CIRM

Troisième Rencontre Internationale sur les Polynômes à Valeurs Entières

Rencontre organisée par :
Sabine Evrard

29 novembre-3 décembre 2010

Carmelo Antonio Finocchiaro and Marco Fontana

Some applications of the ultrafilter topology on spaces of valuation domains, Part I

Vol. 2, n° 2 (2010), p. 107-109.

<http://acirm.cedram.org/item?id=ACIRM_2010__2_2_107_0>

Centre international de rencontres mathématiques
U.M.S. 822 C.N.R.S./S.M.F.
Luminy (Marseille) France

Actes des rencontres du C.I.R.M.
Vol. **2** n^o 2 (2010) 107-109

Some applications of the ultrafilter topology on spaces of valuation domains, Part I

Carmelo Antonio Finocchiaro and Marco Fontana

Abstract

In the present note we introduce the basic definitions and the main results concerning the spaces of valuation domains needed in the subsequent Part II.

General Introduction and Motivations

Let A be an integral domain, let K be its quotient field (sometimes denoted qf(A)), and let Y be a nonempty collection of valuation overrings of A. We say that Y is *a representation* of A if $\bigcap Y := \bigcap\{V \mid V \in Y\} = A$. A representation Y of A is said *irredundant* if, for every $W \in Y$, the ring $\bigcap\{V \in Y \mid V \neq W\}$ is a proper overring of A. It is well known that an integral domain admits a representation if and only if it is integrally closed (W. Krull's Theorem, 1931). For example, a Krull domain always admits an irredundant representation, given by its defining family. If A is a Prüfer domain having an irredundant representation, then it is unique and it is given exactly by $\{A_{\mathfrak{m}} \mid \mathfrak{m} \in \mathcal{T}_A\}$, where $\mathcal{T}_A$ denotes the set of all the maximal ideals $\mathfrak{m}$ of A with the property that there exists a finitely generated ideal $\mathfrak{a}$, such that $\mathfrak{m}$ is the unique maximal ideal containing $\mathfrak{a}$ (see [6]).

Given a nonempty collection of valuation overrings Y of A, we can define an `e.a.b.` semistar operation $\wedge_Y$ on A, by setting $E^{\wedge_Y} := \bigcap\{EV \mid V \in Y\}$, for every $A-$submodule E of K (see [5, Section 32] for more details on `e.a.b.`operations). We call *valuative semistar operation* a semistar operation of the type $\wedge_Y$, for some nonempty set Y of valuation overrings of A. The following questions, related to representations of integral domains and valuative semistar operations, are the starting point of a joint work with K. A. Loper.

Questions. Let A be an integral domain and Y, Y' be two collections of valuation overrings of A.

(1) Is it possible to give a topological interpretation of when $\bigcap\{V \mid V \in Y\}$ coincides with $\bigcap\{V \mid V \in Y'\}$?
(2) Is it possible to provide topological characterizations of when the two valuative semistar operations $\wedge_Y$ and $\wedge_{Y'}$ coincide?

1. Spaces of valuation domains

The motivations for studying from a topological point of view spaces of valuation domains come from various directions and, historically, mainly from Zariski's work on the reduction of singularities of an algebraic surface and, more generally, for establishing new fundations of algebraic geometry by algebraic means (see [16] and [17]). Further motivations come from rigid algebraic geometry started by J. Tate [15], and from real algebraic geometry (see for instance [13] and [9]). For a deeper insight on this topics see [10]. In the following, we want to extend results in the literature

Text presented during the meeting "Third International Meeting on Integer-Valued Polynomials" organized by Sabine Evrard. 29 novembre-3 décembre 2010, C.I.R.M. (Luminy).
Key words. Valuation domain, (semi)star operation, prime spectrum, Zariski topology, constructible topology, filter and ultrafilter, Prüfer domain.
This talk was presented at the meeting by the second named author.

concerning topologies on collections of valuation domains. Let K be a field and A be an *arbitrary* subring of K. Set

$$\mathrm{Zar}(K|A) := \{V \mid V \text{ is a valuation domain and } A \subseteq V \subseteq K = \mathrm{qf}(V)\}$$

When A is the prime subring of K, we will simply denote by $\mathrm{Zar}(K)$ the space $\mathrm{Zar}(K|A)$. Recall that O. Zariski in [16] introduced a topological structure on the set $Z := \mathrm{Zar}(K|A)$ by taking, as a basis for the open sets, the subsets $B_F := \{V \in Z \mid V \supseteq F\}$, for F varying in the family of all finite subsets of K. When no confusion can arise, we will simply denote by B_x the basic open set $B_{\{x\}}$ of Z. This topology is now called *the Zariski topology on* Z and the set Z, equipped with this topology, denoted also by Z^{zar}, is usually called *the (abstract) Zariski-Riemann surface of* K *over* A. Zariski proved the quasi-compactness of Z^{zar} and later it was proven and rediscovered by several authors, with a variety of different techniques, that if A is an integral domain and K is the quotient field of A, then Z^{zar} is a spectral space, in the sense of M. Hochster [8].

