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Eigenvalue distribution for non-self-adjoint operators on compact manifolds with small multiplicative random perturbations

Johannes Sjöstrand[(1)]

Résumé. — Dans ce travail nous étendons un travail précédent sur l'asymptotique de Weyl de la distribution des valeurs propres d'opérateurs différentiels avec des perturbations multiplicatives aléatoires petites, en traitant le cas des opérateurs sur des variétés compactes.

Abstract. — In this work we extend a previous work about the Weyl asymptotics of the distribution of eigenvalues of non-self-adjoint differential operators with small multiplicative random perturbations, by treating the case of operators on compact manifolds.

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1. Introduction

This work is a direct continuation of [15], devoted to semi-classical pseudodifferential operators on $\mathbf{R}^n$ with small multiplicative random perturbations, which was partly based on the work by M. Hager and the author [8]. The main goal in the present work is to obtain the same results as in [15] but with $\mathbf{R}^n$ replaced by a compact smooth n-dimensional manifold X. This extension makes it possible to obtain almost sure Weyl asymptotics for the large eigenvalues of elliptic operators on compact manifolds, see [2]. Such results in the case of $X = S^1$ have previously been obtained by W. Bordeaux-Montrieux [1].

On X we consider an h-differential operator P which in local coordinates takes the form,

$$P = \sum_{|\alpha|\leqslant m} a_\alpha(x;h)(hD)^\alpha, \tag{1.1}$$

where we use standard multiindex notation and let $D = D_x = \frac{1}{i}\frac{\partial}{\partial x}$. (As in [15] we could also consider natural classes of pseudodifferential operators but, as there, the very reason for considering multiplicative random perturbations is that we wish to consider random differential operators.) We assume that the coefficients a_α are uniformly bounded in C^∞ for $h \in]0, h_0]$, $0 < h_0 \ll 1$. (We will also discuss the case when we only have some Sobolev space control of $a_0(x)$.) Assume

$$\begin{aligned} &a_\alpha(x;h) = a_\alpha^0(x) + \mathcal{O}(h) \text{ in } C^\infty, \\ &a_\alpha(x;h) = a_\alpha(x) \text{ is independent of } h \text{ for } |\alpha| = m. \end{aligned} \tag{1.2}$$

Notice that this assumption is invariant under changes of local coordinates.

Also assume that P is elliptic in the classical sense, uniformly with respect to h:

$$|p_m(x,\xi)| \geqslant \frac{1}{C}|\xi|^m, \tag{1.3}$$

for some positive constant C, where

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \tag{1.4}$$

is invariantly defined as a function on T^*X and $|\xi|$ is a norm on the fibers of T^*X, induced by some smooth Riemannian metric. It follows that $p_m(T^*X)$ is a closed cone in $\mathbf{C}$ and we assume that

$$p_m(T^*X) \neq \mathbf{C}. \tag{1.5}$$

If $z_0 \in \mathbf{C} \backslash p_m(T^*X)$, we see that $\lambda z_0 \notin \Sigma(p)$ if $\lambda \geqslant 1$ is sufficiently large and fixed, where $\Sigma(p) := p(T^*X)$ and p is the semiclassical principal symbol

$$p(x,\xi) = \sum_{|\alpha|\leqslant m} a_\alpha^0(x)\xi^\alpha. \tag{1.6}$$

Actually, (1.5) can be replaced by the weaker condition that $\Sigma(p) \neq \mathbf{C}$.

Standard elliptic theory and analytic Fredholm theory now show that if we consider P as an unbounded operator: $L^2(X) \to L^2(X)$ with domain $\mathcal{D}(P) = H^m(X)$ (the Sobolev space of order m), then P has purely discrete spectrum.

We will need the symmetry assumption

$$P^* = \Gamma P \Gamma, \tag{1.7}$$

where P^* denotes the formal complex adjoint of P in $L^2(X, dx)$, with dx denoting some fixed smooth postive density of integration and Γ is the anti-linear operator of complex conjugation; $\Gamma u = \overline{u}$. Notice that this assumption implies that

$$p(x,-\xi) = p(x,\xi), \tag{1.8}$$

and conversely, if p fulfills (1.8), then we get (1.7) if we replace P by $\frac{1}{2}(P + \Gamma P^* \Gamma)$, which has the same semi-classical principal symbol p. Also notice that (1.7) can be rephrased by saying that P should be symmetric with respect to the natural bilinear product on $L^2(X, dx)$.

Let $\Omega \Subset \mathbf{C}$ be open, simply connected, not entirely contained in $\Sigma(p)$. Let $V_z(t) := \mathrm{vol}\,(\{\rho \in \mathbf{R}^{2n};\ |p(\rho) - z|^2 \leqslant t\})$. For $\kappa \in]0,1]$, $z \in \Omega$, we consider the property that

$$V_z(t) = \mathcal{O}(t^\kappa),\ 0 \leqslant t \ll 1. \tag{1.9}$$

Since $r \mapsto p(x, r\xi)$ is a polynomial of degree m in r with non-vanishing leading coefficient, we see that (1.9) holds with $\kappa = 1/(2m)$.

The random potential will be of the form

$$q_\omega(x) = \sum_{0<\mu_k\leqslant L} \alpha_k(\omega)\epsilon_k(x),\ |\alpha|_{\mathbf{C}^D} \leqslant R, \tag{1.10}$$

where ϵ_k is the orthonormal basis of eigenfunctions of $h^2\widetilde{R}$, where $\widetilde{R}$ is an h-independent positive elliptic 2nd order operator on X with smooth coefficients. Moreover, $h^2\widetilde{R}\epsilon_k = \mu_k^2\epsilon_k$, $\mu_k > 0$. We choose $L = L(h)$, $R =$

$R(h)$ in the interval

$$h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}} \ll L \leqslant Ch^{-M}, \qquad M \geqslant \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}, \tag{1.11}$$
$$\frac{1}{C}h^{-(\frac{n}{2}+\epsilon)M+\kappa-\frac{3n}{2}} \leqslant R \leqslant Ch^{-\widetilde{M}}, \qquad \widetilde{M} \geqslant \frac{3n}{2}-\kappa+(\frac{n}{2}+\epsilon)M,$$

for some $\epsilon \in]0, s-\frac{n}{2}[$, $s > \frac{n}{2}$, so by Weyl's law for the large eigenvalues of elliptic self-adjoint operators, the dimension D in (1.10) is of the order of magnitude $(L/h)^n$. We introduce the small parameter $\delta = \tau_0 h^{N_1+n}$, $0 < \tau_0 \leqslant \sqrt{h}$, where

$$N_1 := \widetilde{M} + sM + \frac{n}{2}. \tag{1.12}$$

The randomly perturbed operator is

$$P_\delta = P + \delta h^{N_1} q_\omega =: P + \delta Q_\omega. \tag{1.13}$$

Here the exponent N_1 has been chosen so that we have uniformly for $h \ll 1$ and q_ω as above:

$$\|h^{N_1} q_\omega\|_{L^\infty} \leqslant \mathcal{O}(1) h^{-n/2} \|h^{N_1} q_\omega\|_{H_h^s} \leqslant \mathcal{O}(1),$$

where H_h^s is the semi-classical Sobolev space that we introduce in Section 2.

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$P(d\alpha) = C(h) e^{\Phi(\alpha;h)} L(d\alpha), \tag{1.14}$$

where for some $N_4 > 0$,

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}), \tag{1.15}$$

and $L(d\alpha)$ is the Lebesgue measure. ($C(h)$ is the normalizing constant, assuring that the probability of $B_{\mathbf{C}^D}(0,R)$ is equal to 1.)

We also need the parameter

$$\epsilon_0(h) = (h^\kappa + h^n \ln \frac{1}{h})(\ln \frac{1}{\tau_0} + (\ln \frac{1}{h})^2) \tag{1.16}$$

and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small. Recall that $\Omega \Subset \mathbf{C}$ is open, simply connected, not entirely contained in $\Sigma(p)$. The main result of this work is:

Theorem 1.1. — *Under the assumptions above, let $\Gamma \Subset \Omega$ have smooth boundary, let $\kappa \in]0,1]$ be the parameter in (1.10), (1.11), (1.16) and assume that (1.9) holds uniformly for z in a neighborhood of $\partial\Gamma$. Then there exists a constant $C > 0$ such that for $C^{-1} \geqslant r > 0$, $\widetilde{\epsilon} \geqslant C\epsilon_0(h)$ we have with probability*

$$\geqslant 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} \tag{1.17}$$

that:

$$|\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma))| \leqslant \tag{1.18}$$
$$\frac{C}{h^n}\left(\frac{\widetilde{\epsilon}}{r} + C(r + \ln(\frac{1}{r})\mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r))))\right).$$

Here $\#(\sigma(P_\delta) \cap \Gamma)$ denotes the number of eigenvalues of P_δ in Γ, counted with their algebraic multiplicity.

