On finite $p$-groups minimally of class greater than two

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In memory of Mario Curzio and Guido Zappa

Abstract – Let $G$ be a finite nilpotent group of class three whose proper subgroups and proper quotients are nilpotent of class at most two. We show that $G$ is either a 2-generated $p$-group or a 3-generated 3-group. In the first case the groups of maximal order with respect to a given exponent are all isomorphic except in the cases where $p = 2$ and $\exp(G) = 2^r$, $r \geq 4$. If $G$ is 3-generated, then we show that there is a unique group of maximal order and exponent 3; but a similar result is not valid for exponent 9.


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1. Introduction

Let $\mathcal{K}$ be a class of finite groups. The finite group $G$ is called a minimal non-$\mathcal{K}$-group (we write $G \in \text{Min(}\mathcal{K}\text{)}$), if $G \not\in \mathcal{K}$ but every proper subgroup and every proper quotient of $G$ belongs to $\mathcal{K}$.

For the class $\mathcal{K} = \mathcal{A}$ of all abelian groups, the structure of the groups in $\text{Min(}\mathcal{A}\text{)}$ can easily be derived from results of Miller-Moreno and Rédei (see [1, p. 281] and [1, p. 309]) and Lemma 2.1 below. Indeed, it is easy to see that such a group $G$

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is either a semidirect product $G = [N]Q$ of a minimal normal subgroup $N$ by a complement $Q$ of prime order, or it is one of the following groups:

i) $G_r = \langle a, b \mid a^{p^r} = b^p = 1, a^b = a^{1+p^{r-1}} \rangle$, $r \geq 2$;

ii) $G = \langle a, b \mid a^p = b^p = 1, [a, b] = c, c^p = 1, [a, c] = [b, c] = 1 \rangle$, $p$ odd;

iii) the quaternion group $Q_8$.

$G_r$ is of exponent $p^r$, and from i), ii), and iii) it follows that, for every exponent $p^r \neq 2, 4$, there exists precisely one $p$-group $G \in \text{Min}(\mathcal{A})$ of exponent $p^r$. If $p^r = 4$ we get two groups: the dihedral group $D_4$ and the quaternion group $Q_8$, while the case $p^r = 2$ does not allow any such group.

In this paper, we discuss the minimal non-$\mathcal{N}_2$-groups, where $\mathcal{N}_2$ denotes the class of all nilpotent groups of class $\leq 2$. The structure of non-nilpotent groups $G \in \text{Min}(\mathcal{N}_2)$ follows immediately from the aforementioned results of Miller-Moreno and Rédei. Hence we will restrict attention to finite $p$-groups.

We prove that the $p$-groups in $\text{Min}(\mathcal{N}_2)$ are either 2-generated or 3-generated 2-Engel. In order to give information on the $p$-groups in $\text{Min}(\mathcal{N}_2)$ we determine the structure of the 2-generated free groups in the variety $\mathcal{W}$ of all nilpotent groups of exponent $p^r$ ($r \geq 2$) and class three, satisfying the law $[x, y, z]^{p} = 1$, and the structure of the 3-generated free groups in the variety $\mathcal{V}$ of all 2-Engel groups of exponent $3^r$. We prove that there is a unique 2-generated group of exponent $p$ in $\text{Min}(\mathcal{N}_2)$: its order is $p^4$ with $p \geq 5$. If $G$ is a 2-generated group in $\text{Min}(\mathcal{N}_2)$ of exponent $p^r$ with $r \geq 2$ and $p$ odd we see that $|G| \leq p^{3r}$; if $p = 2$, $r \geq 3$ then $|G| \leq 2^{3(r-1)}$; and if $p = 2$ and $r = 2$ then $|G| \leq 2^{3r-1}$. We give an explicit construction of the groups in $\text{Min}(\mathcal{N}_2)$ of exponent $p^r$ and maximal order and we show that such groups are all isomorphic except in the case $p = 2$ and $r \geq 4$. If $G$ is a 3-generated group of exponent 3 in $\text{Min}(\mathcal{N}_2)$, we show that $|G| = 3^7$ and $G$ is isomorphic to the 3-generated relatively free group in the variety of all groups of exponent 3 but the groups of exponent 9 of maximal order in $\text{Min}(\mathcal{N}_2)$ are not isomorphic.

