quasi-trivial* for "permutation"). Any permutation torus is of the form $R_{E/k}\mathbb{G}_m$ (Weil restriction of scalars) for some étale k-algebra E.

3. Proofs of Theorem 1.2 and Corollary 1.4

Lemma 3.1. — *The exact sequence*

$$0\to T(k)\to G(k)\to A(k)$$

induces an exact sequence

$$0\to T(k)/R\xrightarrow{i} G(k)/R\to A(k).$$

Proof. — Let $f:\mathbf{P}^1\dashrightarrow G$ be a k-rational map defined at 0 and 1. Its composition with the projection $G\to A$ is constant: thus the image of f lies in a T-coset of G defined by a rational point. This implies the injectivity of i, and the rest is clear. □

Let NST denote the category of Nisnevich sheaves with transfers. Recall that $\mathrm{DM}_-^{\mathrm{eff}}$ may be viewed as a localisation of $D^-(\mathrm{NST})$, and that its tensor structure is a descent of the tensor structure on the latter category [19, Prop. 3.2.3].

LEMMA 3.2. — *If G is an invertible torus, there is a canonical isomorphism in $D^-(\mathrm{NST})$*

$$L[0]\otimes\mathbb{G}_m \xrightarrow{\sim} G[0].$$

In particular, $\nu_{\leqslant 0}G[0]=0$.

Proof. — We reduce to the case $T=R_{E/k}\mathbb{G}_m$, where E is a finite extension of k. Let us write more precisely $\mathrm{NST}(k)$ and $\mathrm{NST}(E)$. There is a pair of adjoint functors

$$\mathrm{NST}(k)\xrightarrow{f^*}\mathrm{NST}(E),\quad \mathrm{NST}(E)\xrightarrow{f_*}\mathrm{HI}(k)$$

where $f:\mathrm{Spec}E\to\mathrm{Spec}k$ is the projection. Clearly,

$$f_*\mathbf{Z}=\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E),\quad f_*\mathbb{G}_m=T$$

where $\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E)$ is the Nisnevich sheaf with transfers represented by $\mathrm{Spec}E$. Since $\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E)=L$, this proves the claim. □

Proof of Theorem 1.2. — a) Recall that L_0 is an invertible lattice chosen so that $L_0(E)\to L(E)$ is surjective for any extension E/k. In particular, (1.2) and (1.3) are exact as sequences of Nisnevich sheaves; hence $L[0]$ is isomorphic in $D^-(\mathrm{NST})$ to the complex

$$L.=\ldots\to L_n\to\ldots\to L_0\to 0.$$

(We may view (1.3) as a version of Voevodsky's "canonical resolutions" as in [19, §3.2 p. 206].)

By Lemma 3.2, $L_n[0]\otimes\mathbb{G}_m[0]\simeq T_n[0]$ is homologically concentrated in degree 0 for all n. It follows that the complex

$$T.=\ldots\to T_n\to\ldots\to T_0\to 0$$

is isomorphic to $L[0]\otimes\mathbb{G}_m[0]$ in $D^-(\mathrm{NST})$, hence *a fortiori* in $\mathrm{DM}_-^{\mathrm{eff}}$.

b) For any nonempty open subscheme $U\subseteq X$ we have isomorphisms

$$H_n(\nu_{\leqslant 0}G[0])(X)\xrightarrow{\sim}H_n(\nu_{\leqslant 0}G[0])(U)\xrightarrow{\sim}H_n(\nu_{\leqslant 0}G[0])(K)\tag{3.1}$$

(e.g. [8, p. 912]). By a), the right hand term is the n-th homology group of the complex

$$\ldots\to T_n(K)\to\ldots\to T_0(K)\to G(K)\to 0$$

with $G(K)$ in degree 0. By [4, p. 199, Th. 2], the sequences

$$0 \to S_1(K) \to T_0(K) \to T(K) \to T(K)/R \to 0$$
$$0 \to S_{n+1}(K) \to T_n(K) \to S_n(K) \to S_n(K)/R \to 0$$

are all exact. Using Lemma 3.1 for H_0, the conclusion follows from an easy diagram chase. □

Remark 3.3. — As a corollary to Theorem 1.2, $S_n(K)/R$ only depends on G. This can be seen without mentioning $\mathrm{DM}_-^{\mathrm{eff}}$: in view of the reasoning just above, it suffices to construct a homotopy equivalence between two resolutions of the form (1.3), which easily follows from the definition of coflasque modules.

