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On a Special Class of Non Complete Webs

Julien Sebag[1]

Résumé. — Dans cet article, nous introduisons une classe particulière de tissus incomplets, que nous appelons tissus NN. Nous en étudions les propriétés algébriques et géométriques.

Abstract. — In this article, we introduce a special class of non complete webs, the *NN-webs*. We also study the algebraic and geometric properties of these webs.

1. Introduction

In this article, we consider that a d-web is (*implicitly*) given by a differential equation $F(x, y, y') = 0$, where $F(x, y, p) \in \mathbf{C}\{x, y\}[p]$ is a polynomial of degree (in p) $d \geqslant 3$, with $R_F = Result_p(F, \partial_p F) \neq 0$ (see §2 for the terminology). A main question in web geometry is the question of the "classification" of such objects. A classical result gives a (bounded by $\pi_d := (d-1)(d-2)/2$) discrete invariant: the *rank* of a web. Roughly speaking, for d-webs of rank π_d, the question of classification could amount to the following one: to be or not to be (up to a local analytic isomorphism) an *algebraic web*.

An algebraic web is determined by a polynomial $G \in \mathbf{C}[s, t]$, in two variables s and t, via a Legendre tansformation, *i.e.*, $F(x, y, p) = G(y-px, p)$ (Clairaut's differential equations). Remark that the derivation $\partial_x + p\partial_y$ of

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$\mathbf{C}[x,y,p]$ is *locally nilpotent*, *i.e.*, for every $f \in \mathbf{C}[x,y,p]$, there exists $n \in \mathbf{N}$ such that $(d_F \circ \ldots \circ d_F)(f) = d_F^n(f) = 0$, and that $(\partial_x + p\partial_y)(F) = 0$ in the algebraic case.

If $\mathcal{W}$ is an arbitrary d-web defined by F, one can associate (see, for example, [3]/§3) its *linearization polynomial* V_F, which allows to construct a derivation $D_F := R_F(\partial_x + p\partial_y) - V_F\partial_p$ of $\mathbf{C}\{x,y\}[p]$, which verifies $D_F(F) = U_F F$, for some polynomial U_F. In the case of algebraic webs, $V_F = U_F = 0$. In this way, D_F generalizes, to any web, the derivation $\partial_x + p\partial_y$ (see §2 for details). The author has shown that the local nilpotence property of D_F is shared by many other webs, the *nilpotent webs*, introduced in [10]. A nilpotent web is, again, determined by a polynomial $G \in \mathbf{C}[s,t]$, via a tansformation of Legendre type which depends on the linearization polynomial V_F. Besides, these webs are *algebrizable* (see [10]/Theorem 5.2).

Among webs, *non complete webs* are defined by a polynomial $F \in \mathbf{C}[y,p]$. In this case, we can consider the derivation $D_F = R_F p\partial_y - V_F\partial_p$ of $\mathbf{C}[y,p]$. The subject of the present article is to give a precise answer to the following question: *what means local nilpotence in this context*? In §3, we introduce and study the algebraic-geometric properties of this special class of non complete webs, that we call *NN-webs*. We give precise characterizations (see Theorem 3.6). In particular, we show that such a web is determined by a polynomial $G \in \mathbf{C}[s]$, via a transformation rule that we explicit. We also compute their ranks and we answer to the question of their algebrizability (see Theorem 3.13). Remark that these results can not be obtained by specifying some results of [10].

Notations

We denote by $\mathbf{C}$ the field of complex numbers, and by $\mathbf{C}\{x,y\}$ the ring of convergent power series in variables x and y. By a *derivation* of a $\mathbf{C}$-algebra A, we mean a $\mathbf{C}$-linear map $\delta : A \to A$, which verifies Leibnitz's rule. If $F \in \mathbf{C}\{x,y\}[p]$, the usual partial derivatives are noted ∂_x, ∂_y and ∂_p.

