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Jesse Elliott

Birings and plethories of integer-valued polynomials

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Birings and plethories of integer-valued polynomials

Jesse Elliott

Abstract

Let A and B be commutative rings with identity. An A-B-*biring* is an A-algebra S together with a lift of the functor $\mathrm{Hom}_A(S,-)$ from A-algebras to sets to a functor from A-algebras to B-algebras. An A-*plethory* is a monoid object in the monoidal category, equipped with the composition product, of A-A-birings. The polynomial ring $A[X]$ is an initial object in the category of such structures. The D-algebra $\mathrm{Int}(D)$ has such a structure if $D = A$ is a domain such that the natural D-algebra homomorphism $\theta_n : \bigotimes_{D\,i=1}^{n} \mathrm{Int}(D) \longrightarrow \mathrm{Int}(D^n)$ is an isomorphism for $n = 2$ and injective for $n \leq 4$. This holds in particular if θ_n is an isomorphism for all n, which in turn holds, for example, if D is a Krull domain or more generally a TV PVMD. In these cases we also examine properties of the functor $\mathrm{Hom}_D(\mathrm{Int}(D), -)$ from D-algebras to D-algebras, which we hope to show is a new object worthy of investigation in the theory of integer-valued polynomials.

1. Introduction

This paper is a summary of the results contained in the forthcoming paper [7]. Throughout this paper all rings and algebras are assumed commutative with identity.

For any integral domain D with quotient field K, any set $\mathbf{X}$, and any subset $\mathbf{E}$ of $K^{\mathbf{X}}$, the ring of *integer-valued polynomials on* $\mathbf{E}$ *over* D is the subring

$$\mathrm{Int}(\mathbf{E}, D) = \{f(\mathbf{X}) \in K[\mathbf{X}] : f(\mathbf{E}) \subseteq D\}$$

of the polynomial ring $K[\mathbf{X}]$. In other words, $\mathrm{Int}(\mathbf{E}, D)$ is the pullback of the direct product $D^{\mathbf{E}}$ along the K-algebra homomorphism $K[\mathbf{X}] \longrightarrow K^{\mathbf{E}}$ acting by $f \longmapsto (f(\underline{a}))_{\underline{a}\in\mathbf{E}}$. One writes $\mathrm{Int}(D^{\mathbf{X}}) = \mathrm{Int}(D^{\mathbf{X}}, D)$ and $\mathrm{Int}(D) = \mathrm{Int}(D, D)$. One also writes $\mathrm{Int}(D^n) = \mathrm{Int}(D^{\mathbf{X}})$ if $\mathbf{X}$ is a set of cardinality n.

Much of the theory of integer-valued polynomial rings developed in attempts to generalize results known about $\mathrm{Int}(\mathbb{Z})$ to $\mathrm{Int}(D)$. This paper is concerned with finding such a generalization of a particular result about $\mathrm{Int}(\mathbb{Z})$. To state this result we need a few definitions.

A ring A is said to be *binomial* if A is $\mathbb{Z}$-torsion-free and $\frac{a(a-1)(a-2)\cdots(a-n+1)}{n!} \in A \otimes_{\mathbb{Z}} \mathbb{Q}$ lies in A for all $a \in A$ and all positive integers n. For any set $\mathbf{X}$ the ring $\mathrm{Int}(\mathbb{Z}^{\mathbf{X}})$ is the free binomial ring generated by $\mathbf{X}$, and a $\mathbb{Z}$-torsion-free ring A is binomial if and only if, for every $a \in A$, there exists a ring homomorphism $\mathrm{Int}(\mathbb{Z}) \longrightarrow A$ sending X to a [4]. By the category of binomial rings we will mean the full subcategory of the category of rings whose objects are the binomial rings. By [1, Section 46] and [4, Theorem 9.1] we have the following.

Proposition 1. *There is a functor* Bin *from rings to binomial rings that is left-represented by* $\mathrm{Int}(\mathbb{Z})$ *and is a right adjoint for the inclusion from binomial rings to rings.*

Our motivating problem is to generalize the above result to $\mathrm{Int}(D)$ for further domains D. More specifically, we are interested in the following.

