CIRM

Troisième Rencontre Internationale sur les Polynômes à Valeurs Entières

Rencontre organisée par :
Sabine Evrard

29 novembre-3 décembre 2010

Valentina Barucci and Faten Khouja

Irreducibility of ideals in a one-dimensional analytically irreducible ring

Vol. 2, n° 2 (2010), p. 91-93.

<http://acirm.cedram.org/item?id=ACIRM_2010__2_2_91_0>

Centre international de rencontres mathématiques
U.M.S. 822 C.N.R.S./S.M.F.
Luminy (Marseille) France

Irreducibility of ideals in a one-dimensional analytically irreducible ring

Valentina Barucci and Faten Khouja

Abstract

Let R be a one-dimensional analytically irreducible ring and let I be an integral ideal of R. We study the relation between the irreducibility of the ideal I in R and the irreducibility of the corresponding semigroup ideal $v(I)$. It turns out that if $v(I)$ is irreducible, then I is irreducible, but the converse does not hold in general. We collect some known results taken from [5], [4], [3] to obtain this result, which is new. We finally give an algorithm to compute the components of an irredundant decomposition of a nonzero ideal.

A numerical semigroup is a subsemigroup of $\mathbb{N}$, with zero and with finite complement in $\mathbb{N}$. The numerical semigroup generated by $d_1, \dots, d_\nu \in \mathbb{N}$ is $S =< d_1, \dots, d_\nu >= \{\sum_{i=1}^{\nu} n_i d_i, n_i \in \mathbb{N}\}$. $M = S \backslash \{0\}$ will denote the maximal ideal of S, e the multiplicity of S, that is the smallest positive integer of S, g the Frobenius number of S, that is the greatest integer which does not belong to S. A relative ideal of S is a nonempty subset I of $\mathbb{Z}$ such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$. A relative ideal which is contained in S is an integral ideal of S. If I, J are relative ideals of S, then the following are relative ideals too: $I \cap J$, $I \cup J$, $I + J = \{i + j, i \in I, j \in J\}$, $I \underset{\mathbb{Z}}{-} J = \{z \in \mathbb{Z} | z + J \subseteq I\}$, $I \underset{\mathbb{S}}{-} J = (I \underset{\mathbb{Z}}{-} J) \cap S$. An integral ideal I of a numerical semigroup S is called *irreducible* if it is not the intersection of two integral ideals which properly contain I. Consider the partial order on S given by $s_1 \preccurlyeq s_2 \Leftrightarrow s_1 + s_3 = s_2$, for some $s_3 \in S$, and for $s \in S$, set $B(s) = \{x \in S | x \preccurlyeq s\}$.

Proposition 1. *Let I be a proper integral ideal of S. Then I is irreducible if and only if $I = S \backslash B(s)$, for some $s \in S$.*

Theorem 1. *a) If I is a proper integral ideal of S and if $(I \underset{S}{-} M) \backslash I = \{s_1, \dots, s_n\}$, then $I = (S \backslash B(s_1)) \cap \ldots \cap (S \backslash B(s_n))$ is the unique irredundant decomposition of I in integral irreducible ideals.*
b) I is irreducible if and only if $|(I \underset{S}{-} M) \setminus I| = 1$.

A relative ideal I of a numerical semigroup S is called $\mathbb{Z}-$ *irreducible* if it is not the intersection of two relative ideals which properly contain I. A particular relative ideal of S plays a special role, it is the canonical ideal Ω which is maximal with respect to the property of non containing g, the Frobenius number of S. Thus $\Omega = \{g - x, x \in \mathbb{Z} \backslash S\}$.

Proposition 2. *Let J be a relative ideal of S. Then J is $\mathbb{Z}-$irreducible if and only if $J = \Omega + z$ for some $z \in \mathbb{Z}$, if and only if $|(J \underset{\mathbb{Z}}{-} M) \backslash J| = 1$.*

Theorem 2. *I is a relative ideal of S minimally generated by $i_1, \dots, i_h$ if and only if $\Omega \underset{\mathbb{Z}}{-} I = (\Omega - i_1) \cap \ldots \cap (\Omega - i_h)$ is the unique irredundant decomposition of $\Omega \underset{\mathbb{Z}}{-} I$ in $\mathbb{Z}-$irreducibles ideals.*

Text presented during the meeting "Third International Meeting on Integer-Valued Polynomials" organized by Sabine Evrard. 29 novembre-3 décembre 2010, C.I.R.M. (Luminy).
2000 *Mathematics Subject Classification.* 13A15, 13H10.
Key words. Numerical semigroup, canonical ideal, irreducible ideal.

