Numération : mathématiques et informatique

Rencontre organisée par :
Boris Adamczewski, Anne Siegel and Wolfgang Steiner

23-27 mars 2009

Bernd Sing

Non-Unimodularity

Vol. 1, n° 1 (2009), p. 69-74.

<http://acirm.cedram.org/item?id=ACIRM_2009__1_1_69_0>

Centre international de rencontres mathématiques
U.M.S. 822 C.N.R.S./S.M.F.
Luminy (Marseille) France

Actes des rencontres du C.I.R.M.
Vol. **1** n° 1 (2009) 69-74

Non-Unimodularity

Bernd Sing

1. Introduction

In this talk we explain how and why (and that) things known for unimodular β-integers/Pisot-numbers are also working in the non-unimodular case. In particular, we are interested in the geometric realisation of the β-integers also known as "Rauzy-fractal". In the non-unimodular case, one has to consider the numbers involved also as p-adic numbers. It is our hope that by paralleling the construction in the unimodular and in the non-unimodular case (and by keeping this exposition deliberately informal), we reduce the "fear" of dealing with p-adic numbers and even improve the understanding of the unimodular case.

2. β-Substitutions

A number of talks in this conference deal with β-expansions and β-substitutions, e.g., [1, 4, 6]. Thus, we keep this section short.

Let $\beta > 1$ be a *PV-number* (*Pisot* number) and consider the greedy expansion to base β. Then any (nonnegative) number x has such an expansion $\sum_{n=m}^{\infty} a_n \cdot \beta^{-n}$ with coefficients $a_i \in \{0, 1, \dots, \lceil \beta \rceil - 1\}$. One might be interested in characterising the set of β-integers

$$\mathbb{Z}_\beta = \left\{ x \mid x = \sum_{n=m}^{0} a_n \cdot \beta^{-n} \right\} = \left\{ x \mid x = \sum_{n=0}^{m} a_{-n} \cdot \beta^{n} \right\}.$$

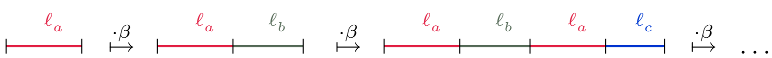
An interesting observation is that $\mathbb{Z}_\beta$ can also be obtained using a substitution rule. We note that there are are two classes of substitutions depending on the β-expansion of 1 being periodic or eventually periodic (i.e., depending on whether β is a *simple* or a *non-simple Parry number*, see [4]). In the "periodic case", where β has minimal polynomial $x^n - d_1 x^{n-1} - \dots - d_n$, this substitution is given by $a \mapsto a^{d_1} b$, $b \mapsto a^{d_2} c$, $\dots$, $z \mapsto a^{d_n}$ (also note that the (periodic) β-expansion of 1 is given by $\frac{d_1}{\beta} + \dots + \frac{d_n - 1}{\beta^n} + \frac{d_1}{\beta^{n+1}} + \dots + \frac{d_n - 1}{\beta^{2n}} + \frac{d_1}{\beta^{2n+1}} + \dots$). We call an algebraic integer β *unimodular* if the constant term in its minimal polynomial is ± 1.

3. A Unimodular Example

3.1. **From Substitutions to Iterated Function Systems.** Earlier this afternoon, Fabien Durand [6, 7] studied the β-substitution associated to the polynomial $p(x) = x^4 - x^3 - x^2 - x - 1$ which has dominant (Pisot) root $\beta \approx 1.928$. Since this is an irreducible polynomial, p is the minimal polynomial of β and the other roots of p are exactly the algebraic conjugates of β. They are here given by $\lambda_r \approx -0.775$ and $\lambda_c, \overline{\lambda}_c \approx -0.076 \pm i \cdot 0.815$. The β-substitution is explicitly given by

$$a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto ad, \quad d \mapsto a. \tag{1}$$

Geometrically, we can describe such a symbolic substitution as a tile substitution: Choose $\ell_a = 1$, $\ell_b = \beta - 1$, $\ell_c = \beta^2 - \beta - 1$ and $\ell_d = \beta^3 - \beta^2 - \beta - 1$ (these are the components of the PF-eigenvector $\underline{\ell}$ of the *substitution/Abelianization* matrix of the substitution). Then inflate and subdivide, and the set of left endpoints of the intervals yields $\mathbb{Z}_\beta$.



