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Levi-flat filling of real two-spheres in symplectic manifolds (I)

HERVÉ GAUSSIER⁽¹⁾, ALEXANDRE SUKHOV⁽²⁾

ABSTRACT. — Let (M, J, ω) be a manifold with an almost complex structure J tamed by a symplectic form ω . We suppose that M has the complex dimension two, is Levi-convex and with bounded geometry. We prove that a real two-sphere with two elliptic points, embedded into the boundary of M can be foliated by the boundaries of pseudoholomorphic discs.

RÉSUMÉ. — Soit (M, J, ω) une variété dont la structure presque complexe J est contrôlée par la forme symplectique ω . On suppose M de dimension complexe deux, Levi-convexe et à géométrie bornée. On démontre que toute 2-sphère possédant deux points elliptiques et plongée dans le bord de M est feuilletée par des bords de disques pseudoholomorphes.

1. Introduction

This expository paper concerns the problem of filling real two-spheres in almost complex manifolds by pseudoholomorphic discs. We prove the following statement :

THEOREM 1.1. — *Let (M, J, ω) be an almost complex manifold of complex dimension 2 with a taming symplectic form, with bounded geometry. Assume that M does not contain any compact J -complex sphere and that*

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the boundary ∂M of M is a smooth Levi-convex hypersurface that does not contain any germ of a nonconstant J -holomorphic disc. Let S^2 be a real 2-sphere with two elliptic points, embedded into ∂M . Then there exists a unique smooth one parameter family of disjoint Bishop discs for S^2 filling a real Levi-flat hypersurface $\Gamma \subset M$ with boundary S^2 . This hypersurface is smooth up to the boundary except at the two elliptic points.

Though results of this type admit important applications and have been obtained by several authors in various forms, it seems difficult to find a reference for a complete proof of the above statement. This is our first motivation. The above result and the techniques of the present work will be used in a forthcoming paper devoted to the problem of filling two-spheres with elliptic and hyperbolic points. The present work allows us to focus the second paper on properties of discs near hyperbolic points.

Recall that according to classical results a real two-sphere does not admit a totally real embedding into a two-dimensional complex manifold : any embedding of S^2 has complex points. In the generic case these points are isolated and are either elliptic or hyperbolic. Parabolic points can be removed by a small deformation. The first related local result goes back to E.Bishop [4] who proved the existence of a family of holomorphic discs with boundaries attached to a neighborhood of an analytic point in a real two-dimensional sphere in \mathbb{C}^2 . The global result is due to E.Bedford-B.Gaveau [2]. They proved the following. Let Ω be a strictly pseudoconvex domain in \mathbb{C}^2 and S^2 be a real 2-sphere embedded into $\partial\Omega$. If S^2 has exactly two elliptic points p, q and is totally real outside these points then S^2 is the boundary of a Levi-flat hypersurface Γ foliated by holomorphic discs with boundaries on $S^2 \setminus \{p, q\}$. The hypersurface Γ can be viewed as a resolution of the Plateau problem and has other important properties related to polynomially convex hulls and envelopes of holomorphy. This result was extended by M.Gromov [13] to the almost complex case under the assumption that the almost complex structure is real analytic and integrable near S^2 . This extra assumption was removed by R.Ye in the paper [25]. These results have many important consequences in complex and contact geometry. One of them is the fundamental observation due to M.Gromov that the characteristic foliation on any 2-sphere embedded into the 3-sphere with the standard contact structure has no closed leaf. Another important application is due to H.Hofer [15] for his proof of the Weinstein's conjecture for contact 3-manifolds M with $\pi_2(M) \neq \{0\}$. Y.Eliashberg discovered several applications in symplectic topology [10].

The above mentioned results were obtained for 2-spheres contained in the boundary of a strictly pseudoconvex domain. In the present paper we

consider the filling of a 2-sphere embedded to the boundary of a pseudoconvex domain assuming that this boundary contains no non-constant holomorphic discs. This condition is automatically satisfied for strictly pseudoconvex domains so the class of domains under consideration is substantially larger. Y.Eliashberg-W.Thurston [11] introduced a new class of structures called confoliations, intermediate objects between contact structures and foliations. One of the most important examples of such structures is provided by the holomorphic tangent bundle of a weakly pseudoconvex hypersurface in \mathbb{C}^2 . This can be viewed as a contact structure with degeneracies. The problem of filling a real 2-sphere in a confoliated 3-manifold by pseudo-holomorphic discs was studied by R.Hind [14]. His approach is based on a result due to E.Bedford-W.Klingenberg [3] on the Levi-flat filling of spheres in the presence of hyperbolic points and on a theorem of Y.Eliashberg-W.Thurston on the approximation of a confoliation by contact structures. However several important arguments of the proof are dropped in [14]. The present paper presents a detailed proof in the elliptic case, based on a quite different approach. Our main idea is to use an almost complex version of the well-known result of Diederich-Fornaess [7] on the existence of bounded exhaustion plurisubharmonic functions for weakly pseudoconvex domains in \mathbb{C}^n . The corresponding almost complex version is obtained in [8]. It removes many technical problems in the proof. In particular this approach allows to give a direct proof without using Eliashberg-Thurston's theory. The results of [8] are valid in any dimension in contrast to the Eliashberg-Thurston theory only working in complex dimension two. This allows to apply our techniques in the study of confoliations in the higher dimensional case for some classes of complex structures. At the end of the paper (see Remark 3 in the last section) we indicate how to use our approach in order to obtain new results on the filling of two-spheres in weakly pseudoconvex boundaries by holomorphic discs. We also point out that the examples due to Y.Eliashberg [10] and J.E.Fornaess-D.Ma [12] show that the condition of pseudoconvexity of ∂M in Theorem 1.1 cannot be dropped.

Keeping in mind forthcoming papers and for convenience of the reader we give a reasonably self contained exposition.

2. Preliminaries

2.1. Almost complex and symplectic structures

All manifolds and almost complex structures are supposed to be of class C^∞ though the main results require a lower regularity. Let (M', J') and (M, J) be almost complex manifolds and let f be a map of class C^1 from \tilde{M}

to M . We say that f is (J', J) -holomorphic if $df \circ J' = J \circ df$. We denote by \mathbb{D} the unit disc in \mathbb{C} and by J_{st} the standard structure on \mathbb{C}^n for every n . If $(M', J') = (\mathbb{D}, \mathbb{J} \sim \approx)$, we call f a J -holomorphic disc in M . Every almost complex manifold (M, J) can be viewed locally as the unit ball \mathbb{B} in \mathbb{C}^n equipped with a small almost complex deformation of J_{st} . Indeed for every point $p \in M$, every real $\alpha \geq 0$ and $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinates diffeomorphism $\phi : U \rightarrow \mathbb{B}$ such that $\phi(p) = 0$, $d\phi(p) \circ J(p) \circ d\phi^{-1}(0) = J_{st}$ and the direct image $\phi_*(J) := d\phi \circ J \circ d\phi^{-1}$ satisfies $\|\phi_*(J) - J_{st}\|_{C^\alpha(\mathbb{B})} \leq \lambda_0$.