In the following the notation is standard. $G = [N]Q$ indicates the semidirect product of the normal subgroup $N$ by the subgroup $Q$, and $d(G)$ indicates the minimal number of generators of $G$. Moreover $o(x)$ is the order of the element $x$. If $\mathcal{V}$ is a variety, $\text{Fr}_n(\mathcal{V})$ denotes the relatively free group of rank $n$ in $\mathcal{V}$.

All groups considered in this paper are finite.
2. Preliminaries

Lemma 2.1. A finite nilpotent group of class \( c \geq 2 \) has all of its proper quotients of class at most \( c - 1 \) if and only if \( Z(G) \) is cyclic and the \( c \)-th term of the lower central series \( \Gamma_c(G) \) is of order \( p \).

Proof. Suppose that \( G \) has class \( c \) and that all proper quotients of \( G \) are of class at most \( c - 1 \). Then \( G \) is monolithic. Indeed, if \( N_1 \) and \( N_2 \) are two distinct minimal normal subgroups, then \( G = G/N_1 \cap N_2 \) is embedded in \( G/N_1 \times G/N_2 \) which is nilpotent of class at most \( c - 1 \). As \( G \) is monolithic, \( Z(G) \) is cyclic. If \( N \) is the minimal normal subgroup of \( G \), then \( G/N \) is nilpotent of class at most \( c - 1 \). So \( \Gamma_c(G) = N \).

Conversely let \( G \) be a nilpotent group of class \( c \) and assume that \( Z(G) \) is cyclic and \( \Gamma_c(G) \) is of order \( p \). Then for every normal subgroup \( K \) of \( G \), we have \( \Gamma_c(G) \trianglelefteq K \). So

\[
\Gamma_c(G/K) = \Gamma_c(G)K/K = 1.
\]

\( \square \)

Lemma 2.2. Let \( G \) be a nilpotent group such that all of its proper subgroups have class at most \( c \) but \( G \) has not class \( c \). Then \( Z(G) \subseteq \Phi(G) \).

Proof. Let \( M \) be a maximal subgroup of \( G \). Then \( M \trianglelefteq G \). Suppose that \( Z(G) \nsubseteq M \). Then \( G = Z(G)M \) and so \( G \) has class \( c \), a contradiction. \( \square \)

Lemma 2.3. Let \( G \) be a \( p \)-group in \( \text{Min}(N_2) \). Then either \( G \) can be generated by two elements, or \( G \) is a 2-Engel 3-group generated by three elements.

Proof. Suppose that \( G \) cannot be generated by two elements. Then for all \( x, y \in G \) we have that \( \langle x, y \rangle \) is a proper subgroup of \( G \). So it is nilpotent of class 2. In particular \( G \) satisfies the 2-Engel condition. If \( p \neq 3 \) then \( G \) is nilpotent of class two ([I, p. 288]), a contradiction. So \( p = 3 \). Moreover \( G \) is generated by three elements, otherwise all subgroups generated by three elements would be proper subgroups of \( G \), and \( G \) would be nilpotent of class two, a contradiction. \( \square \)

We now give a sufficient criterion for a \( p \)-group generated by two elements to have all of its proper subgroups of class two.

