ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE
Mathématiques

ARTURO FERNÁNDEZ-PÉREZ

On Levi-flat hypersurfaces tangent to holomorphic webs

Tome XX, nº 3 (2011), p. 581-597.

<http://afst.cedram.org/item?id=AFST_2011_6_20_3_581_0>

Annales de la Faculté des Sciences de Toulouse Vol. XX, n° 3, 2011
pp. 581–597

On Levi-flat hypersurfaces tangent to holomorphic webs

Arturo Fernández-Pérez[1]

Abstract. — We investigate real analytic Levi-flat hypersurfaces tangent to holomorphic webs. We introduce the notion of first integrals for local webs. In particular, we prove that a k-web with finitely many invariant subvarieties through the origin tangent to a Levi-flat hypersurface has a holomorphic first integral.

Résumé. — Nous étudions les hypersurfaces analytiques réelles Levi-plates tangentes à un tissu holomorphe. Nous introduisons la notion d'intégrale première pour un tissu défini localement. En particulier, nous prouvons qu'un k-tissu admettant un nombre fini de sous-variétés invariantes passant par l'origine et tangentes à une hypersurface Levi-plate possède une intégrale première holomorphe.

1. Introduction

In very general terms, a germ of codimension one k-web is a collection of k germs of codimension one holomorphic foliations in "general position". The study of webs was initiated by Blaschke and his school in the late 1920s. For a recent account of the theory, we refer the reader to [12].

For instance, take $\omega \in Sym^k\Omega^1(\mathbb{C}^2, 0)$ defined by

$$\omega = (dy)^k + a_{k-1}(dy)^{k-1}dx + \ldots + a_0(dx)^k,$$

(*) Reçu le 15/11/2010, accepté le 20/06/2011

Work supported by CNPq-Brazil.

(1) Departamento de Matemática, UFMG, Av. Antônio Carlos, 6627 C.P. 702, 30123-970 – Belo Horizonte – MG, Brazil.
afernan@impa.br

where $a_j \in \mathcal{O}_2$ for all $0 \leqslant j \leqslant k-1$. Then $\mathcal{W} : \omega = 0$, define a non-trivial k-web on $(\mathbb{C}^2, 0)$. In this paper we study webs and its relation with Levi-flat hypersurfaces.

Let M be a germ at $0 \in \mathbb{C}^n$ of a real codimension one irreducible analytic set. Since M is real analytic of codimension one, it can be decomposed into M_{reg} and $\mathsf{Sing}(M)$, where M_{reg} is a germ of smooth real analytic hypersurface in $\mathbb{C}^n$ and $\mathsf{Sing}(M)$, the singular locus, is contained in a proper analytic subvariety of lower dimension. We shall say that M is *Levi-flat* if the complex distribution L on M_{reg}

$$L_p := T_pM \cap iT_pM \subset T_pM, \quad \text{for any } p \in M_{reg} \tag{1.1}$$

is integrable, in Frobenius sense. It follows that M_{reg} is smoothly foliated by immersed complex manifolds of complex dimension $n-1$. The foliation defined by L is called the Levi foliation and will be denoted by $\mathcal{L}_M$.

If M is a real analytic smooth Levi-flat hypersurface, by a classic result of E. Cartan there exists a local holomorphic coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that M can be represented by $M = \{\mathcal{I}m(z_n) = 0\}$. The situation if different if the hypersurface have singularities. Singular Levi-flat real analytic hypersurfaces have been studied by Burns and Gong [1], Brunella [2], Lebl [9], the author [6], [7] and many others.

Recently D.Cerveau and A. Lins Neto [5] have studied codimension one holomorphic foliations tangent to singular Levi-flat hypersurfaces. A codimension one holomorphic foliation $\mathcal{F}$ is tangent to M, if any leaf of $\mathcal{L}_M$ is also a leaf of $\mathcal{F}$. In [5] it is proved that a germ of codimension one holomorphic foliation tangent to a real analytic Levi-flat hypersurface has a non-constant meromorphic first integral. In the same spirit, the authors propose a problem for webs, which is as follows:

Problem. — Let M be a germ at $0 \in \mathbb{C}^n$, $n \geqslant 2$, of real analytic hypersurface Levi-flat. Assume that there exists a singular codimension one k-web, $k \geqslant 2$, such that any leaf of the Levi foliation $\mathcal{L}_M$ on M_{reg} is also a leaf of the web. Does the web has a non-constant meromorphic first integral?.

By a meromorphic first integral we mean something like $f_0(x) + z \cdot f_1(x) + \ldots + z^k \cdot f_k(x) = 0$, where $f_0, f_1, \ldots, f_k \in \mathcal{O}_n$. In this situation, the web is obtained by the elimination of z in the system given by

$$\begin{cases} f_0 + z \cdot f_1 + z^2 \cdot f_2 + \ldots + z^k \cdot f_k = 0 \\ df_0 + z \cdot df_1 + z^2 \cdot df_2 + \ldots + z^k \cdot df_k = 0. \end{cases}$$

In this work, we organize some results on singular Levi-flat hypersurfaces and holomorphic foliations which provide a best approach to study of webs and Levi-flats. Concerning the problem, we obtain an interesting result in a case very special (Theorem 1), the problem remains open in general.

1.1. Local singular webs

It is customary to define a germ of singular holomorphic foliation as an equivalence class $[\omega]$ of germs of holomorphic 1-forms in $\Omega^1(\mathbb{C}^n, 0)$ modulo multiplication by elements of $\mathcal{O}^*(\mathbb{C}^n, 0)$ such that any representative ω is integrable ($\omega \wedge d\omega = 0$) and with singular set $\mathsf{Sing}(\omega) = \{p \in (\mathbb{C}^n, 0) : \omega(p) = 0\}$ of codimension at least two.