Proof of Corollary 1.4. — a) This follows via Theorem 1.2 and Lemma 3.1 from [4, p. 200, Cor. 2] and the Mordell-Weil-Néron theorem. b) We may choose the L_n, hence the S_n split by E/k. The conclusion now follows from Theorem 1.2 and [4, p. 200, Cor. 3]. The last claim is clear. □

Remark 3.4. — In characteristic $p > 0$, all finitely generated perfect fields are finite. To give some contents to Corollary 1.4 a) in this characteristic, one may pass to the perfect [one should say radicial] closure k of a finitely generated field k_0. If G is a semi-abelian k-variety, it is defined over some finite extension k_1 of k_0. If k_2/k_1 is a finite (purely inseparable) subextension of k/k_1, then the composition

$$G(k_2) \xrightarrow{N_{k_2/k_1}} G(k_1) \to G(k_2)$$

equals multiplication by $[k_2 : k_1]$. Hence Corollary 1.4 a) remains true at least after inverting p.

4. Stable birationality

If X is a smooth variety over a field k, we write $\mathrm{Alb}(X)$ for its generalised Albanese variety in the sense of Serre [16]: it is a semi-abelian variety, and a rational point $x_0 \in X$ determines a morphism $X \to \mathrm{Alb}(X)$ which is universal for morphisms from X to semi-abelian varieties sending x_0 to 0.

We also write $\mathrm{NS}(X)$ for the group of cycles of codimension 1 on X modulo algebraic equivalence. This group is finitely generated if k is algebraically closed [9, Th. 3].

4.1. Well-known lemmas

I include proofs for lack of reference.

LEMMA 4.1. — *a) Let G, G' be two semi-abelian k-varieties. Then any k-morphism $f : G \to G'$ can be written uniquely $f = f(0) + f'$, where f' is a homomorphism.*

b) For any semi-abelian k-variety G, the canonical map $G \to \mathrm{Alb}(G)$ sending 0 to 0 is an isomorphism.

Proof. — a) amounts to showing that if $f(0) = 0$, then f is a homomorphism. By an adjunction game, this is equivalent to b). Let us give two proofs: one of a) and one of b).

Proof of a). — We may assume k to be a universal domain. The statement is classical for abelian varieties [15, p. 41, Cor. 1] and an easy computation for tori. In the general case, let T, T' be the toric parts of G and G' and A, A' be their abelian parts. Let $g \in G(k)$. As any morphism from T to A' is constant, the k-morphism

$$\varphi_g : T \ni t \mapsto f(g+t) - f(g) \in G'$$

(which sends 0 to 0) lands in T', hence is a homomorphism. Therefore it only depends on the image of g in $A(k)$. This defines a morphism $\varphi : A \to \underline{\mathrm{Hom}}(T, T')$, which must be constant with value $\varphi_0 = f$. It follows that

$$(g, h) \mapsto f(g+h) - f(g) - f(h)$$

induces a morphism $A \times A \to T'$. Such a morphism is constant, of value 0.

Proof of b). — This is true if G is abelian, by rigidity and the equivalence between a) and b). In general, any morphism from G to an abelian variety is trivial on T. This shows that the abelian part of $\mathrm{Alb}(G)$ is A. Let $T' = \mathrm{Ker}(\mathrm{Alb}(G) \to A)$. We also have the counit morphism $\mathrm{Alb}(G) \to G$, and the composition $G \to \mathrm{Alb}(G) \to G$ is the identity. Thus T is a direct summand of T'. It suffices to show that $\dim T' = \dim T$. Going to the algebraic closure, we may reduce to $T = \mathbb{G}_m$.

Then consider the line bundle completion $\bar{G} \to A$ of the $\mathbb{G}_m$-bundle $G \to A$. It is sufficient to show that the kernel of

$$\mathrm{Alb}(G) \to \mathrm{Alb}(\bar{G}) = A$$

is 1-dimensional. This follows for example from [1, Cor. 10.5.1]. □

LEMMA 4.2. — *Suppose k algebraically closed, and let G be a semi-abelian k-variety. Let A be the abelian quotient of G. Then the map*

$$\mathrm{NS}(A) \to \mathrm{NS}(G) \tag{4.2}$$

is an isomorphism.