The ring $\mathbf{C}[y_0, y_1, \ldots, y_n, \ldots]$, endowed with the derivation $\delta(y_i) = y_{i+1}$, is denoted $\mathbf{C}\langle y\rangle$. We embed $\mathbf{C}[y,p]$ in $\mathbf{C}\langle y\rangle$ (as $\mathbf{C}$-algebras) by $y \mapsto y_0$ and $p \mapsto y_1$. A *differential polynomial* F is just an element of $\mathbf{C}\langle y\rangle$. Its order is the biggest integer n such that y_n appears effectively in F. Let S be a part of $\mathbf{C}\langle y\rangle$, we denote by $[S]$ the differential ideal generated by S, and by $\{S\}$ its radical.

2. Terminology in web geometry

Let $d \geqslant 3$ be an integer. In this article, we look at webs of $\mathbf{C}^2$. In particular we only consider them from a local point of view.

• A *non singular planar d-web* $\mathcal{W}$ is defined by the datum, locally in $\mathbf{C}^2$, of $d \geqslant 1$ holomorphic foliations in $(\mathbf{C}^2, 0)$ in general position. So we can represent such an object by germs $F_i : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}, 0)$ for $1 \leqslant i \leqslant d$, where $F_i \in \mathbf{C}\{x, y\}$, $F_i(0) = 0$ and $dF_i(0) \wedge dF_j(0) \neq 0$ if $i \neq j$. A classical theorem in web geometry asserts that the $\mathbf{C}$-vector space of its *abelian relations*, defined by $\mathcal{A}(d) := \big\{(g_1(F_1), \ldots, g_d(F_d)) \in \mathbf{C}\{x, y\}$ with $g_i \in \mathbf{C}\{t\}$ and $\sum_{i=1}^{d} g_i(F_i)dF_i = 0\big\}$, is of finite dimension. This dimension is called the *rank* of the web. It is bounded by the integer $\pi_d := (d-1)(d-2)/2$, and it is invariant by local analytic isomorphisms (see [1] for some details of the theory).

• Cauchy's theorem and the implicit functions theorem make the datum of a d-web $\mathcal{W}$ correspond (up to an invertible element of $\mathbf{C}\{x, y\}$) to the datum of a differential system $F(x, y, y') = 0$ and $R_F(x, y) := Result(F, \partial_p F)(x, y) \neq 0$, with $F \in \mathbf{C}\{x, y\}[p]$ of degree $d \geqslant 3$ (in p), whose solutions, in a neighborhood of 0, are exactly the d distinct slopes of the F_i as above. Such a polynomial is called a *presentation* of $\mathcal{W}$, and $\mathcal{W}$ is *presented* by F.

This is the *implicit approach* in web geometry. Such a (classical) point of view is well justified, in the study of webs, by Hénaut's work [4] and its consequences (see, for example, [6]/Théorème 4.1 as a non trivial consequence).

Introduced in [6], the *higher linearization polynomials* V_F^i will be used in what follows. Let us recall this construction (see [6]/Chapitre 2). There exist two *associated polynomials* U_F^i and V_F^i, for $0 \leqslant i \leqslant d-3$, respectively of degree at most $d-2$ and $d-1$, verifying the following identity $p^i R_F \cdot (\partial_x F + p\partial_y F) = U_F^i F + V_F^i \partial_p F$. Because of the conditions on the degrees, one can remark that such a couple (U_F^i, V_F^i) is unique. We note $V_F := V_F^0$ and call it the *linearization polynomial* of F.

• Thanks to the implicit approach, we can interpret webs as some objects of differential algebra (see [7], [8], and [9] for specific applications of this point of view). We say that a web $\mathcal{W}$ is *polynomial* if $\mathcal{W}$ admits a presentation $F \in \mathbf{C}\{x, y\}[p]$ which belongs to $\mathbf{C}[x, y, p]$. We call $F \in \mathbf{C}[x, y, p]$ a *W-polynomial* if its degree $d \geqslant 3$ and if $R_F := Result(F, \partial_p F) \in \mathbf{C}[x, y]$ is not identically zero.

3. Non complete webs, and *NN*-webs

For a W-polynomial $F \in \mathbf{C}[y,p]$, with $R_F := Result_p(F, \partial_p F)$, we note $D_F := pR_F\partial_y - V_F\partial_p$ the associated derivation of $\mathbf{C}[y,p]$.