Text presented during the meeting "Numeration: mathematics and computer science" organized by Boris Adamczewski, Anne Siegel and Wolfgang Steiner. 23-27 mars 2009, C.I.R.M. (Luminy).

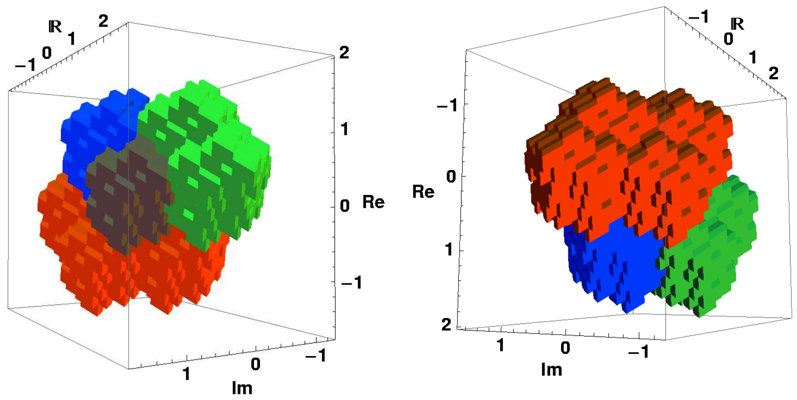


Figure 1: Two views of a polyhedral approximation to the geometric realisation (the "Rauzy fractal") of the β-integers $\mathrm{cl}\,\mathbb{Z}_\beta^\star = \varOmega_a \cup \varOmega_b \cup \varOmega_c \cup \varOmega_d$ associated to the substitution given in Eq. (1).

We denote the set of left endpoints of the i-type intervals (i.e., all intervals that correspond to a letter i in the symbolic sequence) by $\varLambda_i$. We thus have $\mathbb{Z}_\beta = \varLambda_a \cup \varLambda_b \cup \varLambda_c \cup \varLambda_d$. Note that the substitution rule yields a point set equation for the sets $\varLambda_i$:

$$\begin{aligned}
\varLambda_a &= \lambda \cdot \varLambda_a \cup \lambda \cdot \varLambda_b \cup \lambda \cdot \varLambda_c \cup \lambda \cdot \varLambda_d \\
\varLambda_b &= \lambda \cdot \varLambda_a + 1 \\
\varLambda_c &= \lambda \cdot \varLambda_b + 1 \\
\varLambda_d &= \lambda \cdot \varLambda_c + t1
\end{aligned}$$

In the next step, we are looking for a geometric realisation of these β-integers in $\mathbb{C} \times \mathbb{R}$ using the algebraic conjugates of β. To this end, we replace β^k in the expansion of $x \in \mathbb{Z}_\beta$ by

$$\big(\mathrm{Re}(\lambda_c^k),\, \mathrm{Im}(\lambda_c^k),\, \lambda_r^k\big)^\intercal$$

and thus get an object in $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$. In the theory of *model sets* (see remarks in Section 5), this map is called the *star-map* $(\cdot)^\star$.

We are now interested in the image $\mathbb{Z}_\beta^\star$ of $\mathbb{Z}_\beta$ under the star-map, more precisely, the closure of $\mathbb{Z}_\beta^\star$ is of interest. Let $\varOmega_i$ be the closure of $\varLambda_i^\star$; then we have

$$\mathrm{cl}\,\mathbb{Z}_\beta^\star = \varOmega_a \cup \varOmega_b \cup \varOmega_c \cup \varOmega_d.$$

We note that the above substitution rule yields an *iterated function system (IFS)*, for which the sets $\varOmega_i$ are the unique compact solutions: Using the notations

$$\lambda^\star = \begin{pmatrix} \mathrm{Re}\,\lambda_c & -\mathrm{Im}\,\lambda_c & 0 \\ \mathrm{Im}\,\lambda_c & \mathrm{Re}\,\lambda_c & 0 \\ 0 & 0 & \lambda_r \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

we get

$$\begin{aligned}
\varOmega_a &= \lambda^\star \varOmega_a \cup \lambda^\star \varOmega_b \cup \lambda^\star \varOmega_c \cup \lambda^\star \varOmega_d \\
\varOmega_b &= \lambda^\star \varOmega_a + t \\
\varOmega_c &= \lambda^\star \varOmega_b + t \\
\varOmega_d &= \lambda^\star \varOmega_c + t
\end{aligned} \tag{2}$$