An analogous definition can be made for codimension one k-webs. A germ at $(\mathbb{C}^n, 0)$, $n \geqslant 2$ of codimension one k-web $\mathcal{W}$ is an equivalence class $[\omega]$ of germs of k-symmetric 1-forms, that is sections of $Sym^k\Omega^1(\mathbb{C}^n, 0)$, modulo multiplication by $\mathcal{O}^*(\mathbb{C}^n, 0)$ such that a suitable representative ω defined in a connected neighborhood U of the origin satisfies the following conditions:

1. The zero set of ω has codimension at least two.
2. The 1-form ω, seen as a homogeneous polynomial of degree k in the ring $\mathcal{O}_n[dx_1, \ldots, dx_n]$, is square-free.
3. (Brill's condition) For a generic $p \in U$, $\omega(p)$ is a product of k linear forms.
4. (Frobenius's condition) For a generic $p \in U$, the germ of ω at p is the product of k germs of integrable 1-forms.

Both conditions (3) and (4) are automatic for germs at $(\mathbb{C}^2, 0)$ of webs and non-trivial for germs at $(\mathbb{C}^n, 0)$ when $n \geqslant 3$.

We can think k-webs as first order differential equations of degree k. The idea is to consider the germ of web as a meromorphic section of the projectivization of the cotangent bundle of $(\mathbb{C}^n, 0)$. This is a classical point view in the theory of differential equations, which has been recently explored in Web-geometry. For instance see [3], [4], [14].

1.2. The contact distribution

Let us denote $\mathbb{P} := \mathbb{P}T^*(\mathbb{C}^n, 0)$ the projectivization of the cotangent bundle of $(\mathbb{C}^n, 0)$ and $\pi : \mathbb{P}T^*(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ the natural projection. Over a point p the fiber $\pi^{-1}(p)$ parametrizes the one-dimensional subspaces of $T^*_p(\mathbb{C}^n, 0)$. On $\mathbb{P}$ there is a canonical codimension one distribution, the so called contact distribution $\mathcal{D}$. Its description in terms of a system of coordinates $x = (x_1, \dots, x_n)$ of $(\mathbb{C}^n, 0)$ goes as follows: let $dx_1, \dots, dx_n$ be the basis of $T^*(\mathbb{C}^n, 0)$ associated to the coordinate system $(x_1, \dots, x_n)$. Given a point $(x, y) \in T^*(\mathbb{C}^n, 0)$, we can write $y = \sum_{j=1}^{n} y_j dx_j$, $(y_1, \dots, y_n) \in \mathbb{C}^n$. In this way, if $(y_1, \dots, y_n) \neq 0$ then we set $[y] = [y_1, \dots . y_n] \in \mathbb{P}^{n-1}$ and $(x, [y]) \in (\mathbb{C}^n, 0) \times \mathbb{P}^{n-1} \cong \mathbb{P}$. In the affine coordinate system $y_n \neq 0$ of $\mathbb{P}$, the distribution $\mathcal{D}$ is defined by $\alpha = 0$, where

$$\alpha = dx_n - \sum_{j=1}^{n-1} p_j dx_j, \quad p_j = -\frac{y_j}{y_n} \quad (1 \leqslant j \leqslant n-1). \tag{1.2}$$

The 1-form α is called the contact form.

1.3. Webs as closures of meromorphic multi-sections

Let us consider $X \subset \mathbb{P}$ a subvariety, not necessarily irreducible, but of pure dimension n. Let $\pi_X : X \to (\mathbb{C}^n, 0)$ be the restriction to X of the projection π. Suppose also that X satisfies the following conditions:

1. The image under π of every irreducible component of X has dimension n.
2. The generic fiber of π intersects X in k distinct smooth points and at these the differential $d\pi_X : T_pX \to T_{\pi(p)}(\mathbb{C}^n, 0)$ is surjective. Note that $k = \deg(\pi_X)$.
3. The restriction of the contact form α to the smooth part of every irreducible component of X is integrable. We denote $\mathcal{F}_X$ the foliation defined by $\alpha|_X = 0$.

We can define a germ $\mathcal{W}$ at $0 \in \mathbb{C}^n$ of k-web as a triple $(X, \pi_X, \mathcal{F}_X)$. This definition is equivalent to one given in Section 1.1. In the sequel, X will always be the variety associated to $\mathcal{W}$, the singular set of X will be denoted by $\mathsf{Sing}(X)$ and its the smooth part will be denoted by X_{reg}.

Definition 1.1. — *Let R be the set of points $p \in X$ where*

- *either X is singular,*
- *or the differential $d\pi_X : T_pX_{reg} \to T_{\pi(p)}(\mathbb{C}^n, 0)$ is not an isomorphism.*

The analytic set R is called the criminant set of $\mathcal{W}$ and $\Delta_{\mathcal{W}} = \pi(R)$ the discriminant of $\mathcal{W}$. Note that $\mathsf{dim}(R) \leqslant n - 1$.

Remark 1.2. — Let $\omega \in Sym^k\Omega_1(\mathbb{C}^n, 0)$ and assume that it defines a k-web $\mathcal{W}$ with variety X. Then X is irreducible if, and only if, ω is irreducible in the ring $\mathcal{O}_n[dx_1, \ldots, dx_n]$. In this case we say that the web is irreducible.

Let M be a germ at $0 \in \mathbb{C}^n$ of a real analytic Levi-flat hypersurface.

DEFINITION 1.3. — *We say that M is tangent to $\mathcal{W}$ if any leaf of the Levi foliation $\mathcal{L}_M$ on M_{reg} is also a leaf of $\mathcal{W}$.*

1.4. First integrals for webs

DEFINITION 1.4. — *We say that $\mathcal{W}$ a k-web has a meromorphic first integral if, and only if, there exists*

$$P(z) = f_0 + z \cdot f_1 + \ldots + z^k \cdot f_k \in \mathcal{O}_n[z],$$

where $f_0, \ldots, f_k \in \mathcal{O}_n$, such that every irreducible component of the hypersurface $(P(z_0) = 0)$ is a leaf of $\mathcal{W}$, for all $z_0 \in (\mathbb{C}, 0)$.

DEFINITION 1.5. — *We say that $\mathcal{W}$ a k-web has a holomorphic first integral if, and only if, there exists*

$$P(z) = f_0 + z \cdot f_1 + \ldots + z^{k-1} \cdot f_{k-1} + z^k \in \mathcal{O}_n[z],$$

where $f_0, \ldots, f_{k-1} \in \mathcal{O}_n$, such that every irreducible component of the hypersurface $(P(z_0) = 0)$ is a leaf of $\mathcal{W}$, for all $z_0 \in (\mathbb{C}, 0)$.