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Problem 2. *Determine all domains D for which* $\mathrm{Int}(D)$ *left-represents a right adjoint for the inclusion from $\mathcal{C}$ to the category of D-algebras for some full subcategory $\mathcal{C}$ of the category of D-algebras.*

In particular, if D is such a domain, then the functor $\mathrm{Hom}_D(\mathrm{Int}(D), -)$ from D-algebras to sets must lift to a functor from D-algebras to D-algebras in $\mathcal{C}$. If $D = \mathbb{Z}$, then by Proposition 1 this holds for the category $\mathcal{C}$ of binomial rings. Given a domain D, a natural candidate for the category $\mathcal{C}$ is the category of D-torsion-free D-algebras that are "weakly polynomially complete" [5, Section 7], where a D-algebra A is said to be *weakly polynomially complete*, or *WPC*, if for every $a \in A$ there exists a D-algebra homomorphism $\mathrm{Int}(D) \longrightarrow A$ sending X to a. A binomial ring is equivalently a $\mathbb{Z}$-torsion-free WPC $\mathbb{Z}$-algebra, and for any domain D the D-algebra $\mathrm{Int}(D)$ is itself WPC. Our goal, then, is to construct a right adjoint for the inclusion from the category of D-torsion-free WPC D-algebras to the category of D-algebras that is left-represented by $\mathrm{Int}(D)$. In our efforts to do so we found it necessary to utilize the notions of a *biring* and a *plethory*.

Let A and B be rings. An *A-B-biring* is an A-algebra S together with the structure on S of a B-algebra object in the opposite category of the category of A-algebras. Thus an A-B-biring is an A-algebra S equipped with two *binary co-operations* $S \longmapsto S \otimes_A S$, called *co-addition* and *co-multiplication* (both of which are A-algebra homomorphisms), along with a *co-B-linear structure* $B \longrightarrow \mathrm{Hom}_A(S, A)$, satisfying laws dual to those defining the A-algebras. By Yoneda's lemma, an A-B-biring is equivalently an A-algebra S together with a lift of the covariant functor $\mathrm{Hom}_A(S, -)$ it represents to a functor from the category of A-algebras to the category of B-algebras. (See any of [1, 2, 9] for the details.) For example, the polynomial ring $A[X]$ is an A-A-biring as it represents the identity functor from the category of A-algebras to itself. Co-addition acts by $X \longmapsto X \otimes 1 + 1 \otimes X$, co-multiplication by $X \longmapsto X \otimes X$, and the co-linear structure by $a \longmapsto (f \longmapsto f(a))$.

Proposition 3. *Let D be an integral domain.*

1. *The existence of a D-D-biring structure on* $\mathrm{Int}(D)$ *is equivalent to the existence of a lift of the functor* $\mathrm{Hom}_D(\mathrm{Int}(D), -)$ *from D-algebras to sets to a functor from D-algebras to D-algebras.*
2. *A D-D-biring structure on* $\mathrm{Int}(D)$ *is compatible with the D-D-biring structure on $D[X]$, that is, the inclusion $D[X] \longrightarrow$* $\mathrm{Int}(D)$ *is a homomorphism of D-D-birings, if and only if the natural map* $\mathrm{Hom}_D(\mathrm{Int}(D), A) \longrightarrow A$ *given by $\varphi \longmapsto \varphi(X)$ is a D-algebra homomorphism for every D-algebra A.*

Consequently, any solution to Problem 2 would yield conditions on integral domains D under which the D-algebra $\mathrm{Int}(D)$ has a D-D-biring structure. Regarding the latter problem we have the following.

Theorem 4. *Assume that* $\mathrm{Int}(D)$ *is flat over D, or more generally that the n-th tensor power* $\mathrm{Int}(D)^{\otimes n}$ *of* $\mathrm{Int}(D)$ *over D is D-torsion-free for $n \leq 4$. Then the domain* $\mathrm{Int}(D)$ *has a (necessarily unique) D-D-biring structure that is compatible with the D-D-biring structure on $D[X]$ if and only if for every $f \in$* $\mathrm{Int}(D)$ *the polynomials $f(X+Y)$ and $f(XY)$ both can be expressed as sums of polynomials of the form $g(X)h(Y)$ for $g, h \in$* $\mathrm{Int}(D)$.