Lemma 2.4. Let \( G \) be a \( p \)-group which can be generated by two elements. Assume that \( [\Phi(G), G] \leq Z(G) \). Then every proper subgroup of \( G \) is nilpotent of class two.
Proof. It suffices to show that every maximal subgroup $M$ of $G$ is of class two. As $G$ is generated by two elements, we have $G/\Phi(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. So $M = \langle \Phi(G), x \rangle$ for some $x$ in $M$. We get $M' = \Phi(G)' \cdot [\Phi(G), x]$. By hypothesis, both factors are contained in $Z(G)$, so that the class of $M$ is two. \hfill \Box

3. Min($\mathcal{N}_2$)-groups with two generators

We start with the smallest case:

Proposition 3.1. Let $G \in \text{Min}(\mathcal{N}_2)$ be a group of prime exponent $p$. If $d(G) = 2$, then $p \geq 5$, $|G| = p^4$ and $G \cong [N]/(u)$, where $N = \langle v_1 \rangle \times \langle v_2 \rangle \times \langle v_3 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and the action of $u$ on $N$ is given by

$$v_1^u = v_1, \quad v_2^u = v_1v_2, \quad v_3^u = v_2v_3.$$

Proof. As $\exp(G) = p$, we infer that $p \neq 2$ and $|G/G'| = p^2$. Moreover $G'/\Gamma_3(G)$ is cyclic of order $p$ and by Lemma 2.1, we have $|\Gamma_3(G)| = p$. So we get $|G| = p^4$. An inspection of the groups of order $p^4$ (see [1, p. 346]) yields the result. \hfill \Box

A group $G$ in $\text{Min}(\mathcal{N}_2)$ of exponent $p^r$ belongs to the variety $\mathcal{W}$ of all groups of exponent $p^r$ and nilpotent of class three satisfying the law $[x, y, z]^p = 1$ (see Lemma 2.1).

We now collect some information of $Fr_2(\mathcal{W})$.

Proposition 3.2. Let $p^r$ be a power of a prime $p$ and $r \geq 2$. Let $F = Fr_2(\mathcal{W})$ with free generators $x, y$. Then

a) $F/F' \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^r}$ and either $|F'/\Gamma_3(F)| = p^r$ if $p \geq 3$ or $|F'/\Gamma_3(F)| = 2r^{-1}$. Moreover $\Gamma_3(F) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ and hence $|F| = p^{3r+2}$ for $p \geq 3$; and $|F| = 2^{3r+1}$ if $p = 2$;

\begin{equation*}
\begin{cases}
\mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p & \text{if } p \geq 3, \\
\mathbb{Z}_{2^{r-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p = 2 \text{ and } r \geq 3, \\
\mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p = 2 \text{ and } r = 2;
\end{cases}
\end{equation*}

b) $Z(F) \simeq [F^p, F] \leq Z(F)$;

d) every proper subgroup of $F$ is nilpotent of class two.
Proof. a) As \( \exp(F) = p^r \), we infer that \(|F/F'| \leq p^{2r} \). Moreover, 
\( F'/\Gamma_3(F) = \langle [x, y] \Gamma_3(F) \rangle \) is cyclic of exponent dividing \( p^r \) if \( p \neq 2 \) and 
\( 2^{r-1} \) otherwise (we have \( 1 \equiv (xy)^{2r} \equiv x^{2r} y^{2r} [y, x]^{(\frac{2r}{r-1})} \pmod{\Gamma_3(F)} \), so 
\( [y, x]^{2r-1} \equiv 1 \pmod{\Gamma_3(F)} \)). Then \(|F'/\Gamma_3(F)| \leq p^r \) if \( p \neq 2 \) or \( \leq 2^{r-1} \) otherwise. Finally, we have \(|\Gamma_3(F)| \leq p^2 \), because there are only two basic commutators of weight 3. This implies \(|F| \leq p^{3r+2} \) if \( p \neq 2 \), \(|F| \leq 2^{3r+1} \) otherwise.

We now construct a group \( F_0 \), belonging to the variety \( W \), which has order either \( p^{3r+2} \) if \( p \geq 3 \), or \( 2^{3r+1} \). So it will be \( F_0 \simeq F_{r_2}(W) \).