We will prove a result concerning the situation of definitions 1.3 and 1.5.

THEOREM 1. — *Let $\mathcal{W}$ be a germ at $0 \in \mathbb{C}^n$, $n \geqslant 2$ of k-web defined by*

$$\omega = \sum_{\substack{i_1+\ldots+i_n=k \\ i_1,\ldots,i_n \geqslant 0}} a_{i_1,\ldots,i_n}(z) dz_1^{i_1} \ldots dz_n^{i_n},$$

where $a_{i_1,\ldots,i_n} \in \mathcal{O}_n$ and $a_{0,0,\ldots,0,k}(0) \neq 0$. Suppose that $\mathcal{W}$ is tangent to a germ at $0 \in \mathbb{C}^n$ of an irreducible real-analytic Levi-flat hypersurface M. Furthermore, assume that $\mathcal{W}$ is irreducible and has finitely many invariant analytic subvarieties through the origin. Let X be the variety associated to $\mathcal{W}$. Then $\mathcal{W}$ has a non-constant holomorphic first integral, if one of the following conditions is fulfilled :

1. *If $n = 2$.*
2. *If $n \geqslant 3$ and $cod_{X_{reg}}(\mathsf{Sing}(X)) \geqslant 2$.*

Moreover, if $P(z) = f_0 + z \cdot f_1 + \ldots + z^{k-1} \cdot f_{k-1} + z^k \in \mathcal{O}_n[z]$ *is a holomorphic first integral for* $\mathcal{W}$*, then* $M = (F = 0)$*, where* F *is obtained by the elimination of* z *in the system given by*

$$\begin{cases} f_0 + z \cdot f_1 + z^2 \cdot f_2 + \ldots + z^{k-1} \cdot f_{k-1} + z^k = 0 \\ \bar{f}_0 + z \cdot \bar{f}_1 + z^2 \cdot \bar{f}_2 + \ldots + z^{k-1} \cdot \bar{f}_{k-1} + z^k = 0. \end{cases}$$

Remark 1.6. — Under the hypotheses of Theorem 1, if $n = 2$ and $k = 1$, $\mathcal{W}$ is a non-dicritical holomorphic foliation at $(\mathbb{C}^2, 0)$ tangent to a germ of an irreducible real analytic Levi-flat hypersurface M, then Theorem 1 given by Cerveau and Lins Neto [5] assures that $\mathcal{W}$ has a non-constant holomorphic first integral. In this sense, our theorem is a generalization of result of Cerveau and Lins Neto.

Remark 1.7. — Let $\mathcal{W}$ a germ at $0 \in \mathbb{C}^n$, $n \geqslant 2$, of a smooth k-web tangent to a germ at $0 \in \mathbb{C}^n$ of an irreducible real codimension one submanifold M. In other words, $\mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_k$ is a generic superposition of k germs at $0 \in \mathbb{C}^n$ of smooth foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$. In this case the irreducibility and tangency conditions to M implies the existence of a unique $i \in \{1, \ldots, k\}$ such that $\mathcal{F}_i$ is tangent to M. Therefore we can find a coordinates system $z_1, \ldots, z_n$ of $\mathbb{C}^n$ such that $\mathcal{F}_i$ is defined by $dz_n = 0$ and $M = (\mathcal{I}m(z_n) = 0)$.

2. The foliation associated to a web

In this section, we prove a key lemma which will be used in the proof of main theorem.

Since the restriction of $\mathcal{D}$ to X_{reg} is integrable, it defines a foliation $\mathcal{F}_X$, which in general is a singular foliation. Given $p \in (\mathbb{C}^n, 0) \backslash \Delta_{\mathcal{W}}$, $\pi_X^{-1}(p) = \{q_1, \ldots, q_k\}$, where $q_i \neq q_j$, if $i \neq j$, $(\deg(\pi_X) = k)$, denote by $\mathcal{F}_X^i$ the germ of $\mathcal{F}_X$ at q_i, $i = 1, \ldots, k$.

The projections $\pi_*(\mathcal{F}_X^i) := \mathcal{F}_p^i$ define k germs of codimension one foliations at p.

Definition 2.1. — *A leaf of the web* $\mathcal{W}$ *is, by definition, the projection on* $(\mathbb{C}^n, 0)$ *of a leaf of* $\mathcal{F}_X$*.*

Remark 2.2. — Given $p \in (\mathbb{C}^n, 0) \backslash \Delta_{\mathcal{W}}$, and $q_i \in \pi_X^{-1}(p)$, the projection $\pi_X(L_i)$ of the leaf L_i of $\mathcal{F}_X$ through q_i, gives rise to a leaf of $\mathcal{W}$ through p. In particular, $\mathcal{W}$ has at most k leaves through p.

We will use the following proposition (cf. [8] Th. 5, pg. 32). Let $\mathcal{O}(X)$ denote the ring of holomorphic functions on X.

Proposition 2.3. — *Let V be an analytic variety. If $\pi : V \to W$ is a finite branched holomorphic covering of pure order k over an open subset $W \subseteq \mathbb{C}^n$, then to each holomorphic function $f \in \mathcal{O}(V)$ there is a canonically associated monic polynomial $P_f(z) \in \mathcal{O}_n[z] \subseteq \mathcal{O}(V)[z]$ of degree k such that $P_f(f) = 0$ in $\mathcal{O}(V)$.*

We have now the following lemma.

Lemma 2.4. — *Suppose that $(X, \pi_X, \mathcal{F}_X)$ defines a k-web $\mathcal{W}$ on $(\mathbb{C}^n, 0)$, $n \geqslant 2$, where X is an irreducible subvariety of $\mathbb{P}$. If $\mathcal{F}_X$ has a non-constant holomorphic first integral then $\mathcal{W}$ also has a holomorphic first integral.*

Proof. — Let $g \in \mathcal{O}(X)$ be the first integral for $\mathcal{F}_X$. By Proposition 2.3, there exists a monic polynomial $P_g(z) \in \mathcal{O}_n[z]$ of degree k such that $P_g(g) = 0$ in $\mathcal{O}(X)$. Write

$$P_g(z) = g_0 + z \cdot g_1 + \ldots + z^{k-1} \cdot g_{k-1} + z^k,$$

where $g_0, \ldots, g_{k-1} \in \mathcal{O}_n$.