Actually, we shall prove the theorem for the slightly more general operators, obtained by replacing P by $P_0 = P + \delta_0(h^{\frac{n}{2}} q_1^0 + q_2^0)$, where $\|q_1^0\|_{H_h^s} \leqslant 1$, $\|q_2\|_{H^s} \leqslant 1$, $0 \leqslant \delta_0 \leqslant h$. Here, H^s is the standard Sobolev space and H_h^s is the same space with the natural semiclassical h-dependent norm. See Section 3.

As in [8] we also have a result valid simultaneously for a family $\mathcal{C}$ of domains $\Gamma \subset \Omega$ satisfying the assumptions of Theorem 1.1 uniformly in the natural sense: With a probability

$$\geqslant 1 - \frac{\mathcal{O}(1)\epsilon_0(h)}{r^2 h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}}, \tag{1.19}$$

the estimate (1.18) holds simultaneously for all $\Gamma \in \mathcal{C}$.

In the introduction of [15] there is a discussion about the choice of parameters and a corollary which carry over to the present situation without any changes

Remark 1.2. — When $\widetilde{R}$ has real coefficients, we may assume that the eigenfunctions ϵ_j are real. Then (cf Remark 8.3 in [15]) we may restrict α in (1.10) to be in $\mathbf{R}^D$ so that q_ω is real, still with $|\alpha| \leqslant R$, and change $C(h)$ in (1.14) so that P becomes a probability measure on $B_{\mathbf{R}^D}(0,R)$. Then Theorem 1.1 remains valid.

Remark 1.3. — The assumption (1.7) cannot be completely eliminated. Indeed, let $P = hD_x + g(x)$ on $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$ where g is smooth and complex valued. Then (cf Hager [6]) the spectrum of P is contained in the line $\Im z = \int_0^{2\pi} \Im g(x) dx/(2\pi)$. This line will vary only very little under small multiplicative perturbations of P so Theorem 1.1 cannot hold in this case.

The proof follows the general scheme of [15], we will recall the intermediate steps but give proofs only when there is a difference between the case of $\mathbf{R}^n$ and that of compact manifolds. Actually, there will also be some simplifications since we have no support condition on the random potential.

2. Semiclassical Sobolev spaces and multiplication

We let $H_h^s(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$, $s \in \mathbf{R}$, denote the semiclassical Sobolev space of order s equipped with the norm $\| \langle hD \rangle^s u \|$ where the norms are the ones in L^2, ℓ^2 or the corresponding operator norms if nothing else is indicated. Here $\langle hD \rangle = (1 + (hD)^2)^{1/2}$. Let $\widehat{u}(\xi) = \int e^{-ix\cdot\xi} u(x) dx$ denote the Fourier transform of the tempered distribution u on $\mathbf{R}^n$. In [15] we recalled the following result:

Proposition 2.1. — *Let $s > n/2$. Then there exists a constant $C = C(s)$ such that for all $u, v \in H_h^s(\mathbf{R}^n)$, we have $u \in L^\infty(\mathbf{R}^n)$, $uv \in H_h^s(\mathbf{R}^n)$ and*

$$\|u\|_{L^\infty} \leqslant C h^{-n/2} \|u\|_{H_h^s}, \tag{2.1}$$

$$\|uv\|_{H_h^s} \leqslant C h^{-n/2} \|u\|_{H_h^s} \|v\|_{H_h^s}. \tag{2.2}$$

We cover X by finitely many coordinate neighborhoods $X_1, ..., X_p$ and for each X_j, we let $x_1, ..., x_n$ denote the corresponding local coordinates on X_j. Let $0 \leqslant \chi_j \in C_0^\infty(X_j)$ have the property that $\sum_1^p \chi_j > 0$ on X. Define $H_h^s(X)$ to be the space of all $u \in \mathcal{D}'(X)$ such that

$$\|u\|_{H_h^s}^2 := \sum_1^p \|\chi_j \langle hD \rangle^s \chi_j u\|^2 < \infty. \tag{2.3}$$

It is standard to show that this definition does not depend on the choice of the coordinate neighborhoods or on χ_j. With different choices of these quantities we get norms in (2.3) which are uniformly equivalent when $h \to 0$. In fact, this follows from the h-pseudodifferential calculus on manifolds with symbols in the Hörmander space $S_{1,0}^m$, that we quickly reviewed in the appendix in [15]. An equivalent definition of $H_h^s(X)$ is the following: Let

$$h^2 \widetilde{R} = \sum (hD_{x_j})^* r_{j,k}(x) hD_{x_k} \tag{2.4}$$

be a non-negative elliptic operator with smooth coefficients on X, where the star indicates that we take the adjoint with respect to some fixed positive smooth density on X. Then $h^2\widetilde{R}$ is essentially self-adjoint with domain $H^2(X)$, so $(1+h^2\widetilde{R})^{s/2} : L^2 \to L^2$ is a closed densely defined operator for $s \in \mathbf{R}$, which is bounded precisely when $s \leqslant 0$. Standard methods allow to show that $(1+h^2\widetilde{R})^{s/2}$ is an h-pseudodifferential operator with symbol in $S^s_{1,0}$ and semiclassical principal symbol given by $(1+r(x,\xi))^{s/2}$, where $r(x,\xi) = \sum_{j,k} r_{j,k}(x)\xi_j\xi_k$ is the semiclassical principal symbol of $h^2\widetilde{R}$. See the appendix in [15]. The h-pseudodifferential calculus gives for every $s \in \mathbf{R}$:

Proposition 2.2. — *$H^s_h(X)$ is the space of all $u \in \mathcal{D}'(X)$ such that $(1+h^2\widetilde{R})^{s/2}u \in L^2$ and the norm $\|u\|_{H^s_h}$ is equivalent to $\|(1+h^2\widetilde{R})^{s/2}u\|$, uniformly when $h \to 0$.*

Remark 2.3. — From the first definition we see that Proposition 2.1 remains valid if we replace $\mathbf{R}^n$ by a compact n-dimensional manifold X.

Of course, $H^s_h(X)$ coincides with the standard Sobolev space $H^s(X)$ and the norms are equivalent for each fixed value of h, but not uniformly with respect to h. The following variant of Proposition 2.1 is useful when studying the high energy limit in [2].

Proposition 2.4. — *Let $s > n/2$. Then there exists a constant $C = C_s > 0$ such that*

$$\|uv\|_{H^s_h} \leqslant C\|u\|_{H^s}\|v\|_{H^s_h}, \ \forall u \in H^s(\mathbf{R}^n),\ v \in H^s_h(\mathbf{R}^n). \qquad (2.5)$$

The result remains valid if we replace $\mathbf{R}^n$ by X.

Proof. — The adaptation to the case of a compact manifold is immediate by working in local coordinates, so it is enough to prove (2.5) in the $\mathbf{R}^n$-case.

Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be equal to one in a neighborhood of 0. Write $u = u_1 + u_2$ with $u_1 = \chi(hD)u$, $u_2 = (1-\chi(hD))u$. Then, with hats indicating Fourier transforms, we have

$$\langle h\xi\rangle^s \widehat{u_1 v}(\xi) = \frac{1}{(2\pi)^n}\int \frac{\langle h\xi\rangle^s}{\langle h\eta\rangle^s}(\chi(h(\xi-\eta))\widehat{u}(\xi-\eta))\langle h\eta\rangle^s\widehat{v}(\eta)d\eta.$$

Here $\langle h\xi\rangle/\langle h\eta\rangle = \mathcal{O}(1)$ on the support of $(\xi,\eta)\mapsto \chi(h(\xi-\eta))$, so

$$\|u_1 v\|_{H^s_h} \leqslant \mathcal{O}(1)\|\widehat{u}\|_{L^1}\|v\|_{H^s_h} \leqslant \mathcal{O}(1)\|u\|_{H^s}\|v\|_{H^s_h},$$

where we also used that $s > n/2$ in the last estimate.