Proof. — Let $T = \mathrm{Ker}(G \to A)$ and $X(T)$ be its character group. Choosing a basis (e_i) of $X(T)$, we may complete the $\mathbb{G}_m^n$-torsor G into a product of line bundles $\bar{G} \to A$. The surjection

$$\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{Pic}(\bar{G}) \twoheadrightarrow \mathrm{Pic}(G)$$

show the surjectivity of (4.2). Its kernel is generated by the classes of the irreducible components D_i of the divisor with normal crossings $\bar{G} - G$. These components correspond to the basis elements e_i. Since the corresponding $\mathbb{G}_m$-bundle is a group extension of A by $\mathbb{G}_m$, the class of the 0 section of its line bundle completion lies in $\mathrm{Pic}^0(A)$, hence goes to 0 in $\mathrm{NS}(A)$. □

LEMMA 4.3. — *Let X be a smooth k-variety, and let $U \subseteq X$ be a dense open subset. Then there is an exact sequence of semi-abelian varieties*

$$0 \to T \to \mathrm{Alb}(U) \to \mathrm{Alb}(X) \to 0$$

with T a torus. If $\mathrm{NS}(U \otimes_k \bar{k}) = 0$ (this happens if U is small enough), there is an exact sequence of character groups

$$0 \to X(T) \to \bigoplus_{x \in X^{(1)} - U^{(1)}} \mathbf{Z} \to \mathrm{NS}(\bar{X}) \to 0.$$

Proof. — This follows for example from [1, Cor. 10.5.1]. □

LEMMA 4.4. — *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian k-varieties, with G a torus. Then there exists an extension $\tilde{G}$ of G by a permutation torus and a homomorphism $\tilde{f} : \tilde{G} \to G'$ which extends f up to translation in the following sense: there exists a rational section $s : G \dashrightarrow \tilde{G}$ of the projection $\pi : \tilde{G} \to G$ and a rational point $g' \in G'(k)$ such that $f = \tilde{f} s + g'$. If f is defined at 0_G and sends it to $0_{G'}$, then $g' = 0$.*

Proof. — Let U be an open subset of G where f is defined. We define $\tilde{G} = \mathrm{Alb}(U)$. Applying Lemmas 4.3 and 4.1 b) and using $\mathrm{NS}(G \otimes_k \bar{k}) = 0$, we get an extension

$$0 \to P \to \tilde{G} \to G \to 0$$

where P is a permutation torus, as well as a morphism $\tilde{f} = \mathrm{Alb}(f) : \tilde{G} \to G'$.

Let us first assume k infinite. Then $U(k) \neq \emptyset$ because G is unirational. A rational point $g \in U$ defines an Albanese map $s : U \to \tilde{G}$ sending g to $0_{\tilde{G}}$. Since P is a permutation torus, $g \in G(k)$ lifts to $\tilde{g} \in \tilde{G}(k)$ (Hilbert 90) and we may replace s by a morphism sending g to $\tilde{g}$. Then s is a rational section of π. Moreover, $f = \tilde{f}s + g'$ with $g' = f(g) - \tilde{f}(\tilde{g})$. The last assertion follows.

If k is finite, then U has at least a zero-cycle g of degree 1, which is enough to define the Albanese map s. We then proceed as above (lift every closed point involved in g to a closed point of $\tilde{G}$ with the same residue field). □

Lemma 4.5. — *Let G be a finite group, and let A be a finitely generated G-module. Then*

a) There exists a short exact sequence of G-modules $0 \to P \to F \to A \to 0$, with F torsion-free and flasque, and P permutation.

b) Let B be another finitely generated G-module, and let $0 \to P' \to E \to B \to 0$ be an exact sequence with P' an invertible module. Then any G-morphism $f : A \to B$ lifts to $\tilde{f} : F \to E$.

Proof. — a) is the contents of [5, Lemma 0.6, (0.6.2)]. b) The obstruction to lifting f lies in $\mathrm{Ext}^1_G(F, P') = 0$ [4, p. 182, Lemme 9]. □

4.2. Functoriality of $\nu_{\leqslant 0}G$

We now assume k perfect.