Assertion. — P_g define a holomorphic first integral for $\mathcal{W}$.

Let $U \subseteq (\mathbb{C}^n, 0) \backslash \Delta_{\mathcal{W}}$ be an open subset and let $\varphi : X \to (\mathbb{C}^n, 0) \times \mathbb{C}$ be defined by $\varphi = (\pi_X, g)$. Take a leaf L of $\mathcal{W}|_U$. Then there is $z \in \mathbb{C}$ such that the following diagram

$$\begin{array}{ccc} \pi_X^{-1}(U) \cap \varphi^{-1}(L \times \{z\}) & \xrightarrow{\ \varphi\ } & L \times \{z\} \\ & {\scriptstyle \pi_X} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & L & \end{array}$$

is commutative, where pr_1 is the projection on the first coordinate. It follows that L is a leaf of $\mathcal{W}$ if and only if g is constant along of each connected component of $\pi_X^{-1}(L)$ contained in $\varphi^{-1}(L \times \{z\})$.

Consider now the hypersurface $G = \varphi(X) \subset (\mathbb{C}^n, 0) \times \mathbb{C}$ which is the closure of set

$$\{(x, s) \in U \times \mathbb{C} : g_0(x) + s \cdot g_1(x) + \ldots + s^{k-1} \cdot g_{k-1}(x) + s^k = 0\}.$$

Let $\psi : (\mathbb{C}^n, 0) \times \mathbb{C} \to (\mathbb{C}^n, 0)$ be the usual projection and denote by $Z \subset (\mathbb{C}^n, 0)$ the analytic subset such that the restriction to G of ψ not is a finite branched covering. Notice that for all $x_0 \in (\mathbb{C}^n, 0) \backslash Z$, the equation

$$g_0(x) + s \cdot g_1(x) + \ldots + s^{k-1} \cdot g_{k-1}(x) + s^k = 0$$

defines k analytic hypersurfaces pairwise transverse in x_0 and therefore correspond to leaves of $\mathcal{W}$. □

3. Examples

This section is devoted to give some examples of Levi-flat hypersurfaces tangent to holomorphic foliations or webs.

Example 3.1. — Take a non constant holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and set $M = (\mathcal{I}m(f) = 0)$. Then M is Levi-flat and M_{sing} is the set of critical points of f lying on M. Leaves of the Levi foliation on M_{reg} are given by $\{f = c\}$, $c \in \mathbb{R}$. Of course, M is tangent to a singular holomorphic foliation generated by the kernel of df.

Example 3.2 ([5]). — Let $f_0, f_1, \ldots, f_k \in \mathcal{O}_n$, $n \geqslant 2$, be irreducible germs of holomorphic functions, where $k \geqslant 2$. Consider the family of hypersurfaces

$$G := \{G_s := f_0 + sf_1 + \ldots + s^k f_k | s \in \mathbb{R}\}.$$

By eliminating the real variable s in the system $G_s = \bar{G}_s = 0$, we obtain a real analytic germ $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$ such that any complex hypersurface $(G_s = 0)$ is contained in the real hypersurface $(F = 0)$. For instance, in the case $k = 2$, we obtain

$$F = det \begin{pmatrix} f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \\ \bar{f}_0 & \bar{f}_1 & \bar{f}_2 & 0 \\ 0 & \bar{f}_0 & \bar{f}_1 & \bar{f}_2 \end{pmatrix} =$$

$$= f_0^2 \cdot \bar{f}_2^2 + \bar{f}_0^2 \cdot f_2^2 + f_0 \cdot f_2 \cdot \bar{f}_1^2 + \bar{f}_0 \cdot \bar{f}_2 \cdot f_1^2 - |f_1|^2 (f_0 \cdot \bar{f}_2 + \bar{f}_0 \cdot f_2) - 2|f_0|^2 \cdot |f_2|^2. \quad (3.1)$$

which comes from the elimination of s in the system

$$f_0 + s \cdot f_1 + s^2 \cdot f_2 = \bar{f}_0 + s \cdot \bar{f}_1 + s^2 \cdot \bar{f}_2 = 0.$$

We would like to observe that the examples of this type are tangent to singular webs. The web is obtained by the elimination of s in the system given by

$$\begin{cases} f_0 + s \cdot f_1 + s^2 \cdot f_2 + \ldots + s^k \cdot f_k = 0 \\ df_0 + s \cdot df_1 + s^2 \cdot df_2 \ldots + s^k \cdot df_k = 0 \end{cases}$$

In the case we get a 2-web given by the implicit differential equation $\Omega = 0$, where

$$\Omega = det \begin{pmatrix} f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \\ df_0 & df_1 & df_2 & 0 \\ 0 & df_0 & df_1 & df_2 \end{pmatrix}$$

This example shows that, although $\mathcal{L}_M$ is a foliation on $M_{reg} \subset M = (F = 0)$, in general it is not tangent to a germ of holomorphic foliation at $(\mathbb{C}^n, 0)$.

Example 3.3 [Clairaut's equations]. — Clairaut's equations are tangent to Levi-flat hypersurfaces. Consider the first-order implicit differential equation

$$y = xp + f(p), \tag{3.2}$$

where $(x, y) \in \mathbb{C}^2$, $p = \frac{dy}{dx}$ and $f \in \mathbb{C}[p]$ is a polynomial of degree k, the equation (3.2) define a k-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$. The variety S associated to $\mathcal{W}$ is given by $(y - xp - f(p) = 0)$ and the foliation $\mathcal{F}_S$ is defined by $\alpha|_S = 0$, where $\alpha = dy - pdx$. In the chart (x, p) of S, we get $\alpha|_S = (x + f'(p))dp$. The criminant set of $\mathcal{W}$ is given by

$$R = (y - xp - f(p) = x + f'(p) = 0).$$

Observe that $\mathcal{F}_S$ is tangent to S along R and has a non-constant first integral $g(x, p) = p$. Denote by $\pi_S : S \to (\mathbb{C}^2, 0)$ the restriction to S of the usual projection $\pi : \mathbb{P} \to (\mathbb{C}^2, 0)$, then the leaves of $\mathcal{F}_S$ project by π_S in leaves of $\mathcal{W}$. Those leaves are as follows

$$-y + s \cdot x + f(s) = 0, \tag{3.3}$$

where s is a constant. By the elimination of the variable $s \in \mathbb{R}$ in the system

$$\begin{cases} -y + s \cdot x + f(s) = 0 \\ -\bar{y} + s \cdot \bar{x} + \overline{f(s)} = 0, \end{cases}$$

we obtain a Levi-flat hypersurface tangent to $\mathcal{W}$. In particular, Clairaut's equation has a holomorphic first integral.