Definition 3.1. — *Let $d \geqslant 3$, and let $\mathcal{W}$ be a non singular, planar, polynomial d-web, presented by a W-polynomial F. We say that $\mathcal{W}$ is non complete if $F \in \mathbf{C}[y,p]$.*

Remark 3.2. — Note that a non complete web is *never* algebraic. In particular, it follows directly from the definition of the linearization polynomial (and its unicity) that $V_F \neq 0$.

Definition 3.3. — *Let $d \geqslant 3$, and let $\mathcal{W}$ be a d-web, presented by a W-polynomial F. Let $R_F := Result_p(F, \partial_p F)$ be its resultant, and let V_F be its linearization polynomial. We say that $\mathcal{W}$ is an (d-)NN-web if the three following properties are satisfied:*

1. *$\mathcal{W}$ is non complete;*
2. *pR_F divides V_F, i.e., there exists $c \in \mathbf{C}[s,t]$ such that $V_F = pR_F c(y,p)$;*
3. *$d_F := D_F/(pR_F) = \partial_y - c\partial_p$ is locally nilpotent.*

Example 3.4. — Consider $F = (p-y)(p-(y+1))(p-(y-1))$. Its resultant R_F is equal to -4 and its linearization polynomial V_F is equal to $-4p$. We will see, in the next paragraph, that all NN-webs can be obtained by this way.

The technical lemma below will be used and very useful in the following paragraphs.

Lemma 3.5. — *Let $d \geqslant 3$, and let $\mathcal{W}$ be a d-NN-web, presented by a W-polynomial $F = \sum_{i=0}^{d} f_i(y)p^i$. Let $R_F := Result_p(F, \partial_p F)$ be its resultant, and let V_F be its linearization polynomial. Then:*

1. *$V_F \neq 0$;*
2. *$U_F = 0$;*
3. *if $0 \leqslant i \leqslant d-3$ and $V_F^i = \sum_{j=0}^{d-1} v_j^i(y)p^j$, then $v_{d-1}^i = 0$ and $U_F^i = 0$;*
4. *there exists a polynomial $c \in \mathbf{C}[s]\backslash\{0\}$, in one variable s, such that $V_F = pR_F c(y)$;*
5. *there exists a polynomial $C \in \mathbf{C}[s]\backslash\{0\}$, in one variable s, such that $ker(d_F) = \mathbf{C}[p + C(y)]$. In this case, $\partial_y C(y) = c(y)$.*

Proof. — The first point comes from the remark above. The second point is a direct application of [5]/Lemma 1.10. The point (3) can be proved by induction on i, by remarking that, for each $0 \leqslant i \leqslant d-4$, $V_F^{i+1} = pV_F^i - v_{d-1}^i \cdot F/f_d$ and $U_F^{i+1} = pU_F^i + v_{d-1}^i \cdot \partial_p F/f_d$ in $Frac(\mathbf{C}\{x,y\})[p]$, with $U_F^0 := U_F$ (see [6], Proposition 2.5). For (4), by definition, there exists a polynomial $c \in \mathbf{C}[y,p]$ such that $V_F = pR_F c(y,p)$. Since d_F is locally nilpotent, it follows that the divergence of d_F is identically zero (see [2]/Corollary 3.16). But $div(d_F) := \partial_y(d_F(y)) + \partial_p(d_F(p)) = \partial_p(c(y,p))$. It proves (4). For the last point, let $C \in \mathbf{C}[s]\backslash\{0\}$ such that $\partial_y C(y) = c(y)$. A direct computation gives that $\mathbf{C}[p+C(y)] \subset ker(d_F)$. Let $h \in \mathbf{C}[y,p]$ be an element of $ker(d_F)$. If $h = \sum a_{ij} y^i p^j$, there exists $b_{ij} \in \mathbf{C}$ such that $h = \sum b_{ij} y^i (p+C(y))^j$. By assumtion, $d_F(h) = \sum i b_{ij} y^{i-1}(p+C(y))^j = 0$. Since the couple $(y, p+C(y))$ determines a $\mathbf{C}$-automorphism of $\mathbf{C}^2$, we conclude that $\mathbf{C}[y, p+C(y)]$ is a polynomial ring. Thus $h \in \mathbf{C}[p+C(y)]$. □