On the other hand, $\langle h\xi\rangle^s \leqslant Ch^s\langle\xi\rangle^s$ when $1-\chi(h\xi)\neq 0$, so $\|u_2\|_{H_h^s} \leqslant Ch^s\|u\|_{H^s}$. By Proposition 2.1, we get

$$\|u_2 v\|_{H_h^s} \leqslant Ch^{-\frac{n}{2}}\|u_2\|_{H_h^s}\|v\|_{H_h^s} \leqslant \widetilde{C}h^{s-\frac{n}{2}}\|u\|_{H^s}\|v\|_{H_h^s} \leqslant \widetilde{C}\|u\|_{H^s}\|v\|_{H_h^s},$$

when $h \leqslant 1$. $\square$

3. H^s-perturbations and eigenfunctions

This section gives a very straight forward adaptation of the corresponding section in [15]. Let $S^m(T^*X) = S^m_{1,0}(T^*X)$, $S^m(U\times\mathbf{R}^n) = S^m_{1,0}(U\times\mathbf{R}^n)$ denote the classical Hörmander symbol spaces, where $U \subset \mathbf{R}^n$ is open. See for instance [5] and further references given there. As in [7, 8], we can find $\widetilde{p} \in S^m(T^*X)$ which is equal to p outside any given fixed neighborhood of $p^{-1}(\overline{\Omega})$ such that $\widetilde{p} - z$ is non-vanishing, for any $z \in \overline{\Omega}$. Let $\widetilde{P} = P + \mathrm{Op}_h(\widetilde{p} - p)$, where $\mathrm{Op}_h(\widetilde{p} - p)$ denotes any reasonable quantization of $(\widetilde{p} - p)(x, h\xi)$. (See for instance the appendix in [15].) Then $\widetilde{P} - z : H_h^m(X) \to H_h^0(X)$ has a uniformly bounded inverse for $z \in \overline{\Omega}$ and $h > 0$ small enough. As in [8, 15], we see that the eigenvalues of P in Ω, counted with their algebraic multiplicity, coincide with the zeros of the function $z \mapsto \det((\widetilde{P} - z)^{-1}(P - z)) = \det(1 - (\widetilde{P} - z)^{-1}(\widetilde{P} - P))$.

Fix $s > n/2$ and consider the perturbed operator

$$P_\delta = P + \delta(h^{\frac{n}{2}}q_1 + q_2) = P + \delta(Q_1 + Q_2) = P + \delta Q, \tag{3.1}$$

where $q_j \in H^s(X)$,

$$\|q_1\|_{H_h^s} \leqslant 1,\ \|q_2\|_{H^s} \leqslant 1,\ 0 \leqslant \delta \ll 1. \tag{3.2}$$

According to Propositions 2.1, 2.4, $Q = \mathcal{O}(1) : H_h^s \to H_h^s$ and hence by duality and interpolation,

$$Q = \mathcal{O}(1) : H_h^\sigma \to H_h^\sigma,\ -s \leqslant \sigma \leqslant s. \tag{3.3}$$

As in [15], the spectrum of P_δ in Ω is discrete and coincides with the set of zeros of

$$\det((\widetilde{P}_\delta - z)^{-1}(P_\delta - z)) = \det(1 - (\widetilde{P}_\delta - z)^{-1}(\widetilde{P} - P)), \tag{3.4}$$

where $\widetilde{P}_\delta := P_\delta + \widetilde{P} - P$. Here $(\widetilde{P} - z)^{-1} = \mathcal{O}(1) : H_h^\sigma \to H_h^\sigma$ for σ in the same range and as in [15] we get the same conclusion for $(\widetilde{P}_\delta - z)^{-1}$.

Put

$$P_{\delta,z} := (\widetilde{P}_\delta - z)^{-1}(P_\delta - z) = 1 - (\widetilde{P}_\delta - z)^{-1}(\widetilde{P} - P) =: 1 - K_{\delta,z}, \quad (3.5)$$

$$S_{\delta,z} := P^*_{\delta,z} P_{\delta,z} = 1 - (K_{\delta,z} + K^*_{\delta,z} - K^*_{\delta,z} K_{\delta,z}) =: 1 - L_{\delta,z}. \quad (3.6)$$

As in [15] we get

$$K_{\delta,z}, L_{\delta,z} = \mathcal{O}(1) : H_h^{-s} \to H_h^s. \quad (3.7)$$

For $0 \leqslant \alpha \leqslant 1/2$, let $\pi_\alpha = 1_{[0,\alpha]}(S_{\delta,z})$. Then as in [15], we get

$$\pi_\alpha = \mathcal{O}(1) : H_h^{-s} \to H_h^s. \quad (3.8)$$

We also have the corresponding result for $P_\delta - z$. Let

$$S_\delta = (P_\delta - z)^*(P_\delta - z) \quad (3.9)$$

be defined as the Friedrichs extension from $C^\infty(X)$ with quadratic form domain $H_h^m(X)$. For $0 \leqslant \alpha \leqslant \mathcal{O}(1)$, we now put $\pi_\alpha = 1_{[0,\alpha](S_\delta)}$. Then as in [15], we see that this new spectral projection also fulfils (3.9), for $0 \leqslant \alpha \ll 1$.

4. Some functional and pseudodifferential calculus

In this section we derive some results analogous to those of Section 4 in [8]. There we worked on $\mathbf{R}^n$ and by a simple dilation and change of the semi-classical parameter from h to h/α we could reduce ourselves to a situation of more standard h/α-pseudodifferential calculus. On a manifold, this can probably be done also, but appeared to us as quite heavy, so here we take another route and develop directly a slightly exotic pseudodifferential calculus, then use it to study resolvents and functions of certain self-adjoint pseudodifferential operators.

Let P be of the form (1.1) and let p in (1.6) be the corresponding semi-classical principal symbol. Assume classical ellipticity as in (1.3) and let $z \in \mathbf{C}$ be fixed throughout this section. Let

$$S = (P - z)^*(P - z), \quad (4.1)$$

that we realize as a self-adjoint operator in the sense of Friedrichs extensions. Later on we will also consider a different choice of S, namely

$$S = P_z^* P_z, \text{ where } P_z = (\widetilde{P} - z)^{-1}(P - z) \quad (4.2)$$

and $\widetilde{P}$ is defined prior to (3.1). The main goal is to make a trace class study of $\chi(\frac{1}{\alpha}S)$ when $0 < h \leqslant \alpha \ll 1$, $\chi \in C_0^\infty(\mathbf{R})$. With the second choice of S, we shall also study $\ln\det(S+\alpha\chi(\frac{1}{\alpha}S))$, when $\chi\geqslant 0$, $\chi(0)>0$. The main step will be to get enough information about the resolvent $(w-\frac{1}{\alpha}S)^{-1}$ for $w=\mathcal{O}(1)$, $\Im w\neq 0$ and then apply the Cauchy-Riemann-Green-Stokes formula

$$\chi(\frac{1}{\alpha}S)=-\frac{1}{\pi}\int\frac{\partial\widetilde{\chi}(w)}{\partial\overline{w}}(w-\frac{1}{\alpha}S)^{-1}L(dw), \tag{4.3}$$

where $\widetilde{\chi}\in C_0^\infty(\mathbf{C})$ is an almost holomorphic extension of χ, so that

$$\frac{\partial\widetilde{\chi}}{\partial\overline{w}}=\mathcal{O}(|\Im w|^\infty). \tag{4.4}$$

Thanks to (4.4) we can work in symbol classes with some temperate but otherwise unspecified growth in $1/|\Im w|$.

Let

$$s=|p-z|^2 \tag{4.5}$$

be the semiclassical principal symbol of S in (4.1). A basic weight function in our calculus will be

$$\Lambda:=\left(\frac{\alpha+s}{1+s}\right)^{\frac{1}{2}}, \tag{4.6}$$

satisfying $\sqrt{\alpha}\leqslant\Lambda\leqslant 1$.

As a preparation and motivation for the calculus, we first consider symbol properties of $1+\frac{s}{\alpha}$ and its powers.

Proposition 4.1. — *For every choice of local coordinates x on X, let (x,ξ) denote the corresponding canonical coordinates on T^*X. Then for all $\ell\in\mathbf{R}$, $\widetilde{\alpha},\beta\in\mathbf{N}^n$, we have uniformly in ξ and locally uniformly in x:*

$$\partial_x^{\widetilde{\alpha}}\partial_\xi^\beta(1+\frac{s}{\alpha})^\ell=\mathcal{O}(1)(1+\frac{s}{\alpha})^\ell\Lambda^{-|\widetilde{\alpha}|-|\beta|}\langle\xi\rangle^{-|\beta|}. \tag{4.7}$$

Proof. — In the region $|\xi|\gg 1$ we see that $(1+\frac{s}{\alpha})^\ell$ is an elliptic element of the Hörmander symbol class

$$\alpha^{-\ell}S_{1,0}^{2\ell m}=:\alpha^{-\ell}S(\langle\xi\rangle^{2\ell m}),$$

and $\Lambda\asymp 1$ there, so (4.7) holds. In the region $|\xi|=\mathcal{O}(1)$, we start with the case $\ell=1$. Since $s\geqslant 0$, we have $\nabla s=\mathcal{O}(s^{\frac{1}{2}})$, so

$$|\nabla(1+\frac{s}{\alpha})|=\mathcal{O}(\frac{s^{\frac{1}{2}}}{\alpha})\leqslant\mathcal{O}(1)(1+\frac{s}{\alpha})(\alpha+s)^{-\frac{1}{2}}=\mathcal{O}(1)(1+\frac{s}{\alpha})\Lambda^{-1}.$$

For $k \geqslant 2$, we have

$$|\nabla^k(1+\frac{s}{\alpha})| = \mathcal{O}(\frac{1}{\alpha}) = \mathcal{O}(1)(1+\frac{s}{\alpha})\Lambda^{-2} \leqslant \mathcal{O}(1)(1+\frac{s}{\alpha})\Lambda^{-k},$$

and we get (4.7) when $\ell = 1$.

