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Finite character, local stability property and local invertibility property

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Finite character, local stability property and local invertibility property

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Abstract

We illustrate some results relating the finite character property, the local stability property and the local invertibility property of a domain and give a partial answer to two open questions.

Introduction

In the following, R is a domain that is not a field. If I is a fractional ideal of R, we call I simply an *ideal* and if $I \subseteq R$ we say that I is an *integral ideal*.

The *ideal class semigroup* of a domain R, here denoted by $\mathcal{S}(R)$, consists of the isomorphism classes of the nonzero ideals of R. Clearly R is a Dedekind domain if and only if $\mathcal{S}(R)$ is a group. By a well known theorem of Clifford, a commutative semigroup S is a disjoint union of groups if and only if each element $x \in S$ is *(von Neuman) regular*, that is there exists an element $a \in S$ such that $x = x^2a$. Idempotent and invertible elements are regular. S is called a *Clifford semigroup* if its elements are all regular, and R is called a *Clifford regular domain* if $\mathcal{S}(R)$ is a Clifford semigroup. Note that the class of a nonzero ideal I is a regular element of $\mathcal{S}(R)$ if and only if I itself is regular in the semigroup of all nonzero ideals of R, that is $I = I^2J$ for some ideal J (in this case, we say that I is a *regular ideal*). It can be proved that if $I = I^2J$, then $IJ = I(I : I^2)$; it follows that the domain R is Clifford regular if and only if $I = I^2(I : I^2)$ for each nonzero (integral) ideal I [1, Lemma 1.1].

Dedekind domains are trivial examples of Clifford regular domains. Zanardo and Zannier proved that all orders in quadratic fields are Clifford regular domains [20] while Bazzoni and Salce showed that all valuation domains are Clifford regular [5]. The study of Clifford regular domains was then carried on by S. Bazzoni [1, 2, 3, 4].

A particular class of Clifford regular domains is given by stable domains. Recall that a domain is (*finitely*) *stable* if each nonzero (finitely generated) ideal I is *stable*, that is I is invertible in its endomorphism ring $E(I) := (I : I)$. Hence, if I is stable, $I = I^2(E(I) : I) = I^2(I : I^2)$. It follows that a stable ideal is regular. Stable domains have been thoroughly investigated by B. Olberding [13, 14, 15, 16].

Since a valuation domain is stable if and only if it is strongly discrete [13, Proposition 4.1], not all Clifford regular domains are stable. On the other hand, Clifford regular domains are finitely stable [3, Proposition 2.3], so that in the Noetherian case Clifford regularity coincides with stability [3, Theorem 3.1]. Conversely, a local finitely stable domain need not be Clifford regular, as shown in [3, Example 6.6].

Bazzoni characterized integrally closed Clifford regular domains as Prüfer domains with finite character (FC) [3, Theorem 4.5]. To this end, she established an interesting relation between Clifford regularity and the local invertibility property: a domain has the *local invertibility property* (LIP) if each ideal I that is locally invertible (i.e., IR_M is invertible, for each $M \in \mathrm{Max}(R)$) is indeed invertible. Bazzoni proved that

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a Clifford regular domain has the local invertibility property [3, Lemma 4.2] and conjectured that a Prüfer domain with the local invertibility property be of finite character [3, Question 6.2]. This conjecture was then proved in [11] and the local invertibility property and other related properties were later investigated by several authors [12, 18, 6, 7].

Another related concept is the local stability property: a domain has the *local stability property* (LSP) if each ideal I that is locally stable (i.e., IR_M is stable, for each $M \in \mathrm{Max}(R)$) is indeed stable. A Clifford regular domain has this property as well [3, Lemma 5.7].

Finite character implies the local stability property [13, Lemma 4.3] and the local stability property implies the local invertibility property [4, Lemma 4.3]. So that, in general

$$\text{FC} \Rightarrow \text{LSP} \Rightarrow \text{LIP}$$

As proved by Olberding, the arrows can be reversed if each nonzero prime ideal of R is contained in a unique maximal ideal (in particular, if R has dimension one) [13, Lemma 3.9].