In analogy with ordinary polynomial rings, there is for any set $\mathbf{X}$ a canonical D-algebra homomorphism

$$\theta_{\mathbf{X}} : \bigotimes_{X \in \mathbf{X}} \mathrm{Int}(D) \longrightarrow \mathrm{Int}(D^{\mathbf{X}}),$$

where the (possibly infinite) tensor product is over D and is a coproduct in the category of D-algebras. However, we do not know whether or not $\theta_{\mathbf{X}}$ is an isomorphism for every domain D and every set $\mathbf{X}$. There are several large classes of domains for which $\theta_{\mathbf{X}}$ is an isomorphism for all $\mathbf{X}$, such as the Krull domains, the almost Newtonian domains [5, Section 5], and the PVMDs D such that $\mathrm{Int}(D_{\mathfrak{m}}) = \mathrm{Int}(D)_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$, hence the TV PVMDs as well. (See [8] for the definition of a PVMD and a TV PVMD.) We say that the domain D is *polynomially composite* if $\theta_{\mathbf{X}}$ is an isomorphism for every set $\mathbf{X}$. Section 4 of [6] collects several known classes of polynomial composite domains.

Corollary 5. *If D is a polynomially composite domain, and in particular if D is a Krull domain or TV PVMD, then* $\operatorname{Int}(D)$ *has a unique D-D-biring structure such that the inclusion* $D[X] \longrightarrow \operatorname{Int}(D)$ *is a homomorphism of D-D-birings.*

By [4, Proposition 9.3] one has $\operatorname{Bin}(A) \cong \mathbb{Z}_p$ for any integral domain A of characteristic p, where $\mathbb{Z}_p$ denotes the ring of p-adic integers, and in particular one has $\operatorname{Bin}(\mathbb{F}_p) \cong \mathbb{Z}_p$. This generalizes as follows.

Proposition 6. *Let D be a Dedekind domain, and let $\mathfrak{m}$ be a maximal ideal of D with finite residue field. Then the map*

$$\widehat{D}_{\mathfrak{m}} \longrightarrow \operatorname{Hom}_D(\operatorname{Int}(D), D/\mathfrak{m})$$

acting by $\alpha \longmapsto (f \longmapsto f(\alpha) \bmod \mathfrak{m}\widehat{D}_{\mathfrak{m}})$ is a D-algebra isomorphism. More generally, for any domain extension A of D with $\mathfrak{m}A = 0$, the diagram

$$\begin{array}{ccc} \widehat{D}_{\mathfrak{m}} & \longrightarrow & \operatorname{Hom}_D(\operatorname{Int}(D), D/\mathfrak{m}) \\ & \searrow & \downarrow \\ & & \operatorname{Hom}_D(\operatorname{Int}(D), A) \end{array}$$

is a commutative diagram of D-algebra isomorphisms.

By [2, Proposition 1.4], for any A-B-biring S, the lifted functor $\operatorname{Hom}_A(S, -)$ from A-algebras to B-algebras has a left adjoint, denoted $S \odot_A -$. In analogy with the tensor product, the A-algebra $S \odot_A R$ for any B-algebra R is the A-algebra generated by the symbols $s \odot r$ for all $s \in S$ and $r \in R$, subject to the relations [2, 1.3.1–2]. If S and T are A-A-birings, then so is $S \odot_A T$, and the category of A-A-birings equipped with the operation $\odot_A$ is monoidal with unit $A[X]$. An *A-plethory* is a monoid object in that monoidal category, that is, it is an A-A-biring P together with an associative map $\circ : P \odot_A P \longrightarrow P$ of A-A-birings (called *composition*) possessing a unit $e : A[X] \longrightarrow P$. (See any of [1, 2, 9] for details on these constructions.) An A-plethory is also known as an *A-A-biring monad object*, an *A-A-biring triple*, or a *Tall-Wraith monad object* in the category of A-algebras. For example, for any ring A, the polynomial ring $A[X]$ has the structure of an A-plethory and in fact is an initial object in the category of A-plethories.

Proposition 7. *Let D be an integral domain. Any D-D-biring structure on* $\operatorname{Int}(D)$ *compatible with that on $D[X]$ extends uniquely to a D-plethory structure on* $\operatorname{Int}(D)$ *with unit given by the inclusion* $D[X] \longrightarrow \operatorname{Int}(D)$. *Composition* $\circ : \operatorname{Int}(D) \odot_D \operatorname{Int}(D) \longrightarrow \operatorname{Int}(D)$ *acts by ordinary composition on elements of the form $f \odot g$, that is, one has $\circ : f \odot g \longmapsto f \circ g$ for all $f, g \in$* $\operatorname{Int}(D)$.

The following theorem gives a partial solution to Problem 2.

