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Wavelet techniques for pointwise regularity[(*)]

Stéphane Jaffard [(1)]

... she rocks herself to sleep on wavelets of sensation rippling out from the secret grotto at the center of her body.

David Lodge, *Souls and Bodies*, Chap. 1.

Abstract. — Let E be a Banach (or quasi-Banach) space which is shift and scaling invariant (typically a homogeneous Besov or Sobolev space). We introduce a general definition of pointwise regularity associated with E, and denoted by $C^\alpha_E(x_0)$. We show how properties of E are transferred into properties of $C^\alpha_E(x_0)$. Applications are given in multifractal analysis.

Résumé. — Soit E un espace de Banach (ou un quasi-Banach) invariant par translation et dilatation (typiquement un espace de Besov ou de Sobolev homogène). Nous introduisons une définition générale de régularité ponctuelle associée à E, et notée $C^\alpha_E(x_0)$. Nous montrons comment les propriétés de E se traduisent en propriétés de $C^\alpha_E(x_0)$. Nous donnons également des application en analyse multifractale.

1. Introduction

How can be formalized the idea that a function (deterministic or stochastic) satisfies some scaling property in the neighbourhood of a given point x_0? This problem quickly splits in several directions depending whether the setting is deterministic or stochastic (in the latter case, the scaling is required to hold in law rather than sample path by sample path), whether

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the scaling is exact or approximate (i.e. up to higher order correction terms) and, finally, one might not require precise scalings, but only expect bounds which are scaling invariant. Let us mention a few definitions and examples in order to be more specific.

In the stochastic case, the Fractional Brownian Motion of order H $(0 < H < 1)$ satisfies the following exact scaling relation which holds in law at $x_0 = 0$:

$$\forall \lambda > 0, \quad B_H(\lambda t) \overset{\mathcal{L}}{=} \lambda^H B_H(t).$$

A stochastic process is Locally Asymptotically Selfsimilar of order $H \in (0,1)$ at x_0 if $\lambda^{-H}(X(t_0+\lambda t) - X(t_0))$ converges in law towards a non-degenerate process when $\lambda \to 0$ (see [4, 5]).

In the deterministic case, a simple example of scaling invariant function is supplied by the *devil's staircase*; in order to define it, we start by recalling the triadic Cantor set: Let $x \in [0,1]$; x can be written (in base 3) as $x = \sum_{j=1}^{\infty} a_j 3^{-j}$ with $a_j \in \{0,1,2\}$. The triadic Cantor set K is the set of x such that $a_j \in \{0,2\}$ for all j. The devil's staircase is defined as follows: If $x \notin K$, at least one of the a_j is equal to 1. Let $l = \inf\{j : \ a_j = 1\}$; then

$$\mathcal{D}(x) = \sum_{j=1}^{l-1} \frac{a_j}{2^{j+1}} + \frac{a_l}{2^l}.$$

The function $\mathcal{D}(x)$ is thus defined almost everywhere on $[0,1]$. It is then extended by continuity on $[0,1]$. An easy computation show that it satisfies the exact scaling relation $\mathcal{D}(\frac{x}{3}) = \frac{1}{2}\mathcal{D}(x)$. Finally, a function f is approximately selfsimilar of order H at x_0 if there exists a $\lambda < 1$ such that

$$f(x_0+t) - f(x_0) = \lambda^{-H}(f(x_0+\lambda t) - f(x_0)) + o(|t|)^H,$$

see [18] where a wavelet characterization of this property is given. A partial relationship between the deterministic setting and the stochastic one is given for Gaussian processes through their spectral density: If the spectral density is approximately selfsimilar, then the process is Locally Asymptotically Selfsimilar (see [12] for precise statements).

If the function is only required to satisfy upper bounds estimates which are scaling invariant in the neighbourhood of x_0, then the corresponding property is rather reffered to as a *pointwise regularity property.* We will investigate such properties in this paper. The one most widely used is the *Hölder* regularity, which is defined as follows.

Definition 1.1. — *Let $f : \mathbb{R}^d \to \mathbb{R}$, be a locally bounded function, $x_0 \in \mathbb{R}^d$ and $\alpha \geqslant 0$; $f \in C^\alpha(x_0)$ if there exist $R > 0$, $C > 0$, and a polynomial P of degree less than α such that*

$$\text{if } |x - x_0| \leqslant R, \quad \text{then } |f(x) - P(x - x_0)| \leqslant C|x - x_0|^\alpha. \tag{1.1}$$

There are two ways to interpret this definition. The classical one, which is induced by the notation $C^\alpha(x_0)$ itself, consists in interpreting this condition as the usual uniform homogeneous Hölder condition which would just hold at one point rather than uniformly; indeed, recall that, if $0 < s < 1$ then $f \in \dot{C}^\alpha(\mathbb{R}^d)$ if

$$\exists C > 0 \quad \text{such that } \forall x, y \in \mathbb{R}^d, \quad |f(x) - f(y)| \leqslant C|x - y|^\alpha.$$

However, though it is very natural, this point of view has two drawbacks; it does not extend to other settings (for instance, how could one define the Sobolev regularity at one point directly from either the double integral definition or the Fourier definition of Sobolev spaces?) and it does not allow to understand why some stability properties of the global space $\dot{C}^\alpha(\mathbb{R}^d)$ no more hold in the pointwise setting. This second drawback will be explained and dealt with in Section 2.

The other interpretation consists in noticing that (1.1) can actually be rewritten as follows. Let $B(x_0, r)$ denote the open ball centered at x_0 and of radius r; then (1.1) is clearly equivalent to the following condition: There exists a polynomial P and constants $C, R > 0$ such that

$$\forall r \leqslant R \quad \| (f - P) \|_{L^\infty(B(x_0,r))} \leqslant Cr^\alpha. \tag{1.2}$$

In other words, the $C^\alpha(x_0)$ condition describes how the L^∞ norm of f (properly "renormalized" by substracting a polynomial) behaves in small neighbourhoods of x_0. This point of view has two advantages: We will see that it explains why the $C^\alpha(x_0)$ does not have the stability properties of the space $\dot{C}^\alpha(\mathbb{R}^d)$ (it will just be a consequence of the fact that L^∞ does not possess these properties); furthermore, (1.2) can be immediately generalized by replacing the local L^∞ norm by another norm. For instance, using the L^p norm (for $1 \leqslant p < \infty$) one obtains the following definition introduced by Calderón and Zygmund in 1961, see [9].

Definition 1.2. — *Let $p \in [1, +\infty)$; a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ in L^p_{loc} belongs to $T^p_\alpha(x_0)$ if $\exists R, C > 0$ and a polynomial P of degree less than α such that*

$$\forall r \leqslant R, \quad \left(\frac{1}{r^d} \int |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leqslant Cr^\alpha, \tag{1.3}$$

Note that this condition is of the same kind as (1.2) since it can be rewritten

$$\forall r \leqslant R \quad \| (f-P) \|_{L^p(B(x_0,r))} \leqslant C r^{\alpha+d/p}. \tag{1.4}$$

The notion of pointwise regularity can be extended further: It is natural to replace in (1.2) the space L^∞ by an arbitrary function space E; in the following we will work in the setting of Banach or quasi-Banach spaces. Recall that a quasi-norm satisfies the requirements of a norm except for the triangular inequality which is replaced by the weaker condition

$$\exists C > 0, \ \ \forall x, y \in E, \ \ \| x + y \| \leqslant C(\| x \| + \| y \|).$$

A quasi-Banach space is a complete topological vector space endowed with a quasi-norm. The examples we have in mind are the real Hardy spaces H^p, and the Besov spaces $B_p^{s,p}$ with $0 < p < 1$; in these cases $\| x - y \|^p$ defines a distance on E. In the following, if E is a quasi-Banach space, then we will always assume that there exists a $p > 0$ such that this property holds; we will call a space with this property a *quasi-Banach space of type p*. The space E we will work with will be a space of distributions (perhaps defined modulo P_N, the vector space of polynomials of degree at most N) satisfying $\mathcal{S}_0 \hookrightarrow E \hookrightarrow \mathcal{S}_0'$ ($\mathcal{S}_0$ denotes the Schwartz class of C^∞ functions f such that f and all its partial derivatives have fast decay, and all the moments of f vanish).

If B is a ball of $\mathbb{R}^d$, then let

$$\| f \|_{E,B} = \inf_{f=g \ \ on \ \ B} \| g \|_E \ . \tag{1.5}$$

Definition 1.3. — *Let E be a space of distributions which is either a Banach space or a quasi-Banach space defined modulo P_N and satisfying $\mathcal{S}_0 \hookrightarrow E \hookrightarrow \mathcal{S}_0'$. The two-microlocal space of order α associated with E is the space $\mathcal{C}_E^\alpha(x_0)$ defined by*

$$\exists P \textit{ polynomial}, \ \ \exists R, C > 0, \ \forall r \leqslant R \qquad \| f - P \|_{E,B(x_0,r)} \leqslant C r^\alpha. \tag{1.6}$$

Remarks. — The two-microlocal space associated with L^∞ is precisely $C^\alpha(x_0)$; if $E = L^p$, then we obtain the space $T^p_{\alpha+d/p}(x_0)$; in these examples a maximal degree can be imposed on P which implies its uniqueness. We will see that it is also possible to impose a maximal degree on P if E satisfies homogeneity requirements (see Definition 3.6 where these requirements are listed and Theorem 4.1 for the corresponding result).