3.1. Characterizations and first properties

THEOREM 3.6. — *Let $d \geqslant 3$, and let $\mathcal{W}$ be a non complete d-web, presented by a W-polynomial F. Let $R_F := Result_p(F, \partial_p F)$ be its resultant, and let V_F be its linearization polynomial. Then the following assertions are equivalent:*

1. *$\mathcal{W}$ is a NN-web;*
2. *there exists a polynomial $c \in \mathbf{C}[s]\backslash\{0\}$, in one variable s, such that $V_F = pR_F c(y)$;*
3. *there exist two polynomials C, $G \in \mathbf{C}[s]\backslash\{0\}$, in one variable s, such that G is of degree d, $F = G(p + C(y))$. In this case, $V_F = pR_F \partial_y C$.*

Remark 3.7. — As $\mathbf{C}$ is algebraically closed and $d \geqslant 3$, a d-NN-web can not be presented by an irreducible W-polynomial. Remark that the $\mathbf{C}$-algebra $\mathbf{C}[y,p]/(F)$ is $\mathbf{C}$-isomorphic to $(\mathbf{C}[z]/(G(z)))[y]$. Moreover, a presentation F of a NN-d-web is always of the form $F(y,p) = \prod_{i=1}^d (p+C(y)+a_i)$, where $a_i \in \mathbf{C}$ satisfies $a_i \neq a_j$, if $i \neq j$.

Proof. — (2) $\Leftrightarrow$ (1) comes from the relations $d_F(y) = \partial_y(y) = 1$ and $d_F(p) = V_F/R_F$, and Lemma 3.5/(4) above.

By Lemma 3.5/(2), if $\mathcal{W}$ is a NN-web, $d_F(F) = 0$ and we conclude by Lemma 3.5/(5). Conversely, we verify that $(\partial_y - \partial_y C \cdot \partial_p)(F) = 0$. It follows that $pR_F \partial_y F = pR_F \partial_y C(y) \partial_p F$. By the unicity of V_F, $V_F = pR_F \partial_y C$. We have proved that the condition (3) is equivalent to (2). □

Corollary 3.8. — *Let $d \geqslant 3$, and let $\mathcal{W}$ be a d-NN-web, presented by a W-polynomial F. Let $R_F := Result_p(F, \partial_p F)$ be its resultant. Then:*

1. *the general solutions $\gamma \in \mathbf{C}\{x\}$ of the differential equation $F(y, y') = 0$ are solutions of the second order differential equation $y'' - c(y)y' = 0$;*

2. *the differential ideal $\{F, \partial_p F\} = \mathbf{C}\langle y\rangle$. In particular, the equation $F(y, y') = 0$ has no singular solutions, i.e., there is no solution $\gamma \in \mathbf{C}\{x\}$ of the differential system $F(y, y') = \partial_p F(y, y') = 0$.*

Proof. — The point (1) is an easy consequence of Theorem 3.6/(5) (for a general argument see also [7]/Théorème 11). For (2), we have only to prove the first assertion. If $\{F, \partial_p F\} \neq \mathbf{C}\langle y\rangle$, then we can decompose it in a finite number of prime ideals of the form $\{R_\alpha\}$, with R_α some irreducible factor of R_F. If such a differential ideal $\{R_\alpha\}$ is an irreducible component of $\{F\}$, then $\{R_\alpha\} \subset \{y_1 + C(y_0)\}$, with $y_1 + C(y_0)$ dividing F. As $\{R_\alpha\}$ is maximal, we have $\{R_\alpha\} = \{y_1 + C(y_0)\}$. Contradiction. If $\{R_\alpha\}$ contains $\{F\} : \partial_p F$, then it contains $\{y_1 + C(y_0)\}$, with $y_1 + C(y_0)$ dividing F. Thus R_α divides C. As $\{R_\alpha\}$ contains $\partial_p F$, it contains another $\{y_1 + \tilde{C}(y_0)\}$, with $y_1 + \tilde{C}(y_0)$ dividing F. But, by Lemma 3.5/(5), we have $\tilde{C} - C \in \mathbf{C}\backslash\{0\}$. It is again a contradiction. □

3.2. Rank and abelian relations

Recall from [6]/§2.4 the following definition:

Definition 3.9. — *Let $\mathcal{W}$ be a d-web, presented by a W-polynomial F. We say that $r = \sum_{i=0}^{d-3} b_i p^i \in \mathbf{C}\{x, y\}[p]$ is an abelian polynomial associated to $\mathcal{W}$, if the degree (in p) of r is at most $d-3$ and if r satisfies the following differential equation*

$$R_F \cdot (\partial_x r + p\partial_y r) = U_r + \partial_p V_r,$$

where $U_r = \sum_{i=0}^{d-3} b_i U_F^i$ and $V_r = \sum_{i=0}^{d-3} b_i V_F^i$.