Theorem 8. *Let D be an integral domain.*

(1) *Assume that there exists a D-D-biring structure on* $\operatorname{Int}(D)$ *compatible with that on $D[X]$. Then the functors* $\operatorname{Hom}_D(\operatorname{Int}(D), -)$ *and* $\operatorname{Int}(D) \odot_D -$ *are right and left adjoints, respectively, of the inclusion from D-torsion-free WPC algebras to D-algebras if and only if the D-algebras* $\operatorname{Hom}_D(\operatorname{Int}(D), A)$ *and* $\operatorname{Int}(D) \odot_D A$ *are D-torsion-free for any D-algebra A.*

(2) *If D is a PID with all residue fields finite, then the hypotheses (and therefore the conclusion) of statement (1) hold.*

Note that if $\operatorname{Int}(D) = D[X]$, which, for example, holds by [3, Corollary I.3.7] if D has no finite residue fields, then $\operatorname{Hom}_D(\operatorname{Int}(D), A)$ is naturally isomorphic to A and in particular is not D-torsion-free if A is not D-torsion-free. Of course in that case every D-algebra is WPC.

2. WPC D-algebras and tensor powers of $\operatorname{Int}(D)$

As in [5, Section 7] and as in the introduction, we will say that a D-algebra A is *weakly polynomially complete*, or *WPC*, if for every $a \in A$ there exists a D-algebra homomorphism $\operatorname{Int}(D) \longrightarrow A$ sending X to a. A D-torsion-free D-algebra A is WPC if and only if $f(A) \subseteq A$ for all $f \in \operatorname{Int}(D) \subseteq (A \otimes_D K)[X]$. In particular, a domain extension A of D is WPC if and only if $\operatorname{Int}(D) \subseteq \operatorname{Int}(A)$.

For any set $\mathbf{X}$, the smallest subring of $\mathrm{Int}(D^{\mathbf{X}})$ containing $D[\mathbf{X}]$ that is closed under precomposition by elements of $\mathrm{Int}(D)$ is denoted $\mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})$. For any domain D (finite or infinite), the domain $\mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})$ is the free WPC extension of D generated by $\mathbf{X}$ [5, Proposition 7.2]. It is also the *weak polynomial completion* $w_D(D[\mathbf{X}])$ *of* $D[\mathbf{X}]$ *with respect to* D, as defined in [5, Section 8] and in Proposition 11 below.

If $\mathrm{Int}_{\otimes}(D^{\mathbf{X}})$ denotes the image of the D-algebra homomorphism $\theta_{\mathbf{X}} : \bigotimes_{X\in\mathbf{X}} \mathrm{Int}(D) \longrightarrow \mathrm{Int}(D^{\mathbf{X}})$, then we have $\mathrm{Int}_{\otimes}(D^{\mathbf{X}}) \subseteq \mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})$, and equality holds for a given set $\mathbf{X}$ if and only if $\mathrm{Int}_{\otimes}(D^{\mathbf{X}})$ is a WPC extension of D. If equality holds for any set $\mathbf{X}$ then we will say that D is *weakly polynomially composite*.

Proposition 9. *The following conditions are equivalent for any integral domain D.*

(1) *D is weakly polynomially composite.*
(2) $\mathrm{Int}_{\otimes}(D^{\mathbf{X}})$ *is a WPC extension of D for any set* $\mathbf{X}$.
(3) $\mathrm{Int}_{\otimes}(D^{n})$ *is a WPC extension of D for some integer $n > 1$.*
(4) $\mathrm{Int}_{\otimes}(D^{2})$ *is a WPC extension of D.*
(5) *For any element f of* $\mathrm{Int}(D)$, *the polynomials $f(X+Y)$ and $f(XY)$ lie in the image of* $\theta_{\{X,Y\}}$.
(6) *The compositum of any collection of WPC D-algebras of D contained in some D-torsion-free D-algebra is again a WPC D-algebra.*
(7) *The compositum of any collection of WPC extensions of D contained in some domain extension of D is again a WPC extension of D.*

Clearly polynomial compositeness implies weak polynomial compositeness.

At the end of Section 8 of [5] it is noted how to construct the left adjoint of the inclusion functor from WPC domain extensions of D to domain extensions of D. The proof can be easily generalized to establish the following.