3.2. **A Measure Calculation.** The sets Ω_i are compact and for their Lebesgue measure μ we find (using the subadditivity, behaviour under linear transformations and translation-invariance of the Lebesgue measure):

$$\begin{aligned}\mu(\Omega_a) &= \mu(\lambda^\star \Omega_a \cup \lambda^\star \Omega_b \cup \lambda^\star \Omega_c \cup \lambda^\star \Omega_d) \\ &\overset{\text{subadd.}}{\leq} \mu(\lambda^\star \Omega_a) + \mu(\lambda^\star \Omega_b) + \mu(\lambda^\star \Omega_c) + \mu(\lambda^\star \Omega_d) \\ &\overset{\text{lin. trafo}}{=} |\lambda_c|^2 \cdot |\lambda_r| \cdot (\mu(\Omega_a) + \mu(\Omega_b) + \mu(\Omega_c) + \mu(\Omega_d))\end{aligned}$$

and, with $(j_1, j_2) \in \{(b,a), (c,b), (d,c)\}$,

$$\mu(\Omega_{j_1}) = \mu(\lambda^\star \Omega_{j_2} + t) \overset{\text{trans. inv.}}{=} \mu(\lambda^\star \Omega_{j_2}) \overset{\text{lin. trafo}}{=} |\lambda_c|^2 \cdot |\lambda_r| \cdot \mu(\Omega_{j_2}).$$

But $|\lambda_c|^2 \cdot |\lambda_r| = 1/\beta$ and we have component-wise

$$\beta \cdot \begin{pmatrix} \mu(\Omega_a) \\ \mu(\Omega_b) \\ \mu(\Omega_c) \\ \mu(\Omega_d) \end{pmatrix} \leq \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu(\Omega_a) \\ \mu(\Omega_b) \\ \mu(\Omega_c) \\ \mu(\Omega_d) \end{pmatrix}.$$

By construction, β is the Perron-Frobenius eigenvalue of the transpose of the Abelianization matrix on the right, thus equality holds. But this means that all unions on the right-hand side of the IFS in Eq. (2) are disjoint in measure. From the first line of that IFS we deduce that the sets Ω_i are therefore disjoint in measure, and we have a "well-behaved" geometric realisation in $\mathbb{C} \times \mathbb{R}$ of the β-integers in questions. A figure of this "Rauzy-fractal" is shown in Fig. 1.

4. A Non-Unimodular Example

4.1. **Substitution and Measure Calculation.** As a non-unimodular example, we will study the β-substitution given via the polynomial $p(x) = x^3 - 3x^2 - x - \mathbf{2}$, which has dominant (Pisot) root $\beta \approx 3.457$. The algebraic conjugates of β are $\lambda_c, \overline{\lambda}_c \approx -0.228 \pm i \cdot 0.726$, and the associated substitution is given by

$$a \mapsto aaab, \quad b \mapsto ac, \quad c \mapsto aa. \tag{3}$$

We can again describe the β-integers geometrically using a tile substitution. In this case, we choose the interval lengths $\ell_a = 1$, $\ell_b = \beta - 3$ and $\ell_c = \beta^2 - 3\beta - 1$.

If we proceed as before, we would now define a map, say $(\cdot)^\circledast$, that maps β^k to $\big(\mathrm{Re}(\lambda_c^k),\, \mathrm{Im}(\lambda_c^k)\big)^\intercal$. Then, we would look at sets $\Omega'_i = \mathrm{cl}\, \Lambda_i^\circledast$ that are also given by an IFS on $\mathbb{C}$. However, due to the non-unimodularity, the measure calculation goes wrong (note that $\beta\,|\lambda_c|^2 = \mathbf{2}$); using μ' for the Lebesgue measure on $\mathbb{C}$, we obtain

$$\frac{1}{\mathbf{2}}\beta \cdot \begin{pmatrix} \mu'(\Omega'_a) \\ \mu'(\Omega'_b) \\ \mu'(\Omega'_c) \end{pmatrix} \leq \begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu'(\Omega'_a) \\ \mu'(\Omega'_b) \\ \mu'(\Omega'_c) \end{pmatrix}.$$

This additional factor $\frac{1}{2}$ destroys our previous argument using the Perron-Frobenius theorem that established measure-disjointness of the sets in question.