The set of abelian polynomials of $\mathcal{W}$ can be endowed with a structure of $\mathbf{C}$-vector space that we denote $\mathcal{AP}_{\mathcal{W}}$.

Remark 3.10. — Hénaut's work implies that $\mathcal{AP}_{\mathcal{W}}$ is isomorphic to $\mathcal{A}_{\mathcal{W}}$ (see [4] and [6]/ Proposition 2.6).

The next technical lemma is new and complete the description of the abelian polynomials.

LEMMA 3.11. — *Let $\mathcal{W}$ be a non singular, planar d-web, presented by a W-polynomial $F = \sum_{i=0}^{d} f_i(x,y)p^i$. Then:*

1. *if $r \in \mathbf{C}\{x,y\}[p]$ is an abelian polynomial, there exists $\omega \in \mathbf{C}\{x,y\}[f_d^{-1}, p]$ such that $R_F \cdot (\partial_x r + p\partial_y r) - V_F \partial_p r = (U_F + \partial_p V_F)r - F\partial_p \omega$;*

2. *if $\mathcal{W}$ is a NN-web and $V_F = pR_F c(y)$, then $r \in \mathbf{C}\{x,y\}[p]$ is an abelian polynomial if and only if $\partial_x r + p\partial_y r - pc(y)\partial_p r = rc(y)$ (i.e., $\omega = 0$).*

Proof. — For each $0 \leqslant i \leqslant d-4$, $V_F^{i+1} = pV_F^i - v_{d-1}^i \cdot F/f_d$ and $U_F^{i+1} = pU_F^i + v_{d-1}^i \cdot \partial_p F/f_d$ in $\mathbf{C}\{x,y\}[f_d^{-1}, p]$, with $U_F^0 := U_F$ and $V_F^i := \sum_{i=0}^{d-1} v_j^i(x,y)p^i$ (see [6], Proposition 2.5). Let us set $\omega_i := \sum_{j=0}^{i-1} v_{d-1}^j p^{i-j-1}$, for $1 \leqslant i \leqslant d-3$. An induction on i, using the above relations, proves that, for each $1 \leqslant i \leqslant d-3$,

$$V_F^i = p^i V_F - \omega_i F/f_d \quad \text{and} \quad U_F^i = p^i U_F + \omega_i \partial_p F/f_d.$$

Let $r = \sum_{i=0}^{d-3} b_i p^{d-i} \in \mathbf{C}\{x,y\}[p]$. We write $U_r = \sum_{i=0}^{d-3} b_i U_F^i$ and $V_r = \sum_{i=0}^{d-3} b_i V_F^i$. It follows that:

$$U_r = \sum_{i=0}^{d-3} b_i (p^i U_F + \omega_i \partial_p F/f_d),$$

and that $\partial_p V_r$ is equal to:

$$\sum_{i=0}^{d-3} b_i \partial_p (p^i V_F - \omega_i F/f_d) = V_F \partial_p r + r\partial_p V_F - \sum_{i=0}^{d-3} b_i(\omega_i \partial_p F/f_d + F\partial_p \omega_i/f_d).$$

If r is an abelian polynomial, then r is a solution of the differential equation $R_F\,(\partial_x r + p\partial_y r) = U_r + \partial_p V_r$. Let us set $\omega := f_d^{-1} \sum_{i=0}^{d-3} b_i \omega_i$. By substituting the new expressions of U_r and $\partial_p V_r$ in this differential equation, we prove the first statement of the lemma. The second one comes directly from Lemma 3.5/(3) and the fact that $U_r = 0$ and $V_r = rV_F$. □

Remark 3.12. —

1. If $r \in \mathbf{C}\{x,y\}$ is an abelian polynomial, then $R_F(\partial_x r + p\partial_y r) = (U_F + \partial_p V_F)r$. Indeed, consider the degrees (in p) in the relation of Lemma 3.11. Moreover, if R_F divides V_F and if $r \in \mathbf{C}[x,y]$, then either $U_F + \partial_p V_F = 0$, either $r = 0$ by [5]/Lemma 1.10.

