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Towards a more precise understanding of sets of lengths

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Towards a more precise understanding of sets of lengths

Wolfgang A. Schmid

Abstract

This short survey, based on the content of a talk with the same title, summarizes some classical and recent results on the set of differences of an abelian group. We put a certain emphasize on ongoing joint work of A. Plagne and the author. We also briefly review the relevance of this notion in Non-unique Factorization Theory, in particular towards the subject mentioned in the title.

1. Introduction

We present results on the sets $\Delta^*(G)$ were G is a (finite) abelian group, called the set of differences. The aim is to give a short and non-exhaustive overview on the current state of knowledge, including results by S. Chang, S. T. Chapman, W. D. Gao, A. Geroldinger, Y. ould Hamidoune, and W. W. Smith with a certain emphasize on ongoing joint work of A. Plagne and the author. We point out that this is not a historical survey, and we occasionally reference later expository works instead of the original sources and more generally sacrifice historical preciseness for brevity. For undefined or vaguely defined notions we refer to the monograph [8] by A. Geroldinger and F. Halter-Koch.

Before recalling the definition of $\Delta^*(G)$ and relevant notation, we briefly recall the relevance of this set in Non-unique Factorization Theory. Our entire discussion is given for Krull monoids only; yet, leaving trivial technical issues aside, it applies to Dedekind and Krull domains.

There are two main motivations both about two decades old. On the one hand, A. Geroldinger [7] proved the Structure Theorem for Sets of Lengths (we state a mildly non-standard version, a weak form of a refined form due to G. Freiman and A. Geroldinger [5]; also see [8, Chapter 4]).

Theorem 1.1. *Let H be a Krull monoid with finite class group. There exists a constant $M \in \mathbb{N}_0$ and a finite subset $\Delta^* \subset \mathbb{N}$ such that the following holds true. For $a \in H$ let $\mathsf{L}(a) = \{\ell \colon a = u_1 \dots u_\ell,\ u_i \text{ irreducible}\}$ denote the set of lengths of a. There exists a difference $d \in \Delta^*$, a period $\{0, d\} \subset \mathcal{D} \subset \{0, \dots, d\}$, and $y, \ell \in \mathbb{N}_0$, such that*

$$y + \mathcal{D} + \{0, d, \dots, \ell d\} \subset \mathsf{L}(a) \subset y + \mathcal{D} + \{-Md, \dots, (\ell + M)d\}.$$

One also expresses this by saying that there exists a bound M and a finite set of differences Δ^* such that each set of lengths is an almost arithmetical multiprogression with bound M and some difference $d \in \Delta^*$. It is crucial that the bound and the set of differences just depends on the monoid H as opposed to the element a (if they were allowed to depend on a, the assertion is trivial). Now, sets of lengths of a Krull monoid H depend just on the class group G of H and the subset of classes containing prime divisors. And, it is known that the natural choice that just depends on the class group for the set Δ^* is the set $\Delta^*(G)$ that we discuss.

On the other hand, S.T. Chapman and W.W. Smith [3] introduced the notion of congruence half-factorial monoids of order d, where d is a positive integer, and studied this property for Krull monoids. Congruence half-factorial of order d means that for each element a all elements of $\mathsf{L}(a)$

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are congruent modulo d; recall that half-factorial means that $\mathsf{L}(a)$ is a singleton for each a, in other words all elements of $\mathsf{L}(a)$ are congruent modulo 0. In this context, the set $\Delta^*(G)$ determines the set of those d for which there exists a congruence half-factorial Krull monoid with class group G; note that $\Delta^*(G)$ determines this set, and is the natural quantity to study, yet is in general not equal to this set (though to a natural modification of it).

2. Key definition and related notions

It is well-known that many questions on factorizations in Krull monoids, in particular all questions related to lengths of factorizations, can be studied in the associated block monoids, i.e., a monoid of zero-sum sequences over a subset of an abelian group.

Let $(G,+)$ be an abelian group and $G_0 \subset G$ a subset. Let $\mathcal{F}(G_0)$ denote the free abelian monoid over G_0. An element $S = \prod_{g\in G_0} g^{v_g}$, all but finitely many v_g equal to 0, in $\mathcal{F}(G_0)$ is called a sequence over G_0. And, the sequence S is called a zero-sum sequence if $\sum_{g\in G_0} v_g g = 0$. The set of all zero-sum sequences over G_0 is denoted by $\mathcal{B}(G_0)$, and it is a submonoid of $\mathcal{F}(G_0)$; every zero-sum sequence is a product of finitely many minimal zero-sum sequences, i.e., zero-sum sequences not having a proper zero-sum subsequence. It is well-known that for H a Krull monoid with class group G and subset of classes containing prime divisors G_0, studying sets of lengths of elements of H is the same as studying sets of lengths of elements of $\mathcal{B}(G_0)$ (cf. [8, Chapter 3]).