The way we introduce pointwise regularity in a general context differs from Y. Meyer's (Definition 1.1 of [31]). The arbitrary space E introduced in [31] corresponds to our space $\mathcal{C}^\alpha_E(x_0)$. Our motivation here is to emphasize the duality between the "global" space E (which will be assumed to be shift invariant, such as L^p for instance) and the corresponding pointwise regularity space $\mathcal{C}^\alpha_E(x_0)$, in order to show how properties of the second can be derived from properties of the first.

Pointwise regularity differs from the notion of *local* regularity at x_0 which, for Hölder spaces is defined as follows: f belongs to $C^s_{loc}(x_0)$ if there exists φ in $\mathcal{D}(\mathbb{R}^d)$ such that $\varphi(x_0) = 1$ and $f\phi \in C^s(\mathbb{R}^d)$. This notion can be extended to scales of spaces other than C^s. For instance, the case of the spaces $B^{s,p}_p$ is considered in [33, 36].

In order to explain where the terminology of two-microlocalization comes from, we first need to recall the definition of the homogeneous Hölder spaces $\dot{C}^s(\mathbb{R}^d)$.

If $0 < s < 1$ then $f \in \dot{C}^s(\mathbb{R}^d)$ if

$$\exists C > 0 \quad \text{such that } \forall x, y \in \mathbb{R}^d, \quad |f(x) - f(y)| \leqslant C|x - y|^s.$$

If $s = 1$, $\dot{C}^s(\mathbb{R}^d)$ is the Zygmund class of continuous functions satisfying

$$\exists C > 0 \quad \text{such that } \forall x, y \in \mathbb{R}^d, \quad |f(x + y) + f(x - y) - 2f(x)| \leqslant C|y|.$$

If $s > 1$, then $f \in \dot{C}^s(\mathbb{R}^d)$ if $\forall \alpha$ such that $|\alpha| = [s]$, $\partial^\alpha f \in \dot{C}^{s-[s]}(\mathbb{R}^d)$. Finally, if $s < 0$ then the spaces $\dot{C}^s(\mathbb{R}^d)$ are defined by recursion on $[s]$ by

$$f \in \dot{C}^s(\mathbb{R}^d) \quad \text{if} f = \partial_1 f_1 + \cdots + \partial_d f_d \text{ with } \quad f_j \in \dot{C}^{s+1}(\mathbb{R}^d).$$

The two-microlocal spaces $C^{s,s'}_{x_0}$ had been introduced by J.-M. Bony in order to study the propagation of singularities of the solutions of nonlinear evolution equations, see [6]. Yves Meyer showed that these spaces are exactly of the form defined above: If $s' > 0$ then a distribution f belongs to $C^{s,-s'}_{x_0}$ if and only if, using notation (1.5),

$$\exists R, C > 0, \ \forall r \leqslant R \qquad \| f \|_{C^{s-s'}, B(x_0,r)} \leqslant C r^{s'},$$

see [25, 30], and also [24, 32] where two-microlocal conditions are associated with Besov spaces. In the limit case $s' = s$, then the two-microlocal space associated with the space $B^{0,\infty}_\infty$ is the space $C^{s,-s}(x_0)$, see [16].

In Section 2, we will explore the different stability requirements which can naturally be imposed on a function space, and see which implications hold between them. In Section 3, we will investigate the properties of function spaces which satisfy these stability requirements, especially in terms of wavelet characterizations. In Section 4, we will draw the bridge between the properties of E and those of C^α_E. Finally, in Section 5, we will investigate implications of these results in multifractal analysis.

2. Stability conditions

The motivation of Calderón and Zygmund for introducing the $T^p_\alpha(x_0)$ spaces was to understand how pointwise regularity conditions are transformed in the resolution of elliptic PDEs; the standard way to prove such results is to write differential operators as the product of a fractional differentiation and a singular integral transform. Therefore, one has to use pointwise regularity criteria which are preserved under such singular integrals. Calderón and Zygmund introduced the $T^p_\alpha(x)$ spaces because the standard pseudodifferential operators of order 0 are not continuous on $C^\alpha(x_0)$, whereas they are continuous on $T^p_\alpha(x_0)$ if $1 < p < \infty$, see Theorem 6 of [9].

Let us recall how this deficiency of the $C^\alpha(x_0)$ condition can be shown. Consider the simplest possible singular integral operator namely, in dimension 1, the Hilbert transform; it is the convolution with the principal value of $1/x$, i.e. it is defined by

$$\mathcal{H}f(x) = \lim_{\varepsilon\to 0} \int_{I_\varepsilon(x)} \frac{f(y)}{x-y}dy,$$

where $I_\varepsilon(x) = \mathbb{R} - [x-\varepsilon, x+\varepsilon]$. An immediate computation shows that

$$\mathcal{H}(1_{[a,b]})(x) = \log\left|\frac{x-b}{x-a}\right|. \tag{2.1}$$

Let now x_n be a strictly decreasing sequence such that $\lim_{n\to\infty} x_n = 0$. We can pick a positive, strictly decreasing sequence a_n such that $f = \sum a_n 1_{[x_{n+1},x_n]}$ is arbitrarily smooth at x_0. Nonetheless, (2.1) implies that

$$\mathcal{H}f(x) = \sum_{n=1}^{\infty} a_n \log\left|\frac{x-x_n}{x-x_{n+1}}\right| = a_1 \log|x-x_0| + \sum_{n=1}^{\infty}(a_n - a_{n+1}) \log|x-x_n|,$$

which is not locally bounded near the origin, and therefore cannot have any Hölder regularity there. Note that what we really used here is the fact that the Hilbert transform is not continuous on L^∞; the “bad behavior” of the pointwise regularity criterium based on L^∞ follows from the corresponding "bad behavior" of L^∞. On the opposite side, we will see that the continuity of the Hilbert transform on L^p implies its continuity on the $T^p_u(x)$ spaces (as a consequence of Theorem 4.2). This feature also explains why it is better to interpret the $C^\alpha(x_0)$ condition as a local L^∞ condition rather than as a local $C^\alpha(\mathbb{R}^d)$ condition. The fact that properties of the "global space" L^∞ (resp. L^p) are transferred to the "pointwise space" $C^\alpha(x_0)$ (resp. $T^p_\alpha(x_0)$) is an important idea that we will develop (see Theorem 4.1 which shows that one can perform such transfers in a general setting).

Besides the study of PDEs, another motivation recently appeared in completely different areas and led to similar concerns. Many signals or images are now stored, denoised or transmitted via their wavelet coefficients, see [27]. Therefore, if one wants to obtain information on the pointwise regularity of signals, one needs to be able to characterize it in a robust way by conditions bearing on their wavelet coefficients. Recall that, in dimension 1, a wavelet basis is of the form $2^{j/2}\psi(2^jx-k),\quad j,k\in\mathbb{Z}$, where ψ has fast decay and belongs to C^r (one speaks of r-smooth wavelets), for an r picked large enough; the wavelet coefficients are

$$c_{j,k} = 2^j \int f(x)\psi(2^jx-k)dx.$$

What can be meant by a characterization "in a robust way"? It is natural to suppose that the criterium used is not too much perturbed if the size of each wavelet coefficient is slightly altered. The following definition encapsulates these features.

Definition 2.1. — *A norm (or a quasi-norm) on sequences $(c_n)_{n\in\mathbb{N}}$ is robust if it depends only on the moduli $|c_n|$ and if it is an increasing function in each variable $|c_n|$.*

Note that this definition implies the following (more classical) notion: A sequence of vectors $(x_n)_{n\in\mathbb{N}}$ is said to be *monotone* if

$$p \leqslant q \implies \left\|\sum_{n=0}^{p} c_n x_n\right\| \leqslant \left\|\sum_{n=0}^{q} c_n x_n\right\|,$$

see [3]; clearly, if a sequence norm is robust, then the canonical basis is monotone.

Another natural requirement is that the smoothness criterium used does not depend on the particular (smooth enough) wavelet basis which is picked.

This implies that the infinite matrices which map a wavelet basis on another one should act in a continuous way on the spaces of sequences thus defined. Since these infinite matrices are matrices of operators which are very closely related to the pseudo-differential operators of order 0 considered by Calderón and Zygmund, see Chapter 7 of [29], we are essentially led back to our previous requirement.

Let us now be more specific about these different stability requirements. We keep the discussion in dimension 1 for the sake of simplicity. We have met three different continuity requirements

- under the action of the Hilbert transform,
- under changes of wavelet bases,
- under the action of pseudodifferential operators of order 0.

How can such conditions be checked, and what is their hierarchy? It is clear that the first criterium is weaker than the third one. It is also weaker than the second one for the following reason: If the $2^{j/2}\psi(2^jx-k), \quad j,k \in \mathbb{Z}$ form an orthonormal basis of $L^2(\mathbb{R})$, and if $\tilde{\psi}$ denotes the Hilbert transform of ψ, then the $2^{j/2}\tilde{\psi}(2^jx-k), \quad j,k \in \mathbb{Z}$ also form an orthonormal basis of $L^2(\mathbb{R})$, simply because the Hilbert transform is an isometry on $L^2(\mathbb{R})$, and it commutes with translations and dilations (all these properties follow from the fact that, in the Fourier domain, the Hilbert transform is simply a multiplication by $\xi/|\xi|$, which is of modulus one and is homogeneous of degree 0). The second and the third conditions do not really compare, but are both implied by a fourth requirement (as a consequence of Theorem 2.3 below), which is simpler to check in practice, and which we now describe.