2. The constructible topology

Let A be a ring and let $X := \mathrm{Spec}(A)$ denote the collection of all prime ideals of A. The set X can be endowed with the *Zariski topology* which has several attractive properties related to the "geometric aspects" of the set of prime ideals. As it is well known, X^{zar} (that is, the set X with the Zariski topology), is always quasi-compact, but almost never Hausdorff (more precisely, X^{zar} is Hausdorff if and only if $\dim(A) = 0$). Thus, many authors have considered a finer topology, known as *the constructible topology* (see [2], [7]) or as *the patch topology* [8].

In order to introduce such a topology in a more general setting, with a simple set theoretical approach, we need some notation and terminology. Let $\mathcal{X}$ be a topological space. Following [14], we set

$$\mathring{\mathcal{K}}(\mathcal{X}) := \{U \mid U \subseteq \mathcal{X},\, U \text{ open and quasi-compact in } \mathcal{X}\},$$
$$\mathcal{K}(\mathcal{X}) := \text{ the Boolean algebra of the subsets of } \mathcal{X} \text{ generated by } \mathring{\mathcal{K}}(\mathcal{X}).$$

As in [14], we define *the constructible topology on* $\mathcal{X}$ the topology on $\mathcal{X}$ whose basis of open sets is $\mathcal{K}(\mathcal{X})$. We denote by $\mathcal{X}^{\mathrm{cons}}$ the set $\mathcal{X}$, equipped with the constructible topology. Note that, for Noetherian topological spaces, this definition of constructible topology coincides with the classical one given in [2]. When $X := \mathrm{Spec}(A)$, for some ring A, then the set $\mathring{\mathcal{K}}(X^{\mathrm{zar}})$ is a basis of open sets for X^{zar}, and thus the constructible topology on X is finer than the Zariski topology. Moreover, X^{cons} is a compact Hausdorff space and the constructible topology on X is the coarsest topology for which $\mathring{\mathcal{K}}(X^{\mathrm{zar}})$ is a collection of clopen sets (see [7]).

3. The ultrafilter topology

A couple of years ago, the authors in [4] considered "another" natural topology on $X := \mathrm{Spec}(A)$ (see the following Theorem 3.3), by using the notion of an ultrafilter and the following lemma.

Lemma 3.1. (Cahen-Loper-Tartarone, [1]) *Let Y be a subset of $X := \mathrm{Spec}(A)$ and let $\mathscr{U}$ be an ultrafilter on Y. Then $\mathfrak{p}_{\mathscr{U}} := \{f \in A \mid V(f) \cap Y \in \mathscr{U}\}$ is a prime ideal of A called* the ultrafilter limit point of Y, with respect to $\mathscr{U}$.

The notion of ultrafilter limit points of sets of prime ideals has been used to great effect in several recent papers [1], [11], [12]. If $\mathscr{U}$ is a trivial ultrafilter on the subset Y of X, that is, $\mathscr{U} = \{S \subseteq Y \mid \mathfrak{p} \in S\}$, for some $\mathfrak{p} \in Y$, then $\mathfrak{p}_{\mathscr{U}} = \mathfrak{p}$. On the other hand, when $\mathscr{U}$ is a nontrivial ultrafilter on Y, then it may happen that $\mathfrak{p}_{\mathscr{U}}$ does not belong to Y. This fact motivates the following definition.

Definition 3.2. *Let A be a ring. A subset Y of $X := \mathrm{Spec}(A)$ is said* ultrafilter closed *if $\mathfrak{p}_{\mathscr{U}} \in Y$, for every ultrafilter $\mathscr{U}$ on Y.*

It is not hard to see that the ultrafilter closed sets of X are the closed sets for a topology on X. We call it *the ultrafilter topology on* X and we denote by X^{ultra} the set X endowed with the ultrafilter topology. The main result of [4] is the following.

Theorem 3.3. *Let A be a ring. The ultrafilter topology and the constructible topology coincide on* $\mathrm{Spec}(A)$.

Taking as starting point the situation described above for the prime spectrum of a ring, our next goal is to define an ultrafilter topology on the set $Z := \mathrm{Zar}(K|A)$ (where K is a field and A is a subring of K) that is finer than the Zariski topology. We start by recalling the following useful fact.

Lemma 3.4. (Cahen-Loper-Tartarone, [1]) *Let K be a field and A be a subring of K. If Y is a nonempty subset of $Z := \mathrm{Zar}(K|A)$ and $\mathscr{U}$ is an ultrafilter on Y, then*

$$A_{\mathscr{U},Y} := A_{\mathscr{U}} := \{x \in K \mid B_x \cap Y \in \mathscr{U}\}$$

is a valuation domain of K containing A as a subring (that is, $A_{\mathscr{U}} \in Z$), called the ultrafilter limit point of Y in Z , with respect to $\mathscr{U}$.