Let U be an open set in M and $\phi : U \rightarrow \mathbb{C}^n$ be a coordinates diffeomorphism. Then $\phi(U)$ is an open set in \mathbb{C}^n with standard coordinates z . In the sequel we often denote $\phi_*(J)$ by J . If $f : \mathbb{D} \rightarrow \mathbb{U}$ is a J -holomorphic disc then we still denote by f the composition $\phi \circ f$ viewing f as a \mathbb{C}^n -valued map $f : \zeta \mapsto z(\zeta)$. The conditions $J(z)^2 = -I$ and $J(0) = J_{st}$ imply that for every z the endomorphism u of \mathbb{R}^{2n} defined by $u(z) := -(J_{st} + J(z))^{-1}(J_{st} - J(z))$ is anti \mathbb{C} -linear that is $u \circ J_{st} = -J_{st} \circ u$. Thus u is a composition of the complex conjugation and a \mathbb{C} -linear operator. Denote by $A_J(z)$ the complex $n \times n$ matrix such that $u(z)(v) = A_J(z)\bar{v}$ for any $v \in \mathbb{C}^n$. The entries of the matrix $A_J(z)$ are smooth functions of z and

$$A_J(0) = 0. \tag{2.1}$$

The J -holomorphicity condition $J(z) \circ df = df \circ J_{st}$ can be written in the form of the nonlinear Cauchy-Riemann equation :

$$\partial_{\bar{\zeta}} f + A_J(f) \partial_{\bar{\zeta}} \bar{f} = 0. \tag{2.2}$$

The matrix function A_J is called the deformation tensor of J in the coordinates z . Consider the isotropic dilations $d_\lambda : z \mapsto \lambda^{-1}z$ in \mathbb{C}^n . Since the structures $J_\lambda := (d_\lambda)_*(J)$ converge to J_{st} in any C^α -norm as $\lambda \rightarrow 0$, we have $A_{J_\lambda} \rightarrow_{\lambda \rightarrow 0} 0$ in any C^α norm. Thus, shrinking U if necessary and using the isotropic dilations of coordinates we can assume that for given $\alpha > 0$ we have $\|A_J\|_{C^\alpha(\mathbb{B})} < 1$ on the unit ball of \mathbb{C}^n . In particular, the system (2.2) is elliptic.

According to classical results [24], the Cauchy-Green transform

$$Tf(\zeta) = \frac{1}{2\pi i} \int \int_{\mathbb{D}} \frac{f(\tau)}{\tau - \zeta} d\tau \wedge d\bar{\tau}$$

is a continuous linear operator from $C^\alpha(\overline{\mathbb{D}})$ into $C^{\alpha+1}(\overline{\mathbb{D}})$ for every non-integer $\alpha > 0$. Hence the operator

$$\Psi_J : f \rightarrow h = f + TA_J(f) \partial_{\bar{\zeta}} \bar{f} \tag{2.3}$$

maps the space $C^\alpha(\overline{\mathbb{D}})$ into itself and we can write the equation (2.2) in the form $\partial_{\bar{z}}\Psi_J(f) = 0$. This means that a disc f is J -holomorphic if and only if the map $h = \Psi_J(f)$ is J_{st} -holomorphic. If the norm of A_J is small enough, then by the Implicit Function Theorem the operator Ψ_J realizes a one-to-one correspondence between sufficiently small J -holomorphic discs and J_{st} -holomorphic discs. This implies the existence of a J -holomorphic disc in a given tangent direction through a given point, a smooth dependence of such a disc on a deformation of the point, the tangent vector and the almost complex structure, as well as the interior elliptic regularity of discs. This is the content of the classical Nijenhuis-Woolf theorem, see [21] for a short complete proof.

A *symplectic form* ω on a smooth $2n$ -dimensional manifold M is a non-degenerate closed differential two-form. The pair (M, ω) is called a *symplectic manifold*. An almost complex structure J on M is called *tamed* by ω if $\omega(v, Jv) > 0$ for every non-zero tangent vector v . Every ω -tamed almost complex structure J defines a Riemannian metric

$$g_J(u, v) = \frac{1}{2}(\omega(u, Jv) + \omega(v, Ju)).$$

The Gromov Compactness Theorem used in our paper requires some restriction on (M, J, ω) called *the bounded geometry condition*. This means that g_J has uniformly bounded sectional curvatures and positive injectivity radius, uniformly separated from zero, and that J is uniformly continuous with respect to g_J . It is shown in [21] that the bounded geometry condition can be stated in the following form which imposes less regularity restrictions:

(M, J) admits a complete Riemannian metric g such that there exist positive constants r_0, C_1, C_2 with the following properties:

- (i) For every $p \in M$ the $\exp_p : B(0, r_0) \rightarrow B(p, r_0)$ is a diffeomorphism. Here B denote the corresponding balls.
- (ii) Every loop γ in M contained in the ball $B(p, r)$, $r \leq r_0$, bounds a disc in $B(p, r)$ of area less than $C_1 \text{length}(\gamma)^2$,
- (iii) On every ball $B(p, r_0)$ there exists a symplectic form ω_p such that $\|\omega_p\| \leq 1$ and $|X|^2 \leq C_2 \omega_p(X, JX)$ (taming property).

Properties (i) and (ii) always hold if M is closed and the complete metric g is of class C^2 on M . Property (iii) holds if J is additionally uniformly continuous with respect to g . In what follows we always assume that the symplectic tamed almost complex manifold (M, J, ω) satisfies the bounded geometry condition.

2.2. Levi form and plurisubharmonic functions

The proof of the statements of this Subsection can be found in [8]. Let r be a C^2 function on (M, J) . We denote by J^*dr the differential form acting on a vector field X by $J^*dr(X) := dr(JX)$. For example, if $J = J_{st}$ on \mathbb{R}^2 , then $J^*dr = r_y dx - r_x dy$. The value of the *Levi form* of r at a point $p \in M$ and at a vector $t \in T_p(M)$ is defined by

$$L_r^J(p; t) := -d(J^*dr)(X, JX)$$

where X is an arbitrary smooth vector field in a neighborhood of p satisfying $X(p) = t$. This definition is independent of the choice of vector fields. For instance, if $J = J_{st}$ in \mathbb{R}^2 , then $-d(J^*dr) = \Delta r dx \wedge dy$ (Δ denotes the Laplacian). In particular, $L_r^{J_{st}}(0, \frac{\partial}{\partial x}) = \Delta r(0)$.

The following properties of the Levi form are fundamental :

PROPOSITION 2.1. — *Let r be a real function of class C^2 in a neighborhood of a point $p \in M$.*

- (i) *If $F : (M, J) \rightarrow (M', J')$ is a (J, J') -holomorphic map, and φ is a real function of class C^2 in a neighborhood of $F(p)$, then for any $t \in T_p(M)$ we have $L_{\varphi \circ F}^J(p; t) = L_{\varphi}^{J'}(F(p), dF(p)(t))$.*
- (ii) *If $f : \mathbb{D} \rightarrow \mathbb{M}$ is a J -holomorphic disc satisfying $f(0) = p$, and $df(0)(e_1) = t$ (here e_1 denotes the vector $\frac{\partial}{\partial \text{Re } \zeta}$ in \mathbb{R}^2), then $L_r^J(p; t) = \Delta(r \circ f)(0)$.*

Property (i) expresses the invariance of the Levi form with respect to bi-holomorphic maps. Property (ii) is often useful to compute the Levi form if a vector t is given.

DEFINITION 2.2. — *A real C^2 function r on (M, J) is called plurisubharmonic if :*

$$L_r^J(p, t) \geq 0 \text{ for any } p \in M \text{ and } t \in T_p(M).$$

A C^2 function r is strictly plurisubharmonic on M if $L_r^J(p, t) > 0$ for any $p \in M$ and $t \in T_p(M) \setminus \{0\}$.

According to Proposition 2.1, r is plurisubharmonic if its composition with any J -holomorphic disc is subharmonic.

A useful observation due to E.Chirka [6] is that the Levi form of a function r at a point p in an almost complex manifold (M, J) coincides with the Levi form with respect to the standard structure J_{st} of \mathbb{R}^{2n} if *suitable* local coordinates near p are chosen. See the details in [8].

Let p be a boundary point of a domain Ω in an almost complex manifold (M, J) . Assume that $\partial\Omega$ is of class C^2 in a neighborhood U of p . Then $\Omega \cap U = \{q \in U : r(q) < 0\}$ where r is a real function of class C^2 on U , $dr(p) \neq 0$.

DEFINITION 2.3. — (i) Ω is called *Levi J -convex* at $p \in \partial\Omega$ if $L_r^J(p; t) \geq 0$ for any $t \in T_p(\partial\Omega) \cap J(T_p(\partial\Omega))$ and *strictly Levi J -convex* at p if $L_r^J(p; t) > 0$ for any non-zero $t \in T_p(\partial\Omega) \cap J(T_p(\partial\Omega))$. If Ω is a relatively compact domain with C^2 boundary in an almost complex manifold (M, J) , then Ω is called *Levi J -convex* if it is *Levi J -convex* at every boundary point. (ii) A real hypersurface $\Gamma = \{r = 0\}$ in an almost complex manifold (M, J) is called *Levi -flat* if $L_r^J(p, t) = 0$ for every $p \in \Gamma$ and every $t \in T_p(\Gamma) \cap J(T_p(\Gamma))$.