Corollary 1. *Each relative ideal J of S has a unique irredundant decomposition as intersection of $\mathbb{Z}-$irreducible ideals. The number of components is the cardinality of a minimal set of generators of $\Omega \underset{\mathbb{Z}}{-} J$, which is also equal to $|(J \underset{\mathbb{Z}}{-} M)\backslash J|$.*

Applications to one-dimensional Rings: As usual, an integral ideal I of a ring R is called *irreducible* if it is not the intersection of two proper overideals. A fractional ideal F of a ring R with total ring of fractions K is called $K-$*irreducible* if it is not the intersection of two strictly larger fractional ideals.

Using the following lemma, we recover a known result with Proposition 3 below [4, Proposition 3.1.6, p.67].

Lemma 1. *Let I be an ideal of a local ring (R, m) and J an irreducible ideal such that $I \subsetneq J$. Then $l_R((I \underset{R}{:} m)/J \cap (I \underset{R}{:} m)) \leq 1$.*

Proposition 3. *Let (R, m) be a Noetherien local ring and I be an m-primary ideal. Then the number $n(I)$ of components of an irredundant decomposition of I is*

$$n(I) = l_R((I \underset{R}{:} m)/I) = dim_{R/m} Socle(R/I)$$

Corollary 2. *Let (R, m) be a Noetherien local ring and I be an m-primary ideal. Then I is irreducible if and only if $l_R((I \underset{R}{:} m)/I) = 1$.*

Let R be an integral domain with field of fractions K. A fractional ideal ω of R is called an *m-canonical* ideal if for any nonzero fractional ideal I of R, we have $I = \omega \underset{K}{:} (\omega \underset{K}{:} I)$. We fix from here on the following notation: (R, m) is a one-dimensional analytically irreducible Noetherian domain. This is a domain for which the integral closure $V = \overline{R}$ in the field of fractions K of R is a rank-one discrete valuation domain and is a finitely generated $R-$module.
Let $v : K\backslash\{0\} \to \mathbb{Z}$ be the normalized valuation associated to V. Thus, if $t \in V$ generates the maximal ideal of V, then $v(t) = 1$. Moreover, we assume that $R/m \simeq V/M$, where $M = tV$ is the maximal ideal of V, i.e. R is residually rational. A one-dimensional analytically irreducible Noetherian domain has an m-canonical ideal, cf. e.g. [2]. Observe that: $S = v(R) = \{v(r)|r \in R \setminus \{0\}\}$ is a numerical semigroup. We denote by Ω the canonical ideal of $v(R)$.

Proposition 4. *Let F be a fractional ideal of R. Then F is $K-$irreducible if and only if $l_R(F \underset{K}{:} m/F) = 1$ if and only if $v(F) = \Omega + z$, for some $z \in \mathbb{Z}$.*

Corollary 3. *Let F be a fractional ideal of R. Then F is $K-$irreducible if and only if $v(F)$ is $\mathbb{Z}-$irreducible.*

It is a natural question to ask whether a result similar to Corollary 3 holds for integral ideals.

Theorem 3. *Let I be a non zero integral ideal of R such that $v(I)$ is irreducible, then I is irreducible.*

Proof: Now I is m-primary, so $I \subset (I \underset{R}{:} m)$. Since $v(I \underset{R}{:} m) \setminus v(I) \subseteq (v(I) \underset{S}{-} v(m)) \setminus v(I)$, we have $l_R((I \underset{R}{:} m)/I) = |v(I \underset{R}{:} m) \setminus v(I)| \leq |(v(I) \underset{S}{-} v(m) \setminus v(I)| = 1$, where the last equality follows from Theorem 1 b). So by Corollary 2, I is irreducible.

The converse of Theorem 3 does not hold, as the following example shows.

Example: $S =< 2, 5 >= \{0, 2, 4, \to\}$, $R = k[[t^2, t^5]]$, $I = (t^4 + t^5, t^7)$, we have: $v(I) \underset{S}{-} v(m)\backslash v(I) = \{2, 5\}$, then by Theorem 1, a) $v(I) = (S\backslash B(2)) \cap (S\backslash B(5))$. So $v(I)$ is not irreducible. But, $l_R((I \underset{R}{:} m)/I) = (v(I) \underset{R}{:} v(m))\backslash v(I) = 1$. In fact, consider $f = a_2t^2 + a_4t^4 + a_5t^5 + \ldots \in R$, with $a_2 \neq 0$. If $ft^2 \in I$, $ft^2 = h(t^4 + t^5) + gt^7$, such that $h = b_0 + b_2t^2 + b_4t^4 + \ldots$, then $0 = b_0 = a_2$, so that $ft^2 \notin I$. Thus $f \notin (I \underset{R}{:} m)$. Hence I is irreducible.