4. Lifting of Levi-flat hypersurfaces to the cotangent bundle

In this section we give some remarks about the lifting of a Levi-flat hypersurface to the cotangent bundle of $(\mathbb{C}^n, 0)$.

Let $\mathbb{P}$ be as before, the projectivized cotangent bundle of $(\mathbb{C}^n, 0)$ and M an irreducible real analytic Levi-flat at $(\mathbb{C}^n, 0)$, $n \geqslant 2$. Note that $\mathbb{P}$ is a $\mathbb{P}^{n-1}$-bundle over $(\mathbb{C}^n, 0)$, whose fiber $\mathbb{P}T^*_z\mathbb{C}^n$ over $z \in \mathbb{C}^n$ will be thought of as the set of complex hyperplanes in $T^*_z\mathbb{C}^n$. Let $\pi : \mathbb{P} \to (\mathbb{C}^n, 0)$ be the usual projection.

The regular part M_{reg} of M can be lifted to $\mathbb{P}$: just take, for every $z \in M_{reg}$, the complex hyperplane

$$T^{\mathbb{C}}_z M_{reg} = T_z M_{reg} \cap i(T_z M_{reg}) \subset T_z\mathbb{C}^n. \tag{4.1}$$

We call

$$M'_{reg} \subset \mathbb{P} \tag{4.2}$$

this lifting of M_{reg}. We remark that it is no more a hypersurface: its (real) dimension $2n-1$ is half of the real dimension of $\mathbb{P}T^*\mathbb{C}^n$. However, it is still "Levi-flat", in a sense which will be precised below.

Take now a point y in the closure $\overline{M'_{reg}}$ projecting on $\mathbb{C}^n$ to a point $x \in \overline{M}$. Now, we shall consider the following results, which are adapted from [2].

Lemma 4.1. — *There exist, in a germ of neighborhood $U_y \subset \mathbb{P}T^*(\mathbb{C}^n, 0)$ of y, a germ of real analytic subset N_y of dimension $2n-1$ containing $M'_{reg} \cap U_y$.*

Proposition 4.2. — *Under the above conditions, in a germ of neighborhood $V_y \subset U_y$ of y, there exists a germ of complex analytic subset Y_y of (complex) dimension n containing $N_y \cap V_y$.*

5. Proof of Theorem 1

The proof will be divided in two parts. First, we give the proof for $n = 2$. The proof in dimension $n \geqslant 3$ will be done by reduction to the case of dimension two.

First of all, we recall some results (cf. [5]) about foliations and Levi-flats. Let M and $\mathcal{F}$ be germs at $(\mathbb{C}^2, 0)$ of a real analytic Levi-flat hypersurface and of a holomorphic foliation, respectively, where $\mathcal{F}$ is tangent to M. Assume that:

(i) $\mathcal{F}$ is defined by a germ at $0 \in \mathbb{C}^2$ of holomorphic vector field X with an isolated singularity at 0.

(ii) M is irreducible.

Let us assume that 0 is a reduced singularity of X, in the sense of Seidenberg [13]. Denote the eigenvalues of $DX(0)$ by λ_1, λ_2.

PROPOSITION 5.1. — *Suppose that X has a reduced singularity at $0 \in \mathbb{C}^2$ and is tangent to a real analytic Levi-flat hypersurface M. Then $\lambda_1, \lambda_2 \neq 0$, $\lambda_2/\lambda_1 \in \mathbb{Q}_-$ and X has a holomorphic first integral.*
In particular, in a suitable coordinates system (x, y) around $0 \in \mathbb{C}^2$, $X = \phi.Y$, where $\phi(0) \neq 0$ and

$$Y = q.x\partial_x - p.y\partial_y \ , g.c.d(p, q) = 1. \tag{5.1}$$

In this coordinate system, $f(x, y) := x^p.y^q$ is a first integral of X.

We call this type of singularity of $\mathcal{F}$ a saddle with first integral, (cf. [10], pg. 162). Now we have the following lemma.

LEMMA 5.2. — *For any $z_0 \in M_{reg}$, the leaf L_{z_0} of $\mathcal{L}_M$ through z_0 is closed in M_{reg}.*

5.1. Planar webs

A k-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$ can be written in coordinates $(x, y) \in \mathbb{C}^2$ by

$$\omega = a_0(x, y)(dy)^k + a_1(x, y)(dy)^{k-1}(dx) + \ldots + a_k(x, y)(dx)^k = 0,$$

where the coefficients $a_j \in \mathcal{O}_2$, $j = 1, \ldots, k$. We set

$$U = \{(x, y, [adx + bdy]) \in \mathbb{P}T^*(\mathbb{C}^2, 0) : a \neq 0\}$$

and

$$V = \{(x, y, [adx + bdy]) \in \mathbb{P}T^*(\mathbb{C}^2, 0) : b \neq 0\}.$$

Note that $\mathbb{P}T^*(\mathbb{C}^2, 0) = U \cup V$. Suppose that $(S, \pi_S, \mathcal{F}_S)$ define $\mathcal{W}$, in the coordinates $(x, y, p) \in U$, where $p = \frac{dy}{dx}$, we have

1. $S \cap U = \{(x, y, p) \in \mathbb{P}T^*(\mathbb{C}^2, 0) : F(x, y, p) = 0\}$, where

   $$F(x, y, p) = a_0(x, y)p^k + a_1(x, y)p^{k-1} + \ldots + a_k(x, y).$$

   Note that S is possibly singular at 0.