Lemma 4.6. — *Let*

$$0 \to P \to G \to H \to 0 \tag{4.3}$$

be an exact sequence of semi-abelian varieties, with P an invertible torus. Then $\nu_{\leqslant 0}G[0] \xrightarrow{\sim} \nu_{\leqslant 0}H[0]$.

Proof. — As P is invertible, (4.3) is exact in NST hence defines an exact triangle

$$P[0] \to G[0] \to H[0] \xrightarrow{+1}$$

in $\mathrm{DM}^{\mathrm{eff}}_{-}$. The conclusion then follows from Lemma 3.2. □

Proposition 4.7. — *Let G, G' be two semi-abelian k-varieties, with G a torus. Then a rational map $f : G \dashrightarrow G'$ induces a morphism $f_* : \nu_{\leqslant 0}G[0] \to \nu_{\leqslant 0}G'[0]$, hence a homomorphism $f_* : G(K)/R \to G'(K)/R$ for any extension K/k. If K is infinite, f_* agrees up to translation with the morphism induced by f via the isomorphism $U(K)/R \xrightarrow{\sim} G(K)/R$ from [4, p. 196 Prop. 11], where U is an open subset of definition of f.*

Proof. — By Lemma 4.4, f induces a homomorphism $\tilde{G} \to G'$ where $\tilde{G}$ is an extension of G by a permutation torus. By Lemma 4.6, the induced morphism

$$\nu_{\leqslant 0}\tilde{G}[0] \to \nu_{\leqslant 0}G'[0]$$

factors through a morphism $f_* : \nu_{\leqslant 0}G[0] \to \nu_{\leqslant 0}G'[0]$.

The claims about R-equivalence classes follow from Theorem 1.2 b) and Lemma 4.4. □

Remark 4.8. — The proof shows that $f'_* = f_*$ if f' differs from f by a translation by an element of $G(k)$ or $G'(k)$.

Corollary 4.9. — *If T and T' are birationally equivalent k-tori, then $\nu_{\leqslant 0}T[0] \simeq \nu_{\leqslant 0}T'[0]$. In particular, the groups $T(k)/R$ and $T'(k)/R$ are isomorphic.*

Proof. — The proof of Proposition 4.7 shows that $f \mapsto f_*$ is functorial for composable rational maps between tori. Let $f : T \dashrightarrow T'$ be a birational isomorphism, and let $g : T' \dashrightarrow T$ be the inverse birational isomorphism. Then we have $g_*f_* = 1_{\nu_{\leqslant 0}T[0]}$ and $f_*g_* = 1_{\nu_{\leqslant 0}T'[0]}$. The last claim follows from Theorem 1.2. □

Remark 4.10. — It is proven in [4] that a birational isomorphism of tori $f : T \dashrightarrow T'$ induces a set-theoretic bijection $f_* : T(k)/R \xrightarrow{\sim} T'(k)/R$ (p. 197, Cor. to Prop. 11) and that the *group* $T(k)/R$ is abstractly a birational invariant of T (p. 200, Cor. 4). The proof above shows that f_* is an isomorphism of groups if f respects the origins of T and T'. This solves the question raised in [4, mid. p. 397]. The proofs of Lemma 4.4 and Proposition 4.7 may be seen as dual to the proof of [4, p. 189, Prop. 5], and are directly inspired from it.

4.3. Faithfulness and fullness

Proposition 4.11. — *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian varieties, with G a torus. Assume that the map $f_* : G(K)/R \to G'(K)/R$ from Proposition 4.7 is identically 0 when K runs through the finitely generated extensions of k. Then there exists a permutation torus P and a factorisation of f as*

$$G \overset{\tilde{f}}{\dashrightarrow} P \xrightarrow{g} G'$$

where $\tilde{f}$ is a rational map and g is a homomorphism. If f is a morphism, we may choose $\tilde{f}$ to be a homomorphism.