This definition does not depend on the choice of defining functions. We simply write a Levi convex domain dropping J in the notations when an almost complex structure is prescribed.

Example. — Consider in (\mathbb{C}^2, J_{st}) the smoothly bounded domain $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$, where $m \geq 1$ is an integer. This domain is strictly Levi convex for $m = 1$ and Levi convex for $m \geq 1$. Its boundary contains no non-constant J_{st} -holomorphic discs for all integer $m \geq 1$.

A strictly Levi convex domain admits near every boundary point a strictly plurisubharmonic defining function. One could expect that Levi convex domains have a similar property i.e. admit local defining plurisubharmonic functions. However, it is well-known that there are smoothly bounded domains in \mathbb{C}^n with Levi convex boundary (for the standard complex structure) which do not admit defining plurisubharmonic functions, see [7]. Fortunately, a weaker property always holds : a smoothly bounded Levi convex domain in \mathbb{C}^n admits a bounded exhaustion strictly plurisubharmonic function [7].

The following result obtained in [8] is an almost complex analog of the classical result [7]. It is a key for our approach.

THEOREM 2.4. — *Let (M, J) be an almost complex manifold and let $\Omega \subset M$ be a relatively compact Levi convex domain with C^3 boundary, such that*

there exists a C^2 strictly plurisubharmonic function ψ in a neighborhood U of $\partial\Omega$. Let r be any C^3 defining function for $\Omega \cap U$. Then there exist a neighborhood U' of $\partial\Omega$ and constants $A > 0$, $0 < \eta_0 < 1$, such that for any $0 < \eta \leq \eta_0$ the function $\rho = -(-re^{-A\psi})^\eta$ is strictly plurisubharmonic on $\Omega \cap U'$. If U is a neighborhood of $\overline{\Omega}$, then ρ is strictly plurisubharmonic on Ω . Furthermore for a fixed point $p \in \partial\Omega$ and any given $0 < \eta < 1$ there is a neighborhood U of p , a strictly plurisubharmonic function ψ in U and $A > 0$ such that ρ is strictly plurisubharmonic in $\Omega \cap U$.

As a direct consequence we obtain that no J -holomorphic disc can touch $\partial\Omega$ from inside.

DEFINITION 2.5. — Let E be a real submanifold in (M, J) . A Bishop disc for E is a J -holomorphic disc $f : \mathbb{D} \rightarrow \mathbb{M}$ continuous on $\overline{\mathbb{D}}$ and such that $f(\partial\mathbb{D}) \subset E$.

We will use the following properties of Bishop discs.

PROPOSITION 2.6. — (i) Let Ω be a smooth Levi convex domain. For any point $p \in \partial\Omega$ there exists a neighborhood U of p with the following property: if $f : \mathbb{D} \rightarrow U$ is a Bishop disc for $\partial\Omega$ then $f(\mathbb{D}) \subset \overline{\Omega} \cap U$.

(ii) Under the hypothesis of Theorem 2.4 for any point $p \in \partial\Omega$ there exists a neighborhood U of p with the following property: if $f : \mathbb{D} \rightarrow U$ is a Bishop disc for $\partial\Omega$ then either $f(\mathbb{D}) \subset \partial\Omega$ or the disc $f(\mathbb{D})$ is contained in Ω and is transverse to the holomorphic tangent space of $\partial\Omega$ at every boundary point.

Proof. — Part (i) is proved in [8]. Let us prove part (ii). Suppose that f is not contained in $\partial\Omega$. The statement is local so according to Theorem 2.4 η in the construction of an exhaustion plurisubharmonic function can be chosen arbitrarily close to 1. Fix $\eta = 3/4$. We can assume that in local coordinates $p = 0$, the condition (2.1) is satisfied and $f(1) = 0$. Applying the Hopf lemma to the subharmonic function $u(\zeta) = -(-re^{-A\psi} \circ f(\zeta))^{3/4}$ on the unit disc we obtain the estimate

$$|r \circ f(\zeta)| \geq C(1 - |\zeta|)^{4/3}.$$

We point out that u does not vanish identically since f is not contained in $\partial\Omega$ so the constant C is strictly positive. On the other hand the function r is smooth so near the point 1 we have $r \circ f(\zeta) = a(1 - |\zeta|) + O((1 - |\zeta|)^2)$. Hence $a \neq 0$. In particular $f(\partial\mathbb{D})$ is a real curve in $\partial\Omega$. Let v be a vector

tangent to $f(\partial\mathbb{D})$ at a point p . Since $f(\mathbb{D})$ is a J -curve then Jv is tangent to $f(\mathbb{D})$ at p and does not belong to the real tangent space of $\partial\Omega$ at p by the expression of $r \circ f$. Hence v is transverse to the holomorphic tangent space of $\partial\Omega$ at p . \square

According to Y.Eliashberg - W.Thurston [11] a tangent hyperplane field $\xi = \{\alpha = 0\}$, where α is a 1-form, on a $(2n + 1)$ -dimensional manifold Γ is called a *positive confoliation* if there exists an almost complex structure J on the bundle ξ such that

$$d\alpha(X, JX) \geq 0$$

for any vector $X \in \xi$. The 1-form α is defined up to the multiplication by a nonvanishing function. Thus, the confoliation condition for ξ is equivalent to the existence of a compatible Levi convex CR-structure (in general, non-integrable). In other words, if $\Gamma = \{r = 0\}$ is a smooth Levi convex hypersurface in an almost complex manifold (M, J) , then the distribution of its holomorphic tangent spaces $\xi = T\Gamma \cap J(T\Gamma)$ is a confoliation: we can set $\alpha = J^*dr$. In particular, if Γ is a strictly Levi convex hypersurface, then $\xi = \{J^*dr = 0\}$ is a contact structure. Recall that a tangent hyperplane field $\xi = \{\alpha = 0\}$ on Γ is called a contact structure if $\alpha \wedge (d\alpha)^n \neq 0$ on Γ . One of the main questions considered by Y.Eliashberg - W.Thurston concerns the possibility to deform a given confoliation to a contact structure or approximate a confoliation by contact structures. Combining the contact topology techniques with the geometric foliation theory they obtained several results of this type in the case where Γ is of real dimension 3. The next result of [8] holds in any dimension.

THEOREM 2.7. — *Let Ω be a relatively compact pseudoconvex domain with C^∞ boundary in an almost complex manifold (M, J) . Assume that there exists a smooth strictly plurisubharmonic function ψ in a neighborhood of $\partial\Omega$. Then the confoliation of holomorphic tangent spaces $T(\partial\Omega) \cap J(T(\partial\Omega))$ can be approximated in any C^k norm by contact structures.*

In a suitable neighborhood of a fixed point $p \in \partial\Omega$ there exists a smooth strictly plurisubharmonic function. So every confoliation can be approximated locally by contact structures.

3. Complex and totally real points of real surfaces

Let (M, J, ω) be a real four-dimensional symplectic manifold with an almost complex structure J tamed by the symplectic form ω . Let E be

a real smooth submanifold in M . A point $p \in E$ is called *totally real* if $T_p E \cap J(p)T_p E = \{0\}$. A submanifold E is called *totally real* if it is totally real at every point. Let S^2 be a smooth real surface diffeomorphic to the real two-sphere, embedded into M . A point $p \in S^2$ is called *complex* if the tangent space $T_p S^2$ is a $J(p)$ -invariant subspace of $T_p M$. We consider the generic case where S^2 admits a finite set Σ of complex points so that $S^2 \setminus \Sigma$ is a totally real submanifold of M .