We will state precise definitions in the d-dimensional setting. Let $r \in \mathbb{N}$; an r-smooth wavelet basis of $\mathbb{R}^d$ is composed of 2^d-1 wavelets $\psi^{(i)}$ which belong to C^r and satisfy the following properties:

- $\forall i, \ \ \forall \alpha$ such that $|\alpha| \leqslant r, \quad \partial^\alpha \psi^{(i)}$ has fast decay,
- The set of functions $2^{dj/2}\psi^{(i)}(2^jx-k)$, $j\in\mathbb{Z}$, $k\in\mathbb{Z}^d$, $i \in \{1, ..., 2^d-1\}$ is an orthonormal basis of $L^2(\mathbb{R})^d$.

Thus any function f in $L^2(\mathbb{R}^d)$ can be written

$$f = \sum c_{j,k}^{(i)} \psi^{(i)}(2^jx-k) \tag{2.2}$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int f(x)\psi^{(i)}(2^j x - k)dx.$$

(Note that, in (2.2), wavelets are not normalized for the L^2 norm but for the L^∞ norm, which will simplify some notations.)

Wavelets will be indexed by dyadic cubes as follows: We can consider that i takes values among all dyadic subcubes λ_i of $[0,1)^d$ of width 1/2 except for $[0,1/2)^d$; thus, the set of indices (i,j,k) can be relabelled using dyadic cubes: λ denotes the cube $\{x : \ 2^j x - k \in \lambda_i\}$; we note $\psi_\lambda(x) = \psi^{(i)}(2^j x - k)$ (an L^∞ normalization is used), and $c_\lambda = 2^{dj} \int \psi_\lambda(x) f(x)dx$. We will use the notations $c_{j,k}^{(i)}$ or c_λ indifferently for wavelet coefficients. Note that the index λ indicates where the wavelet is localized; for instance, if the wavelets $\psi^{(i)}$ are compactly supported then $\exists C : supp(\psi_\lambda) \subset C\lambda$ where $C\lambda$ denotes the cube of same center as λ and C times larger; thus the indexation by the dyadic cubes is more than a simple notation: The wavelet ψ_λ is "essentially" localized around the cube λ.

The following classes of infinite matrices acting on sequences indexed by Λ were introduced by Y. Meyer.

Definition 2.2. — *Let $\gamma > 0$. An infinite matrix $M(\lambda,\lambda')$ indexed by the dyadic cubes belongs to $\mathcal{M}^\gamma$ if*

$$|M(\lambda,\lambda')| \leqslant \frac{C \; 2^{-(\frac{d}{2}+\gamma)|j-j'|}}{(1+(j-j')^2)(1+2^{\inf(j,j')} dist(\lambda,\lambda'))^{d+\gamma}}.$$

The following result was proved by Y. Meyer, see Chapter 8 of [29].

Theorem 2.3. — *If $\gamma > 0$, then $\mathcal{M}^\gamma$ is an algebra. Furthermore, if (ψ_λ) and $(\tilde{\psi}_\lambda)$ are two r-smooth wavelet bases, then the matrix $M(\lambda,\lambda') = \langle \psi_\lambda | \tilde{\psi}_{\lambda'} \rangle$ belongs to $\mathcal{M}^\gamma$ as soon as $\gamma < r$.*

We denote by $\mathcal{O}p(\mathcal{M}^\gamma)$ the space of operators whose matrix on a r-smooth wavelet basis (for a $r > \gamma$) belongs to $\mathcal{M}^\gamma$. This definition makes sense precisely because Theorem 2.3 implies that this notion does not depend on the (smooth enough) wavelet basis which is used. Pseudodifferential operators of order 0, such as the Hilbert transform in dimension 1, or the Riesz transforms in higher dimensions, belong to the algebras $\mathcal{O}p(\mathcal{M}^\gamma)$ for any γ, see Chap. 7 of [29]. In practice, in order to check that a criterium based on wavelet coefficients does not depend on the particular wavelet basis which is chosen, one checks the stronger requirement that it is invariant under the action of $\mathcal{M}^\gamma$ for a γ large enough.

Definition 2.4. — *Let E be a Banach space (or a quasi-Banach space); E is γ-stable if the operators of $\mathcal{O}p(\mathcal{M}^\gamma)$ are continuous on E.*

3. The space E

The first problem we will consider is to find natural conditions on the space E, which are not too restrictive, and imply that the pointwise regularity condition supplied by Definition 1.3 can be characterized by a robust condition on the wavelet coefficients (in the sense of Definition 2.1). We start with a few simple considerations concerning the relationships between a robust characterization and the existence of bases. Let us recall the two standard definitions of bases, depending whether E is separable or not.

Definition 3.1. — *Let E be a Banach, or a quasi-Banach space. A sequence e_n is a basis of E if the following condition holds: For any element f in E, there exists a unique sequence c_n such that the partial sums $\sum_{n\leqslant N} c_n e_n$ converge to f in E. It is an unconditional basis if furthermore*

$$\exists C>0,\ \ \forall \varepsilon_n \ \ \text{such that}\ |\varepsilon_n|\leqslant 1,\ \forall c_n,\ \| \sum c_n\varepsilon_n e_n \|_E \leqslant C \| \sum c_n e_n \|_E\ . \tag{3.1}$$

Remark. — The definition of a basis is usually given in the context of Banach spaces, see [3, 35, 38]. However it extends to the non-locally convex case of quasi-Banach spaces, which will be useful in the following.

If the space E is not separable then, of course, it cannot have a basis in the previous sense. In this case, the following weaker notion often applies.

Definition 3.2. — *Assume that E is the dual of a separable space F; then a sequence e_n is a weak* basis of E if, $\forall f \in E$, there exists a unique sequence c_n such that the partial sums $\sum_{n\leqslant N} c_n e_n$ converge to f in the weak* topology. It is unconditional if furthermore (3.1) holds.*

In all cases, we will always assume in the following that,

$$\text{if}\ \ f=\sum c_n e_n, \quad \text{then}\ \ c_n=\langle f|g_n\rangle \quad \text{with}\ \ g_n\in F, \tag{3.2}$$

where F is either the dual of E (in the basis setting of Definition 3.1) or a predual of E (in the weak* basis setting). The g_n are called the biorthogonal system of the e_n. Note that if E is a Banach space, if $F = E^*$ and if the e_n form a basis according to Definition 3.1, then (3.2) is automatically verified, see [3, 35]; it is also verified if the e_n are a wavelet basis, in which case $g_n = e_n$ for L^2 orthonomal wavelet bases, (or g_n is another wavelet basis in the wavelet biorthogonal case). Note that, for wavelets, the L^2 biorthogonal system is also the biorthogonal system for the (E, F) duality; indeed, by uniqueness if $\mathcal{S}_0$ is dense in either E or F, then the $(\mathcal{S}_0,\ \mathcal{S}'_0)$ duality, the $(L^2,\ L^2)$ duality and the $(E\ ,F)$ duality coincide for finite linear combinations of wavelets; therefore (3.2) holds for all functions of E by density, and the duality product $\langle f|g_n\rangle$ in (3.2) can be understood in any of the three settings.

Examples of non-separable spaces for which wavelets are weak* bases include the Hölder spaces $\dot{C}^s(\mathbb{R}^d)$, and, more generally, the Besov spaces $\dot{B}^{s,q}_p$ with $p = +\infty$ or $q = +\infty$. Properties of weak* bases have been studied by I. Singer and J. R. Retherford, see [35] and references therein. In the settings supplied by Definition 3.1 and Definition 3.2, if e_n is unconditional, then the norm (resp. quasi-norm) on E induces a norm (resp. quasi-norm) on a sequence space $S(E)$, which can be defined as follows.

Definition 3.3. — *Let E be a Banach space (resp. a quasi-Banach space) and let $(e_n)_{n\in\mathbb{N}}$ be an unconditional basis or an unconditional weak* basis of E. Then the sequence space $S(E)$ is the Banach space (resp. quasi-Banach space) of sequences endowed with the norm (resp. quasi-norm)*

$$\| (c_n)_{n\in\mathbb{N}} \|_{S(E)} = \sup_{|\varepsilon_n|\leqslant 1} \left\| \sum \varepsilon_n c_n e_n \right\|_E , \tag{3.3}$$

where the supremum is taken on all sequences (ε_n) satisfying $|\varepsilon_n| \leqslant 1$.

The sequence norm thus defined clearly is a robust norm and satisfies: There exist $C_1, C_2 > 0$ such that

$$\forall (c_n)_{n\in\mathbb{N}},\ \ C_1 \| (c_n)_{n\in\mathbb{N}} \|_{S(E)} \leqslant \left\| \sum c_n e_n \right\|_E \leqslant C_2 \| (c_n)_{n\in\mathbb{N}} \|_{S(E)} . \tag{3.4}$$

Note that the sequence space norm associated with a basis e_n is usually defined by

$$\| (c_n)_{n\in\mathbb{N}} \| = \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} c_n e_n \right\|_E ,$$

see [3, 35, 38]; in the unconditional case, the norm we defined is clearly equivalent to this one; we prefer it because it satisfies obviously the robustness property given by Definition 2.1 and, because of its symmetry, it does not require a particular ordering of the basis.