We now make a little detour to the realm of p-adic numbers; our goal is to establish that one has to consider the geometric realisation in $\mathbb{C} \times \mathbb{Q}_2$ and not just $\mathbb{C}$ for this non-unimodular example.

4.2. **p-adic Fields $\mathbb{Q}_p$.** The p-adic integers $\mathbb{Z}_p$ are a complete discrete valuation ring. An element $x \in \mathbb{Z}_p$ can be written as Taylor series in powers of p, i.e.,

$$x = \sum_{n=0}^{\infty} s_n\, p^n = .s_0 s_1 \dots \qquad \text{with } s_n \in \{0, \dots, p-1\}.$$

The p-adic numbers $\mathbb{Q}_p$ are the field of fractions of $\mathbb{Z}_p$. An element $x \in \mathbb{Q}_p$ can be written as Laurent series

$$x = \sum_{n=m}^{\infty} s_n\, p^n = s_m \dots s_{-1}.s_0 s_1 \dots \qquad \text{with } s_n \in \{0, \dots, p-1\}.$$

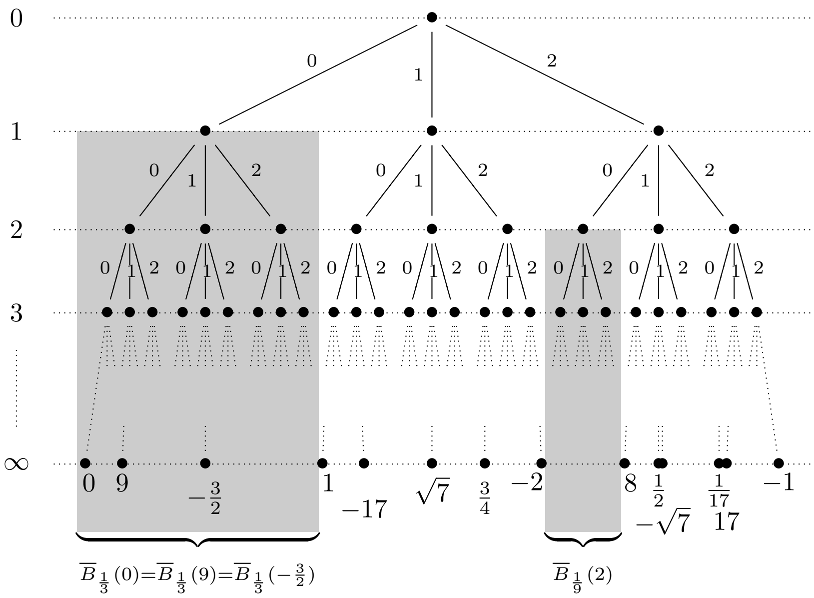


Figure 2: A "tree-picture" for the 3-adic integers, the digits in the 3-adic expansion determine the path starting from the root at the top. This leads to an "embedding" of the p-adic numbers into the reals (respectively, of the p-adic integers into the interval $[0,1]$) that lines up with the p-adic metric: closed balls $\overline{B}_r(x)$ of radius r around x correspond to closed intervals of length r (containing the image of the 3-adic number x). Details and references may be found in [13, Chapter 3c].

As an example of this p-adic numeration, we consider some 3-adic integers:

$$0 = .\overline{0}, \quad 1 = .1\overline{0}, \quad 8 = .22\overline{0}, \quad -1 = .\overline{2}, \quad -13 = .211\overline{2},$$
$$\frac{3}{4} = .01\overline{20}, \qquad \sqrt{7} = .2112022\ldots \quad \text{and/or} \quad .1110200\ldots.$$

A nice way to think these 3-adic numbers and "visualise" them in a way that (kind of) lines up with the 3-adic metric, is shown in Fig. 2 using a tree-picture that leads to an "embedding" into the real line.

4.3. **Geometric Realisation in $\mathbb{C}\times\mathbb{Q}_2$.** We now consider the polynomial $p(x) = x^3 - 3\,x^2 - x - 2$ in $\mathbb{Q}_2$! This polynomial has a[1] 2-adic root $\lambda_2 = .010111\ldots$ of 2-adic absolute value $\frac{1}{2}$ Moreover, the Haar measure μ_H on $\mathbb{Q}_2$ has the property

$$\mu_H(\lambda_2\, A) = \frac{1}{2}\cdot \mu_H(A)$$

for any Haar-measurable set $A \subset \mathbb{Q}_2$ (the Haar measure is the generalisation of the Lebesgue measure in a locally compact topological group).