If $\ell \in \mathbf{R}$, then $\partial_x^{\widetilde{\alpha}}\partial_\xi^\beta(1+\frac{s}{\alpha})^\ell$ is a finite linear combination of terms

$$(1+\frac{s}{\alpha})^{\ell-k}(\partial_x^{\widetilde{\alpha}_1}\partial_\xi^{\beta_1}(1+\frac{s}{\alpha}))\cdots(\partial_x^{\widetilde{\alpha}_k}\partial_\xi^{\beta_k}(1+\frac{s}{\alpha})),$$

with $\widetilde{\alpha} = \widetilde{\alpha}_1 + ... + \widetilde{\alpha}_k$, $\beta = \beta_1 + ... + \beta_k$, and we get (4.7) in general. □

We next notice that when $w = \mathcal{O}(1)$,

$$\frac{|\Im w|}{C}(1+\frac{s}{\alpha}) \leqslant |w - \frac{s}{\alpha}| \leqslant C(1+\frac{s}{\alpha}). \tag{4.8}$$

In fact, the second inequality is obvious, and so is the first one, when $\frac{s}{\alpha} \gg 1$. When $\frac{s}{\alpha} \leqslant \mathcal{O}(1)$, it follows from the fact that

$$1+\frac{s}{\alpha} = \mathcal{O}(1), \quad |w-\frac{s}{\alpha}| \geqslant |\Im w|.$$

From (4.7), (4.8), we get

$$|\partial_x^{\widetilde{\alpha}}\partial_\xi^\beta(w-\frac{s}{\alpha})| \leqslant \mathcal{O}(1)(w-\frac{s}{\alpha})\Lambda^{-|\widetilde{\alpha}|-|\beta|}\langle\xi\rangle^{-|\beta|}|\Im w|^{-1}. \tag{4.9}$$

When passing to $(w - \frac{s}{\alpha})^\ell$ and applying the proof of Proposition 4.1, we loose more powers of $|\Im w|$ that can still be counted precisely, but we refrain from doing so and simply state the following result:

Proposition 4.2. — *For all $\ell \in \mathbf{R}$, $\widetilde{\alpha}, \beta \in \mathbf{N}^n$, there exists $J \in \mathbf{N}$, such that*

$$\partial_x^{\widetilde{\alpha}}\partial_\xi^\beta(w-\frac{s}{\alpha})^\ell = \mathcal{O}(1)(1+\frac{s}{\alpha})^\ell\Lambda^{-|\widetilde{\alpha}|-|\beta|}\langle\xi\rangle^{-|\beta|}|\Im w|^{-J}, \tag{4.10}$$

uniformly in ξ and locally uniformly in x.

We now define our new symbol spaces.

Definition 4.3. — *Let $\widetilde{m}(x,\xi)$ be a weight function of the form $\widetilde{m}(x,\xi) = \langle\xi\rangle^k\Lambda^\ell$. We say that the family $a = a_w \in C^\infty(T^*X)$, $w \in D(0,C)$, belongs to $S_\Lambda(\widetilde{m})$ if for all $\widetilde{\alpha}, \beta \in \mathbf{N}^n$ there exists $J \in \mathbf{N}$ such that*

$$\partial_x^{\widetilde{\alpha}}\partial_\xi^\beta a = \mathcal{O}(1)\widetilde{m}(x,\xi)\Lambda^{-|\widetilde{\alpha}|-|\beta|}\langle\xi\rangle^{-|\beta|}|\Im w|^{-J}. \tag{4.11}$$

Here, as in Proposition 4.2, it is understood that that the estimate is expressed in canonical coordinates and is locally uniform in x and uniform in ξ. Notice that the set of estimates (4.11) is invariant under changes of local coordinates in X.

Let $U \subset X$ be a coordinate neighborhood that we shall view as a subset of $\mathbf{R}^n$ in the natural way. Let $a \in S_\Lambda(T^*U, \widetilde{m})$ be a symbol as in Definition 4.3 so that (4.11) holds uniformly in ξ and locally uniformly in x. For fixed values of α, w the symbol a belongs to $S^k_{1,0}(T^*U)$, so the classical h-quantization

$$Au = \mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot\eta} a(x,\eta;h)u(y)dyd\eta \tag{4.12}$$

is a well-defined operator $C_0^\infty(U) \to C^\infty(U)$, $\mathcal{E}'(U) \to \mathcal{D}'(U)$. In order to develop our rudimentary calculus on X we first establish a pseudolocal property for the distribution kernel $K_A(x,y)$:

Proposition 4.4. — *For all $\widetilde{\alpha}, \beta \in \mathbf{N}^n$, $N \in \mathbf{N}$, there exists $J \in \mathbf{N}$ such that*

$$\partial_x^{\widetilde{\alpha}} \partial_y^\beta K_A(x,y) = \mathcal{O}(h^N |\Im w|^{-J}), \tag{4.13}$$

locally uniformly on $U \times U \setminus \mathrm{diag}(U \times U)$.

Proof. — If $\gamma \in \mathbf{N}^n$, then $(x-y)^\gamma K_A(x,y)$ is the distribution kernel of $\mathrm{Op}_h((-hD_\xi)^\gamma a)$ and $(-hD_\xi)^\gamma a \in S_\Lambda\left(\widetilde{m}\left(\frac{h}{\Lambda\langle\xi\rangle}\right)^{|\gamma|}\right)$ and we notice that $h/\Lambda \leqslant h/\alpha^{\frac{1}{2}} \leqslant h^{\frac{1}{2}}$. Thus for any $N \in \mathbf{N}$, we have

$$(x-y)^\gamma K_A(x,y) = \mathcal{O}(h^N|\Im w|^{-J}) \text{ if } |\gamma| \geqslant \gamma(N)$$

is large enough. From this we get (4.13) when $\widetilde{\alpha} = \beta = 0$. Now, $\partial_x^{\widetilde{\alpha}}\partial_y^\beta K_A$ can be viewed as the distribution kernel of a new pseudodifferential operator of the same kind, so we get (4.13) for all $\widetilde{\alpha}, \beta$. □

This means that if $\phi, \psi \in C_0^\infty(U)$ have disjoint supports, then for every $N \in \mathbf{N}$, there exists $J \in \mathbf{N}$ such that $\phi A \psi : H^{-N}(\mathbf{R}^n) \to H^N(\mathbf{R}^n)$ with norm $\mathcal{O}(h^N|\Im w|^{-J})$, and this leads to a simple way of introducing pseudodifferential operators on X: Let $U_1, ..., U_s$ be coordinate neighborhoods that cover X. Let $\chi_j \in C_0^\infty(U_j)$ form a partition of unity and let $\widetilde{\chi}_j \in C_0^\infty(U_j)$ satisfy $\chi_j \prec \widetilde{\chi}_j$ in the sense that $\widetilde{\chi}_j$ is equal to 1 near $\mathrm{supp}\,(\chi_j)$. Let $a = (a_1, ..., a_s)$, where $a_j \in S_\Lambda(\widetilde{m})$. Then we quantize a by the formula:

$$A = \sum_1^s \widetilde{\chi}_j \circ \mathrm{Op}_h(a_j) \circ \chi_j. \tag{4.14}$$

This is not an invariant quantization procedure but it will suffice for our purposes.

We next study the composition to the left with non-exotic pseudodifferential operators. Let U be a coordinate neighborhood, viewed as an open set in $\mathbf{R}^n$, and take $A = \mathrm{Op}_h(a)$, $a \in S_{1,0}(m_1)$, $m_1 = \langle \xi \rangle^r$, $B = \mathrm{Op}_h(b)$, $b \in S_\Lambda(m_2)$ with $m_2 = \langle \xi \rangle^k \Lambda^\ell$ as in Definition 4.3. We will assume that $\mathrm{supp}\,(b) \subset K \times \mathbf{R}^n$, where $K \subset U$ is compact. We are interested in $C = A \circ B$.