Let \( N = [A] \langle x \rangle \) be the semidirect product of the abelian group 
\[ A = \langle u \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle, \]
with the cyclic group \( \langle x \rangle \) of order \( p^r \); where 
\[ o(v_1) = o(v_2) = p \]
and 
\[ o(u) = \begin{cases} p^r & \text{if } p \geq 3, \\ 2^{r-1} & \text{otherwise.} \end{cases} \]

The action of \( x \) on \( A \) is given by 
\[ u^x = u v_1, \quad v_1^x = v_1, \quad v_2^x = v_2. \]

Then we consider the group \( F_0 = [N] \langle y \rangle \), where \( y \) is a cyclic group of order \( p^r \) and the action of \( y \) on \( N \) is given by 
\[ x^y = x u, \quad u^y = u v_2, \quad v_1^y = v_1, \quad v_2^y = v_2. \]

We can immediately verify that 
\[ u = [x, y], \quad v_1 = [u, x] = [x, y, x], \quad v_2 = [u, y] = [x, y, y]. \]

So \( F_0 = \langle x, y \rangle \). Moreover 
\[ F'_0 = A, \quad F_0/F'_0 = \langle x F'_0 \rangle \times \langle y F'_0 \rangle \simeq \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^r}, \quad F'_0/\Gamma_3(F_0) = \langle u \Gamma_3(F_0) \rangle, \]
\[ \Gamma_3(F_0) = \langle v_1 \rangle \times \langle v_2 \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p, \quad \Gamma_3(F_0) \leq Z(F_0). \]

We observe that, if \( p \geq 3 \), then \( \langle u \Gamma_3(F_0) \rangle \simeq \mathbb{Z}_{p^r} \), while if \( p = 2 \), then \( \langle u \Gamma_3(F_0) \rangle \simeq \mathbb{Z}_{2^{r-1}} \). By the above conditions we deduce that \( F_0 \) is nilpotent of class three with \( |F_0| = p^{3r+2} \) if \( p \geq 3 \) while, if \( p = 2 \) then \( |F_0| = 2^{3r+1} \).

It remains to show that the exponent of \( F_0 \) is \( p^r \) for all \( p \).
First of all we prove that the exponent of $N$ is $p^r$ for all $p$. (We note that for $p \geq 3$ we have $\exp(N) = p^r$, and for $p \geq 5$ we have $\exp(F_0) = p^r$ by the regularity of these groups).

Let $w \in N$ where $w = ax^k$ with $a \in A$. Since $N$ is of class two we have

$$w^n = (ax^k)^n = a^n x^{kn} [x^k, a]^{\binom{n}{2}}.$$  

Since $[x^k, a] \in \Gamma_3(F_0)$ which has exponent $p$ and $r \geq 2$, we have that $[x^k, a]^{\binom{r}{2}} = 1$. So $(ax^k)^{p^r} = 1$.

If $w$ is an element of $N$ we set

$$[w, y^h] = a_1 \in A, \quad [a_1, w] = c_1 \in \Gamma_3(F_0), \quad [a_1, y^h] = c_2 \in \Gamma_3(F_0).$$

For $n \geq 2$ it is easy to prove by induction the following results

(1) \[ [w, y^{hn}] = a^n c_2^{\binom{n}{2}} \]

and

(2) \[ (w y^h)^n = w^n y^{hn} a_1^{-\binom{n}{2}} c_1^{-\binom{n}{3}} c_2^{-2\binom{n}{3} - \binom{n}{2}}. \]

Since $N$ is of exponent $p^r$ and $a$ has order $2^{r-1}$ for $p = 2$, we have by (2) that the exponent of $F_0$ is $p^r$ for all $p$.