As before, when $V \in Y$ and $\mathscr{U} := \{S \subseteq Y \mid V \in S\}$ is the trivial ultrafilter of Y generated by V, then $A_{\mathscr{U}} = V$. But, in general, it is possible to construct nontrivial ultrafilters on Y whose ultrafilter limit point are not elements of Y. This leads to:

Definition 3.5. *Let K be a field and A be a subring of K. A subset Y of $Z := \mathrm{Zar}(K|A)$ is said to be* ultrafilter closed *if $A_{\mathscr{U}} \in Y$, for every ultrafilter $\mathscr{U}$ on Y.*

We observe that the ultrafilter closed sets of Z are the closed sets for a topology. We call it *the ultrafilter topology on Z* and we denote by $Z^{\mathtt{ultra}}$ the set Z equipped with the ultrafilter topology. With this notation we have:

Theorem 3.6. *Let K be a field, A be a subring of K, and $Z := \mathrm{Zar}(K|A)$. The following statements hold.*

(1) *The space Z^{ultra} is a compact Hausdorff topological space.*
(2) *The ultrafilter topology is the coarsest topology for which the basic open sets B_F of the Zariski topology of Z are clopen. In particular, the ultrafilter topology on Z is finer than the Zariski topology and coincides with the constructible topology.*
(3) *The surjective map $\gamma : \mathrm{Zar}(K|A)^{ultra} \to \mathrm{Spec}(A)^{ultra}$, mapping a valuation domain to its center on A, is continuous and closed.*
(4) *If A is a Prüfer domain, the map $\gamma : \mathrm{Zar}(K|A)^{ultra} \to \mathrm{Spec}(A)^{ultra}$ is a homeomorphism.*

References

[1] Paul-Jean Cahen, Alan Loper, and Francesca Tartarone, Integer-valued polynomials and Prüfer $v-$multiplication domains, *J. Algebra* **226** (2000), 765–787.

[2] Claude Chevalley et Henri Cartan, Schémas normaux; morphismes; ensembles constructibles, *Séminaire Henri Cartan* **8** (1955-1956), Exp. No. 7, 1–10.

[3] M. Fontana, K. A. Loper, Cancellation properties in ideal systems: a classification of `e.a.b.` semistar operations, *J. Pure Appl. Algebra* **213** (2009), no. 11, 2095–2103.

[4] Marco Fontana and K. Alan Loper, The patch topology and the ultrafilter topology on the prime spectrum of a commutative ring, *Comm. Algebra* **36** (2008), 2917–2922.

[5] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, New York 1972.

[6] R. Gilmer and W. Heinzer, Irredundant intersections of valuation rings, *Math. Z.* **103** (1968), 306–317.

[7] Alexander Grothendieck et Jean Dieudonné, *Éléments de Géométrie Algébrique I*, IHES 1960; Springer, Berlin, 1970.

[8] Melvin Hochster, Prime ideal structure in commutative rings, *Trans. Amer. Math. Soc.* **142** (1969), 43–60.

[9] Roland Huber, *Bewertungsspektrum und rigide Geometrie, Regensburger Mathematische Schriften*, vol. 23, Universität Regensburg, Fachbereich Mathematik, Regensburg, 1993.

[10] Roland Huber and Manfred Knebusch, On valuation spectra, in *"Recent advances in real algebraic geometry and quadratic forms: proceedings of the RAGSQUAD year"*, Berkeley, 1990-1991, *Contemp. Math.* **155**, Amer. Math. Soc., Providence, RI, 1994.

[11] K. Alan Loper, Sequence domains and integer-valued polynomials, *J. Pure Appl. Algebra* **119** (1997), 185–210.

[12] K. Alan Loper, A classification of all D such that $\mathrm{Int}(D)$ is a Prüfer domain, *Proc. Amer. Math. Soc.* **126** (1998), 657–660.

[13] Niels Schwartz, Compactification of varieties, *Ark. Mat.* **28** (1990), 333–370.

[14] Niels Schwartz and Marcus Tressl, Elementary properties of minimal and maximal points in Zariski spectra, *J. Algebra* **323** (2010), 698–728.

[15] John Tate, Rigid analytic spaces, *Invent. Math.* **12** (1971), 257–269.

[16] O. Zariski, The compactness of the Riemann manifold of an abstract field of algebraic functions, *Bull. Amer. Math. Soc* **50** (1944), 683–691.

[17] O. Zariski, P. Samuel, *Commutative Algebra, Volume 2*, Springer Verlag, Graduate Texts in Mathematics **29**, New York, 1975 (First Edition, Van Nostrand, Princeton, 1960).