The symbol c of this composition is given by

$$\begin{aligned} c(x,\xi;h) &= e^{-\frac{i}{h}x\cdot\xi} A(b(\cdot,\xi)e^{\frac{i}{h}(\cdot)\cdot\xi})(x) \\ &= \frac{1}{(2\pi h)^n} \iint a(x,\eta) b(y,\xi) e^{\frac{i}{h}(x-y)\cdot(\eta-\xi)} dy d\eta \end{aligned} \tag{4.15}$$

In the region $|\eta - \xi| \geqslant \frac{1}{C}\langle \xi \rangle$ we can make repeated integrations by parts in the y-variables and see that the contribution from this region is a symbol $d(x,\xi;h)$ satisfying

$$\begin{aligned} &\forall N \in \mathbf{N}, \widetilde{\alpha}, \beta \in \mathbf{N}^n, \exists J \in \mathbf{N}, \forall K \Subset U, \exists C > 0; \\ &|\partial_x^{\widetilde{\alpha}} \partial_\xi^\beta d(x,\xi;h)| \leqslant C \frac{h^N \langle \xi \rangle^{-N}}{|\Im w|^J},\ (x,\xi) \in K \times \mathbf{R}^n. \end{aligned} \tag{4.16}$$

Up to such a term d, we may assume that with $\chi \in C_0^\infty(B(0,\frac{1}{2}))$ equal to 1 near 0,

$$\begin{aligned} c(x,\xi;h) &\equiv \frac{1}{(2\pi h)^n} \iint a(x,\eta) b(y,\xi) \chi(\frac{\eta-\xi}{\langle \xi \rangle}) e^{\frac{i}{h}(x-y)\cdot(\eta-\xi)} dy d\eta \\ &= \left(\frac{\langle \xi \rangle}{2\pi h}\right)^n \iint a(x, \langle \xi \rangle(\eta + \frac{\xi}{\langle \xi \rangle})) b(x+y,\xi) \chi(\eta) e^{\frac{-i\langle \xi \rangle}{h} y\cdot\eta} dy d\eta. \end{aligned} \tag{4.17}$$

In order to expand this, we review a slight variant of the method of stationary phase with explicit remainder: For $u \in \mathcal{S}(\mathbf{R}^{2n})$ and $s > 0$, put

$$J(s,u) = \frac{1}{(2\pi s)^n} \iint u(y,\eta) e^{-\frac{i}{s} y\cdot\eta} dy d\eta,$$

which by stationary phase has an asymptotic expansion in the limit $s \to 0$ and in particular

$$J(s,u) \to u(0,0),\ s \to 0.$$

On the other hand, $\partial_s J(s,u) = J(s, \partial_\eta \cdot D_y u)$, so J extends to a smooth function of $s \in [0, \infty[$, and has the Taylor expansion

$$\begin{aligned} J(s,u) &= \sum_0^{N-1} \frac{s^k}{k!} \partial_s^k J(0,u) + s^N \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} (\partial_s^N J)(ts, u) ds \\ &= \sum_0^{N-1} \frac{s^k}{k!} (\partial_\eta \cdot D_y)^k u(0,0) + s^N \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} J(ts, (\partial_\eta \cdot D_y)^N u) ds. \end{aligned}$$

Applying this to the last integral in (4.17), we get

$$c(x,\xi;h) = \sum_{|\beta|<N} \frac{h^{|\beta|}}{\beta!} \partial_\xi^\beta a D_x^\beta b + R_N, \tag{4.18}$$

where,

$$\begin{aligned} R_N &= \left(\frac{h}{\langle\xi\rangle}\right)^N \frac{1}{(N-1)!} \times \qquad (4.19) \\ & \int_0^1 (1-t)^N J\left(t\frac{h}{\langle\xi\rangle}, (\partial_\eta \cdot D_y)^N (a(x, \langle\xi\rangle(\eta + \frac{\xi}{\langle\xi\rangle}))b(x+y,\xi)\chi(\eta))\right) dt. \end{aligned}$$

Noting that

$$|J(s,u)| \leqslant C \sum_{|\widetilde{\alpha}|+|\beta| \leqslant 2n+1} \| \partial_y^{\widetilde{\alpha}} \partial_\eta^\beta u \|_{L^1},$$

we see that there exist exponents N_2, N_3 independent of N, such that

$$|R_N| \leqslant C \left(\frac{h}{\langle\xi\rangle}\right)^N m_1(\xi) \langle\xi\rangle^{N_2} \alpha^{N_3 - \frac{N}{2}} |\Im w|^{-J(N)}.$$

Similar estimates hold for the derivatives and we conclude:

Proposition 4.5. — *Let $A = \mathrm{Op}_h(a)$, $a \in S_{1,0}(m_1)$, $B = \mathrm{Op}_h(b)$, $b \in S_\Lambda(m_2)$ and assume that b has uniformly compact support in x. Then $A \circ B = \mathrm{Op}_h(c)$, where c belongs to $S_\Lambda(m_1 m_2)$ and has the asymptotic expansion*

$$c \sim \sum \frac{h^{|\beta|}}{\beta!} \partial_\xi^\beta a(x,\xi) D_x^\beta b(x,\xi),$$

in the sense that for every $N \in \mathbf{N}$,

$$c = \sum_{|\beta|<N} \frac{h^{|\beta|}}{\beta!} \partial_\xi^\beta a(x,\xi) D_x^\beta b(x,\xi) + r_N(x,\xi;h),$$

where $r_N \in S_\Lambda(\frac{m_1 m_2}{(\Lambda\langle\xi\rangle)^N} h^N)$.

We next make a parametrix construction for $w - \frac{1}{\alpha}S$, still with S as in (4.1), and most of the work will take place in a coordinate neighborhood U, viewed as an open set in $\mathbf{R}^n$. The symbol of $w - \frac{1}{\alpha}S$ is of the form

$$F = F_0 + F_{-1}, \quad F_0 = w - \frac{1}{\alpha}s,\ F_{-1} = \frac{h}{\alpha}s_{-1} \in S(\frac{h}{\alpha}\langle\xi\rangle^{2m-1}). \tag{4.20}$$

Put

$$E_0 = \frac{1}{w - \frac{1}{\alpha}s} \in S_\Lambda(\frac{\alpha}{\Lambda^2\langle\xi\rangle^{2m}}). \tag{4.21}$$

With Proposition 4.5 in mind, we first consider the formal composition

$$\begin{aligned} F\#E_0 \quad &\sim \quad \sum \frac{h^{|\beta|}}{\beta!}(\partial_\xi^\beta F)(D_x^\beta E_0) \\ &\sim \quad 1 + \sum_{|\beta|\geqslant 1} \frac{h^{|\beta|}}{\beta!}(\partial_\xi^\beta F_0)(D_x^\beta E_0) + F_{-1}\#E_0. \end{aligned} \tag{4.22}$$

Here,

$$F_{-1}\#E_0 \in S_\Lambda(\frac{h}{\alpha}\langle\xi\rangle^{2m-1}\frac{\alpha}{\Lambda^2\langle\xi\rangle^{2m}}) = S_\Lambda(\frac{h}{\Lambda^2\langle\xi\rangle}).$$

Since F_0 also belongs to $S_\Lambda(\frac{1}{\alpha}\Lambda^2\langle\xi\rangle^{2m})$, we see that for $|\beta| \geqslant 1$,

$$h^{|\beta|}(\partial_\xi^\beta F_0)(D_x^\beta E_0) \in S_\Lambda(\frac{h^{|\beta|}}{\Lambda^{2|\beta|}\langle\xi\rangle^{|\beta|}}) \subset S_\Lambda(\frac{h}{\Lambda^2\langle\xi\rangle}),$$

and this can be improved for $|\beta| \geqslant 2$, using that $F \in S_{1,0}(\frac{1}{\alpha}\langle\xi\rangle^{2m})$. Hence,

$$F\#E_0 = 1 + r_1,\ r_1 \in S_\Lambda(\frac{h}{\Lambda^2\langle\xi\rangle}).$$

Now put $E_1 = E_0 - r_1/(w - s/\alpha)$. Then by the same estimates with an extra power of $h\Lambda^{-2}\langle\xi\rangle^{-1}$, we get

$$F\#E_1 = 1 + r_2,\ r_2 \in S_\Lambda((\frac{h}{\Lambda^2\langle\xi\rangle})^2),$$

and iterating the procedure we get

$$E_N \equiv \frac{1}{w - \frac{s}{\alpha}} \bmod S_\Lambda(\frac{\alpha}{\Lambda^2\langle\xi\rangle^{2m}}\frac{h}{\Lambda^2\langle\xi\rangle}), \tag{4.23}$$

such that

$$F\#E_N = 1 + r_N,\ r_N \in S_\Lambda((\frac{h}{\Lambda^2\langle\xi\rangle})^{N+1}). \tag{4.24}$$

Actually, in this construction we can work with finite sums instead of asymptotic ones and then

$$E_N \text{ is a holomorphic function of } w, \text{ for } |\xi| \geqslant C, \tag{4.25}$$

where C is independent of N.