Let $p \in S^2$ be a complex point. We may choose local coordinates centered at p so that $p = 0$ and the deformation tensor A_J satisfies Condition (2.1). The complex 2×2 -matrix function A can be written in the form $A_J = [A_J^1, A_J^2]$ where the columns of A_J , A_J^j , $j = 1, 2$, are smooth maps from a neighborhood of the origin in \mathbb{C}^2 to \mathbb{C}^2 . We need the following additional normalization of A_J .

LEMMA 3.1. — (*Adapted coordinates near a complex point*). *After a suitable local change of coordinates at p we have :*

- (i) *the deformation tensor A_J satisfies the normalization condition (2.1) and*

$$A_J^1(z) = L(z_2) + O(|z|^2) \tag{3.1}$$

where $L : \mathbb{C} \rightarrow \mathbb{C}^2$ is an \mathbb{R} -linear map,

- (ii) *the sphere S^2 is given near the origin by the equation*

$$z_2 = z_1 \bar{z}_1 + \gamma \operatorname{Re}(z_1^2) + o(|z_1|^2) \tag{3.2}$$

Such a system of local coordinates is called *adapted*. We use the notation

$$\rho(z) = z_2 - z_1 \bar{z}_1 - \gamma \operatorname{Re}(z_1^2) + o(|z_1|^2)$$

We have used above the standard notation $\phi(x) = o(\psi(x))$ for the functions ϕ and ψ defined in a neighborhood of the origin in \mathbb{R} and satisfying

$$\lim_{x \rightarrow 0} \frac{\phi(x)}{\psi(x)} = 0.$$

We also use the usual notation $\phi(x) = O(\psi(x))$ if there exists a constant $C > 0$ such that

$$|\phi(x)| \leq C|\psi(x)|$$

in a neighborhood of the origin.

Proof. — First, choose coordinates so that $\rho(z) = z_2 + O(z_1^2)$. Keeping this expression we may achieve Condition (3.1). Then consider a change of coordinates $(z_1, z_2) \mapsto (az_1, z_2 + Q(z))$, $a \neq 0$, with a holomorphic homogeneous second degree polynomial Q to obtain the expression (3.2). It follows from the transformation rule for the deformation tensor A_J that Condition (3.1) still holds in the new coordinates. \square

The number $\gamma \in [0, +\infty[$, $\gamma \neq 1$, is a local invariant of S^2 : it is independent of a choice of coordinates satisfying Conditions (2.1), (3.1), (3.2).

DEFINITION 3.2. — *A complex point is called elliptic if $0 \leq \gamma < 1$, parabolic if $\gamma = 1$ and hyperbolic if $\gamma > 1$.*

In the generic case (i.e. after an arbitrary small perturbation) S^2 contains no parabolic point. Thus, in what follows we consider the only case where the set Σ of complex points consists of elliptic and hyperbolic points. Denote by e the number of elliptic points and by h the number of hyperbolic points. Since the Euler characteristic of S^2 is equal to 2 we have, according to well-known results : $e = h + 2$. In particular if S^2 contains no hyperbolic point it contains precisely two elliptic points.

4. Generation of Bishop discs near an elliptic point

A local study of Bishop discs near an elliptic point was performed by several authors [4, 15, 17, 23, 25]. The exposition of this Section is based on the approach developed by A.Sukhov-A.Tumanov in [23] in any dimension. Here we adapt it to the two-dimensional case.

4.1. Bishop's equation

Denote by $\bar{\partial}_J$ the Gromov operator

$$\bar{\partial}_J f := df + J \circ df \circ J_{st}.$$

A smooth map f defined on \mathbb{D} is J -holomorphic if it satisfies the non-linear Cauchy-Riemann equation $\bar{\partial}_J f = 0$ on \mathbb{D} .

A smooth map f defined on \mathbb{D} and continuous on $\bar{\mathbb{D}}$ is a Bishop disc if and only if it satisfies the following non-linear boundary problem of the Riemann-Hilbert type for the quasi-linear operator $\bar{\partial}_J$:

$$(RH) : \begin{cases} \bar{\partial}_J f(\zeta) = 0, \zeta \in \mathbb{D}, \\ \rho(f)(\zeta) = 0, \zeta \in \partial\mathbb{D}. \end{cases}$$

First we derive Riemann-Hilbert type boundary problems describing pseudoholomorphic Bishop discs near generic elliptic points of a real submanifold in an almost complex manifold.

Since our considerations are local, we suppose that local coordinates are chosen as in Section 1, namely that Conditions (2.1), (3.1) and (3.2) are satisfied near an elliptic point in S^2 .

Denote by $P(z_1) = z_1\bar{z}_1 + \gamma Re(z_1^2)$ the quadratic part of the Taylor expansion of S^2 near the origin. Consider the non-isotropic dilations $\Lambda_\delta : z = (z_1, z_2) \mapsto z' = (\delta^{-1/2}z_1, \delta^{-1}z_2)$. In the new z -variables (we drop the primes) the image $S_\delta^2 := \Lambda_\delta(S^2)$ is defined by the equation $\rho_\delta(z) := \delta^{-1}\rho((\Lambda_\delta)^{-1}z) = 0$. The functions ρ_δ converge in any C^k norm on every compact subset of \mathbb{C}^2 to the function

$$\rho^0(Z) = z_2 - P(z_1)$$

as $\delta \rightarrow 0$. Hence the surfaces S_δ^2 converge for the local Hausdorff distance to the model quadric manifold $S_0^2 := \{\rho^0(Z) = 0\}$, \mathbb{C}^2 , as $\delta \rightarrow 0$.

Suppose additionally that S^2 is contained in a strictly J -Levi convex hypersurface Γ near the origin. Then the Taylor expansion of the defining function ϕ of Γ is

$$\phi(z) = \alpha Re z_2 + \beta Im z_2 + O(|z|^2).$$

Let $f(\zeta) = (z_1(\zeta), z_2(\zeta)) = (a\zeta, 0) + O(\zeta^2)$ be a J -holomorphic disc tangent to Γ at the origin. Since $z_2 = O(\zeta^2)$, the J -holomorphicity equations for f in the adapted coordinates imply that there exist $b_1, b_2 \in \mathbb{C}$ such that :

$$f(\zeta) = (a\zeta + b_1\zeta^2, b_2\zeta^2) + O(\zeta^3).$$

Hence the Levi form of Γ at the origin with respect to J coincides with the Levi form of Γ with respect to J_{st} . In particular :

$$\frac{\partial^2 \phi}{\partial z_1 \partial \bar{z}_1}(0) > 0.$$

The dilated hypersurface $\Gamma_\delta := \{\rho_\delta(z) = 0\}$ converges to a real quadric hypersurface Γ_0 which is J_{st} -strictly pseudoconvex near the origin.

Consider the pushed-forward structures $J_\delta := (\Lambda_\delta)_*(J) = d\Lambda_\delta \circ J \circ (d\Lambda_\delta)^{-1}$.

LEMMA 4.1 (*Structure deformation near a complex point*). —

- (i) For every positive integer k and every compact subset $K \subset \mathbb{C}^n$ we have $\|J_\delta - J_{st}\|_{C^k(K)} \rightarrow 0$ as $\delta \rightarrow 0$. Thus S^2 and J are small deformations of S_0^2 and J_{st} , respectively, near the origin.
- (ii) Suppose that S^2 is contained in a strictly J -Levi convex hypersurface Γ near the origin. Then Γ is a small deformation of a strictly J_{st} -pseudoconvex hypersurface Γ_0 containing S_0^2 .