From now on we identify $F_0$ with $F$.

b) By the structure of $F$ we can write an element $z \in F$ in the form

$$z = u^k v_1^i v_2^m x^i y^j.$$  

We have $z \in Z(F)$ if and only if $[z, x] = [z, y] = 1$. So

$$1 = [z, y] = [u^k x^i y^j, y] = [u^k, y][u^k, x^i][x^i, y]$$  

(3) \[ = [u, y]^k [x, y]^i v_1^{\binom{i}{2}} = u^i v_1^{\binom{i}{2}} v_2^{k+ij}. \]

Similarly we have

(4) \[ 1 = [z, x] = u^{-j} v_1^k v_2^{-\binom{j}{2}}. \]

Therefore, for $p \geq 3$ we have $i \equiv j \equiv 0 \pmod{p^r}$ and $k \equiv 0 \pmod{p}$. It follows $z = u^{pk_1} v_1^i v_2^m$ with $k = pk_1$. This implies

$$Z(F) = \langle u^p \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \simeq \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p.$$
If \( p = 2 \) we must have \( i \equiv j \equiv 0 \pmod{2^{r-1}} \) and \( k \equiv 0 \pmod{2} \). So we have \( z = u^{2k_1}v_1^{i_1}v_2^{m}x^{2^{r-1}i_1}y^{2^{r-1}j_1} \) where \( k = 2k_1 \), \( i = 2^{r-1}i_1 \), \( j = 2^{r-1}j_1 \). Then, if \( r \geq 3 \) we get

\[
Z(F) = \langle u^2 \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \times \langle x^{2^{r-1}} \rangle \times \langle y^{2^{r-1}} \rangle \cong \mathbb{Z}_{2^{r-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.
\]

If \( p = 2 \) and \( r = 2 \), we have \( u^2 = 1 \), so \( z = v_1^j v_2^m x^{2i_1} y^{2j_1} \) with \( i = 2i_1 \), \( j = 2j_1 \). But the condition \( \binom{j}{i} \equiv 0 \pmod{2} \) implies \( i_1(2i_1 - 1) \equiv 0 \pmod{2} \). So \( i_1 \equiv 0 \pmod{2} \). Similarly we obtain \( j_1 \equiv 0 \pmod{2} \). Therefore \( z = v_1^i v_2^m \) and

\[
Z(F) = \langle v_1 \rangle \times \langle v_2 \rangle = \Gamma_3(F) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.
\]

c) Observe that for all \( a, b, c \in F \) we have \( [a^p, b, c] = [a, b, c]^p = 1 \). So \( [F^p, F] \leq Z(F) \).

d) We have \( \Phi(F), F = \left[F', F^p, F\right] \). Since \( [F', F] = \Gamma_3(F) \leq Z(F) \) and \( [F^p, F] \leq Z(F) \) by Part c), it follows that \( \Phi(F), F \leq Z(F) \). So by Lemma 2.4 every proper subgroup of \( F \) is nilpotent of class two.

**Theorem 3.3.** Let \( p \) be a prime and \( r \geq 2 \).

a) Let \( G \) be a 2-generator group in \( \text{Min}(\mathcal{N}_2) \) with \( \exp G = p^r \). Then

\[
|G| \leq \begin{cases} 
  p^{3r} & \text{if } p \geq 3, \\
  2^{3(r-1)} & \text{if } p = 2, r \geq 3, \\
  2^{3r-1} & \text{if } p = 2, r = 2.
\end{cases}
\]

b) For each one of the above three cases, there is a group of exponent \( p^r \) in \( \text{Min}(\mathcal{N}_2) \) whose order attains the upper bound.