Now we return to the manifold situation and denote by $E_N^{(j)}$, $r_N^{(j)}$ the corresponding symbols on T^*U_j, constructed above. Denote the operators by the same symbols, and put on the operator level:

$$E_N = \sum_{j=1}^{s} \widetilde{\chi}_j E_N^{(j)} \chi_j, \tag{4.26}$$

with χ_j, $\widetilde{\chi_j}$ as in (4.14). Then

$$\begin{aligned} (w - \frac{1}{\alpha} S) E_{N-1} &= 1 - \sum_{j=1}^{s} \frac{1}{\alpha} [S, \widetilde{\chi}_j] E_{N-1}^{(j)} \chi_j + \sum_{j=1}^{s} \widetilde{\chi}_j r_N^{(j)} \chi_j \quad (4.27) \\ &=: 1 + R_N^{(1)} + R_N^{(2)} \\ &=: 1 + R_N. \end{aligned}$$

Proposition 4.4 implies that for every $\widetilde{N}$, there exists an $\widetilde{J}$ such that the trace class norm of $R_N^{(1)}$ satisfies

$$\| R_N^{(1)} \|_{\mathrm{tr}} \leqslant \mathcal{O}(h^{\widetilde{N}} |\Im w|^{-\widetilde{J}}). \tag{4.28}$$

As for the trace class norm of $R_N^{(2)}$, we review some easy facts about such norms for pseudodifferential operators:

If $A = a(x, D)$ is a pseudodifferential operator on $\mathbf{R}^n$, either in the Weyl or in the classical quantization, then A is of trace class and we have

$$\|A\|_{\mathrm{tr}} \leqslant C \iint \sum_{|\beta| \leqslant 2n+1} |\partial_{x,\xi}^{\beta} a| dx d\xi,$$

provided that the integral is finite. In that case we also know that

$$\mathrm{tr}\,(A) = \frac{1}{(2\pi)^n} \iint a(x, \xi) dx d\xi.$$

See Robert [12], and also [3] for a sharper statement. If instead we consider an h-pseudodifferential operator $A = a(x, hD)$, then it is unitarily equivalent to $\widetilde{A} = a(h^{\frac{1}{2}} x, h^{\frac{1}{2}} D_x)$, so

$$\|A\|_{\mathrm{tr}} \leqslant \frac{C}{h^n} \iint \sum_{|\beta| \leqslant 2n+1} |(h^{\frac{1}{2}} \partial_{x,\xi})^{\beta} a| dx d\xi,$$

where the factor h^{-n} is the Jacobian, when passing from $h^{1/2}x, h^{1/2}\xi$ to x, ξ.

Now, let $a \in S_\Lambda(m)$ be a symbol on T^*U with uniformly compact support in x. Then for $|\beta| \leqslant 2n+1$, we have

$$h^{\frac{|\beta|}{2}}\partial_{x,\xi}^{\beta} a = \mathcal{O}(1) m \left(\frac{h}{\alpha}\right)^{\frac{|\beta|}{2}} |\Im w|^{-J(\beta)}.$$

Thus there exists $J \geqslant 0$ such that $a(x, hD_x)$ is of trace class and

$$\|a(x,hD)\|_{\mathrm{tr}} \leqslant Ch^{-n} \iint_{U\times\mathbf{R}^n} m(x,\xi) dx d\xi \, |\Im w|^{-J}, \tag{4.29}$$

provided that the integral converges.

From (4.27), (4.24), we now get

$$\|R_N^{(2)}\|_{\mathrm{tr}} \leqslant Ch^{-n}|\Im w|^{-J(N)} \iint \left(\frac{h}{\Lambda^2\langle\xi\rangle}\right)^N dx d\xi,$$

and (4.28) then shows that we have the same estimate for R_N:

$$\|R_N\|_{\mathrm{tr}} \leqslant Ch^{-n}|\Im w|^{-J(N)} \iint \left(\frac{h}{\Lambda^2\langle\xi\rangle}\right)^N dx d\xi. \tag{4.30}$$

The contribution to this expression from the region where $\Lambda \geqslant 1/C$ is $\mathcal{O}(h^{N-n})|\Im w|^{-J(N)}$.

The volume growth assumption (1.9), that we now assume for our fixed z, says that

$$V(t) := \mathrm{vol}\left(\{\rho \in T^*X;\, s \leqslant t\}\right) = \mathcal{O}(t^\kappa),\ 0 \leqslant t \ll 1, \tag{4.31}$$

for $0 < \kappa \leqslant 1$. The contribution to the integral in (4.30) from the region $0 \leqslant s \leqslant t_0$, $0 < t_0 \ll 1$, is equal to some negative power of $|\Im w|$ times

$$\begin{aligned}
&\mathcal{O}(1)\int_0^{t_0}\left(\frac{h}{\alpha+t}\right)^N dV(t)\\
=\ &\mathcal{O}(1)\left[\left(\frac{h}{\alpha+t}\right)^N V(t)\right]_{t=0}^{t_0} + \mathcal{O}(1)\int_0^{t_0}\frac{h^N}{(\alpha+t)^{N+1}}V(t)dt\\
=\ &\mathcal{O}(1)h^N + \mathcal{O}(1)h^N\int_0^{t_0}\frac{t^\kappa}{(\alpha+t)^{N+1}}dt.
\end{aligned}$$

The last integral is equal to

$$\int_0^{t_0/\alpha} \frac{(\alpha s)^\kappa}{\alpha^{N+1}(1+s)^{N+1}}\alpha ds \leqslant \alpha^{\kappa-N}\int_0^\infty \frac{s^\kappa}{(1+s)^{N+1}}ds.$$

Thus,

$$\|R_N\|_{\mathrm{tr}} \leqslant \mathcal{O}(1)h^{-n}\alpha^\kappa\left(\frac{h}{\alpha}\right)^N |\Im w|^{-J(N)}. \tag{4.32}$$

From (4.27), we get

$$(w-\frac{1}{\alpha}S)^{-1} = E_{N-1} - (w-\frac{1}{\alpha}S)^{-1}R_N.$$

Write

$$E_{N-1} = \frac{1}{w-\frac{s}{\alpha}} + F_{N-1}, \quad F_{N-1}\in S_\Lambda(\frac{\alpha h}{\Lambda^4\langle\xi\rangle^{2m+1}}).$$

More precisely we do this for each $E_{N-1}^{(j)}$ in (4.26). Then quantize and plug this into (4.3):

$$\begin{aligned}\chi(\frac{1}{\alpha}S) \;=\; &-\frac{1}{\pi}\int\frac{\partial\widetilde{\chi}}{\partial\overline{w}}\mathrm{Op}_h(\frac{1}{w-\frac{s}{\alpha}})L(dw) - \frac{1}{\pi}\int\frac{\partial\widetilde{\chi}}{\partial\overline{w}}F_{N-1}L(dw)\\ &-\frac{1}{\pi}\int\frac{\partial\widetilde{\chi}}{\partial\overline{w}}(w-\frac{1}{\alpha}S)^{-1}R_NL(dw) =: \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned} \tag{4.33}$$

Here by definition,

$$\mathrm{Op}_h\left(\frac{1}{w-\frac{s}{\alpha}}\right) = \sum_{j=1}^{s}\widetilde{\chi}_j\mathrm{Op}_h\left(\frac{1}{w-\frac{s}{\alpha}}\right)\chi_j$$

with the coordinate dependent quantization appearing to the right.

$$\mathrm{tr}\,(-\frac{1}{\pi}\int\frac{\partial\widetilde{\chi}}{\partial\overline{w}}(w)\widetilde{\chi}_j\mathrm{Op}_h\left(\frac{1}{w-\frac{s}{\alpha}}\right)\chi_jL(dw))$$

is equal to

$$\begin{aligned}&\frac{1}{(2\pi h)^n}\iint -\frac{1}{\pi}\int\frac{\partial\widetilde{\chi}}{\partial\overline{w}}(w)\frac{1}{w-\frac{s}{\alpha}}L(dw)\chi_j(x)dxd\xi\\ &\qquad = \frac{1}{(2\pi h)^n}\iint\chi(\frac{s(x,\xi)}{\alpha})\chi_j(x)dxd\xi,\end{aligned}$$

so

$$\mathrm{tr}\,(\mathrm{I}) = \frac{1}{(2\pi h)^n}\iint\chi(\frac{s(x,\xi)}{\alpha})dxd\xi. \tag{4.34}$$

As at the last estimate in the proof of Proposition 4.4 in [8] we see that this quantity is $\mathcal{O}(\alpha^\kappa h^{-n})$ and more generally,

$$\|\mathrm{I}\|_{\mathrm{tr}} = \mathcal{O}(\alpha^\kappa h^{-n}).$$

For II, we get, using the fact that the symbol is holomorphic in w for large ξ,

$$\begin{aligned}
\|\mathrm{II}\|_{\mathrm{tr}} &= \mathcal{O}(h^{-n}) \iint_{|\xi|\leqslant C} \frac{h\alpha}{(\alpha+s)^2} dx d\xi \\
&= \mathcal{O}(1) h^{-n} \frac{h}{\alpha} \int_0^{t_0} \frac{1}{(1+\frac{t}{\alpha})^2} dV(t) \\
&= \mathcal{O}(1) h^{-n} \frac{h}{\alpha} \left(\left[(1+\frac{t}{\alpha})^{-2} V(t) \right]_0^{t_0} + \int_0^{t_0} (1+\frac{t}{\alpha})^{-3} V(t) \frac{dt}{\alpha} \right) \\
&= \mathcal{O}(1) h^{-n} \frac{h}{\alpha} (\alpha^2 + \alpha^\kappa \int_0^{t_0} (1+\frac{t}{\alpha})^{-3} \left(\frac{t}{\alpha}\right)^\kappa \frac{dt}{\alpha}) \\
&= \mathcal{O}(1) \frac{\alpha^\kappa}{h^n} \frac{h}{\alpha}.
\end{aligned}$$