So, if we define a star-map $\mathbb{Z}_\beta \to \mathbb{C}\times\mathbb{Q}_2$ by replacing each power β^k by $(\mathrm{Re}(\lambda_c^k),\, \mathrm{Im}(\lambda_c^k),\, \lambda_2^k)^\intercal$, then one can check that the measure calculation (using the Haar measure on $\mathbb{C}\times\mathbb{Q}_2$) works out! Concluding as in the previous unimodular example, we therefore obtain a "well-behaved" geometric realisation in $\mathbb{C}\times\mathbb{Q}_2$ of the β-integers in questions. A figure of this "Rauzy-fractal" is depicted in Fig. 3; as before in Fig. 1, the sets Ω_i yield "nice" tiles.

[1] It also has two roots in the extension field $\mathbb{Q}_2(\sqrt{-3}) \cong \mathbb{Q}_2(\sqrt{5})$, namely

$$1.001001\ldots + \sqrt{5}\cdot 1.010001\ldots \qquad \text{and} \qquad 1.001001\ldots + \sqrt{5}\cdot 1.101110\ldots;$$

but they are of 2-adic absolute value 1.

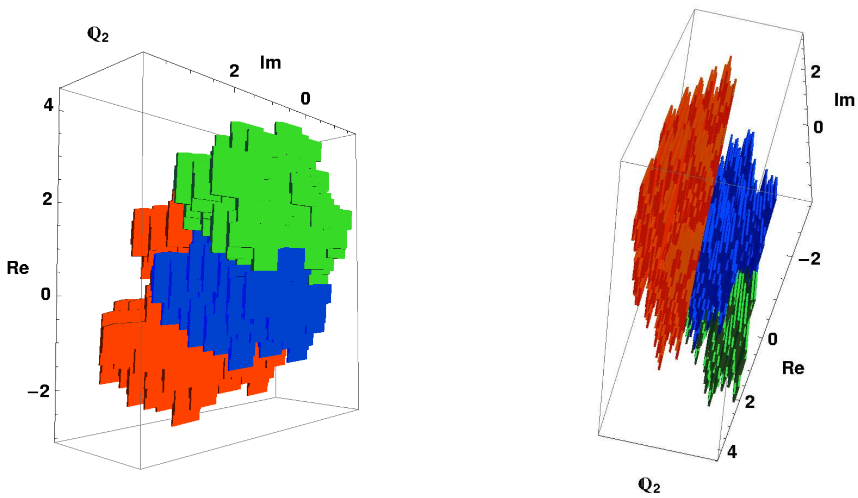


Figure 3: Two views of a polyhedral approximation to the geometric realisation (the "Rauzy fractal") in $\mathbb{C}\times\mathbb{Q}_2$ of the β-integers $\mathrm{cl}\,\mathbb{Z}_\beta^\star = \boldsymbol{\Omega_a} \cup \Omega_b \cup \boldsymbol{\Omega_c}$ associated to the substitution given in Eq. (3).

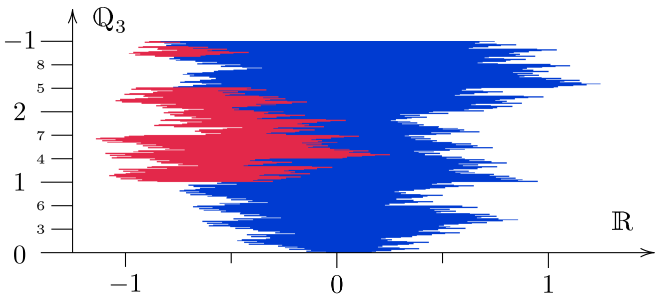


Figure 4: A further example: The geometric realisation in $\mathbb{R} \times \mathbb{Q}_3$ for the β-substitution given by $a \mapsto aaaab$, $b \mapsto ab$.