**Proof.** a) Every 2-generator group \( G \in \text{Min}(\mathcal{N}_2) \) of exponent \( p^r \) is a quotient \( F/H \) of \( F \) where \( H \cap Z(F) \) does not contain \( \Gamma_3(F) \) because \( G \cong F/H \) is of class three. As \( Z(G) \) is cyclic by Lemma 2.1, also \( Z(F)/(H \cap Z(F)) \) must be cyclic. Then \( H \cap Z(F) \) is abelian of rank \( \geq 2 \) if \( p \neq 2 \); of rank \( \geq 4 \) if \( p = 2 \) and \( r \geq 3 \); of rank 1 if \( p = 2 \) and \( r = 2 \). Thus if \( p \geq 3 \) we have \( |H| \geq p^2 \) and \( |G| \leq p^{3r} \); if \( p = 2 \) and \( r \geq 3 \) we have \( |H| \geq 2^4 \) and \( |G| \leq 2^{3(r-1)} \). Finally we observe that, if \( p = 2 \) and \( r = 2 \), no quotient of \( F \) by a proper subgroup of \( Z(F) \) is in \( \text{Min}(\mathcal{N}_2) \). In fact, there are only three proper subgroups of \( Z(F) \), namely \( H_1 = \langle v_1 \rangle \), \( H_2 = \langle v_2 \rangle \), \( H_3 = \langle v_1 v_2 \rangle \). We see that in each quotient \( F/H_i \), \( (i = 1, 2, 3) \) there are couples of independent elements of \( Z(F/H_i) \); for example, \( x^2 H_1, v_2 H_1 \) in \( Z(F/H_1) \); \( y^2 H_2, v_1 H_2 \) in \( Z(F/H_2) \) and \((xy)^2 H_3, v_1 v_2 H_3 \) in \( Z(F/H_3) \). So no \( F/H_i \) belongs to \( \text{Min}(\mathcal{N}_2) \) and therefore \( |G| \leq 2^{3r-1} \).
b) For the first two cases of a) we consider respectively the subgroups of \(Z(F)\):

\[
\begin{align*}
R_1 &= \langle v_2, u^{r-1}v_1 \rangle \quad \text{if } p \geq 3, \; r \geq 2, \\
R_2 &= \langle v_2, v_1u^{2r-2}, v_1x^{2r-1}, v_1y^{2r-1} \rangle \quad \text{if } p = 2, \; r \geq 4, \\
R_3 &= \langle v_2, u^2, x^4, v_1y^4 \rangle \quad \text{if } p = 2, \; r = 3.
\end{align*}
\]

We want to show that \(G_t = F/R_t \in \text{Min}(\mathcal{N}_2) \; (t = 1, 2, 3)\). First, since \(R_t\) does not contain \(\Gamma_3(F)\) it follows that \(G_t\) is of class three. Moreover, as every proper subgroup of \(F\) is of class two, the same holds for \(G_t\). By definition of \(G_t\), we also have \(|\Gamma_3(G_t)| = p\). Therefore, by Lemma 2.1, it is sufficient to show that \(Z(G_t)\) is cyclic.

Let us consider a typical element \(zR_t \in G_t\) with \(z = u^kv_1^l v_2^m x^i y^j \in F\). Then \(zR_t \in Z(F/R_t)\) if and only if \([z, y] \in R_t\) and \([z, x] \in R_t\). By (3) and (4), this holds if and only if

\[
\begin{align*}
u^i v_1^{(l)} v_2^{k+ij} &\in R_t \\
u^{-j} v_1^k v_2^{-(l)} &\in R_t
\end{align*}
\]

For \(p \geq 3\) this happens if and only if there are \(\alpha, \beta \in \mathbb{Z}\) such that

\[
\begin{align*}
u^i v_1^{(l)} &= (u^{p^{r-1}}v_1)^\alpha, \\
u^{-j} v_1^k &= (u^{p^{r-1}}v_1)^\beta.
\end{align*}
\]

By equation (5) we obtain that \(i \equiv \alpha p^{r-1} \pmod{p^r}\) and \(i(\frac{r-1}{2}) \equiv \alpha \pmod{p}\). So

\[
\begin{align*}
i \left(1 - \frac{i - 1}{2}p^{r-1}\right) &\equiv 0 \pmod{p^r}
\end{align*}
\]

which gives \(i \equiv 0 \pmod{p^r}\).