It is also clear that

$$\|\mathrm{III}\|_{tr} = \mathcal{O}(1) \frac{\alpha^\kappa}{h^n} \left(\frac{h}{\alpha}\right)^N .$$

Summing up our estimates, we get under the assumption (4.31) (equivalent to (1.9)) the following result:

Proposition 4.6. — *Let $\chi \in C_0^\infty(\mathbf{R})$. For $0 < h \leqslant \alpha < 1$, we have*

$$\|\chi(\frac{1}{\alpha} S)\|_{\mathrm{tr}} = \mathcal{O}(1) \frac{\alpha^\kappa}{h^n}, \tag{4.35}$$

$$\operatorname{tr} \chi(\frac{1}{\alpha} S) = \frac{1}{(2\pi h)^n} \iint \chi(\frac{s(x,\xi)}{\alpha}) dx d\xi + \mathcal{O}(\frac{\alpha^\kappa}{h^n} \frac{h}{\alpha}). \tag{4.36}$$

Remark 4.7. — Using simple h-pseudodifferential calculus (for instance as in the appendix of [15], we see that if we redefine S as in (4.2), then in each local coordinate chart, $S = \mathrm{Op}_h(S)$, where $S \equiv s \bmod S_{1,0}(h\langle\xi\rangle^{-1})$ and s is now redefined as

$$s(x,\xi) = \left(\frac{|p(x,\xi) - z|}{|\widetilde{p}(x,\xi) - z|} \right)^2 . \tag{4.37}$$

The discussion goes through without any changes (now with $m = 0$) and we still have Proposition 4.6 with the new choice of S, s.

In the remainder of this section, we choose S, s as in (4.2), (4.37). In this case we notice that S is a trace class perturbation of the identity, whose symbol is $1+\mathcal{O}(h^\infty/\langle\xi\rangle^\infty)$ and similarly for all its derivatives, in a region $|\xi|\geqslant$ Const.

Let $0\leqslant\chi\in C_0^\infty([0,\infty[)$ with $\chi(0)>0$ and let $\alpha_0>0$ be small and fixed. Using standard pseudodifferential calculus in the spirit of [11], we get

$$\ln\det(S+\alpha_0\chi(\frac{1}{\alpha_0}S))=\frac{1}{(2\pi h)^n}(\iint\ln(s+\alpha_0\chi(\frac{1}{\alpha_0}s))dxd\xi+\mathcal{O}(h)). \tag{4.38}$$

Extend χ to be an element of $C_0^\infty(\mathbf{R};\mathbf{C})$ in such a way that $t+\chi(t)\neq 0$ for all $t\in\mathbf{R}$. As in [8], we use that

$$\frac{d}{dt}\ln(E+t\chi(\frac{E}{t}))=\frac{1}{t}\psi(\frac{E}{t}), \tag{4.39}$$

where

$$\psi(E)=\frac{\chi(E)-E\chi'(E)}{E+\chi(E)}, \tag{4.40}$$

so that $\psi\in C_0^\infty(\mathbf{R})$. By standard functional calculus for self-adjoint operators, we have

$$\frac{d}{dt}\ln\det(S+t\chi(\frac{S}{t}))=\operatorname{tr}\frac{1}{t}\psi(\frac{S}{t}). \tag{4.41}$$

Using (4.36), we then get for $t\geqslant\alpha\geqslant h>0$:

$$\frac{d}{dt}\ln\det(S+t\chi(\frac{1}{t}S))=\frac{1}{(2\pi h)^n}(\iint\frac{1}{t}\psi(\frac{s}{t})dxd\xi+\mathcal{O}(ht^{\kappa-2})).$$

Integrating this from $t=\alpha_0$ to $t=\alpha$ and using (4.38), (4.39), we get

$$\ln\det(S+\alpha\chi(\frac{1}{\alpha}S))=\frac{1}{(2\pi h)^n}(\iint\ln(s+\alpha\chi(\frac{s}{\alpha}))dxd\xi+\mathcal{O}(\frac{h}{\alpha})r_\kappa(\alpha)), \tag{4.42}$$

where $r_\kappa(\alpha)=\alpha^\kappa$ when $\kappa<1$, and $r_1(\alpha)=\alpha\ln\alpha$.

Improving the calculation prior to (4.22) in [8], we get

$$\begin{aligned}\iint\ln(s+\alpha\chi(\frac{s}{\alpha}))dxd\xi &= \iint\ln(s)dxd\xi+\int_0^\alpha\iint\frac{1}{t}\psi(\frac{s}{t})dxd\xi\,dt\\ &= \iint\ln(s)dxd\xi+\int_0^\alpha t^{\kappa-1}dt\\ &= \iint\ln(s)dxd\xi+\mathcal{O}(\alpha^\kappa).\end{aligned}$$

and together with (4.42) this leads to

PROPOSITION 4.8. — *If* $0 \leqslant \chi \in C_0^\infty([0,\infty[)$, $\chi(0) > 0$, *we have uniformly for* $0 < h \leqslant \alpha \ll 1$

$$\ln\det(S + \alpha\chi(\frac{1}{\alpha}S)) = \frac{1}{(2\pi h)^n}(\iint \ln s(x,\xi)dxd\xi + \mathcal{O}(\alpha^\kappa \ln\alpha)). \quad (4.43)$$

Here the remainder term can be replaced by $\mathcal{O}(\alpha^\kappa)$ *when* $\kappa < 1$ *and by* $\mathcal{O}(\alpha + h\ln\alpha)$ *when* $\kappa = 1$.

5. End of the proof

Having established in Section 4 the analogues for manifolds of the results in Section 4 of [8], the remainder of the proof of Theorem 1.1 is basically identical to the proof in [15] for the $\mathbf{R}^n$ case. For that reason, we will only give a brief outline.

Let P be as in (1.1), (1.2), classically elliptic as in (1.3) and let p be the semiclassical principal symbol. Assume (1.5). To start with, we let $z \in \Omega$ be fixed and assume (4.31), where $s = |p - z|^2$ is the semiclassical principal symbol of $S = (P - z)^*(P - z)$. Also, put

$$S_z = P_z^* P_z,\ P_z = (\widetilde{P} - z)^{-1}(P - z). \quad (5.1)$$

We fix the choice of an operator $P_0 = P_{\delta_0}$ as in (3.1), (3.2) (with $\delta = \delta_0$ still depending on h) and define $\widetilde{P}_{\delta_0}$, $P_{\delta_0,z}$, $S_{\delta_0,z}$, S_{δ_0} as in that section. As in [15], $\mathcal{D}(S_{\delta_0}) = \{u \in H^m(X);\ P_{\delta_0}u \in H^m(X)\}$. As there, we also introduce the self-adjoint operator $T_{\delta_0} = (P_{\delta_0} - z)(P_{\delta_0} - z)^*$ with domain $\mathcal{D}(T_{\delta_0}) = \{u \in H^m(X);\ P_{\delta_0}^* u \in H^m(X)\}$. In some fixed ($h$-independent) neighborhood of 0 the spectra of S_{δ_0}, T_{δ_0} are discrete and coincide. If $0 < \alpha \ll 1$, denote the (common) eigenvalues in $[0,\alpha[$ by $t_1^2, t_2^2, ..., t_N^2$, where $0 \leqslant t_1 \leqslant t_2 \leqslant ... \leqslant t_N$. Then, there are orthonormal families of eigenfunctions, $e_1, ..., e_N \in \mathcal{D}(S_{\delta_0})$, $f_1, ..., f_N \in \mathcal{D}(T_{\delta_0})$ such that

$$(P_{\delta_0} - z)e_j = t_j f_j,\ (P_{\delta_0} - z)^* f_j = t_j e_j. \quad (5.2)$$

Define $R_+ : L^2(X) \to \mathbf{C}^N$, $R_- : \mathbf{C}^N \to L^2(X)$ by

$$R_+u(j) = (u|e_j),\ R_-u_- = \sum_1^N u_-(j)f_j. \quad (5.3)$$

The Grushin problem

$$(P_{\delta_0} - z)u + R_-u_- = v,\ R_+u = v_+, \quad (5.4)$$

has a unique solution $u = E^0 v + E^0_+ v_+ \in H^m(X)$, $u_- = E^0_- v + E^0_{-+} v_+ \in \mathbf{C}^N$ for every $(v, v_+) \in L^2(X) \times \mathbf{C}^N$, and $E^0_\pm$ and E^0_{-+} can be given explicitly. In particular, $E^0_{-+} = -\mathrm{diag}\,(t_j)$.