Conversely, if there is such a factorisation, then $f_* : \nu_{\leqslant 0}G[0] \to \nu_{\leqslant 0}G'[0]$ *is the* 0 *morphism.*

Proof. — By Lemma 4.4, we may reduce to the case where f is a homomorphism. Let $K = k(G)$. By hypothesis, the image of the generic point $\eta_G \in G(K)$ is R-equivalent to 0 on $G'(K)$. By a lemma of Gille [6, Lemme II.1.1 b)], it is directly R-equivalent to 0: in other words, there exists a rational map $h : G \times \mathbf{A}^1 \dashrightarrow G'$, defined in the neighbourhood of 0 and 1, such that $h_{|G\times\{0\}} = 0$ and $h_{|G\times\{1\}} = f$.

Let $U \subseteq G \times \mathbf{A}^1$ be an open set of definition of h. The 0 and 1-sections of $G \times \mathbf{A}^1 \to G$ induce sections

$$s_0, s_1 : G \to \mathrm{Alb}(U)$$

of the projection $\pi : \mathrm{Alb}(U) \to \mathrm{Alb}(G \times \mathbf{A}^1) = G$ such that $\mathrm{Alb}(h) \circ s_0 = 0$ and $\mathrm{Alb}(h) \circ s_1 = f$. If $P = \mathrm{Ker}\pi$, then $s_1 - s_0$ induces a homomorphism $\tilde{f} : G \to P$ such that the composition

$$G \xrightarrow{\tilde{f}} P \to \mathrm{Alb}(U) \xrightarrow{\mathrm{Alb}(h)} G'$$

equals f. Finally, P is a permutation torus by Lemma 4.3.

The last claim follows from Lemma 3.2. □

Proof of Theorem 1.5. — a) Take $K = k(G)$. The image of the generic point η_G by f_K lifts to a (non unique) rational map $f : G \dashrightarrow G'$. Using Lemma 4.4, we may extend f to a homomorphism

$$\tilde{f} : \tilde{G} \to G'$$

where $\tilde{G}$ is an extension of G by a permutation torus P. Since $\tilde{G}(K)/R \xrightarrow{\sim} G(K)/R$, we reduce to $\tilde{G} = G$ and $\tilde{f} = f$.

Let L/k be a fonction field, and let $g \in G(L)$. Then g arises from a morphism $g : X \to G$ for a suitable smooth model X of L. By assumption on $K \mapsto f_K$, the diagram

$$\begin{array}{ccc} G(K)/R & \xrightarrow[f_K]{} & G'(K)/R \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ G(L)/R & \xrightarrow[f_L]{} & G'(L)/R \end{array}$$

commutes. Applying this to $\eta_K \in G(K)$, we find that $f_L([g]) = [g \circ f]$, which means that f_L is the map induced by f.

b) The hypothesis implies that $G'(E)/R = 0$ for any algebraically closed extension E/k, which in turn implies that G' is also a torus. Applying a), we may, and do, convert f into a true homomorphism by replacing G by a suitable extension by a permutation torus. Applying Lemma 4.5 a) to the cocharacter group of G, we then get a resolution $0 \to P_1 \to Q \to G \to 0$ with Q coflasque and P_1 permutation. Hence we may (and do) further assume G coflasque.

Let $K = k(G')$ and choose some $g \in G(K)$ mapping modulo R-equivalence to the generic point of G'. Then g defines a rational map $g : G' \dashrightarrow G$ such that fg is R-equivalent to $1_{G'}$. It follows that the induced map

$$1 - fg : G'/R \to G'/R \tag{4.4}$$

is identically 0.

Reapplying Lemma 4.4, we may find an extension $\tilde{G}'$ of G' by a suitable permutation torus which converts g into a true homomorphism. Since G is coflasque, Lemma 4.5 b) shows that $f : G \to G'$ lifts to $\tilde{f} : G \to \tilde{G}'$. Then (4.4) is still identically 0 when replacing (G', f) by $(\tilde{G}', \tilde{f})$.

Summarising: we have replaced the initial G and G' by suitable extensions by permutation tori, such that f lifts to these extensions and there is a homomorphism $g : G' \to G$ such that (4.4) vanishes identically. Hence $1-fg$ factors through a permutation torus P thanks to Proposition 4.11. Write $u : G' \to P$ and $v : P \to G'$ for homomorphisms such that $1 - fg = vu$. Let $G_1 = G \times P$ and consider the maps

$$f_1 = (f, v) : G_1 \to G', \qquad g_1 = (\, g \,)\, u : G' \to G_1.$$

Then $f_1 g_1 = 1$ and G' is a direct summand of G_1 as requested. □

5. Some open questions

Question 5.1. — Are lemma 4.4 and Proposition 4.7 still true when G is not a torus?