Consider the particular case of a wavelet basis. The family of matrices indexed by Λ, which are diagonals of ε_λ with $|\varepsilon_\lambda| \leqslant 1$, are obviously elements of $\mathcal{M}^\gamma$; thus, if E is γ-stable, then each of these matrices is bounded on E. The Banach-Steinhaus theorem implies that they form an equi-continuous family of bounded operators, which is precisely what (3.1) means. The fact that, for wavelet bases, γ-stability implies the unconditionality property (3.1) has a direct practical consequence in the definition of new function spaces through wavelet conditions. Initially, the fact that Sobolev spaces have a robust characterization played an important role in PDEs (see [11, 17] for instance). It was later the case for Besov spaces in statistics, see [13]. But afterwards, there came situations where traditional spaces which had been introduced before wavelet bases appeared (i.e. before the mid 80s) did not supply the right framework. New spaces, defined directly through their wavelet characterization, had to be introduced. Let us mention the *weak Besov spaces*, which come up naturally in sharp embeddings problems, see [10], or the *oscillation spaces*, which allow to determine the upper box dimension of the graph of a function, see [19, 20], give the correct "function-space formulation " of the multifractal formalism, see [21], and also found a recent use in statistics, see [2] (we will consider extensions of these spaces in Section 5); other examples will probably pop up in the short future. Of course, if one wants the definition of such a space to be consistent, it should not depend on the particular wavelet basis which is chosen. As we saw before, in practice, the only way to ensure this property is to check that the space is γ-stable, which implies (3.1); therefore one falls in one of the two situations described in Proposition 3.4 below. (This does not mean that wavelets are never conditional bases, as shown by $L^1(\mathbb{R})$ if one uses the functions $\varphi(x-k)$ and the wavelets for $j \geqslant 0$ as basis.)

Since wavelets have vanishing moments, it is natural to assume that the space E satisfies

$$\mathcal{S}_0 \hookrightarrow E \hookrightarrow \mathcal{S}_0'. \tag{3.5}$$

These embeddings also are the right requirement attached to the homogeneity hypothesis stated in Definition 3.5. The following proposition yields a simple criterium in order to check that a family of vectors is a basis.

Proposition 3.4. — *Let E be a Banach or a quasi-Banach space satisfying (3.5), and let e_n be a sequence in $\mathcal{S}_0$ satisfying (3.1) and such that $Span\{(e_n)_{n\in\mathbb{N}}\}$ is dense in $\mathcal{S}_0$.*

- *If $\mathcal{S}_0$ is dense in E, then (e_n) is an unconditional basis of E.*
- *If $\mathcal{S}_0$ is dense in F, and $F^* = E$, then (e_n) is an unconditional weak* basis of E.*

Note that, in the case of a wavelet basis in the Schwartz class, the density requirement in $\mathcal{S}_0$ is satisfied and the hypothesis (3.1) can be replaced by the stronger requirement that E is γ-stable. The first part of the proposition is standard, see Chap. 2 of [3] for instance, and we prove it only for the sake of completeness.

Proof. — If $f = \sum_{n=1}^{l} a_n e_n$, let $P_N(f) = \sum_{n=1}^{\inf(l,N)} a_n e_n$. Then, using (3.4) and the robustness of the norm in $S(E)$,

$$\| P_N(f) \|_E \leqslant C_2 \| (a_n)_{n \leqslant inf(l,N)} \|_{S(E)} \leqslant C_2 \| (a_n)_{n \leqslant l} \|_{S(E)} \leqslant \frac{C_2}{C_1} \| f \|_E ;$$

thus P_N extends into a linear continuous operator of norm less than C_2/C_1.

If $Span\{(e_n)_{n\in\mathbb{N}}\}$ is dense in $\mathcal{S}_0$, hence in E, then $\forall \varepsilon$, $\exists g = \sum_{n=1}^{K} a_n e_n$ such that $\| f - g \| \leqslant \varepsilon$. If $N > K$ then $P_N(g) = g$ so that

$$\| P_N f - f \| \leqslant \| P_N f - P_N g \| + \| P_N g - g \| + \| g - f \| \leqslant \left(1 + \frac{C_2}{C_1}\right) \varepsilon$$

for N large enough; hence the first point of Proposition 3.4.

In order to prove the second point, we have to check that

$$\forall g \in F, \ \ \langle g | P_N f \rangle \longrightarrow \langle g | f \rangle .$$

Since the P_N are uniformly bounded, it is enough to check it for g in a dense subspace of F; but it is obviously true if g is a finite linear combination of the g_n (the biorthogonal system of the e_n).

Let τ_a denote the shift operator $(\tau_a(f))(x) = f(x-a)$, and σ_λ the dilation operator $(\sigma_\lambda(f))(x) = f(\lambda x)$.

Definition 3.5. — *A Banach (or quasi-Banach) space of distributions (perhaps defined modulo polynomials up to degree N) is homogeneous of order H if it satisfies*

$$\exists C \ \forall a \in \mathbb{R}^d, \quad \| \tau_a(f) \| \leqslant C \| f \|, \tag{3.6}$$

and

$$\exists C \ \forall \lambda > 0, \quad \| \sigma_\lambda(f) \| \leqslant C\lambda^H \| f \| . \tag{3.7}$$

Examples of homogeneous spaces are supplied by the spaces L^p, where $H = -d/p$, the homogeneous Besov spaces $\dot{B}^{s,q}_p$, and the homogeneous Sobolev spaces $\dot{L}^{p,s}$ where $H = s - d/p$.

Requiring the space E to be homogeneous is a very natural requirement; indeed, the shift invariance implies that the definition of pointwise regularity is the same at every point, and the dilation invariance is an implicit requirement in the motivation we gave of pointwise regularity through scaling invariance. Furthermore, in practice, norms which are not homogeneous usually are the sum of several terms: A main part, which describes the "high frequency" behavior and is homogeneous, and a "low frequency" part which usually can be written under several alternative forms, and ensures that the space is not a quotient space; locally, the norm on E is equivalent to the homogeneous high frequency part, so that using the non-homogeneous norm would lead to the same definition of pointwise regularity. A typical example is supplied by the Sobolev spaces H^s for $s > 0$, whose norm is the sum of the "low frequency" L^2 norm and the "high frequency" part supplied by the homogeneous $\dot{H}^s$ semi-norm. We will come back to the problems which may appear when E is a quotient space at the end of this section.

The several requirements that were derived are now collected into the following definition.

Definition 3.6. — *A function space E is a gentle space of order H if*

- *E is a Banach or quasi-Banach space defined modulo polynomials of degree N,*
- $\mathcal{S}_0 \hookrightarrow E \hookrightarrow \mathcal{S}_0'$,
- *if E is separable, then $\mathcal{S}_0$ is dense in E, and if E is the dual of a separable space F, then $\mathcal{S}_0$ is dense in F,*
- *E is homogeneous of order H,*
- $\exists \gamma > 0$ *such that E is γ-stable.*

It follows from Proposition 3.4 that, if E is gentle, then wavelets are either unconditional bases or unconditional weak* bases of E, depending whether E is separable or not. Note that, a priori, wavelets are required to belong to $\mathcal{S}_0$, but the γ-stability implies that, once wavelets in $\mathcal{S}_0$ have been shown to be bases of E, then any r-smooth wavelet basis (for $r > \gamma$) is also a basis of E. In particular, we can use compactly supported wavelet bases, which will be useful in the following.

Recall that a function f belongs to the homogeneous Besov space $\dot{B}_p^{s,q}$ if

$$\sum_{j\in \boldsymbol{Z}} \left(\sum_{\lambda\in\Lambda_j} \left[2^{(s-d/p)j} |c_\lambda|\right]^p \right)^{q/p} \leqslant C,$$

where Λ_j denote the set of dyadic cubes of width 2^{-j}.

Proposition 3.7. — *If E is a gentle Banach space of order H, then*

$$\dot{B}_1^{H+d,1} \hookrightarrow E \hookrightarrow \dot{C}^H.$$

If E is a gentle quasi-Banach space of order H and of type p, then

$$\dot{B}_p^{H+d/p,p} \hookrightarrow E \hookrightarrow \dot{C}^H.$$

Proof. — Let $f = \sum c_\lambda \psi_\lambda$; in both settings, using the fact that the sequence space norm (or quasi-norm) is robust (so that it gets smaller if we set to 0 all wavelet coefficients except one) we get

$$\| f \|_E \geqslant \frac{C_1}{C_2} \sup_\lambda \| c_\lambda \psi_\lambda \|_E \geqslant C' \sup_\lambda (2^{Hj} |c_\lambda|),$$

which implies that $E \hookrightarrow \dot{C}^H$.

Conversely, if E is a Banach space, then

$$\| \sum c_\lambda \psi_\lambda \|_E \leqslant \sum_\lambda |c_\lambda| \, \| \psi_\lambda \|_E \leqslant C' \sum_\lambda 2^{Hj} |c_\lambda|;$$

it follows that, if $f \in \dot{B}_1^{H+d,1}$, then $f \in E$.

If E is a quasi-Banach space of type p, then

$$\| \sum c_\lambda \psi_\lambda \|_E^p \leqslant \sum_\lambda |c_\lambda|^p \, \| \psi_\lambda \|_E^p \leqslant C' \sum_\lambda 2^{Hpj} |c_\lambda|^p,$$

hence the last statement of Proposition 3.7 follows.