2. $\mathcal{F}_S$ is defined by $\alpha|_S = 0$, where $\alpha = dy - pdx$.

3. The criminant set R is defined by the equations

   $$F(x, y, p) = F_p(x, y, p) = 0.$$

In V the coordinate system is $(x, y, q) \in \mathbb{C}^3$, where $q = \frac{1}{p}$, the equations are similar.

5.2. Proof in dimension two

Let $\mathcal{W}$ be a k-web tangent to M Levi-flat and let us consider S, π be as before. The idea is to use Lemma 2.4, assume that $\mathcal{W}$ is defined by

$$\omega = a_0(x,y)(dy)^k + a_1(x,y)(dy)^{k-1}dx + \ldots + a_k(x,y)(dx)^k = 0, \quad (5.2)$$

where the coefficients $a_j \in \mathcal{O}_2$, $j = 1, \ldots, k$ and $a_0(0,0) = 1$.

Lemma 5.3. — *Under the hypotheses of Theorem 1 and above conditions, the surface S is irreducible and $S \cap \pi^{-1}(0)$ contains just a number finite of points. See figure 1.*

Proof. — Since $\mathcal{W}$ is irreducible so is S. On the other hand, $S \cap \pi^{-1}(0)$ is finite because $\mathcal{W}$ has a finite number of invariant analytic leaves through the origin and is defined as in (5.2). □

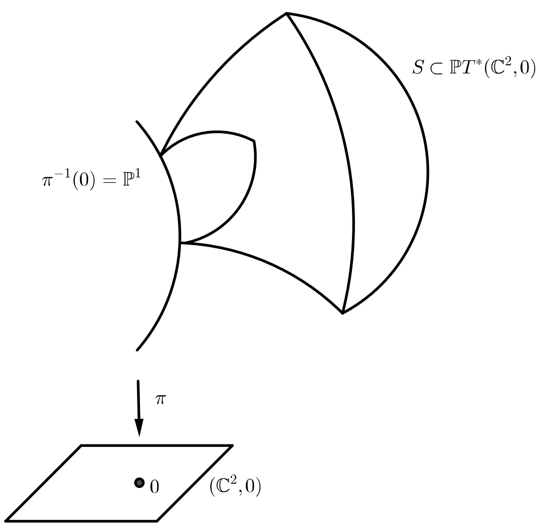

Figure 1. — $S \cap \pi^{-1}(0)$.

We can assume without lost of generality that $S \cap \pi^{-1}(0)$ contains just one point, in the case general, the idea of the proof is the same. Then in the coordinate system $(x,y,p) \in \mathbb{C}^3$, where $p = \frac{dy}{dx}$, we have $\pi^{-1}(0) \cap S = \{p_0 = (0,0,0)\}$, which implies that S must be singular at $p_0 \in \mathbb{P}T^*(\mathbb{C}^2, 0)$. In particular, (S, p_0) the germ of S at p_0 is defined by $F^{-1}(0)$, where

$$F(x,y,p) = p^k + a_1(x,y)p^{k-1} + \ldots + a_k(x,y),$$

and $a_1, \ldots, a_k \in \mathcal{O}_2$. Let $\mathcal{F}_S$ be the foliation defined by $\alpha|_S = 0$. The assumptions implies that $\mathcal{F}_S$ is a non-dicritical foliation with an isolated singularity at p_0.

Recall that a germ of foliation $\mathcal{F}$ at $p_0 \in S$ is dicritical if it has infinitely many analytic separatrices through p_0. Otherwise it is called non-dicritical.

Let M'_{reg} be the lifting of M_{reg} by π_S, and denote by $\sigma : (\tilde{S}, D) \to (S, p_0)$ the resolution of singularities of S at p_0. Let $\tilde{\mathcal{F}} = \sigma^*(\mathcal{F}_S)$ be the pull-back of $\mathcal{F}_S$ under σ. See figure 2.

Lemma 5.4. — *In the above situation. The foliation $\tilde{\mathcal{F}}$ has only singularities of saddle with first integral type in D.*

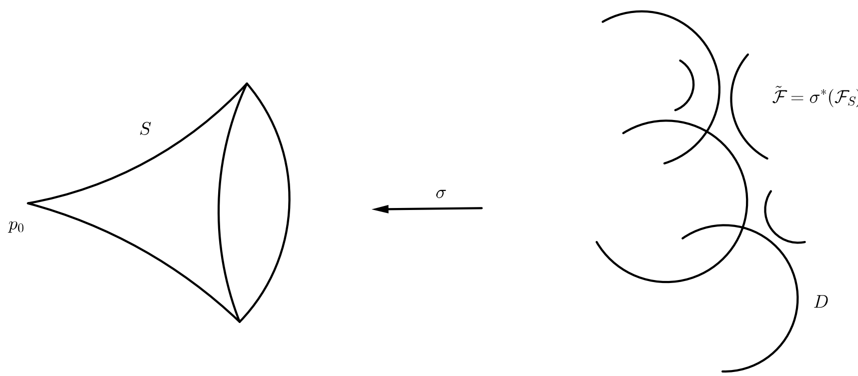

Figure 2. — Resolution of singularities of S at p_0.

Proof. — Let $y \in \overline{M'_{reg}}$, it follows from Lemma 4.1 the existence, in a neighborhood $U_y \subset \mathbb{P}T^*(\mathbb{C}^2, 0)$ containing y, of a real analytic subset N_y of dimension 3 containing $M'_{reg} \cap U_y$. Then by Proposition 4.2, there exists, in a neighborhood $V_y \subset U_y$ of y, a complex analytic subset Y_y of (complex) dimension 2 containing $N_y \cap V_y$. As germs at y, we get $Y_y = S_y$ then $N_y \cap V_y \subset S_y$, we have that $N_y \cap V_y$ is a real analytic hypersurface in S_y, and it is Levi-flat because each irreducible component contains a Levi-flat piece (cf. [1], Lemma 2.2).