Corollary 4. *Let I be a monomial ideal of $k[[t^{n_1}, \ldots, t^{n_k}]]$. Then,*
I is irreducible if and only if $v(I)$ is irreducible.

In other terms, the non trivial deduction of Corollary 4 says that, if I is a monomial ideal which is not the intersection of two strictly larger monomial ideals, then I is not the intersection of two strictly larger ideals, even if non monomial ideals are allowed. This is indeed known in a more general context [6, Proposition 11, p.41].

Algorithm: The following algorithm is a method for computing the components of an irredundant decomposition of a non zero ideal I of R.

(1) Compute the length of $(I \underset{R}{:} m/I)$ as $R-$module, $l_R(I \underset{R}{:} m/I) = n$

(2) Look at a set of generators of $(I \underset{R}{:} m/I)$ as R/m vector space.
$$(I \underset{R}{:} m/I) =< f_1 + I, \dots, f_n + I > .$$

(3) Let for $i = 1, \dots, n$,
$$J_i = (I, f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$$

(4) For each J_i, we will construct another ideal J_i' such that $J_i \subseteq J_i'$, J_i' is irreducible and $\bigcap J_i' = I$.
Compute the length of $(J_i \underset{R}{:} m/J_i)$.
If $l_R(J_i \underset{R}{:} m/J_i) = 1$, then we take $J_i' = J_i$.
If not, look at a set of generators $(g_1 + J_i, \dots, g_j + J_i)$ of $(J_i \underset{R}{:} m/J_i)$ as R/m vector space. Since $I \underset{R}{:} m \subseteq J_i \underset{R}{:} m$ and $f_i \notin J_i$, we can take $g_1 = f_i$.

(5) Iterating the construction above we will obtain:
$J_{i1} = (J_i, g_2, \dots, g_s)$
$J_{i2} = (J_i, f_i, g_3, \dots, g_s)$
$\vdots$
$J_{ij} = (J_i, f_i, \dots, g_{s-1})$
Yet we are interested only in the ideal J_{i1} which does not contain $f_i = g_1$.

(6) Compute the length of $(J_{i1} \underset{R}{:} m/J_{i1})$.
If $l_R(J_{i1} \underset{R}{:} m/J_{i1}) = 1$, then we take $J_i' = J_{i1}$. If not we proceed in the same way. After at most $k-2$ steps, where $k = l_R(R/I)$, we find an irreducible ideal J_i'.
It is easy to see that $\bigcap_{i \neq j} J_j' \nsubseteq J_i'$, because $f_i \in \bigcap_{j \neq i} J_j'$ but $f_i \notin J_i'$. We claim that $I = \bigcap_{i=1}^n J_i'$ is an irredundant intersection of I into irreducible ideals. In fact suppose that we have $I \subset \bigcap_{i=1}^n J_i'$. Then
$$I \subset \bigcap_{i=1}^n J_i' \subset J_2' \cap \ldots \cap J_n' \subset \ldots \subset J_{n-1}' \cap J_n' \subset J_n' \cap (I \underset{R}{:} m) \subset (I \underset{R}{:} m),$$
a contradiction since $l_R = (I \underset{R}{:} m/I) = n$, so $I = \bigcap_{i=1}^n J_i'$.

References

[1] V. Barucci, *Decompositions of ideals into irreducible ideals in numerical semigroups.* Journal of Commutative Algebra 2 (2010), 281-294.

[2] V. Barucci and R. Fröberg, *One-dimensional almost Gorenstein rings.* J. Algebra 188 (1997), 418-442.

[3] J. Jäger, *Langenberechnung und kanonische Ideale in eindimensionalen Ringen.* Arch. Math. 29 (1977), 504-512.

[4] W. Vasconcelos, *Computational Methods in Commutative Alegebra and Algebraic Geometry.* Springer-Verlag, 1998.

[5] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry.* Birkhauser, 1984.

[6] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra.* Springer-Verlag, 2005.

Dipartimento di Matematica, Sapienza, Università di Roma, Piazzale A. Moro 2, 00185 Rome, ITALY • barucci@mat.uniroma1.it

Department of Mathematics, Faculty of Sciences, 5000 Monastir, TUNISIA. • koja-faten@yahoo.fr