This is far from clear in general, starting with the case where G is an abelian variety and G' a torus. Let me give a positive answer in the case of an elliptic curve.

PROPOSITION 5.2. — *The answer to Question 5.1 is yes if the abelian part A of G is an elliptic curve.*

Proof. — Arguing as in the proof of Proposition 4.7, we get for an open subset $U \subseteq G$ of definition for f an exact sequence

$$0 \to \mathbb{G}_m \to P \to \mathrm{Alb}(U) \to G \to 0$$

where P is a permutation torus. Here we used that $\mathrm{NS}(\bar{G}) \simeq \mathbf{Z}$, which follows from Lemma 4.2.

The character group $X(P)$ has as a basis the geometric irreducible components of codimension 1 of $G - U$. Up to shrinking U, we may assume that $G - U$ contains the inverse image D of $0 \in A$. As the divisor class of 0 generates $\mathrm{NS}(\bar{A})$, D provides a Galois-equivariant splitting of the map $\mathbb{G}_m \to P$. Thus its cokernel is still a permutation torus, and we conclude as before. □

Question 5.3. — Can one formulate a version of Theorem 1.5 and Corollary 1.7 providing a description of the groups $\mathrm{Hom}_{\mathrm{DM}_-^{\mathrm{eff}}}(\nu_{\leqslant 0}G[0], \nu_{\leqslant 0}G'[0])$ and $\mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R)$ (at least when G and G' are tori)?

The proof of Theorem 1.5 suggests the presence of a closed model structure on the category of tori (or lattices), which might provide an answer to this question.

For the last question, let G be a semi-abelian variety. Forgetting its group structure, it has a motive $M(G) \in \mathrm{DM}_-^{\mathrm{eff}}$. Recall the canonical morphism

$$M(G) \to G[0]$$

induced by the "sum" maps

$$c(X, G) \xrightarrow{\sigma} G(X) \tag{5.1}$$

for smooth varieties X ([17, (6), (7)], [1, §1.3]).

The morphism (5.1) has a canonical section

$$G(X) \xrightarrow{\gamma} c(X, G) \tag{5.2}$$

given by the graph of a morphism: this section is functorial in X but is not additive.

Consider now a smooth equivariant compactification $\bar{G}$ of G. It exists in all characteristics. For tori, this is written up in [3]. The general case reduces to this one by the following elegant argument I learned from M. Brion: if G is an extension of an abelian variety A by a torus T, take a smooth projective equivariant compactification Y of T. Then the bundle $G \times^T Y$ associated to the T-torsor $G \to A$ also exists: this is the desired compactification.

Then we have a diagram of birational motives

$$\begin{array}{ccc} \nu_{\leqslant 0} M(G) & \xrightarrow{\sim} & \nu_{\leqslant 0} M(\bar{G}) \\ {\scriptstyle \nu_{\leqslant 0}\sigma} \downarrow & & \\ \nu_{\leqslant 0} G[0]. & & \end{array} \tag{5.3}$$

By [11], we have $H_0(\nu_{\leqslant 0} M(\bar{G}))(X) = CH_0(\bar{G}_{k(X)})$ for any smooth connected X. Hence the above diagram induces a homomorphism

$$CH_0(\bar{G}_{k(X)}) \to G(k(X))/R \tag{5.4}$$

which is natural in X for the action of finite correspondences (compare Corollary 1.3). One can probably check that this is the homomorphism of [12, (17) p. 78], reformulating [4, Proposition 12 p. 198]. Similarly, the set-theoretic map

$$G(k(X))/R \to CH_0(\bar{G}_{k(X)}) \tag{5.5}$$

of [4, p. 197] can presumably be recovered as a birational version of (5.2), using perhaps the homotopy category of schemes of Morel and Voevodsky [14].

In [12], Merkurjev shows that (5.4) is an isomorphism for G a torus of dimension at most 3. This suggests:

Question 5.4. — Is the map $\nu_{\leqslant 0}\sigma$ of Diagram (5.3) an isomorphism when G is a torus of dimension $\leqslant 3$?