Proof. — (i) Consider the Taylor expansion of $J(z)$ near the origin: $J(z) = J_{st} + L(z) + R(z)$ where $L(z)$ is the linear part of the expansion and $R(z) = O(|z|^2)$. Clearly, $\Lambda_\delta \circ R(\Lambda_\delta^{-1}(z)) \circ \Lambda_\delta^{-1}$ converges to 0 as $\delta \rightarrow 0$. Fix $j, k \in \{1, 2\}$ and denote by $L_{kj}^\delta(z)$ (respectively, by $L_{kj}(z)$) an entry of the real matrix $\Lambda_\delta \circ L(\Lambda_\delta^{-1}(z)) \circ \Lambda_\delta^{-1}$ (respectively, of $L(z)$). We have $J_{12}^\delta(z) = \delta^{1/2} L_{12}(\delta^{1/2} z_1, \delta z_2)$ and $J_{jj}^\delta(z) = L_{jj}(\delta^{1/2} z_1, \delta z_2)$, $j = 1, 2$. Hence, these terms tend to 0 as $\delta \rightarrow 0$. Furthermore $J_{21}^\delta(z) = \delta^{-1/2} L_{21}(\delta^{1/2} z_1, \delta z_2) \rightarrow L_{21}(z_1, 0)$ as $\delta \rightarrow 0$, uniformly on K . However, it follows from (3.1) that $L_{kj}(z_1, 0) = 0$ for $k, j = 1, 2$. This implies that L_{kj}^δ converges to 0 as $\delta \rightarrow 0$, for every $k, j = 1, 2$. Now Part (ii) follows from Part (i). \square

Let f be a J_δ -holomorphic disc in a neighborhood of the origin in \mathbb{C}^2 . The J_δ -holomorphicity condition for f has the form (2.2) with the deformation tensor A_{J_δ} associated to J_δ .

Considering the operator Ψ_{J_δ} defined by (2.3) we can replace the non-linear Riemann-Hilbert problem (RH) by the Bishop equation

$$\rho_\delta(\Psi_{J_\delta}^{-1}(h))(\zeta) = 0, \zeta \in \partial\mathbb{D} \tag{4.1}$$

for an unknown function h , holomorphic in \mathbb{D} and continuous on $\overline{\mathbb{D}}$.

If h is a solution of the boundary problem (4.1) then $f := \Psi_{J_\delta}^{-1}(h)$ is a Bishop disc with boundary attached to S_δ^2 . Since the manifold S_δ^2 is biholomorphic via non-isotropic dilations to the initial manifold S^2 , the solutions of Equation (4.1) allow to describe the Bishop discs attached to the sphere S^2 near an elliptic point.

4.2. Geometry of the Bishop discs near an elliptic point

Our goal is to prove the following statement :

THEOREM 4.2. —

- (i) Let p be an elliptic point in S^2 . Given a positive integer k and $0 < \alpha < 1$ there exists a family $(f^t)_t$ of J -holomorphic Bishop discs for S^2 , $C^{k,\alpha}$ -smoothly depending on one real parameter $t \in [0, t_0]$. These discs foliate a real hypersurface E such that (E, S^2) is a $C^{k,\alpha}$ smooth manifold with boundary, outside p .
- (ii) Suppose additionally that S^2 is contained in the boundary $\partial\Omega$ of a Levi J -convex domain. Then the generated family of discs is contained in $\overline{\Omega}$.

Part (ii) of Theorem 4.2 follows from part (i) of Proposition 2.6 so we prove the part (i).

We begin with the description of the Bishop discs attached to the model quadric manifold S_0^2 in \mathbb{C}^2 with the standard structure J_{st} . They are the solutions of the boundary problem (4.1) for $\delta = 0$.

For $r > 0$ consider the ellipse $D_r := \{\zeta \in \mathbb{C} : P(\zeta) < r\}$ and denote by $z_{1,r}$ the biholomorphism $z_{1,r} = z_1(r, \bullet) : \mathbb{D} \rightarrow D_r$ satisfying $z_{1,r}(0) = 0$, $(\partial z_{1,r} / \partial \zeta)(0) > 0$. Then $P \circ z_{1,r}|_{\partial\mathbb{D}} \equiv r$ and we set $z_2(r, \zeta) \equiv r$. The maps $\zeta \mapsto (z_1(r, \zeta), z_2(r, \zeta))$ provide a one parameter family of J_{st} -holomorphic Bishop discs. Their boundaries are disjoint and fill a pointed neighborhood of the origin in $S_0^2 \setminus \{0\}$. For $r = 0$ these discs degenerate to the constant map $\zeta \mapsto (0, 0)$. Their images form a foliation of a real hypersurface E such that (E, S_0^2) is a smooth manifold with boundary outside the origin.

We claim that in the general case of an almost complex structure J the Bishop discs have similar properties. Indeed, for a sufficiently small real positive number δ let $p^\delta := (0, \delta)$ be a point on the real “normal” to S^2 . The image of p^δ by the non-isotropic dilation Λ_δ coincides with $p^0 := (0, 1)$. There exists a unique Bishop disc $f^0 = (z_1^0, z_2^0)$ of the described above family centered at p^0 ; it corresponds to the parameter $r = 1$. The parametrizing map $F : (r, \zeta) \rightarrow (z_1(r, \zeta), z_2(r, \zeta))$ has maximal rank when the parameter r is in a neighborhood U of the point 1 and $\zeta \in \mathbb{D}$. Linearize the equation (4.1) for $\delta = 0$, at the disc f^0 . We claim that the linearized operator is surjective between the corresponding Banach spaces. Indeed, the linearized Bishop equation

$$Re(\partial_z \rho_0(f^0)\dot{z}) = \psi$$

has the form

$$\begin{cases} -Re(z_1^0 - \gamma z_1^0)(e^{i\theta})\dot{z}_1(e^{i\theta}) + (1/2)Re\dot{z}_2(e^{i\theta}) & = \psi_1(e^{i\theta}), \\ Re(1/2i)\dot{z}_2(e^{i\theta}) & = \psi_2(e^{i\theta}). \end{cases}$$

The second equation admits a one-parameter solution given by the Schwarz integral

$$(1/2i)\dot{z}_2(\zeta) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\tau + \zeta}{\tau - \zeta} \frac{\psi_2(\tau)}{\tau} d\tau + ic_1$$

with $c_1 \in \mathbb{R}$. Since every ellipse D_r is homotopic to the unit disc, the winding number of the function $\zeta \mapsto (z_1^0 - \gamma z_1^0)$ is equal to the winding number of $z_1^0(\zeta) \equiv \zeta$ i.e. is equal to -1 . Solving then the boundary problem for \dot{z}_1 we obtain a general solution depending on four real parameters. If ψ_j are of class $C^{k,\alpha}(\partial\mathbb{D})$ for some positive integer k and $0 < \alpha < 1$, then $\dot{z} \in C^{k,\alpha}(\overline{\mathbb{D}})$ by the classical regularity properties of the Cauchy type integral [24]. Applying the Implicit Function Theorem to the operator equation (4.1) we obtain for every sufficiently small positive real number δ a J_{st} -holomorphic solution $h(r, \bullet)$, smoothly depending on four real parameters. Three of these four parameters may be removed by fixing a parametrization of the discs. Thus, we obtain a one-parameter family of disjoint discs. Then $f(r) = \Psi_{J_\delta}^{-1}(h)$ is a family of J -holomorphic Bishop discs with boundaries attached to S_δ^2 . This family is a small deformation of the above Bishop discs attached to S_0^2 . Since δ is small enough, the parametrizing map $(r, \zeta) \mapsto f(r, \zeta)$ has maximal rank. So these discs swept a real smooth hypersurface Σ_δ with smooth boundary. In order to conclude the proof of theorem it remains to show that this hypersurface is foliated by the Bishop discs $f(r, \bullet)$. This is a consequence of the general uniqueness result which we establish in the next subsection. It will also have further applications in our approach.

4.3. Uniqueness of Bishop discs

Uniqueness results for Bishop discs were obtained by several authors in different forms (see for instance [2, 3, 25]). Here we follow the exposition of [25].

PROPOSITION 4.3. — *Let Ω be a Levi convex domain, relatively compact in (M, J) . Let S^2 be an embedded sphere in $\partial\Omega$ and p an elliptic point in S^2 . Suppose that $\partial\Omega$ contains no germ of a nonconstant J -holomorphic disc near p . Then there exists a neighborhood U of p in M with the following property : if $f : \mathbb{D} \rightarrow \Omega \cap U$ is a J -holomorphic disc with boundary attached to $S^2 \setminus \{p\}$, then the image of f coincides with one of the discs of the family constructed in Theorem 4.2.*

Proof. — We proceed in several steps.