Another result of Olberding says that a domain is stable if and only if it is locally stable and has finite character [16, Theorem 3.3]. An open question is to decide whether a similar result holds for Clifford regular domains [3, Question 6.8]. Note that a finitely stable domain need not have finite character, but nevertheless a domain is finitely stable if and only if it is locally finitely stable [4, Lemma 2.2(1)].

One direction is true; in fact Bazzoni proved that any localization of a Clifford regular domain is Clifford regular [3, Proposition 2.8] and that a finitely stable domain with the local stability property (in particular a Clifford regular domain) has finite character [4, Theorems 4.5 and 4.7]. It is not known whether the hypothesis of local stability property can be weakened to the local invertibility property [4, Question 4.6].

Summarizing:

$$\begin{array}{l}\text{Clifford regular} \Rightarrow \text{finitely stable + LSP} \Leftrightarrow \text{finitely stable + FC} \Rightarrow \text{finitely stable + LIP}\\ \Downarrow \\ \text{locally Clifford regular + FC}\end{array}$$

Since the integrally closed finitely stable domains are precisely the Prüfer domains, all these implications are equivalences in the integrally closed case ([3, Theorem 4.5] and [4, Proposition 3.3]) and, since a Clifford regular Noetherian domain is stable [3, Theorem 3.1], they are all equivalent to stability in the Noetherian case.

In this short note, we prove that a finitely stable domain has finite character if and only if each fractional overring has the local invertibility property (Theorem 1.2) and that the local invertibility property is equivalent to finite character if each maximal ideal of R is stable (Theorem 1.4).

We also show that, if each prime ideal of R is contained in a unique maximal ideal (in particular, if R has dimension one), then R is Clifford regular if and only if it is locally Clifford regular and has finite character (Theorem 1.5).

In my conference talk, based on [8, 9, 10], I showed that many properties of stability and Clifford regularity can be generalized and improved in the set of (semi)star operations. In particular, by using the techniques developed in [9], the results illustrated in the next section can be proved for star regularity with respect to any star operation that is spectral and of finite type (see [9] for the relevant definitions).

1. Finite character

We will use the following facts.

Theorem 1.1. (1) *(*[7, Proposition 1.6] *or* [6, Corollary 4]*) R has finite character if and only if, for each nonzero $x \in R$, any family of pairwise comaximal finitely generated ideals containing x is finite.*
(2) [18, Theorem 1] *If R has the local invertibility property and x is a nonzero element of R, then any family of pairwise comaximal invertible ideals containing x is finite.*

Theorem 1.2. *Let R be a finitely stable domain. The following conditions are equivalent:*
(i) *R has finite character;*
(ii) *R has the local stability property;*
(iii) *Each fractional overring of R has the local stability property;*

(iv) *Each fractional overring of R has the local invertibility property.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are always true by [13, Lemma 4.3] and [4, Lemma 4.3].

(ii) $\Rightarrow$ (iii) Let E be a fractional overring of R. Given a maximal ideal M of R, R_M is finitely stable and so E_M is semi-quasilocal; in fact any overring of a finitely stable domain is finitely stable and any overring of a local finitely stable domain is semi-quasilocal [16, Lemma 2.4 and Corollary 2.5]. It follows that if I is a locally stable ideal of E, then IE_M is stable for each $M \in \mathrm{Max}(R)$. Hence, by the local stability property of R, the ideal $I = IE$ is stable.

(iv) $\Rightarrow$ (i) Assume that R does not have finite character. Then, by Theorem 1.1(1), there exists a nonzero element $x \in R$ which is contained in infinitely many pairwise comaximal finitely generated ideals I_α. Set $E_\alpha := (I_\alpha : I_\alpha)$ and consider the R-module $E := \sum_\alpha E_\alpha$. Since E is contained in the integral closure of R and R is finitely stable, E is an overring of R [17, Proposition 2.1]. In addition, since $x \in I_\alpha$ for each α, $xE \subseteq \sum_\alpha I_\alpha \subseteq R$ and so E is a fractional overring of R.