Remark. — If E is a Banach space and $H = -d$, then we obtain the embedding $\dot{B}_1^{0,1} \hookrightarrow E$, which is a consequence of the famous minimality property of the Bloch space $\dot{B}_1^{0,1}$ (see for example [28] and Chap. 6 of [29]); this minimality property states that any Banach space homogeneous of degree $H = -d$ and satisfying $\mathcal{S}_0 \hookrightarrow E \hookrightarrow \mathcal{S}_0'$ necessarily contains $\dot{B}_1^{0,1}$ (see also Chap. 3 of [15] and, for general considerations on minimal spaces, [14]).

Before studying the pointwise regularity spaces $C^\alpha_E(x_0)$, let us come back to the requirement that E is homogeneous, which may be felt as a problem since it often implies that E is a quotient space defined modulo polynomials. One possible way to turn this difficulty is to replace E by another space obtained through a *realization* of E at x_0. A realization of E is a continuous embedding σ: $E \to \mathcal{S}'$ such that $\forall u \in E$, the equivalence class of $\sigma(u)$ in E is u. This means that the "floating" polynomial in the definition of E is fixed in a way which is continuous and compatible with the vector space structure. Of course, one does not want to lose the scaling invariance in this procedure, so that one also requires that the norm on E restricted to its image by σ still satisfies $\parallel (f(\lambda\ \cdot\ ) \parallel = \lambda^H \parallel (f(\lambda\ \cdot\ ) \parallel$. On the other hand, we do not need to keep the translation invariance. In the Sobolev and Besov cases, if $s - d/p \in \mathbb{R}^+ - \mathbb{N}$, such a realization can be obtained by substracting the Taylor expansion of f of degree $[s - d/p]$ at x_0, see [7]. This provides a coherent definition of pointwise smoothness and allows to obtain the uniqueness of the polynomial of degree less than $\alpha + H$ in (1.6). It would be interesting to determine if, in the general setting supplied by gentle spaces of positive order H, realizations can always be obtained by substracting the Taylor expansion of f of degree $[H]$ at x_0. However, even if it were the case, we wouldn't follow this path because wavelets are usually not bases of realizations of homogeneous spaces (simply because they don't all vanish at x_0). Bases of these spaces can be obtained by some simple modifications of wavelet bases, see [31, 37]; however, such bases depend on the point x_0, which is an inacceptable drawback if one wants to analyze pointwise regularity simultaneously at several points, such as in Section 5.

Another possibility is to use instead of E another space $\tilde{E}$ which is a non-homogeneous version of E. Let us be more specific. Recall that there are two possible ways to write the wavelet expansion of a function of L^2, see [29]: Either

$$f(x) = \sum_{j \in \boldsymbol{Z}} \sum_{k \in \boldsymbol{Z}^d} \sum_i c^i_{j,k} \psi^{(i)}(2^j x - k),$$

or

$$f(x) = \sum_{k \in \boldsymbol{Z}^d} C_k \varphi(x-k) + \sum_{j=0}^{\infty} \sum_{k \in \boldsymbol{Z}^d} \sum_i c^i_{j,k} \psi^{(i)}(2^j x - k); \tag{3.8}$$

where φ is the function associated with the multiresolution analysis construction, see [27, 29] and

$$C_k = \int_{\mathbb{R}^d} f(x)\varphi(x-k)dx;$$

φ has the same smoothness and localization properties as the $\psi^{(i)}$, but $\int \varphi(x)dx = 1$. Following the examples supplied by Sobolev or Besov spaces, we assume that a non-homogeneous space $\tilde{E}$ is associated with E and is characterized by the condition

$$(C_k)_{k\in \boldsymbol{Z}^d} \in l^p, \quad \text{and} \quad \sum_{j\geqslant 0}\sum_{k\in \boldsymbol{Z}^d}\sum_i c^i_{j,k}\psi^{(i)} \in E,$$

for a $p>0$. One easily checks that the proof of Proposition 3.7 yields that, if $\tilde{E}$ is a gentle Banach space, then

$$B_1^{H+d,1} \hookrightarrow \tilde{E} \hookrightarrow C^H,$$

and, if E is a gentle quasi-Banach space of order H and of type p, then

$$B_p^{H+d/p,p} \hookrightarrow \tilde{E} \hookrightarrow C^H.$$

Since all results that will be obtained below deal with wavelet coefficients of f for $j \geqslant 0$, it follows that all results proved in the following sections are valid in this setting.

4. The $C^\alpha_E(x_0)$ spaces

If A is a subset of Λ, then, by definition

$$\| \{c_\lambda\}_{\lambda\in A} \|_{S(E)} \quad \text{denotes} \quad \| \{d_\lambda\}_{\lambda\in \Lambda} \|_{S(E)} \quad \text{where} \begin{cases} d_\lambda & = c_\lambda \quad \text{if } \lambda \in A \\ & = 0 \quad \text{else.}\end{cases}$$

If $x_0 \in \mathbb{R}^d$, then $\lambda_j(x_0)$ denotes the unique dyadic cube of width 2^{-j} which contains x_0 and

$$A_j(x_0) = \| \{c_\lambda\}_{\lambda\subset 3\lambda_j(x_0)} \|_{S(E)},$$

where $3\lambda_j(x_0)$ denotes the cube of same center as $\lambda_j(x_0)$ and three times wider.

Note that, if E is γ-stable, any wavelet basis which is r-smooth for an $r>\gamma$ can be used to characterize the norm in E, and we can use in particular a compactly supported wavelet basis.

Theorem 4.1. — *Let E be a gentle space of order H, let $f \in E$ and $\alpha > 0$. We assume that the wavelet basis is r-smooth with $r > |H|+2d+2\alpha$.*

If $f \in C^\alpha_E(x_0)$, then $\exists C \geqslant 0$ such that $\forall j \geqslant 0$,

$$A_j(x_0) \leqslant C 2^{-\alpha j}. \tag{4.1}$$

Conversely, if (4.1) holds and if $\alpha + H \notin \mathbb{N}$, then $f \in C^\alpha_E(x_0)$ and the polynomial in (1.6) can be chosen of degree less than $\alpha + H$.

Remarks. — Since E may be a space defined modulo polynomials, we cannot expect uniqueness of the polynomial P. However the degree $[\alpha + H]$ is optimal in the cases which have been worked out before (L^p or C^s).

Y. Meyer proved the characterization supplied by Theorem 4.1 if $E = \dot{C}^s(\mathbb{R}^d)$, see [25, 30], and if $E = L^2$ (personal communication); the cases where E is either L^p (for $1 < p < \infty$), the real Hardy spaces H^p or BMO are treated in [22].

Theorem 4.1 will be proved in two steps. Recall that, since E is γ-stable, we can use compactly supported wavelets as a basis of E. Thus, we will first show that Theorem 4.1 holds if the wavelets used are compactly supported, and afterwards, we will show that the elements of $\mathcal{M}^\gamma$ are continuous on the space defined by (4.1); using Theorem 2.3, this will imply that the characterization actually holds for any (smooth enough) wavelet basis.

If the wavelets are compactly supported, then the direct part of the theorem is straightforward: Let D be a large enough constant and g be a distribution which coincides with $f(x) - P(x - x_0)$ on $B(x_0, D2^{-j})$ and is such that the norm in $S(E)$ of its wavelet coefficients is bounded by $C2^{-\alpha j}$ (by hypothesis, such a g exists). Since this norm is robust, its restriction to the indices λ satisfying $\lambda \subset 3\lambda_j(x_0)$ is also bounded by $C2^{-\alpha j}$; but, if D is large enough and $\lambda \subset 3\lambda_j(x_0)$, then the corresponding wavelet coefficients of f and g coincide so that (4.1) holds.

Let us now prove the converse part, still in the case of compactly supported wavelets. We can forget the "low frequency component" of f corresponding to $j < 0$ in its wavelet decomposition for the following reason: It belongs locally to $C^r(\mathbb{R}^d)$ (for r-smooth wavelets); therefore it belongs to $C^\alpha_F(x_0)$ with $F = \dot{C}^{r-\alpha}(\mathbb{R}^d)$; but

$$\dot{C}^{r-\alpha}(\mathbb{R}^d) \hookrightarrow \dot{B}^{r-\alpha,1}_1 \hookrightarrow E \quad \text{if} \quad r - \alpha \geqslant H;$$

therefore f belongs to $C^\alpha_E(x_0)$ if $r - \alpha \geqslant H$.

Let $\Delta_j f = \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda$, and let $P_j(x - x_0)$ denote the Taylor polynomial of $\Delta_j f$ of degree $[\alpha + H]$ at x_0. If $\alpha + H < 0$, we pick $P_J = 0$ (so that, there are no more Taylor expansions in the following, in which case the reader should read the proof which follows using the convention $[\alpha + H] = -1$).