Proposition 10. *Let D be a domain with quotient field K, and let A be a D-torsion-free D-algebra.*

(1) *A is contained in a smallest D-torsion-free WPC D-algebra, denoted $w_D(A)$, equal to the intersection of all WPC D-algebras containing A and contained in $A \otimes_D K$.*
(2) *One has $w_D(A) = A$ if and only if A is WPC, and $w_D(A)$ is a domain if and only if A is a domain.*
(3) *One has* $w_D(A) \cong \mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})/((\ker\varphi)K \cap \mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}}))$ *for any surjective D-algebra homomorphism $\varphi : D[\mathbf{X}] \longrightarrow A$.*
(4) *The association $A \longmapsto w_D(A)$ defines a functor from the category of D-torsion-free D-algebras to the category of D-torsion-free WPC D-algebras—both categories with morphisms as D-algebra homomorphisms—that is a left adjoint for the inclusion functor.*

Assuming that D is weakly polynomially composite, we can also construct the right adjoint of the inclusion functor from D-torsion-free WPC D-algebras to D-torsion-free D-algebras.

Proposition 11. *Let D be a weakly polynomially composite domain, and let A be a D-torsion-free D-algebra.*

(1) *A contains a largest WPC D-algebra, denoted $w^D(A)$, equal to the compositum of all WPC D-algebras contained in A.*
(2) *One has $w^D(A) = A$ if and only if A is WPC.*
(3) *One has* $w^D(A) = \{a \in A : a = \varphi(X) \text{ for some } \varphi \in \mathrm{Hom}_D(\mathrm{Int}(D), A)\}$.
(4) *The association $A \longmapsto w^D(A)$ defines a functor from the category of D-torsion-free D-algebras to the category of D-torsion-free WPC D-algebras—both categories with morphisms as D-algebra homomorphisms—that is a right adjoint for the inclusion functor.*

3. Biring and plethory structure on $\mathrm{Int}(D)$

The following result implies Theorem 4 and Corollary 5 of the introduction.

Theorem 12. *Let D be an integral domain.*

(1) *If the domain* $\mathrm{Int}(D)$ *has a D-D-biring structure such that the inclusion* $D[X] \longrightarrow \mathrm{Int}(D)$ *is a homomorphism of D-D-birings, then D is weakly polynomially composite.*

(2) *Assume that the n-th tensor power* $\mathrm{Int}(D)^{\otimes n}$ *of* $\mathrm{Int}(D)$ *over D is D-torsion-free for $n \leq 4$. Then* $\mathrm{Int}(D)$ *has a unique D-D-biring structure such that the inclusion* $D[X] \longrightarrow \mathrm{Int}(D)$ *is a homomorphism of D-D-birings if D is weakly polynomially composite.*

The plethory $A[X]$ is an initial object in the category of A-plethories. Like $A[X]$, and in particular like the domain $D[X]$, the domain $\mathrm{Int}(D)$ has its own "internal" operation of composition. This leads to the following result.

Proposition 13. *Let D be an integral domain. Any D-D-biring structure on* $\mathrm{Int}(D)$ *such that the inclusion* $D[X] \longrightarrow \mathrm{Int}(D)$ *is a homomorphism of D-D-birings extends uniquely to a D-plethory structure on* $\mathrm{Int}(D)$ *with unit given by the inclusion* $D[X] \longrightarrow \mathrm{Int}(D)$*. Composition* $\circ : \mathrm{Int}(D) \odot_D \mathrm{Int}(D) \longrightarrow \mathrm{Int}(D)$ *acts by ordinary composition on elements of the form $f \odot g$, that is, one has $\circ : f \odot g \longmapsto f \circ g$ for all $f, g \in \mathrm{Int}(D)$.*

Corollary 14. *If D is a polynomially composite domain, and in particular if D is a Krull domain or TV PVMD, then* $\mathrm{Int}(D)$ *has a unique D-plethory structure with unit given by the inclusion* $D[X] \longrightarrow \mathrm{Int}(D)$*.*

Let A be a ring and P an A-plethory. A *P-ring* is an A-algebra R together with an A-algebra homomorphism $\circ : P \odot_A R \longrightarrow R$ such that $(\alpha \circ \beta) \circ r = \alpha \circ (\beta \circ r)$ and $e \circ r = e$ for all $\alpha, \beta \in P$ and all $r \in R$, where e is the image of X in the unit $A[X] \longrightarrow P$ [2, 1.9]. Such a map $\circ$ is said to be a *left action of P on R*. For example, P itself has a structure of a P-ring, as do the A-algebras $P \odot_A R$ and $\mathrm{Hom}_A(P, R)$ for any A-algebra R [2, 1.10], with left actions given by