2. If $3 \leqslant d < 5$, and by considering the degrees (in p), we deduce also from the construction of ω that, if $r \in \mathbf{C}\{x,y\}[p]$ is an abelian polynomial, then $R_F(\partial_x r + p\partial_y r) - V_F\partial_p r = (U_F + \partial_p V_F)r$. In particular, abelian polynomials $r \in \mathbf{C}[x,y,p]$ can be interpreted as associated Darboux polynomials in $\mathbf{C}[x,y,p]$.

Theorem 3.13. — *Let $d \geqslant 3$ and let $\mathcal{W}$ be a d-NN-web, presented by $F \in \mathbf{C}[y,p]$. Then:*

1. *the rank of $\mathcal{W}$ is egal to $\pi_d := (d-1)(d-2)/2$ if and only if $V_F = pR_Fc$, with $c \in \mathbf{C}\backslash\{0\}$. In the other case, the rank of $\mathcal{W}$ is zero;*

2. *$\mathcal{W}$ is algebrizable if and only if $V_F = pR_Fc$, with $c \in \mathbf{C}\backslash\{0\}$.*

Proof. — For $d \geqslant 4$, the second assertion is a consequence of the first one and of [3]/Théorème 1. We have to show that the $\mathbf{C}$-vector space $\mathcal{PA}_{\mathcal{W}}$ is of dimension π_d if $V_F = pR_Fc$ with $c \in \mathbf{C}\backslash\{0\}$, and of dimension 0 if $V_F = pR_Fc(y)$ with $c(y) \in \mathbf{C}[y]\backslash\mathbf{C}$. Let $r = \sum_{i=0}^{d-3} b_i p^i \in \mathbf{C}\{x,y\}[p]$ be an abelian polynomial. By Lemma 3.11, r satisfies the differential equation $\partial_x r + p\partial_y r - pc(y)\partial_p r = rc(y)$. We can compare the degrees (in p) in this equation.

Assume that $d = 3$. Thus the coefficient b_0 must satisfy the following differential system:

$$\mathcal{M}(3) := \left\{ \begin{array}{ccc} \partial_y b_0 & = & 0 \\ \partial_x b_0 - c(y)b_0 & = & 0 \end{array} \right. .$$

If $c \in \mathbf{C}\backslash\{0\}$, then $b_0 = \lambda exp(cx)$, with $\lambda \in \mathbf{C}$, is the (general) solution of this system. The rank of $\mathcal{W}$ is then 1. Besides, $\mathcal{W}$ is algebrizable. If $c \in \mathbf{C}[y]\backslash\mathbf{C}$, $\mathcal{M}(3)$ has no non trivial solution, and the rank of $\mathcal{W}$ is zero.

Assume that $d \geqslant 4$. Thus the coefficients $(b_i)_{0\leqslant i\leqslant d-3}$ are the solutions of the following differential system:

$$\mathcal{M}(d) := \left\{ \begin{array}{ccc} \partial_y b_{d-3} & = & 0 \\ \partial_x b_{d-3} + \partial_y b_{d-4} - (d-2)c(y)b_{d-3} & = & 0 \\ \partial_x b_{d-4} + \partial_y b_{d-5} - (d-3)c(y)b_{d-4} & = & 0 \\ \vdots & \vdots & \vdots \\ \partial_x b_1 + \partial_y b_0 - 2c(y)b_{d-1} & = & 0 \\ \partial_x b_0 - c(y)b_0 & = & 0 \end{array} \right. .$$

Let ν the degree of c. Assume firstly that $\nu \geqslant 1$. Remark that $b_{d-3} \in \mathbf{C}\{x\}$. By integrating with respect to y, we show that b_{d-4} is of the form