Proposition 3.7 and (4.1) imply that,

$$\text{if } dist(\lambda, x_0) \leqslant D2^{-j}, \text{ then } \quad |c_\lambda| \leqslant C2^{-(\alpha+H)j}. \tag{4.2}$$

Let $\rho > 0$ be fixed and let J be defined by $2^{-J} \leqslant \rho < 2 \cdot 2^{-J}$ and L be a constant which will be fixed later, but depends only on the size of the support of the wavelets. If $j \leqslant J+L$, then at most C of the $(\psi_\lambda)_{\lambda\in\Lambda_j}$ have a support intersecting $B = B(x_0,\rho)$ and each of them satisfies (4.2). It follows from Taylor's formula that, if $x \in B$ and $j \leqslant J+L$, then

$$|\Delta_j f(x) - P_j(x-x_0)| \leqslant C\rho^{[\alpha+H]+1} 2^{j([\alpha+H]+1-\alpha-H)};$$

each function $\Delta_j f(x) - P_j(x-x_0)$ can be extended outside B and written under the form

$$\rho^{[\alpha+H]+1} 2^{j([\alpha+H]+1-\alpha-H)} \tilde{\psi}_j(x) = \omega(\rho,j)\tilde{\psi}_j(x)$$

where $\tilde{\psi}_j(x)$ is a compactly supported "vaguelette" of scale ρ in the terminology introduced by Y. Meyer, i.e. has all its moments of order less than $r+2$ vanishing and is supported by $B(x_0, C\rho)$. Therefore

$$\begin{aligned}\left\| \sum_{j=0}^{J+L} \Delta_j f(x) - P_j(x-x_0) \right\|_{E,B(x_0,\rho)} &= \left\| \sum_{j=0}^{J+L} \omega(\rho,j)\tilde{\psi}_j(x) \right\|_{E,B(x_0,\rho)} \\ &\leqslant \sum \omega(\rho,j)\rho^{-H}\end{aligned}$$

(because the $\tilde{\psi}_j(x)$ are vaguelettes of scale ρ, and E is γ-stable). Therefore

$$\begin{aligned}\left\| \sum_{j=0}^{J+L} \Delta_j f(x) - P_j(x-x_0) \right\|_{E,B(x_0,\rho)} &\leqslant \sum_{j=0}^{J+L} \rho^{[\alpha+H]+1-H} 2^{j([\alpha+H]+1-\alpha-H)} \\ &\leqslant C\rho^{\alpha}. \end{aligned} \tag{4.3}$$

It follows also from (4.2) that,

$$\text{if} \quad |k| \leqslant [\alpha+H]+1, \quad \text{then} \quad \forall j \geqslant 0, \;\; |(\Delta_j f)^{(k)}(x_0)| \leqslant C.2^{(|k|-\alpha-H)j}; \tag{4.4}$$

therefore the series

$$P(x-x_0) := \sum_{j=0}^{\infty} P_j(x-x_0) = \sum_{j=0}^{\infty} \sum_{|k|<\alpha+H} \frac{(\Delta_j f)^{(k)}(x_0)}{k!}(x-x_0)^k$$

converges; (4.4) implies that

$$R_J(x-x_0) = \sum_{j=J+L}^{\infty} P_j(x-x_0)$$

is a polynomial of the form

$$\sum_{|k|<\alpha+H} \omega_k(J,x_0)(x-x_0)^k \quad \text{where } |\omega_k(J,x_0)| \leqslant C2^{(|k|-\alpha-H)j},$$

therefore

$$\forall \beta, \quad |\partial_\beta R_j(x-x_0)| \leqslant C\rho^{\alpha+H-|\beta|}$$

it follows that R_j can be extended outside $B(x_0,\rho)$ as a function of the form $\rho^{\alpha+H}\Psi_J(x)$ where Ψ_J is a vaguelette at scale 2^{-J} and supported by $B(x_0,D\rho)$. Using the same γ-stability argumant as above, it follows that $\| \Psi_J \|_E \leqslant C2^{-HJ}$.

Let now $g_J(x) = \sum_{j=J+L}^{\infty} \Delta_j f(x)$; g_J and

$$\sum_{j=J+L}^{\infty} \sum_{\lambda \subset B(x_0,2\rho)} c_\lambda \psi_\lambda$$

coincide on B if L has been picked large enough; furthermore, $\| g_J \|_E \leqslant A_{J+L}(x_0)$, which, by hypothesis, is bounded by $C2^{-\alpha(J+L)} \leqslant C\rho^\alpha$. Adding up the estimates we obtained, it follows that $\| f-P \|_{E,B(x_0,\rho)} \leqslant C\rho^\alpha$.

Remark. — Assume now that $\alpha + H \in \mathbb{N}$. We can come back to the previous proof and substract only Taylor's expansions of degree $\alpha+H-1$ (i.e. reproduce exactly the same proof with the convention $[\alpha+H] = \alpha+H-1$). Then all points of the proof run the same except for the derivation of (4.3); indeed, each term of the sum is now bounded by a constant. It follows that the bound obtained there is $C\rho^\alpha \log(1/\rho)$. Thus, if $\alpha + H \in \mathbb{N}$ and if (4.1) holds then f satisfies

$$\exists P \text{ polynomial}, \quad C>0,\ \forall r \leqslant 1/2 \qquad \| f-P \|_{E,B(x_0,r)} \leqslant Cr^\alpha \log(1/r).$$

In order to end the proof of the theorem, we still have to prove the following theorem, which has its own interest, as will be shown below.

THEOREM 4.2. — *Let E be a gentle space of order H; if $\gamma > |H| + 2d + 2\alpha$ and $\alpha > 0$ then $C^\alpha_E(x_0)$ is γ-stable.*

Proof of Theorem 4.2. — We have to prove that, if M is a matrix in $\mathcal{M}^\gamma$ and if a sequence $C = (c_\lambda)$ satisfies

$$\forall j \geqslant 0, \quad A_j(x_0) \leqslant C2^{-\alpha j}, \tag{4.5}$$

then MC satisfies the same estimate. If μ_j denotes the set of indices

$$\mu_j = 3\lambda_j(x_0) - 3\lambda_{j+1}(x_0),$$

then, the μ_j form a partition of Λ. Let

$$\begin{cases} d_\lambda^j & = c_\lambda \quad \text{if } \lambda \subset 3\lambda_j(x_0) \\ & = 0 \quad \text{else,} \end{cases} \quad \text{and} \quad \begin{cases} e_\lambda^j & = c_\lambda \quad \text{if } \lambda \in \mu_j \\ & = 0 \quad \text{else;} \end{cases}$$

we denote by capital letters the corresponding vectors

$$C = (c_\lambda)_{\lambda\in\Lambda}, \quad D_j = (d_\lambda^j)_{\lambda\in\Lambda}, \quad E_j = (e_\lambda^j)_{\lambda\in\Lambda}.$$

Then (4.5) can be rewritten $||D_j||_{S(E)} \leqslant C2^{-\alpha j}$.

Lemma 4.3. — *Let $(c_\lambda)_{\lambda\in\Lambda}$ be a vector; then, (4.5) is equivalent to*

$$\forall j \geqslant 0, \quad ||E_j||_{S(E)} \leqslant C2^{-\alpha j}. \tag{4.6}$$

Indeed, (4.6) is weaker than (4.5) since the sum in (4.6) bears on less terms. Assume now that (4.6) holds; note that $D_j = \sum_{j'\geqslant j} E_j$. If E is a Banach space, then

$$\| D_j \| \leqslant \sum_{j'\geqslant j} \| E_j \| \leqslant C \sum_{j'\geqslant j} 2^{-\alpha j'} \leqslant C2^{-\alpha j}.$$

If E is a quasi-Banach space of type p, then, one applies the same argument to $\| D_j \|^p$, which satisfies the triangular inequality.

Let us come back to the proof of Theorem 4.2. Let $M_{\lambda,\lambda'}$ be an infinite matrix in $\mathcal{M}^\gamma$. We note

$$\tilde{c}_\lambda = \sum_{\lambda'\in\Lambda} M_{\lambda,\lambda'} c_{\lambda'} = \sum_{j'} \tilde{c}_\lambda^{j'} \quad \text{where} \quad \tilde{c}_\lambda^{j'} = \sum_{\lambda'\in\mu_{j'}} M_{\lambda,\lambda'} c_{\lambda'}.$$

Because of Lemma 4.3, we only have to estimate the norm of $\tilde{C} = (\tilde{c}_\lambda)$ restricted to μ_j; for that, we will estimate the norm of each $\tilde{C}^{j',j} = (\tilde{c}_\lambda^{j'})_{\lambda\in\mu_j}$. Let j and j' be fixed; if $|j-j'| \leqslant 3$ then, by assumption,

$$\| (c_{\lambda'})_{\lambda'\in\mu_{j'}} \| \leqslant C2^{-\alpha j'}.$$

By continuity of the matrices of $\mathcal{M}^\gamma$ on $S(E)$, it follows that

$$\| \tilde{C}^{j',j} \| \leqslant C2^{-\alpha j'} \leqslant C2^{-\alpha j}.$$

In order to deal with the case $|j-j'| > 3$, we first prove the following lemma.

Lemma 4.4. — *Let $M_{\lambda,\lambda'}$ be an infinite matrix in $\mathcal{M}^\gamma$. If $|j-j'|>3$, $\lambda \in \mu_j$ and $\lambda' \in \mu_{j'}$ then*

$$M_{\lambda,\lambda'} = 2^{-\frac{\gamma}{4}|j-j'|} N_{\lambda,\lambda'},$$

where $N \in \mathcal{M}^{\gamma/2}$, and the norm of N in $\mathcal{M}^{\gamma/2}$ is bounded independently of j and j'.

Proof of Lemma 4.4. — Since the estimates required are symmetric, we can assume that $j \leqslant j'$ (and therefore that $j \leqslant j'-4$). Denote by 2^{-l} the width of λ and by $2^{-l'}$ the width of λ'. Since $\lambda \in \mu_j$ and $\lambda' \in \mu_{j'}$ it follows that $l \geqslant j-1$ and $l' \geqslant j'-1$.