Step 1. Transversality to the boundary. Since $\partial\Omega$ is Levi convex, it follows from the Hopf lemma that every J -holomorphic disc in Ω with boundary glued to $\partial\Omega$ intersects $\partial\Omega$ transversally at all points of the boundary.

Step 2. Stability of intersections. This is the content of the following

LEMMA 4.4. — *Let f and g be two distinct J -holomorphic Bishop discs glued to $S^2 \setminus \{p\}$ such that they intersect at a boundary point $q \in S^2$. If \tilde{f} is a Bishop disc close enough to f in the C^2 -norm, then $\tilde{f}(\overline{\mathbb{D}})$ has a non-empty intersection with $g(\overline{\mathbb{D}})$. The same holds if they intersect at an interior point where at least one of them is immersed.*

Proof. — If f and g intersect transversally at q then their boundaries at q are transverse and the assertion is obvious. Assume that they are tangent at q . We can suppose after a reparametrization of the maps f and g that they are defined on the half-disc $\mathbb{D}^+ = \{\zeta : \text{Im } \zeta > 0\}$ and are smooth up to the boundary. Then we can choose local coordinates near q such that $q = 0$, $J(0) = J_{st}$, $S^2 = \{z \in \mathbb{C}^2 : \text{Im } z = 0\}$ and $g(\zeta) = (\zeta, 0)$. Then f and \tilde{f} are the graphs over $g(\mathbb{D}^+) = \mathbb{D}^+ \times \{0\}$ of the functions h and \tilde{h} respectively : $f(\zeta) = (\zeta, h(\zeta))$ and $\tilde{f}(\zeta) = (\zeta, \tilde{h}(\zeta))$ for $\zeta \in \mathbb{D}^+$. The tangency condition implies

$$h(\zeta) = a\zeta^k + o(\zeta^k) \tag{4.2}$$

for some integer $k \geq 2$, with $a \in \mathbb{R}$. Since the boundary of f is glued to S^2 , we have $\text{Im } h|_{[-1,1]} \equiv \text{Im } \tilde{h}|_{[-1,1]} \equiv 0$. Then we may extend h and \tilde{h} continuously to \mathbb{D} setting $h(\zeta) = \bar{h}(\bar{\zeta})$ for $\zeta \in \mathbb{D} \setminus \mathbb{D}^+$. The extension of h still satisfies (4.2) and by the mapping degree theory \tilde{h} has exactly k zeros, counted with their multiplicities, near the origin. But since it is obtained by reflection over the real axis, it admits at least one zero in $\overline{\mathbb{D}^+}$. The proof for an interior point is similar with obvious simplifications. \square

Step 3. We prove Proposition 4.3. Consider the one parameter family $(f^t)_t$ of Bishop discs constructed in Theorem 4.2. For t sufficiently close to 0 the Bishop disc f^t does not intersect f . Let $\tau > 0$ be the infimum of $t \in [0, t_0]$ such that f intersects f^t . Then by continuity f intersects f^τ at an interior point or at a boundary point (notice that the discs f^t are embedded). If f and f^τ are distinct then Lemma 4.4 gives a contradiction to the minimality of τ . \square

5. Deformation of discs with totally real boundaries

The results of this Section can be deduced from general Theorems due to H.Hofer [15], H.Hofer-V.Lizan-J.C.Sikorav [16] and R.Ye [25]. Since our situation is rather special we give a direct approach.

Consider a J -holomorphic embedding $f^0 : \mathbb{D} \rightarrow (M, J)$ of class $C^{k,\alpha}(\overline{\mathbb{D}})$ for some $k \geq 1$ and $0 < \alpha < 1$. We suppose that $f^0(\partial\mathbb{D})$ is contained in

a totally real submanifold E of dimension 2 in M . Furthermore we assume that E is contained in a smooth real hypersurface Γ . We always assume that f^0 is transverse to Γ . The aim of this Section is to generate Bishop's discs attached to E by deforming f^0 . This will be a direct application of the Implicit Function Theorem. We proceed in several steps.

Step 1. Choice of coordinates. Let $h_\zeta : \mathbb{D} \rightarrow M$ be a family of J -holomorphic discs, smoothly depending on a parameter $\zeta \in \partial\mathbb{D}$, such that $h_\zeta(0) = f(\zeta)$ for every $\zeta \in \partial\mathbb{D}$ and h_ζ is tangent to Γ at the point $f(\zeta)$. Since f^0 is an embedding there exists a neighborhood U of $f^0(\overline{\mathbb{D}})$ and a coordinate diffeomorphism H from U to $H(U) \subset \mathbb{C}^2$ such that

$$(H \circ f^0)(\zeta) = (\zeta, 0), \zeta \in \mathbb{D}$$

and

$$J|_{H \circ f^0(\mathbb{D})} = J_{st}$$

i.e. the deformation tensor A_J of J satisfies

$$A_J(\zeta, 0) = \partial_{z_1} A_J(\zeta, 0) = 0 \tag{5.3}$$

(see [22]). Furthermore for every $\zeta \in \partial\mathbb{D}$ the map $\phi \circ h_\zeta : \tau \in \mathbb{D} \mapsto (H \circ h_\zeta)(\tau)$ has the form $(H \circ h_\zeta)(\tau) = (\zeta, \tau)$. We call these coordinates *normal coordinates* along the disc f^0 .

For simplicity of notations we still denote by f^0 the composition $H \circ f^0$ and by h_ζ the composition $H \circ h_\zeta$.

Step 2. Winding number. The manifold E , in a neighborhood of the circle $\partial\mathbb{D} \times \{0\}$, forms a bundle over this circle with as fibers smooth curves γ_ζ which are tangent to the the complex plane $\{\zeta\} \times \mathbb{C}$ at $(\zeta, 0)$, $\zeta = e^{i\theta} \in \partial\mathbb{D}$. Denote by $X_1(\zeta)$ the vector field $(i\zeta, 0)$ tangent to the circle $\partial\mathbb{D} \times \{0\}$. Let also $X^2(\zeta)$, $\zeta \in \partial\mathbb{D}$, be a vector field tangent to the fiber γ_ζ at the point $(\zeta, 0)$. We assume that E is orientable along $f^0(\partial\mathbb{D})$; this assumption will always hold in our applications. Then $X^2(\zeta) = (0, ie^{i\phi(\theta)})$ where ϕ is a smooth function on $[0, 2\pi]$ such that $\phi(0) = \phi(2\pi)$. Let $\mu := \mu(f^0)$ be the winding number of the function $\theta \mapsto e^{i\phi(\theta)}$, or equivalently the number of times which the tangent vector to γ_ζ turns around the origin in the plane $\{\zeta\} \times \mathbb{C}$ when ζ runs over $\partial\mathbb{D}$. Denote by L_ζ the real tangent space of E at $f(\zeta)$, generated by $X^1(\zeta)$ and $X^2(\zeta)$.

It is worth pointing out that μ is defined independently of a choice of normal coordinates. Obviously μ is invariant with respect to the homotopy of Bishop discs. Namely, if f^t is a family of Bishop discs for E continuously depending on a real parameter t , then normal coordinates corresponding to

f^t can be chosen smoothly depending in t . Therefore $\mu(f^t)$ continuously depends in t and so is a constant function in t .

Let us return to the family of Bishop discs constructed near an elliptic point.

LEMMA 5.1. — *The winding number $\mu(f)$ of a Bishop disc f near an elliptic point is equal to 0.*

Proof. — By the non-isotropic rescaling every Bishop disc is homotopic to the disc $(z_r^1(\zeta), r)$ glued to the model quadric S_0^2 . Its winding number is equal to 0 which implies the statement. \square

Hence, for all t we have $\mu(f^t) = 0$.

Step 3. Linearization. Now we are able to prove the main result of this section.