5. Remarks

One is often (or usually) interested whether the sets Ω_j are the prototiles of a specific aperiodic tiling since this implies that the underlying dynamical system is pure point, see the remarks in the talk by Akiyama [1] and compare [5, 8, 13]. This condition can be checked algorithmically by making use of the above mentioned measure-disjointness on the right-hand side of the associated IFS, compare [12] (also see [13, Sections 6.9 & 6.10]). Another way to phrase this is by so-called *coincidence conditions* and/or *(weak) finiteness conditions*, see Akiyama's talk [1] and [5, 8, 13] and references therein.

We also mention that behind "geometric realisation" the concepts of a *cut and project scheme* and *model sets* are hidden that played an important part in understanding diffractive properties of "quasicrystals". See [10, 9, 2] (or [13, Chapter 5] and references therein) on more on these concepts.

Also, one has to mention that this method at looking at additional p-adic components has its predecessors in [3, 11]. That and how behind this all actually the ring of adeles is working can be found in [13, Section 6.5].

We conclude this article with the picture of a further example, see Fig. 4.

References

1. S. Akiyama. "Finiteness and periodicity of beta expansions – number theoretical and dynamical open problems". Talk given at the Conference "Numeration: Mathematics and Computer Science", held at the CIRM, Luminy, France, from March 23 to 27, 2009; see this volume.
2. M. Baake. "A guide to mathematical quasicrystals". In: J.-B. Suck, M. Schreiber, and P. Häussler (editors), *Quasicrystals: An Introduction to Structure, Physical Properties, and Applications*, Springer Series in Materials Science **55**. Springer, Berlin, 2002, pages 17–48. `math-ph/9901014`.
3. M. Baake, R.V. Moody, and M. Schlottmann. "Limit-(quasi)periodic point sets as quasicrystals with p-adic internal spaces". *J. Phys. A: Math. Gen.* **31**(27):5755–5765 (1998). `math-ph/9901008`.
4. L'. Balková. "Resemblance and difference between beta-integers and ordinary integers". Talk given at the Conference "Numeration: Mathematics and Computer Science", held at the CIRM, Luminy, France, from March 23 to 27, 2009; see this volume.
5. M. Barge and J. Kwapisz. "Geometric theory of unimodular Pisot substitutions". *Amer. J. Math.* **128**(5):1219–1282 (2006).
6. F. Durand. "Rauzy fractal in $\mathbb{R} \times \mathbb{C}$ and points with multiple expansions". Talk given at the Conference "Numeration: Mathematics and Computer Science", held at the CIRM, Luminy, France, from March 23 to 27, 2009; see this volume.
7. F. Durand and A. Messaoudi. "Boundary of the Rauzy fractal sets in $\mathbb{R} \times \mathbb{C}$ generated by $P(x) = x^4 - x^3 - x^2 - x - 1$". *Preprint.*
8. S. Ito and H. Rao. "Atomic surfaces, tilings and coincidence I. Irreducible case". *Israel J. Math.* **153**:129–156 (2006).
9. R.V. Moody. "Model sets: a survey". In: F. Axel, F. Dénoyer, and J.P. Gazeau (editors), *From Quasicrystals to More Complex Systems.* Winter School "Order, Chance and Risk: Aperiodic Phenomena from Solid State to Finance", held at Centre de Physique Les Houches, France, from February 23 to March 6, 1998. Centre de Physique des Houches **13**. Springer, Berlin, 2000, pages 145–166. `math.MG/0002020`.
10. M. Schlottmann. "Cut-and-project sets in locally compact Abelian groups". In: J. Patera (editor), *Quasicrystals and Discrete Geometry.* 1995 Fall Programme at The Fields Institute, Toronto, ON, Canada. Fields Institute Monographs **10**. American Mathematical Society, Providence, RI, 1998, pages 247–264.
11. A. Siegel. "Represéntation des systèmes dynamiques substitutifs non unimodulaires". *Ergodic Theory Dynam. Systems* **23**(4):1247–1273 (2003).
12. A. Siegel. "Pure discrete spectrum dynamical system and periodic tiling associated with a substitution". *Ann. Inst. Fourier (Grenoble)* **54**(2):288–299 (2004).
13. B. Sing. *Pisot Substitutions and Beyond.* PhD thesis, Universität Bielefeld, 2007. Available at `http://nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:hbz:361-11555`.

Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, Somerset, BA1 6BA, UK • bs259@bath.ac.uk