Let now $P_\delta = P_{\delta_0} + \delta Q$ be a small perturbation (in a suitable sense) of P_{δ_0}. Then we still have a wellposed problem after replacing P_{δ_0} by P_δ in (5.4) with the solution $u = E^\delta v + E^\delta_+ v_+ \in H^m$, $u_- = E^\delta_- v + E^\delta_{-+} v_+ \in \mathbf{C}^N$ and the new solution operators have Neumann series expansions. In particular,

$$E^\delta_{-+} = E^0_{-+} - \delta E^\delta_- Q E^0_+ + \delta^2 E^0_- Q E^0 Q E^0_+ - ..., \tag{5.5}$$

where we can write the leading perturbation $-\delta E^0_- Q e^0_+ = -\delta M$, where $M = (M_{j,k})_{1 \leqslant j,k \leqslant N}$, $M_{j,k} = (Qe_k | f_j)$. When Q is a multiplication operator, $Qu(x) = q(x)u(x)$, then

$$M_{j,k} = \int q(x) e_k(x) \overline{f}_j(x) dx. \tag{5.6}$$

Now, adopt the symmetry assumption (1.7). Then we can replace the orthonormal family f_j by the new orthonormal family of eigenfunctions $\widetilde{f}_j = \overline{e}_j$ without changing the singular values of E^0_{-+}, E^δ_{-+} and we get

$$M_{j,k} = \int q(x) e_k(x) e_j(x) dx. \tag{5.7}$$

In [15] we showed how to find admissible potentials q as in (1.10), (1.11), such that M gets at least $N/2$ "large" singular values and this was used in an iteration procedure in order to find perturbations of the form P_δ where q is an admissible potential, for which the small singular values are not "too small".

Strengthen the assumption on δ_0 to

$$\delta_0 \leqslant h. \tag{5.8}$$

Then combining Proposition 4.6 with the perturbative functional calculus in Section 4 of [15], we obtain that for $0 < h \leqslant \alpha \ll 1$, the number of eigenvalues of S_{δ_0} in $[0, \alpha]$ satisfies $N = \mathcal{O}(\alpha^\kappa h^{-n})$. The iteration scheme in Sections 5 to 7 in [15] now works without any changes and we get the following analogue of Proposition 7.3 there:

Proposition 5.1. — *We make the assumptions above (with z fixed). Let $s > \frac{n}{2}$, $0 < \epsilon < s - \frac{n}{2}$, $N_1 = \widetilde{M} + sM + \frac{n}{2}$, $N_2 = 2(N_1 + n) + \epsilon_0$, where $\epsilon_0 > 0$ and $M, \widetilde{M}$ are as in (1.11). Let L, R be h-dependent parameters* *as in (1.11). Let* $0 < \tau_0 \leqslant \sqrt{h}$ *and let* $N^{(0)} = \mathcal{O}(h^{\kappa-n})$ *be the number of singular values of* $P_{\delta_0} - z$ *in* $[0, \tau_0[$. *Let* $0 < \theta < \frac{1}{4}$ *and let* $N(\theta) \gg 1$ *be sufficiently large. Define* $N^{(k)}$, $1 \leqslant k \leqslant k_1$ *iteratively in the following way. As long as* $N^{(k)} \geqslant N(\theta)$, *we put* $N^{(k+1)} = [(1-\theta)N^{(k)}]$ *(the integer part of* $(1-\theta)N^{(k)}$*). Let* $k_0 \geqslant 0$ *be the last* k *value we get in this way. For* $k > k_0$ *put* $N^{(k+1)} = N^{(k)} - 1$ *until we reach the value* k_1 *for which* $N^{(k_1)} = 1$.

Put $\tau_0^{(k)} = \tau_0 h^{kN_2}$, $1 \leqslant k \leqslant k_1 + 1$. *Then there exists* $q = q_h(x)$ *of the form (1.10), satisfying (1.11), so that by the choice of* L,

$$\|q\|_{H_h^s} \leqslant \mathcal{O}(1) h^{-N_1 + \frac{n}{2}}, \ \|q\|_{L^\infty} \leqslant \mathcal{O}(1) h^{-N_1},$$

such that if $P_\delta = P_{\delta_0} + \frac{1}{C}\tau_0 h^{2N_1+n} q = P + \delta Q$, $\delta = \frac{1}{C} h^{N_1+n}\tau_0$, $Q = h^{N_1} q$, *we have the following estimates on the singular values of* $P_\delta - z$*:*

- *If* $\nu > N^{(0)}$, *we have* $t_\nu(P_\delta - z) \geqslant (1 - \frac{h^{N_1+n}}{C}) t_\nu(P - z)$.
- *If* $N^{(k)} < \nu \leqslant N^{(k-1)}$, $1 \leqslant k \leqslant k_1$, *then* $t_\nu(P_\delta - z) \geqslant (1 - \mathcal{O}(h^{N_1+n}))\tau_0^{(k)}$.
- *Finally, for* $\nu = N^{(k_1)} = 1$, *we have* $t_1(P_\delta - z) \geqslant (1 - \mathcal{O}(h^{N_1+n}))\tau_0^{(k_1+1)}$.

As shown in [15] we have an equivalence between lower bounds for the small singular values of $P_\delta - z$ in the above proposition and for the singular values of E_{-+}^δ appearing in the solution of the Grushin problem for $P_\delta - z$ (and that is used in the proof of the proposition). We also have an equivalence between lower bounds for the small singular values of $P_\delta - z$ and those of $P_{\delta,z}$. For the latter operator we have a well posed Grushin problem analogous to (5.4) and an equivalence between lower bounds for the small singular values of $P_{\delta,z}$ and for the singular values of $E_{-+}^{\delta,z}$, appearing in the solution of the new Grushin problem. Using perturbative functional calculus we also have an asymptotic formula for $\ln\det \mathcal{P}_{\delta,z}$, where

$$\mathcal{P}_{\delta,z} = \begin{pmatrix} P_{\delta,z} & R_-^z \\ R_+^z & 0 \end{pmatrix}$$

is the matrix associated to the new Grushin problem. As showed in [8] by means of calculations from [16], we have

$$\det P_{\delta,z} = \det \mathcal{P}_{\delta,z} \det E_{-+}^{\delta,z}. \tag{5.9}$$

The perturbative functional calculus gives a general upper bound on $\ln\det P_{\delta,z}$, and for the special admissible perturbation in Proposition 5.1, we have a lower bound on $\ln|\det E_{-+}^{\delta,z}|$ (using the lower bound on the singular values of $E_{-+}^{\delta,z}$ and the fact the modulus of the determinant is equal to the product of the singular values). We get as in [15]:

PROPOSITION 5.2. — *For the special admissible perturbation P_δ in Proposition 5.1, we have*

$$\ln|\det P_{\delta,z}| \geqslant \frac{1}{(2\pi h)^n}\left(\iint \ln|p_z|dxd\xi - \mathcal{O}\left((h^\kappa + h^n \ln\frac{1}{h})(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2)\right)\right). \tag{5.10}$$

On the other hand, for more general operators of the form $P_\delta = P_{\delta_0} + \tau_0 h^{2N_1+n} q$ with q admissible as in (1.10), (1.11) we get as in [15] the upper bound:

$$\ln|\det P_{\delta,z}| \leqslant \frac{1}{(2\pi h)^n}\left(\iint \ln|p_z|dxd\xi + \mathcal{O}\left(h^\kappa \ln\frac{1}{h}\right)\right). \tag{5.11}$$

Section 8 of [15] (based on Jensen type arguments in the α-variables) now applies and shows that with probability close to 1, we have

$$\ln|\det P_{\delta,z}| \approx \frac{1}{(2\pi h)^n}\iint \ln|p_z|dxd\xi. \tag{5.12}$$

So, far z was fixed and we now let it vary in a neighborhood of Γ, recalling that the eigenvalues P_δ in this region coincide with the zeros of the holomorphic function $F_\delta(z) = \det P_{\delta,z}$. Assuming now that (1.9) holds uniformly for z in a neighborhood of $\partial\Gamma$, we can then conlude the proof as in Section 9 of [15], by applying a general result about the number of zeros of holomorphic functions with exponential growth, from [8]. Recall that this result (applied to $F_\delta(z)$) requires an upper bound on $\ln|F_\delta(z)|$ in a fixed neighborhood of $\partial\Gamma$, in our case provided by (5.11), as well as a corresponding lower bound at finitely many points along $\partial\Gamma$. The latter is provided by the lower bound part of (5.12) and holds with probability close to 1.

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