For $\{\ell_1 < \ell_2 < \ell_3 \dots\} \subset \mathbb{N}_0$, let $\Delta(L) = \{\ell_2-\ell_1, \ell_3-\ell_2, \dots\}$. The set $\Delta(G_0) = \bigcup_{B\in\mathcal{B}(G_0)} \Delta(\mathsf{L}(B))$ is called the set of distances of G_0, where $\mathsf{L}(B)$ is defined in analogy with $\mathsf{L}(a)$ in Theorem 1.1.

The set of differences $\Delta^*(G)$ is defined as

$$\Delta^*(G) = \{\min \Delta(G_0) \colon G_0 \subset G,\ \Delta(G_0) \neq \emptyset\}.$$

In the following two sections we recall a variety of results on it; for various reasons, it is common to focus on its large elements.

3. General results and groups of large rank

Let $(G,+)$ be an abelian group. Being essentially equivalent to a result of L. Carlitz [1], it is well-known that $\Delta^*(G) \neq \emptyset$ if and only if $|G| \geq 3$. Moreover, it is not hard to see that for finite G the set $\Delta^*(G)$ is finite; more specifically, $|G|-2$ is a fairly simple to obtain bound for its maximum. Finally, by a recent result of S. T. Chapman, W. W. Smith and the author [4] it is known that $\Delta^*(G) = \mathbb{N}$ for G infinite. Thus, we assume below that $3 \leq |G| < \infty$.

For various elements it can be shown, by constructing an appropriate set G_0, that they are contained in $\Delta^*(G)$ (cf. [8, Proposition 6.8.2]).

Proposition 3.1. *Let r denote the rank of G, and let n denote the exponent of G.*

(1) $1 \in \Delta^*(G)$.
(2) $\{1, \dots, r-1\} \subset \Delta^*(G)$.
(3) *For each $3 \leq d \mid n$, one has $d-2 \in \Delta^*(G)$.*

There are more results in this directions known. In particular, the results of S. Chang, S.T. Chapman, and W.W. Smith [2] allow to expand point three of the proposition. Yet, it should be noted that the size of the additional elements obtainable in this way does never exceed $n-2$, and typically is small relative to n.

Regarding general upper bounds the following is known; the second part is due to W.D. Gao and A. Geroldinger [6], for the first see [11].

Theorem 3.2. *Let s denote the total rank of G, and let n denote the exponent of G.*

(1) $\max \Delta^*(G) \leq \max\{n-2, \mathsf{K}(G)-1, s-1\}$, *where $\mathsf{K}(G)$ denotes the cross number of G and one has, in particular,* $\mathsf{K}(G) \leq 1/2 + \log |G|$.
(2) *Suppose that $s \geq (n-1) + (n-1)^2(n-2)/2$. Then* $\max \Delta^*(G) \leq s-1$.

Note that it is conjectured that $\mathsf{K}(G)$ never exceeds the total rank of G; if this were known, the second point would be a special case of the first one. Yet, as this conjecture is open and most likely is very difficult to settle, the second point remains relevant.

This result, in combination with results on the cross number and Proposition 3.1, yields the following precise result for p-groups (see [11] for details).

Corollary 3.3. *Let G be a p-group of rank r. Then*

$$\Delta^*(G) = \{1, \dots, r-1\}.$$

Moreover, it follows from Theorem 3.2 that if $|G|$ is not too large relative to n, the exponent of G, then $\max \Delta^*(G) = n-2$, and we discuss refinements of this result for cyclic groups in the next section; there are also some weaker refinements for non-cyclic groups, which we do not discuss. Finally, no counter example to the equality, again n the exponent and r the rank, $\max \Delta^*(G) = \max\{n-2, r-1\}$ is known.

4. Cyclic groups

Let C_n denote a cyclic group of order n. It is known by a result of A. Geroldinger and Y. ould Hamidoune [9] that for $n \geq 4$

$$\max \Delta^*(C_n) = n-2 \text{ and } \max(\Delta^*(C_n) \setminus \{n-2\}) = \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

i.e., the two largest elements of $\Delta^*(C_n)$ are known.

In ongoing joint work of A. Plagne and the author [10] this result was improved (for sufficiently large n).

Theorem 4.1. *For all sufficiently large n the elements in $\Delta^*(C_n)$ of size at least $n/5$ are precisely the integers in the set* $\{n-2, (n-2)/2, (n-3)/2, (n-4)/2, (n-4)/3, (n-6)/3, (n-4)/4, (n-5)/4, (n-6)/4, (n-8)/4\}$.

The proof of the result gives directly an explicit value for ‘sufficiently large,’ e.g., 2000 is more than sufficient. Moreover, actually a result for general n to give all elements of size as small as $n/10$ (not only $n/5$) was obtained; yet, without specialized notation even the formulation becomes cumbersome. Finally, restricting to the investigation of cyclic groups of prime power order, a similar result for all elements of size greater than $(2n^2)^{1/3}$ was obtained; we do not state it, for the same reason.

An important difference, yet not the only one, between the case of general n and that of prime power n is that only in the former case weakly half-factorial yet non-half-factorial subset of C_n exist.

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