Assume first that $l = j-1$ or $l = j$; then $l'-l \geqslant j'-j-2$ and therefore

$$2^{-(\frac{d}{2}+\gamma)|j-j'|} \leqslant 2^{-(\frac{d}{2}+\frac{\gamma}{2})|j-j'|} 2^{-(\frac{\gamma}{2}+\gamma)|l-l'|},$$

Hence Lemma 4.4 in this case.

If $l \geqslant j+1$, since $\lambda \notin 3\lambda_{j+1}(x_0)$, it follows that $dist(\lambda, x_0) \geqslant 2^{-j-1}$; since $\lambda' \in 3\lambda_{j+3}(x_0)$, it follows that $dist(\lambda,\lambda') \geqslant 2^{-j-2}$; but $\inf(l,l') - (j+2) \geqslant \frac{j'-j}{2} - 2$ (consider separately the cases $l \leqslant (j+j')/2$ and $l > (j+j')/2$). Therefore

$$2^{\inf(l,l')} dist(\lambda,\lambda') \geqslant 2^{-\frac{j'-j}{2}-2}$$

and Lemma 4.4 follows in this case.

We come back to the proof of Theorem 4.2. Let us show that $\| \tilde{C}^{j',j} \| \leqslant C2^{-\alpha j}$. Since $\| E_{j'} \| \leqslant C2^{-\alpha j'}$, by continuity of N on $S(E)$, it follows that $\| N(E_{j'}) \|_p \leqslant C2^{-\alpha j'}$. But if λ' belongs to $\mu_{j'}$ and λ belongs to μ_j, then Lemma 4.4 implies that

$$\| M\tilde{C}^{j',j} \| \leqslant C2^{-\frac{\gamma}{2}|j-j'|} 2^{-\alpha j'},$$

and therefore, in the Banach case,

$$\| (MC)_{\lambda \in \mu_j} \| \leqslant \sum_{j'} C2^{-\frac{\gamma}{2}|j-j'|} 2^{-\alpha j'},$$

which is bounded by $C2^{-\alpha j}$ as soon as $\gamma/2 > \alpha$. The proof in the quasi-Banach case is similar: One considers $\| \ . \ \|_{S(E)}^p$ to which the triangular inequality applies.

Note that wavelets are not bases of the space $C^\alpha_E(x_0)$ in the sense of Definition 3.1, even if E has an inconditional basis. The reason is that the characterization given by (4.1) is an l^∞-type condition in j and therefore $\mathcal{S}_0$ is not dense in E; a counterexample is given by the following function f constructed through its wavelet coefficients (we use a wavelet basis in $\mathcal{S}_0$): All wavelet coefficients of f vanish except one for each scale $j \geqslant 0$, the coefficient indexed by $\lambda_j(x_0)$, in which case

$$c_{\lambda_j(x_0)} = 2^{-(\alpha+H)j}.$$

The function f clearly satisfies (4.1) because (in the Banach case),

$$\left|\left|\{c_\lambda\}_{\lambda\subset 3\lambda_j(x_0)}\right|\right| \leqslant \sum_{j'=j}^{\infty} C 2^{-\alpha j^{-1}} \leqslant C 2^{-\alpha j}.$$

Let $g = \sum c_\lambda \psi_\lambda$ be a finite linear combination of wavelets; if j is picked large enough, then

$$||f-g||_{C^\alpha_E(x_0)} \leqslant 2^{\alpha j} \left|\left|\{c_\lambda\}_{\lambda\subset 3\lambda_j(x_0)}\right|\right| \leqslant 2^{\alpha j} \left|\left|c_{\lambda_j(x_0)}\right|\right| = 1,$$

because of the robustness of the sequence norm. The proof in the quasi-Banach case is similar.

It would be interesting to identify a predual of $C^\alpha_E(x_0)$, in order to determine if wavelets are weak* bases of $C^\alpha_E(x_0)$. This would probably involve the construction of pointwise regularity spaces associated with a negative α, which, in particular cases, has been performed by Y. Meyer in [31].

The following result states that local regularity is a stronger requirement that pointwise regularity.

Corollary 4.5. — *Let E be a gentle space and $f \in E$ be such that $(-\Delta)^\alpha f \in E$; then $\forall x_0 \in \mathbb{R}^d$, $f \in C^\alpha_E(x_0)$.*

Proof. — The matrix in a wavelet basis of the operator $(-\Delta)^\alpha$ can be written as the product of a diagonal matrix of $2^{-\alpha j}$ by a matrix in $\mathcal{M}^\gamma$, see Chap. 8 of [29]. Therefore, the condition $(-\Delta)^\alpha f \in E$ can be rewritten

$$(2^{\alpha j} c_\lambda)_{\lambda\in\Lambda} \in S(E).$$

But, since the norm of $S(E)$ is robust, this implies that

$$\forall j \geqslant 0, \quad \| (2^{\alpha j'} c_{\lambda'})_{\lambda'\subset 3\lambda_j(x_0)} \| \leqslant C,$$

so that, since $j' \geqslant j-1$,

$$\forall j \geqslant 0, \quad \| (2^{\alpha j} c_{\lambda'})_{\lambda' \subset 3\lambda_j(x_0)} \| \leqslant C,$$

which means that $f \in C^\alpha_E(x_0)$.

The following result is more surprising: In contradistinction with Hölder regularity, the fact that $f \in C^\alpha_E(x_0)$ uniformly in x_0 does not necessarily imply that $(-\Delta)^\alpha f \in E$, even locally, as shown by the following counterexample: We pick $E = \dot{B}^{1,1}_1(\mathbb{R})$, so that $\| f \|_E = \sum |c_\lambda|$. The function f is defined by its wavelet coefficients:

$$\begin{aligned} c_\lambda &= j^{-2} 2^{-j} && \text{if } \lambda \subset [0,1], \quad \text{and } \; j > 0 \\ &= 0 && \text{else.} \end{aligned}$$

Clearly,

$$\sum_{\lambda' \subset \lambda} |c_{\lambda'}| \leqslant j^{-1} 2^{-j}$$

so that $f \in C^1_E(x_0)$ uniformly in x_0, but $\forall \alpha > 0$, $(-\Delta)^\alpha f \notin \dot{B}^{1,1}_1(\mathbb{R})$.

5. Implications in multifractal analysis

Multifractal analysis in concerned with the determination of the dimensions of the sets of points where a function has a given pointwise regularity. The dimension mostly used in multifractal analysis is the *Hausdorff dimension.* Let us recall its definition. Let A be a subset of $\mathbb{R}^d$. For each $\varepsilon > 0$, let

$$M^d_\varepsilon = \inf_{\mathcal{R}} \sum_i \varepsilon_i^d,$$

where $\mathcal{R}$ denotes a generic covering of A by balls B_i of diameter $\varepsilon_i \leqslant \varepsilon$; then

$$dim(A) = \sup\{d : \lim_{\varepsilon \to 0} M^d_\varepsilon = +\infty\} = \inf\{d : \lim_{\varepsilon \to 0} M^d_\varepsilon = 0\}.$$

Definition 5.1. — *Let f be a distribution which belongs to E. The E-exponent of f at x_0 is $h^E_f(x_0) = \sup\{\alpha : f \in C^\alpha_E(x_0)\}$.*

The E-spectrum of singularities of f is

$$d^E_f(H) = dim(\{x : h^E_f(x) = H\}). \tag{5.1}$$

If $E = L^\infty$, then $d_f^E(H)$ is simply called the spectrum of singularities of f and is denoted by $d_f(H)$; If $E = L^p$, it is called the p-spectrum of singularities of f and is denoted by $d_f^p(H)$. Properties of the p-spectrum are investigated in [22, 24].

One cannot expect to compute the spectrum of singularities of an experimental signal by following the algorithm implicit in Definition 5.1 step by step. Indeed, the computation of a regularity exponent leads to numerical instabilities if it jumps from point to point; the determination of the level sets of a complicated function is also a problem; finally, computing one Hausdorff dimension involves considering all possible coverings of the corresponding set, which is not numerically feasible...and in the case of a multifractal function, we expect to deal with an infinite number of such sets! The purpose of the multifractal formalism is to derive the spectrum of singularities from quantities effectively computable on experimental signals. First, we will show that the multifractal formalism can be heuristically derived using a remarkable idea which G. Parisi and U. Frisch introduced in the setting of Hölder regularity, see [34]; it was later adapted using wavelets (see [1, 21, 26] and references therein), and we present it in the C_E^α regularity setting: We consider global quantities obtained by averaging the quantity

$$A_f(\lambda) = \| \{c_{\lambda'}\}_{\lambda' \subset \lambda} \|_{S(E)} \ .$$

In order to keep as much information as possible, one actually computes averages of $A_f(\lambda)^p$ for all (positive and negative) values of the parameter p; one obtains the *structure functions*

$$\Sigma_f^j(p) = 2^{-dj} \sum_{\lambda \in \Lambda_j} (A_f(\lambda))^p.$$

The behavior of these quantities when $j \to +\infty$ is described by the *scaling function* of f

$$\eta_f^E(p) = \liminf_{j \to +\infty} \frac{\log(\Sigma_f^j(p))}{\log(2^{-j})}.$$

Thus Σ_f^j is of the order of magnitude of $2^{-\eta_f^E(p)j}$ in the limit of small scales. The fundamental idea of the multifractal formalism is to estimate the contribution to $\Sigma_f^j(p)$ of the points x_0 where the E-exponent takes the value h. Indeed, if the cube λ contains such a point, then Theorem 4.1 asserts that $A_f(\lambda)$ is of the order of magnitude of 2^{-hj}. Coming back to the definition of the dimension, we need about $2^{d_f^E(h)j}$ such cubes to cover the set $\{x_0 : \ h_f^E(x_0) = h\}$ by cubes of size 2^{-j}; thus the contribution we look for is, for each value of j,

$$2^{-dj} 2^{d_f^E(h)j} 2^{-hpj} = 2^{-(d - d_f^E(h) + hp)j}.$$

When $j \to +\infty$, the contribution given by the smallest possible exponent $d - d_f^E(h) + hp$ becomes preponderent; thus, we expect that

$$\eta_f^E(p) = \inf_h (d - d_f^E(h) + hp).$$

If $d - d_f^E(h)$ is a convex function, then $-\eta_f^E(p)$ is the Legendre transform of $d - d_f^E(h)$ (in the sense of convex functions duality, see Chap. 1.3 of [8]). The inversion formula allows to recover $d_f^E(h)$:

$$d_f^E(h) = \inf_p (hp - \eta_f^E(p) + d). \tag{5.2}$$

When (5.2) holds, one says that f satisfies the multifractal formalism. The heuristic argument we described cannot be turned into a mathematical proof; the following result shows that, however, the right hand side of (5.2) always yields an upper bound for the spectrum.