$(d-2)C(y)b_{d-3} - y\partial_x b_{d-3} + \gamma(x)$, with $\gamma \in \mathbf{C}\{x\}$ and $\partial_y C = c$. By deriving this expression with respect to x, we can compute $\partial_y b_{d-5}$ in terms of b_{d-3}, and iterate this process. Then b_0 can be expressed as a polynomial in y with coefficients in $\mathbf{C}\{x\}$. The leading term of b_0 is of the following form:

$$\alpha b_{d-3} y^{g(\nu)},$$

with $g : \mathbf{N} \to \mathbf{N}$, $g(\nu) \geqslant \nu$ and $\alpha \in \mathbf{C}\backslash\{0\}$. The last equation of the system $\partial_x b_0 = c(y) b_0$ implies that

$$c_\nu y^\nu b_{d-3} = \partial_x b_{d-3},$$

with $c_\nu \in \mathbf{C}\backslash\{0\}$ the leading coefficient of c. So $b_{d-3} = 0$, because $\nu > 0$ by assumption, and $b_{d-3} \in \mathbf{C}\{x\}$. Thus the coefficients $(b_i)_{0 \leqslant i \leqslant d-4}$ are the solutions of a system $\mathcal{M}(d-1)$ (with $c \in \mathbf{C}[y]\backslash\mathbf{C}$). We conclude by induction on d, since the result is true for $d = 3$.

Assume now that $\nu = 0$, *i.e.*, $c \in \mathbf{C}\backslash\{0\}$. By integrating with respect to y, we show that $b_{d-4} = ((d-2)cb_{d-3} - \partial_x b_{d-3})y + \gamma(x)$, with $\gamma \in \mathbf{C}\{x\}$. By deriving this expression with respect to x, we obtain that $\partial_y b_{d-5}$ is equal to:

$$\big((d-3)c\left((d-2)cb_{d-3} - \partial_x b_{d-3}\right) - \big((d-2)c\partial_x b_{d-3} - \partial_x^2 b_{d-3}\big)\big)\, y$$
$$+(d-3)c\gamma(x) - \partial_x\gamma(x).$$

Equivalently, $\partial_y b_{d-5}$ is equal to:

$$\big((d-3)(d-2)c^2 b_{d-3} + (d-1)c\partial_x b_{d-3} + \partial_x^2 b_{d-3}\big)\, y + (d-3)c\gamma(x) + \partial_x\gamma(x).$$

By iterating this process, it is clear that the last equation of the system $\partial_x b_0 = c(y) b_0$ can be rewritten under the following form:

$$G_{d-3}(\gamma_{d-3})y^{d-3} + G_{d-4}(\gamma_{d-4})y^{d-4} + \ldots + G_0(\gamma_0(x)) = 0,$$

where $G_i \in \mathbf{C}\langle y\rangle$, for $0 \leqslant i \leqslant d-3$, is a linear homogeneous differential polynomial of order $i+1$ and γ_i, for $0 \leqslant i \leqslant d-3$, is an element $\mathbf{C}\{x\}$, that we want to determine. Note that $\gamma_{d-3} := b_{d-3}$. This equation is of course equivalent to the differential system $G_i(\gamma_i) = 0$, for all $0 \leqslant i \leqslant d-3$. We conclude that the dimension of the space of solutions is equal to the sum of the dimensions of the spaces of solutions of all differential equations $G_i = 0$. As the G_i are linear of order $i+1$, we conclude that this dimension is equal to $1 + 2 + \ldots + (d-2) = \pi_d$. □

Remark 3.14. — Theorem 3.13 above can be proved using more sophisticated machinery. If $d = 3$, a non singular, planar, polynomial 3-web $\mathcal{W}$

is algebrizable if and only if it is of rank 1. This last condition is again equivalent to $K_{\mathcal{W}} = 0$, where $K_{\mathcal{W}}$ is the Blaschke curvature of $\mathcal{W}$. If $\mathcal{N}$ is a 3-NN-web presented by F and if $V_F = pR_F c(y)$, then $K_{\mathcal{W}} = \partial_y c(y) dx \wedge dy$. For $d \geqslant 4$, Hénaut's formalism (see [4]), and precisely [6]/Théorème 4.1, allows (in particular) to compute the rank of a d-NN-web.

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