PROPOSITION 5.2. — *Suppose that the winding number of f^0 is equal to 0. Then there exists a one parameter family $(f^t)_t$ of Bishop discs for E , close to f^0 in the $C^{k,\alpha}(\mathbb{D})$ -norm, such that the boundaries of $(f^t)_t$ fill a neighborhood of $f^0(\partial\mathbb{D})$ in E .*

If $f : \zeta \mapsto (z_1(\zeta), z_2(\zeta))$ is a J -holomorphic disc close to f^0 , the Cauchy-Riemann equations satisfied by f have the form

$$\begin{cases} \partial_{\bar{\zeta}} z_1 - a_{11}(z) \partial_{\bar{\zeta}} \bar{z}_1 - a_{12}(z) \partial_{\bar{\zeta}} \bar{z}_2 = 0, & \zeta \in \mathbb{D} \\ \partial_{\bar{\zeta}} z_2 - a_{21}(z) \partial_{\bar{\zeta}} \bar{z}_1 - a_{22}(z) \partial_{\bar{\zeta}} \bar{z}_2 = 0, & \zeta \in \mathbb{D}. \end{cases}$$

Therefore in order to prove that the linearized Cauchy-Riemann operator is surjective we must solve the following system of linear PDEs :

$$\begin{cases} \partial_{\bar{\zeta}} \dot{z}_1 - b_{11}(\zeta) \dot{z}_2 - b_{12}(\zeta) \bar{\dot{z}}_2 = h_1, & \zeta \in \mathbb{D} \\ \partial_{\bar{\zeta}} \dot{z}_2 - b_{21}(\zeta) \dot{z}_2 - b_{22}(\zeta) \bar{\dot{z}}_2 = h_2, & \zeta \in \mathbb{D}. \end{cases}$$

Here $h = (h_1, h_2)$ is a given map of class $C^{k-1,\alpha}(\mathbb{D})$, the coefficients b_{ij} are of class $C^{k,\alpha}(\mathbb{D})$ and $\dot{z} : \mathbb{D} \rightarrow \mathbb{C}^2$ denotes an unknown map of class $C^{k,\alpha}(\mathbb{D})$. The above special form of the linearized equations along f^0 is a consequence of the normalization condition (5.3).

Moreover if E is defined by the equation $\rho := (\rho_1, \rho_2) = 0$, the linearization of the boundary condition $\rho \circ z|_{\partial\mathbb{D}} = 0$ has the form

$$\dot{z}(\zeta) \in L_\zeta, \quad \zeta \in \partial\mathbb{D}.$$

This is equivalent to the following conditions for $\theta \in [0, 2\pi]$:

$$\operatorname{Re}(e^{-i\theta} \dot{z}_1(e^{i\theta})) = 0, \quad \operatorname{Re}(e^{-i\phi(\theta)} \dot{z}_2(e^{i\theta})) = 0.$$

In particular the non-homogeneous boundary conditions are for $\theta \in [0, 2\pi]$:

$$\operatorname{Re}(e^{-i\theta} \dot{z}_1(e^{i\theta})) = g_1(e^{i\theta}), \quad (5.4)$$

$$\operatorname{Re}(e^{-i\phi(\theta)} \dot{z}_2(e^{i\phi(\theta)})) = g_2(e^{i\theta}). \quad (5.5)$$

The linearized boundary value problem splits into two parts. Consider first the problem for \dot{z}_2 . Since $\mu = 0$, it follows from classical results (see [24]) that the boundary value problem

$$\begin{cases} \partial_{\bar{z}} \dot{z}_2 - b_{21}(\zeta) \dot{z}_2 - b_{22}(\zeta) \overline{\dot{z}_2} &= h_2, \quad \zeta \in \mathbb{D} \\ \operatorname{Re}(e^{-i\phi(\theta)} \dot{z}_2(e^{i\theta})) &= g_2(e^{i\theta}) \end{cases}$$

is solvable in the class $C^{k,\alpha}(\overline{\mathbb{D}})$ for any $h_2 \in C^{k-1,\alpha}(\mathbb{D})$ and $g_2 \in C^{k,\alpha}(\partial\mathbb{D})$. Furthermore the corresponding homogeneous problem admits precisely one linearly independent solution. Considering then \dot{z}_2 as a known function, the boundary problem for \dot{z}_1 admits a general solution of class $C^{k,\alpha}$ depending on 3 real parameters. Thus the linearized non-homogeneous problem always admits a solution. By the Implicit Function Theorem there exists a family of discs, depending on 4 real parameters, glued to E in a neighborhood of f^0 . Three parameters correspond to a parametrization of \mathbb{D} and must be removed if we seek for discs with different images. In particular if the winding number μ is equal to zero we obtain a one-dimensional family of discs forming a one-dimensional submanifold \mathcal{M} in the Banach space $C^{k,\alpha}(\mathbb{D})$. Consider the evaluation map $ev : \mathcal{M} \rightarrow E$ defined by $ev(z) = z(1)$. We claim that it has maximal rank equal to 1 at $z = (\zeta, 0)$. Indeed, the tangent map \dot{ev} at the disc $f^0 = (\zeta, 0)$ is defined on $T_{f^0}(\mathcal{M})$ by $\dot{ev} : \dot{z} \mapsto \dot{z}(1)$. In order to give a suitable parametrization of the tangent space $T_{f^0}(\mathcal{M})$ consider the following auxiliary problem:

$$\begin{cases} \partial_{\bar{z}} w - b_{22}(\zeta) w - b_{22}(\zeta) \overline{w} = h, \quad \zeta \in \mathbb{D} \\ \operatorname{Re} w(e^{i\theta}) = 0 \end{cases}$$

where h is a given function of class $C^{k-1,\alpha}(\overline{\mathbb{D}})$ and w is an unknown function of class $C^{k,\alpha}(\overline{\mathbb{D}})$.

By the classical results of Bojarski [5] that boundary problem admits a unique solution satisfying the condition $Im w(1) = 0$. Then given $c \in \mathbb{R}$ set $h = b_{22}(\zeta)ic - b_{22}(\zeta)ic$. If w is a solution of the boundary problem with $Im w(0) = 0$, then $\dot{z}_2 = w + ic$ satisfies the linearized homogeneous problem and $\dot{z}_2(0) = ic$. This implies that the evaluation map has rank 1. Therefore the boundaries of the discs in \mathcal{M} fill a neighborhood of the circle $f^0(\partial\mathbb{D})$ on E .

6. Filling of a sphere with two elliptic points

This Section is devoted to the proof of Theorem 1.1. Since the proof relies on the Gromov Compactness Theorem we recall related notions following [20, 21]. Let S be a compact Riemann surface with (possibly empty) smooth boundary ∂S . We use the canonical identification of the complex plane \mathbb{C} with $\mathbf{CP} \setminus \{\infty\}$. Let (M, J, ω) be a symplectic manifold with a tamed almost complex structure. We assume that M has bounded geometry. Let E be a smooth compact totally real submanifold of maximal dimension in M .

Consider a sequence $f_n : S \rightarrow M$ of J -holomorphic maps.

Let $g : \mathbf{CP} \rightarrow M$ be a non-constant J -holomorphic map. We say that g occurs as a sphere bubble for the sequence (f_n) if there exists a sequence of holomorphic charts $\phi_n : R_n\mathbb{D} \rightarrow S$ with $R_n \rightarrow \infty$ converging uniformly on compacts subsets of \mathbb{C} to a constant map $\phi_\infty = p \in S$ and such that

$$f_n \circ \phi_n \rightarrow g$$

uniformly on compact subsets of \mathbb{C} .

Let $g : \mathbb{D} \rightarrow M$ be a J -holomorphic map, continuous on $\overline{\mathbb{D}}$, with $\partial\mathbb{D} \subset E$. We say that g occurs as a disc bubble for the sequence $(f_n)_n$ if there exists a sequence of holomorphic charts $\phi_n : \mathbb{D} \setminus (-1 + \delta_n\mathbb{D}) \rightarrow M$, smooth on $\overline{\mathbb{D}} \setminus (-1 + \delta_n\mathbb{D})$ with $\phi_n(\partial\mathbb{D} \setminus (-1 + \delta_n\mathbb{D})) \subset E$ and $\delta_n \rightarrow 0$, such that (ϕ_n) converge uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{-1\}$ to a constant map point $\phi_\infty = p \in S \cup \partial S$ and

$$f_n \circ \phi_n \rightarrow g$$

uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{-1\}$.