THEOREM 5.2. — *Let E be a gentle space, let $f \in E$ and assume that the wavelet basis which is used belongs to the Schwartz class; then*

$$d_f^E(h) \leqslant \inf_{p \neq 0} (d - \eta_f^E(p) + hp). \tag{5.3}$$

When p is positive, the scaling function can be given a functional interpretation. For that, we introduce the *oscillation spaces* associated with E which are defined by the following condition on the wavelet expansion (using the expansion on the $\psi^{(i)}(2^j x - k)$ for $j \geqslant 0$ and the $\varphi(x-k)$ as in (3.8)).

DEFINITION 5.3. — *Let $s \in \mathbb{R}$ and $p > 0$; a distribution f belongs to $\mathcal{O}_E^{s,p}(\mathbb{R}^d)$ if it satisfies*

$$\sum_k |C_k|^p \leqslant C$$

and

$$\exists C > 0 \quad \forall j \geqslant 0 \quad 2^{(sp-d)j} \sum_k A_f(\lambda)^p \leqslant C. \tag{5.4}$$

Remarks. — Particular examples of oscillation spaces were already introduced, for $E = B_\infty^{0,\infty}$, see [20, 21], for $E = C^s$, see [20, 23] and for $E = L^p$, see [22].

It follows immediately from Definition 5.3 that

$$\eta_f^E(p) = \sup\{s : \ f \in \mathcal{O}_E^{s/p,p}\},$$

therefore, when $p > 0$, the function $\eta_f^E(p)$ indicates which spaces $\mathcal{O}_E^{s,p}$ contain the function f.

It would be interesting to determine if oscillation spaces are gentle.

Proof of Theorem 5.2. — First, we consider the case where p is positive. We will estimate the dimensions of the sets G_h, which are defined by

$$G_h = \{x_0 \in \mathbb{R}^d : \ \ f \notin C_E^h(x_0)\}. \tag{5.5}$$

PROPOSITION 5.4. — *Let $p > 0$; if $f \in \mathcal{O}_E^{s,p}$, then*

$$\forall h \geqslant s - \frac{d}{p}, \quad dim(G_h) \leqslant d + hp - sp. \tag{5.6}$$

Furthermore, if $s - \frac{d}{p} > 0$, then $G_h = \emptyset$ for any $h < s - \frac{d}{p}$.

Proof. — If $h < s - \frac{d}{p}$ and $f \in \mathcal{O}_E^{s,p}$ then, for any λ of width less than 1,

$$A_f(\lambda) \leqslant C 2^{-(s-\frac{d}{p})j}$$

so that $\forall x_0$, $h_f^E(x_0) \geqslant s - d/p$.

Let us now prove the first assertion of the proposition. Let

$$G_{j,h} = \left\{\lambda : \ |A_f(\lambda)| \geqslant 2^{-hj}\right\},$$

and denote by $N_{j,h}$ the cardinality of $G_{j,h}$. By hypothesis, $f \in \mathcal{O}_E^{s,p}$ so that

$$2^{(sp-d)j} \sum |A_f(\lambda)|^p \leqslant C;$$

therefore

$$2^{(sp-d)j} N_{j,h} 2^{-hpj} \leqslant C,$$

so that

$$N_{j,h} \leqslant C 2^{(d-sp+hp)j}.$$

Two dyadic cubes λ_1 and λ_2 are called adjacent if they are at the same scale and if $dist(\lambda_1, \lambda_2) = 0$. Denote by $F_{j,h}$ the set of cubes λ of scale j such that either $\lambda \in G_{j,h}$, or λ is adjacent to a cube of $G_{j,h}$. Clearly,

$$Card(F_{j,h}) \leqslant 3^d Card(G_{j,h}) \leqslant 3^d C 2^{(d-sp+hp)j}.$$

Denote by $F_h = \limsup_{j\to+\infty} F_{j,h}$ the set of points that belong to an infinite number of $F_{j,h}$. If $x_0 \notin F_h$, then there exists j_0 $(= j_0(x))$ such that, for any $j \geqslant j_0$, $A_j(x_0) \leqslant 2^{-hj}$; thus we can choose C ($= C(x)$) large enough so that

$$\forall j \geqslant 0, \quad A_j(x_0) \leqslant C2^{-hj}.$$

Theorem 4.1 implies that $f \in C^E_\alpha(x_0)$; so that $G_h \subset F_h$.

It remains to bound the dimension of F_h. Let $\varepsilon > 0$, and

$$j_0 = \inf\{j : \sqrt{d}2^{-j} \leqslant \varepsilon\}.$$

We choose for ε-covering of F_h all the cubes λ such that $j \geqslant j_0$ and $\lambda \in F_{j,h}$. Clearly,

$$\sum Diam(B_\lambda)^\delta \leqslant C \sum_{j=j_0}^{\infty} Card(F_{j,h}) \left(\sqrt{d}2^{-j}\right)^\delta \leqslant C \sum_{j=j_0}^{\infty} 2^{(d-sp+hp-\delta)j},$$

which is finite if $\delta > d+hp-sp$; hence the first part of the proposition holds.

Let us now check that the upper bound in Theorem 5.2 holds for $p > 0$. If $x_0 \in E_h$, then $h_f(x_0) = h$, and $\forall h' > h$, $x_0 \in G_{h'}$; so that $E_h \subset G_{h'}$. Let $p > 0$; by definition of $\eta^E_f(p)$ we have $\forall \varepsilon > 0$, $f \in \mathcal{O}_E^{(\eta^E_f(p)-\varepsilon)/p,p}$, so that

$$\forall h' > h,\ \forall \varepsilon > 0 \ \ d^E_f(h) = dim(E_h) \leqslant dim(G_{h'}) \leqslant d + h'p - \eta^E_f(p) + \varepsilon,$$

and thus

$$d^E_f(h) \leqslant d + hp - \eta^E_f(p).$$

Since this upper bound is valid for all $p > 0$, (5.3) follows (with the infimum taken on $\mathbb{R}^+$).

We consider now the case where p is negative. In this case, we will obtain a result which is stronger than Theorem 5.2 since it yields a bound for the *packing dimension* of the Hölder singularities. Let us recall its definition.

Definition 5.5. — *Let $A \subset \mathbb{R}^d$; if $\varepsilon > 0$, let $N_\varepsilon(A)$ be the smallest number of sets of radius ε required to cover A. The upper box dimension of A is*

$$\overline{dim}_B(A) = \limsup_{\varepsilon\to 0} \frac{\log N_\varepsilon(A)}{-\log \varepsilon}.$$

The packing dimension of A is

$$dim_p(A) = \inf \left\{ \sup_i \left(\overline{dim}_B A_i : A \subset \bigcup_{i=1}^{\infty} A_i \right) \right\}$$

(the infimum is taken over all possible partitions of A into a countable collection A_i).

The packing dimension of a set is always larger than its Hausdorff dimension. Denote by B_h the set of points x_0 such that $f \in C^h_E(x_0)$.

PROPOSITION 5.6. — *Let $p < 0$; if f satisfies*

$$\exists C > 0 \quad \forall j \geqslant 0 \quad 2^{(sp-d)j} \sum_k A_f(\lambda)^p \leqslant C, \tag{5.7}$$

then the packing dimension of B_h is bounded by $d - sp + hp$.

Proof. — Let $\delta > 0$ and J such that $2^{-J} \leqslant \delta$. If $f \in C^E_h(x_0)$, then there exists $A > 0$ such that

$$\forall j \geqslant J, \quad A_f(\lambda) \leqslant A 2^{-hj};$$

so that, since $p < 0$,

$$(A_f(\lambda))^p \geqslant A^p 2^{-hpj}. \tag{5.8}$$

Denote by Ω_A the set of points x where (5.8) holds for any $j \geqslant J$. Clearly,

$$B_h \subset \bigcup_{A>0} \Omega_A,$$

where the union can be written as a countable union. Since f satisfies (5.7), there are at most $CA^p 2^{(d-sp+hp+\varepsilon)j}$ cubes λ satisfying (5.8), so that the upper box dimension of each set Ω_A is bounded by $d-sp+hp$. The proposition follows by countable union, and the upper bound (5.3) for $p < 0$ also follows by the same argument as for $p > 0$. Therefore Theorem 5.2 is completely proved.

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