$$\begin{array}{rcl} P \odot_A (P \odot_A R) & \longrightarrow & P \odot_A R \\ \alpha \odot (\beta \odot r) & \longmapsto & (\alpha \circ \beta) \odot r \end{array}$$

and

$$\begin{array}{rcl} P \odot_A \mathrm{Hom}_A(P, R) & \longrightarrow & \mathrm{Hom}_A(P, R) \\ \alpha \odot \varphi & \longmapsto & (\beta \longmapsto \varphi(\beta \circ \alpha)), \end{array}$$

respectively. Moreover, the functors $P \odot_A -$ and $W_P = \mathrm{Hom}_A(P, -)$ from A-algebras to P-rings are left and right adjoints, respectively, for the forgetful functor from P-rings to A-algebras [2, 1.10].

For any A-plethory P, the P-ring $W_P(R) = \mathrm{Hom}_A(P, R)$ of any A-algebra R is called the *P-Witt ring* of R. This terminology comes from the fact that, if P is the $\mathbb{Z}$-plethory Λ of [2, 2.11], then a P-ring is equivalently a λ-ring, and the functor W_P is isomorphic to the universal λ-ring functor Λ. If P is the $\mathbb{Z}$-plethory $\mathrm{Int}(\mathbb{Z})$, then a P-ring is equivalently a binomial ring, and the functor W_P is isomorphic to the functor Bin. The latter fact generalizes to the following result, which implies Theorem 8 of the introduction.

Theorem 15. *Let D be an integral domain such that* $\mathrm{Int}(D)$ *has a D-plethory structure with unit given by the inclusion* $D[X] \longrightarrow \mathrm{Int}(D)$*, and let A be a D-algebra.*

(1) *If there exists an* $\mathrm{Int}(D)$*-ring structure on A, then A is WPC.*
(2) *If A is D-torsion-free, then there exists a (necessarily unique)* $\mathrm{Int}(D)$*-ring structure on A if and only if A is WPC.*
(3) *If A is D-torsion-free, then the D-algebra homomorphism $W_{\mathrm{Int}(D)}(A) \longrightarrow A$ is an inclusion with image equal to $w^D(A)$, and the functor w^D is therefore isomorphic to the functor $W_{\mathrm{Int}(D)}$ restricted to the category of D-torsion-free D-algebras.*
(4) *If A is D-torsion-free, then the D-algebra homomorphism* $\mathrm{Int}(D) \odot_D A \longrightarrow A \otimes_D K$ *acting by $f \odot a \longmapsto f(a)$ has image equal to $w_D(A)$, and the functor w_D is therefore isomorphic to the functor $T \circ (\mathrm{Int}(D) \odot_D -)$ restricted to the category of D-torsion-free D-algebras, where $T(B)$ for any D-algebra B denotes the image of B in $B \otimes_D K$, where K is the quotient field of D.*
(5) *If A is D-torsion-free and WPC, then the natural D-algebra homomorphisms $W_{\mathrm{Int}(D)}(A) \longrightarrow A$ and $A \longrightarrow T(\mathrm{Int}(D) \odot_D A)$ are isomorphisms.*
(6) *The functor $T \circ (\mathrm{Int}(D) \odot_D -)$ is a left adjoint for the inclusion from D-torsion-free WPC D-algebras to D-algebras.*

(7) *The functors* $W_{\mathrm{Int}(D)}$ *and* $\mathrm{Int}(D) \odot_D -$ *are right and left adjoints, respectively, for the inclusion from* D*-torsion-free WPC* D*-algebras to* D*-algebras if and only if the* $\mathrm{Int}(D)$*-rings* $W_{\mathrm{Int}(D)}(R)$ *and* $\mathrm{Int}(D) \odot_D R$ *are* D*-torsion-free for every* D*-algebra* R*.*

(8) *Every* $\mathrm{Int}(D)$*-ring is* D*-torsion-free if* D *is a PID with finite residue fields.*

We end with the following problem.

Problem 16. *Determine equivalent conditions on an integral domain* D *so that the* D*-algebra* $\mathrm{Int}(D)$ *has a* D*-plethory structure with unit given by the inclusion* $D[X] \longrightarrow \mathrm{Int}(D)$ *and so that the* D*-algebras* $W_{\mathrm{Int}(D)}(R)$ *and* $\mathrm{Int}(D) \odot_D R$ *are* D*-torsion-free for every* D*-algebra* R*.*

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Department of Mathematics
California State University, Channel Islands
Camarillo, California 93012 • jesse.elliott@csuci.edu