Let us denote $M'_y = N_y \cap V_y$. The hypotheses implies that $\mathcal{F}_S$ is tangent to M'_y. These local constructions are sufficiently canonical to be patched together, when y varies on $\overline{M'_{reg}}$: if $S_{y_1} \subset V_{y_1}$ and $S_{y_2} \subset V_{y_2}$ are as above, with $M'_{reg} \cap V_{y_1} \cap V_{y_2} \neq \emptyset$, then $S_{y_2} \cap (V_{y_1} \cap V_{y_2})$ and $S_{y_1} \cap (V_{y_1} \cap V_{y_2})$ have some common irreducible components containing $M'_{reg} \cap V_{y_1} \cap V_{y_2}$, so that M'_{y_1}, M'_{y_2} can be glued by identifying those components. In this way, we obtain a Levi-flat hypersurface N on S tangent to $\mathcal{F}_S$.

By doing additional blowing-ups if necessary, we can suppose that $\tilde{\mathcal{F}}$ has reduced singularities. Since $\mathcal{F}_S$ is non-dicritical, all irreducible components of D are $\tilde{\mathcal{F}}$-invariants. Let $\tilde{N}$ be the strict transform of N under σ, then $\tilde{N} \supset D$. In particular, $\tilde{N}$ contains all singularities of $\tilde{\mathcal{F}}$ in D. It follows from Proposition 5.1 that all singularities of $\tilde{\mathcal{F}}$ are saddle with first integral. □

5.3. End of the proof of Theorem 1 in dimension two

The idea is to prove that $\mathcal{F}_S$ has a holomorphic first integral. Since D is invariant by $\tilde{\mathcal{F}}$, i.e., it is the union of leaves and singularities of $\tilde{\mathcal{F}}$, we have $S := D \backslash \mathsf{Sing}(\tilde{\mathcal{F}})$ is a leaf of $\tilde{\mathcal{F}}$. Now, fix $p \in S$ and a transverse section $\sum$ through p. By Lemma 5.4, the singularities of $\tilde{\mathcal{F}}$ in D are saddle with first integral types. Therefore the transverse section $\sum$ is complete, (see [10], pg. 162). Let $G \subset \mathsf{Diff}(\sum, p)$ be the Holonomy group of the leaf S of $\tilde{\mathcal{F}}$. It follows from Lemma 5.2 that all leaves of $\mathcal{F}_S$ through points of N_{reg} are closed in N_{reg}. This implies that all transformations of G have finite order and G is linearizable. According to [11], $\mathcal{F}_S$ has a non-constant holomorphic first integral. Finally from Lemma 2.4, $\mathcal{W}$ has a first integral as follows:

$$P(z) = f_0(x,y) + z \cdot f_1(x,y) + \ldots + z^{k-1} \cdot f_{k-1}(x,y) + z^k,$$

where $f_0, f_1, \ldots, f_{k-1} \in \mathcal{O}_2$.

5.4. Proof in dimension $n \geqslant 3$

Let us give an idea of the proof. First of all, we will prove that there is a holomorphic embedding $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ with the following properties:

(i) $i^{-1}(M)$ has real codimension one on $(\mathbb{C}^2, 0)$.

(ii) $i^*(\mathcal{W})$ is a k-web on $(\mathbb{C}^2, 0)$ tangent to $i^{-1}(M)$.

Set $E := i(\mathbb{C}^2, 0)$. The above conditions and Theorem 1 in dimension two imply that $\mathcal{W}|_E$ has a non-constant holomorphic first integral, say $g = f_0 + z \cdot f_1 + \ldots + z^{k-1} \cdot f_{k-1} + z^k$, where $f_0, \ldots, f_{k-1} \in \mathcal{O}_2$. After that we will use a lemma to prove that g can be extended to a holomorphic germ g_1, which is a first integral of $\mathcal{W}$.

Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geqslant 3$, of a holomorphic codimension one foliation, tangent to a real analytic hypersurface M. Let us suppose that $\mathcal{F}$ is defined by $\omega = 0$, where ω is a germ at $0 \in \mathbb{C}^n$ of an integrable holomorphic 1-form with $cod_{\mathbb{C}^n}(\mathsf{Sing}(\omega)) \geqslant 2$. We say that a holomorphic embedding $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ is transverse to ω if $cod_{\mathbb{C}^n}(\mathsf{Sing}(\omega)) = 2$, which means in fact that, as a germ of set, we have $\mathsf{Sing}(i^*(\omega)) = \{0\}$. Note that the definition is independent of the particular germ of holomorphic 1-form which represents $\mathcal{F}$. Therefore, we will say that the embedding i is transverse to $\mathcal{F}$ if it is transverse to some holomorphic 1-form ω representing $\mathcal{F}$.

We will use the following lemma of [5].

LEMMA 5.5. — *In the above situation, there exists a 2-plane $E \subset \mathbb{C}^n$, transverse to $\mathcal{F}$, such that the germ at $0 \in E$ of $M \cap E$ has real codimension one.*

We say that a embedding i is transverse to $\mathcal{W}$ if it is transverse to all k-foliations which defines $\mathcal{W}$. Now, one deduces the following

LEMMA 5.6. — *There exists a 2-plane $E \subset \mathbb{C}^n$, transverse to $\mathcal{W}$, such that the germ at $0 \in E$ of $M \cap E$ has real codimension one.*

Proof. — First of all, note that outside of the discriminant set of $\mathcal{W}$, we can suppose that $\mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_k$, where $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are germs of codimension one smooth foliations. Since $\mathcal{W}$ is tangent to M, there is a foliation $\mathcal{F}_j$ such that is tangent to a Levi foliation $\mathcal{L}_M$ on M_{reg}. Lemma 5.5 implies that we can find a 2-plane E_0 transverse to M and to $\mathcal{F}_j$. Clearly the set of linear mappings transverse to $\mathcal{F}_1, \ldots, \mathcal{F}_k$ simultaneously is open and dense in the set of linear mappings from $\mathbb{C}^2$ to $\mathbb{C}^n$, by Transversality theory, there exists a linear embedding i such that $E = i(\mathbb{C}^2, 0)$ is transverse to M_{reg} and to $\mathcal{W}$ simultaneously. □

Let E be a 2-plane as in Lemma 5.6. It easy to check that $\mathcal{W}|_E$ satisfies the hypotheses of Theorem 1. By the two dimensional case $\mathcal{W}|_E$ has a non-constant first integral:

$$g_0 + z \cdot g_1 + \ldots + z^{k-1} \cdot g_{k-1} + z^k, \tag{5.3}$$

where $g_0, \ldots, g_{k-1} \in \mathcal{O}_2$.