We claim that $\{I_\alpha E\}$ is an infinite family of pairwise comaximal invertible ideals of E containing x. Since, by hypothesis, E has the local invertibility property, this contradicts Theorem 1.1(2).

First of all, since R is finitely stable, each ideal I_α is stable, so that $I_\alpha E$ is invertible in E. Then, since E is integral over R, each maximal ideal of R is contained in a maximal ideal of E. It follows that $I_\alpha E \neq E$ for each α. In addition, for $\alpha \neq \beta$ the ideals $I_\alpha E$ and $I_\beta E$ are comaximal, since the contraction of a prime ideal of E is a prime ideal of R. □

We observe that each fractional overring E of a domain R is the endomorphism ring of some nonzero ideal $I \subseteq R$. In fact, for some nonzero $d \in R$, $I := dE$ is an integral ideal of R and $E = (I : I)$.

Corollary 1.3. [4, Theorems 4.5 and 4.7] *A finitely stable domain with the local stability property (e.g., a Clifford regular domain) has finite character.*

We now prove that the local invertibility property is equivalent to finite character when each maximal ideal of R is stable. If each maximal ideal of R is invertible, this result follows directly from Theorem 1.1(2). We remark that a domain whose prime ideals are all stable is not necessarily stable, even if it is finitely stable [15, Section 3].

Theorem 1.4. *Assume that each maximal ideal of R is stable. The following conditions are equivalent:*

(i) *R has finite character;*
(ii) *R has the local stability property;*
(iii) *R has the local invertibility property.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are always true by [13, Lemma 4.3] and [4, Lemma 4.3].

(iii) $\Rightarrow$ (i) Let M be a maximal ideal of R. Since M is stable, $MR_M = x(MR_M : MR_M)$ for some $x \in M$ [16, Lemma 3.1]. Then $M^2R_M \subseteq xMR_M \subseteq xR_M$. It follows that the ideal $I := xR_M \cap R$, containing M^2, is not contained in any maximal ideal of R different from M. Since $IR_M = xR_M$, I is locally principal and so it is invertible by the local invertibility property.

Now, suppose that y is a nonzero element of R contained in infinitely many maximal ideals M_α. For each α, consider the ideal I_α constructed in the preceding paragraph. Then $y^2 \in M_\alpha^2 \subseteq I_\alpha$, for each α. Since $\{I_\alpha\}$ is an infinite family of pairwise comaximal invertible ideals, this contradicts Theorem 1.1(2). □

Recall that a domain is *h-local* provided it has finite character and each nonzero prime ideal is contained in a unique maximal ideal.

Theorem 1.5. [9, Proposition 5.7] *Assume that each nonzero prime ideal of R is contained in a unique maximal ideal (e.g., R has dimension one). The following conditions are equivalent:*

(i) *R is Clifford regular;*
(ii) *R_M is Clifford regular for each $M \in \mathrm{Max}(R)$, and R has finite character.*

Proof. (i) $\Rightarrow$ (ii) is always true. In fact, when R is Clifford regular, R_M is Clifford regular for each $M \in \mathrm{Max}(R)$ by [3, Proposition 2.8], and R has finite character by Theorem 1.2.

(ii) ⇒ (i) By definition, R is h-local. Hence $(I : I^2)_M = (I_M : I^2_M)$, for each nonzero ideal I and maximal ideal M [13, Lemma 3.8]. Then, since R is locally Clifford regular,

$$I^2(I : I^2) = \cap_{M\in\mathrm{Max}(R)} I^2(I : I^2)_M = \cap_{M\in\mathrm{Max}(R)} I^2_M(I_M : I^2_M) = \cap_{M\in\mathrm{Max}(R)} I_M = I.$$

We conclude that each nonzero ideal I of R is regular. □

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