We have the following simple version of Gromov's Compactness Theorem:

PROPOSITION 6.1. — *Let $f_n : \mathbb{D} \rightarrow M$ be a sequence of J -holomorphic discs continuous on $\bar{\mathbb{D}}$ satisfying the boundary conditions $f_n(\partial\mathbb{D}) \subset E$, intersecting a fixed compact subset $K \subset M$ and such that*

$$\text{area}(f_n) := \int_{\mathbb{D}} f_n^* \omega \leq c$$

where $c > 0$ is a constant. Then there exists a finite set Σ in $\bar{\mathbb{D}}$, eventually empty, such that after extraction :

- (i) *The sequence $(f_n)_n$ converges uniformly on compact subsets of $\bar{\mathbb{D}} \setminus \Sigma$ to a J -holomorphic map $f_\infty : \mathbb{D} \rightarrow M$,*
- (ii) *A bubbling of some J -holomorphic sphere occurs at every point in $\Sigma \cap \mathbb{D}$,*
- (iii) *A bubbling of some J -holomorphic disc occurs at every point in $\Sigma \cap \partial\mathbb{D}$.*

Proof of Theorem 1.1. — We proceed in several steps.

Step 1. Characteristic foliation. Let p and q be the elliptic points in S^2 and let f^t, g^t be the families of Bishop discs near these points, given by Theorem 4.2. Denote by χ the bundle over S^2 formed by $T(\partial M) \cap JT(\partial M) \cap T(S^2)$. The integral curves of this bundle form a foliation (the characteristic foliation) of $S^2 \setminus \{p, q\}$ with two singular points p and q . Every Bishop disc is transverse to χ by part (ii) of Proposition 2.6. In particular, the characteristic foliation cannot have any closed trajectories and the boundary of a Bishop disc intersects every leaf of the characteristic foliation at a single point. Fix such an integral curve parametrized by a parameter $t \in [0, 1]$. We may assume that the family of Bishop discs (f^t) near p is parametrized by t close to 0.

Step 2. Area estimate. Let T be the supremum of the parameters t for which the family $(f^t)_t$ is defined. The areas of the discs f^t , $t \in [0, T]$, are uniformly bounded with respect to t . Indeed for a fixed t the boundary of f_t divides S^2 into two discs, $S_+^2(t)$ and $S_-^2(t)$. Since ω is closed, the Stokes Theorem implies

$$\int_{f_t(\mathbb{D})} \omega = \int_{S_+^2(t)} |\omega| \leq \int_{S^2} |\omega|.$$

Hence it follows from Proposition 6.1 that there is a sequence $t_k \rightarrow T$ as $k \rightarrow \infty$ and a finite set $\Sigma \subset \bar{\mathbb{D}}$ such that the sequence $(f_k) = (f^{t_k})_k$

converges uniformly to a J -holomorphic disc on every compact subset of $\mathbb{D} \setminus \Sigma$. Moreover if $\Sigma \neq \emptyset$ then every point in Σ corresponds to a disc bubble or to a sphere bubble. Since M does not contain any holomorphic sphere by assumption then $\Sigma \cap \mathbb{D} = \emptyset$.

Step 3. Non appearance of bubbles. We have the following

LEMMA 6.2. — *There is a sequence of Möbius transformations $\phi_k : \mathbb{D} \rightarrow \mathbb{D}$ such that (after extraction) the discs $f_k \circ \phi_k$ converge uniformly on \mathbb{D} to a J -holomorphic disc $f_\infty : \mathbb{D} \rightarrow M$.*

Proof. — Fix 3 distinct points ζ_j , $j = 1, 2, 3$ on the unit circle and 3 distinct leaves L_j of the characteristic foliation. Consider a sequence ϕ_k of Möbius transformations such that $f_k \circ \phi_k(\zeta_j) = L_j$ for $j = 1, 2, 3$ and $k = 1, 2, \dots$. We claim that there are no disc bubbles for this sequence. By contradiction, a boundary disc bubble g is attached to S^2 and everywhere transverse to χ by part (ii) of Proposition 2.6. Furthermore, by Gromov's Compactness Theorem (see [20, 25]) the boundary $g|_{\partial\mathbb{D}}$ of g represents the trivial homological class on $S^2 \setminus \{p, q\}$. Every $f_t(\partial\mathbb{D})$ is a closed curve transverse to χ and bounding a disc on S^2 . Hence the curve $g(\partial\mathbb{D})$ bounds on S^2 a totally real disc D (or a finite number of discs) and the line field χ does not vanish on its boundary. Then it vanishes at an interior point of D according to [9], P.123, Cor. 14.5.2 : a contradiction. \square

Step 4. Gluing Bishop's families. The sequence (f^{t_k}) converges to a Bishop disc f^T . We point out that by the adjunction inequality (see [18, 19]) the limit disc f^T is embedded. If $f^T(\partial\mathbb{D})$ does not intersect a sufficiently small neighborhood of q on S^2 (to be precised below), it is contained in the totally real part of S^2 . Moreover according to Lemma 5.1 and to the fact that the winding number is a homotopy invariant, the winding number of f^T is equal to 0. It follows from Proposition 5.2 that the disc f^T generates a one-parameter family of nearby Bishop discs which contradicts the maximality of T . Hence $f^T(\overline{\mathbb{D}})$ intersects a sufficiently small neighborhood of q , meaning that it intersects a boundary of some Bishop disc from the family (g^t) . By the uniqueness Proposition 4.3 the image of f^T coincides with the image of one of the discs of this family. We recall that the Bishop discs as maps are defined up to three real parameters. Hence after a suitable reparametrization the two families of Bishop's discs are glued smoothly into a global one parameter family of discs with distinct images. This proves Theorem 1.1. \square

7. Concluding remarks

In this section we address several remarks concerning the results discussed in the present work.

Remark 1. — For simplicity of notations we suppose everywhere that manifolds are C^∞ -smooth. However Theorem 1.1 remains true under weaker assumptions. Indeed, it suffices to suppose that J is of class C^2 and that the boundary of M is of class C^3 for Theorem 2.4 and other results, like the generation of discs near elliptic points or the transversality statements, to hold. The positivity of intersection in [18] requires only C^2 -smoothness of J .

Remark 2. — In Theorem 1.1 it suffices to suppose that ∂M contains no germs of non-constant J -holomorphic discs in a neighborhood of the sphere S^2 . The proof does not require any change. We point out that in the smooth category this assumption is weaker than finite type assumptions. However, they are essentially equivalent in the real analytic case. See a more detailed discussion in [1]. Using Theorem 2.4 it is easy to prove (see [8]) that if ∂M is Levi convex and if there exists a strictly pseudoconvex function in a neighborhood of ∂M , then M can be exhausted by a sequence of domains M_j with strictly Levi-convex boundaries. Under this global assumption the condition that ∂M contains no non-constant holomorphic discs can be dropped. Namely, we can consider a sequence S_j^2 of spheres in ∂M_j , their Levi-flat fillings, and pass to the limit. This gives a Levi-flat filling of S^2 . However, if ∂M contains non-constant holomorphic discs, the behaviour of boundaries of the limit discs can be complicated as show examples from [14].

Remark 3. — The assumption that M contains no J -holomorphic spheres can be weakened. A simple topological argument of [25] shows that if a spherical bubble arises, then it has a negative self-intersection index. Thus it suffices to require that M contains no sphere of this class. One can also use a more subtil description of cusp-curves arising as bubbles (see [25]) and require that M contains no such curves. This result is new for manifolds with Levi-convex boundaries. However, this condition is difficult to verify. Finally, our method of proof allows to obtain similarly to [25] a Levi-flat filling with singularities if M contains holomorphic spheres and ∂M is Levi-convex. Thus Theorems 4,5,6 and 7 from [25] obtained there for strictly Levi-convex manifolds remain true in the case where the boundary of M is Levi-convex. We point out that these results are new in the Levi-convex case. We do not state them here since they require a rather long description of generic assumption on an almost complex structure J (rational regularity in the terminology of [25]) irrelevant to our forthcoming study of spheres with hyperbolic points. We leave the details to an interested reader.

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