Let X be the variety associated to $\mathcal{W}$ and set S be the surface associated to $\mathcal{W}|_E$. Observe that $\mathcal{F}_S$ has a non-constant holomorphic first integral g defined on S.

LEMMA 5.7. — *In the above situation, we have $\mathcal{F}_X|_S = \mathcal{F}_S$ and $\mathcal{F}_X$ has a non-constant holomorphic first integral g_1 on X, such that $g_1|_S = g$.*

Proof. — It is easily seen that $S \subset X$ which implies that $\mathcal{F}_X|_S = \mathcal{F}_S$. Let us extend g to X. Fix $p \in X_{reg} \backslash \mathsf{Sing}(\mathcal{F}_X)$. It is possible to find a small neighborhood $W_p \subset X$ of p and a holomorphic coordinate chart $\varphi : W_p \to \triangle$, where $\triangle \subset \mathbb{C}^n$ is a polydisc, such that:

(i) $\varphi(S \cap W_p) = \{z_3 = \ldots = z_n = 0\} \cap \triangle$.

(ii) $\varphi_*(\mathcal{F}_X)$ is given by $dz_n|_\triangle = 0$.

Let $\pi_n : \mathbb{C}^n \to \mathbb{C}^2$ be the projection defined by $\pi_n(z_1, \ldots, z_n) = (z_1, z_2)$ and set $\tilde{g}_p := g \circ \varphi^{-1} \circ \pi_n|_\triangle$. We obtain that $\tilde{g}$ is a holomorphic function defined in $\triangle$ and is a first integral of $\varphi_*(\mathcal{F}_X)$. Let $g_p = \tilde{g}_p \circ \varphi$. Notice that, if $W_p \cap W_q \neq \emptyset$, p and q being regular points for $\mathcal{F}_X$, then we have $g_p|_{W_p \cap W_q} = g_q|_{W_p \cap W_q}$. This follows easily form the identity principle for holomorphic functions. In particular, g can be extended to

$$W = \bigcup_{p \in X_{reg} \backslash \mathsf{Sing}(\mathcal{F}_X)} W_p,$$

which is a neighborhood of $X_{reg} \backslash \mathsf{Sing}(\mathcal{F}_X)$. Call g_W this extension.

Since $cod_{X_{reg}} \mathsf{Sing}(\mathcal{F}_X) \geqslant 2$, by a theorem of Levi (cf. [15]), g_W can be extended to X_{reg}, as $cod_{X_{reg}}(\mathsf{Sing}(X)) \geqslant 2$ this allows us to extend g_W to g_1 as holomorphic first integral for $\mathcal{F}_X$, in whole X. □

5.5. End of the proof of Theorem 1 in dimension $n \geqslant 3$

Since $\mathcal{F}_X$ has a non-constant holomorphic first integral on X, Lemma 2.4 imply that $\mathcal{W}$ has a non-constant holomorphic first integral.

Acknowledgments. — The author is greatly indebted to Alcides Lins Neto for suggesting the problem and for many stimulating conversations. This is part of the author's Ph.D. thesis, written at IMPA. I would like to acknowledge D. Cerveau for their useful comments for improvement the exposition. I would also like to acknowledge M. Brunella, for pointing out a problem with the proof of Lemma 5.3. Finally, I would like to thank the referee for his comments and suggestions.

Bibliography

[1] Burns (D.), Gong (X.). — Singular Levi-flat real analytic hypersurfaces, Amer. J. Math. 121, p. 23-53 (1999).

[2] Brunella (M.). — Singular Levi-flat hypersurfaces and codimension one foliations. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 6, no. 4, p. 661-672 (2007).

[3] Cavalier (V.), Lehmann (D.). — Introduction à l'étude globale des tissus sur une surface holomorphe. Ann. Inst. Fourier (Grenoble) 57, no. 4, p. 1095-1133 (2007).

[4] Cavalier (V), Lehmann (D.). — Global structure of holomorphic webs on surfaces. Geometry and topology of caustics-CAUSTICS '06, 35-44, Banach Center Publ., 82, Polish Acad. Sci. Inst. Math., Warsaw (2008).

[5] Cerveau (D.), Lins Neto (A.). — Local Levi-flat hypersurfaces invariants by a codimension one holomorphic foliation, To appear in Amer. J. Math.

[6] Fernández-Pérez (A.). — On normal forms of singular Levi-flat real analytic hypersurfaces. Bull. Braz. Math. Soc. (N.S.) 42, no. 1, p. 75-85 (2011).

[7] Fernández-Pérez (A.). — Singular Levi-flat hypersurfaces. An approach through holomorphic foliations. Ph.D Thesis IMPA, Brazil (2010).

[8] Gunning (R.). — Introduction to holomorphic functions of several variables. Vol. II. Local theory. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA (1990).

[9] Lebl (J.). — Singularities and complexity in CR geometry. Ph.D. Thesis, University of California at San Diego, Spring (2007).

[10] Loray (F.). — Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux. ; http://hal.archives-ouvertures.fr/ccsd-00016434

[11] Mattei (J.F.), Moussu (R.). — Holonomie et intégrales premières, Ann. Ec. Norm. Sup. 13, p. 469-523 (1980).

[12] Pereira (J.V.), Pirio (L.). — An invitation to web geometry. From Abel's addition theorem to the algebraization of codimension one webs. Publicações Matemáticas do IMPA. Rio de Janeiro (2009).

[13] Seidenberg (A.). — Reduction of singularities of the differential equation $A\,dy = B\,dx$. Amer. J. Math. 90, p. 248-269 (1968).

[14] Yartey (J.N.A). — Number of singularities of a generic web on the complex projective plane. J. Dyn. Control Syst. 11, no. 2, p. 281-296 (2005).

[15] Siu (Y.T.). — Techniques of extension of analytic objects. Lecture Notes in Pure and Applied Mathematics, Vol. 8. Marcel Dekker, Inc., New York (1974).