In [13], Merkurjev gives examples of tori G for which (5.5) is not a homomorphism; hence its (additive) left inverse (5.4) cannot be an isomorphism. Merkurjev's examples are of the form $G = R^1_{K/k}\mathbb{G}_m \times R^1_{L/k}\mathbb{G}_m$, where K and L are distinct biquadratic extensions of k. This suggests:

Question 5.5. — Can one study Merkurjev's examples from the above viewpoint? More generally, what is the nature of the map $\nu_{\leqslant 0}\sigma$ of Diagram (5.3)?

We leave all these questions to the interested reader.

Bibliography

[1] Barbieri-Viale (L.), Kahn (B.). — On the derived category of 1-motives, arXiv:1009.1900.

[2] Borovoi (M.), Kunyavskĭi (B. E'), Lemire (N.), Reichstein (Z.). — Stably Cayley groups in characteristic zero, IMRN (2013).

[3] Colliot-Thélène (J.-L.), Harari (D.), Skorobogatov (A.). — Compactification équivariante d'un tore (d'après Brylinski et Künnemann), Expo. Math. 23, p. 161-170 (2005).

[4] Colliot-Thélène (J.-L.), Sansuc (J.-J.). — La R-équivalence sur les tores, Ann. Sci. Éc. Norm. Sup. 10, p. 175-230 (1977).

[5] Colliot-Thélène (J.-L.), Sansuc (J.-J.). — Principal homogeneous spaces under flasque tori; applications, J. Alg. 106, p. 148-205 (1987).

[6] Gille (P.). — La R-équivalence pour les groupes algébriques réductifs définis sur un corps global, Publ. Math. IHÉS 86, p. 199-235 (1997).

[7] Gille (P.). — Spécialisation de la R-équivalence pour les groupes réductifs, Trans. Amer. Math. Soc. 356, p. 4465-4474 (2004).

[8] Huber (A.) and Kahn (B.). — The slice filtration and mixed Tate motives, Compositio Math. 142, p. 907-936 (2006).

[9] Kahn (B.). — Sur le groupe des classes d'un schéma arithmétique (avec un appendice de Marc Hindry), Bull. Soc. Math. France 134 (2006), p. 395-415.

[10] Kahn (B.), Yamazaki (T.). — Voevodsky's motives and Weil reciprocity, Duke Math. J. 162, p. 2751-2796 (2013).

[11] Kahn (B.), Sujatha (R.). — Birational motives, I (preliminary version), preprint, 2002, http://www.math.uiuc.edu/K-theory/0596/.

[12] Merkurjev (A. S.). — R-equivalence on three-dimensional tori and zero-cycles, Algebra Number Theory 2, p. 69-89 (2008).

[13] Merkurjev (A. S.). — Zero-cycles on algebraic tori, in The geometry of algebraic cycles, 119-122, Clay Math. Proc., 9, Amer. Math. Soc., Providence, RI (2010).

[14] Morel (F.), Voevodsky (V.). — $\mathbf{A}^1$-homotopy theory of schemes, Publ. Math. IHÉS 90, p. 45-143 (1999).

[15] Mumford (D.). — Abelian varieties (corrected reprint), TIFR - Hindustan Book Agency (2008).

[16] Serre J.-P.. — Morphismes universels et variétés d'Albanese, in Exposés de séminaires, 1950-1989, Doc. mathématiques 1, SMF, p. 141-160 (2001).

[17] Spiess (M.), T. Szamuely (T.). — On the Albanese map for smooth quasi-projective varieties, Math. Ann. 325, p. 1-17 (2003).

[18] Voevodsky (V.). — Cohomological theory of presheaves with transfers, in E. Friedlander, A. Suslin, Voevodsky (V.) Cycles, transfers and motivic cohomology theories, Ann. Math. Studies 143, Princeton University Press, 88-137 (2000).

[19] Voevodsky (V.). — Triangulated categories of motives over a field, in E. Friedlander, A. Suslin, Voevodsky (V.) Cycles, transfers and motivic cohomology theories, Ann. Math. Studies 143, Princeton University Press, p. 188-238 (2000).