By Equation (6) we get \(-j \equiv p^{r-1}\beta \pmod{p^r}\) and \(k \equiv \beta \pmod{p}\). So

\[
\begin{align*}j &\equiv -p^{r-1}k \pmod{p^r}.
\end{align*}
\]

Therefore, we have that \(zR_1 \in Z(F/R_1)\) if and only if

\[
z = u^k v_1^l y^{p^{r-1}} = (u^{p^{r-1}}v_1^l).
\]

We observe that

\[
(u^{p^{r-1}})^{-p^{r-1}} = u^{-p^{r-1}} y^{p^{2r-2}} = u^{-p^{r-1}}.
\]
Since \( u^{p^{r-1}}v_1 \in R_1 \), we have \( v_1 R_1 = u^{-p^{r-1}}R_1 = (uy^{-p^{r-1}})^{-p^{r-1}}R_1 \). Then 
\( zR_1 = (uy^{-p^{r-1}})^{k-p^{r-1}}R_1 \). Thus \( Z(F/R_1) = \langle uy^{-p^{r-1}} \rangle R_1 \) is cyclic.

If \( p = 2 \) and \( r \geq 3 \) an analogous calculation yields

\[
(9) \quad u^i v_1^{(i/2)} = (u^{2^{r-2}} v_1)^\alpha
\]
and

\[
(10) \quad u^{-j} v_1^k = (u^{2^{r-2}} v_1)^\beta
\]

By (9) and (10) we obtain

\[
i(1 - (i - 1)2^{r-3}) \equiv 0 \pmod{2^{r-1}}
\]
and

\[
j \equiv -2^{r-2}k \pmod{2^{r-1}}.
\]

So if \( r \geq 4 \), we obtain \( i \equiv 0 \pmod{2^{r-1}} \); while if \( r = 3 \) we have \( i \equiv 0 \pmod{2} \).

In the case \( p = 2 \), \( r = 4 \) it follows that \( zR_2 \in Z(F/R_2) \) if and only if 
\( z = (uy^{-2^{r-1}})^k v_1^{r-1} x^{2^{r-1}}y^i \) with \( i = 2^{r-1} - 1 \). Since 
\( (uy^{-2^{r-1}})^2 = u^{-2^{r-2}} \), we have 
\( u^{-2^{r-2}} R_2 = v_1 R_2 = x^{2^{r-1}} R_2 = y^{2^{r-1}} R_2 \). Thus 
\( zR_2 = (uy^{-2^{r-1}})^{k-2^{r-2}(i+1)} \) and 
\( Z(F/R_2) = \langle uy^{-2^{r-2}} \rangle R_2 \) is cyclic.

In the case \( p = 2 \), \( r = 3 \) we have \( zR_3 \in Z(F/R_3) \) if and only if 
\( z = (uy^{-2})^k v_1^i \). Since 
\( (uy^{-2})^2 = u^{-2} y^4 = u^{-2} \) and \( u^{-2} R_3 = v_1 R_3 \), we have 
\( zR_3 = (uy^{-2})^{k-2l} R_3 \). Thus, \( Z(F/R_3) = \langle uy^{-2} R_3 \rangle \) is cyclic.

Finally, in the case \( p = 2 \) and \( r = 2 \), we consider the normal (non central) subgroup 
\( R_4 = \langle v_2, y^2 \rangle \). Then 
\( zR_4 \in Z(F/R_4) \) if and only if 
\( z = u^k v_1^i x^j y^j \) with \( k \equiv 0 \pmod{2} \), 
\( j \equiv 0 \pmod{2} \), \( i \equiv 0 \pmod{2} \) and 
\( j(i-1)2 \equiv 0 \pmod{2} \). The last two conditions implies \( i \equiv 0 \pmod{4} \). Then 
\( zR_4 = v_1^i R_4 \) and thus \( Z(F/R_4) = \langle v_1 \rangle R_4 \) is cyclic.

\[ \square \]

**Theorem 3.4.** Let \( p \) be a prime and \( r \geq 2 \). If \( p \geq 3 \) or \( p = 2 \) and either \( r = 3 \) or \( r = 2 \), then all 2-generator groups in \( \text{Min}(\mathcal{N}_2) \) of exponent \( p^r \) and maximal order are isomorphic.

**Proof.** Using the same notation as in the proof of Theorem 3.3, let \( F/H \in \text{Min}(\mathcal{N}_2) \) be of exponent \( p^r \) \((p \geq 3)\) and maximal order \(|F/H| = p^{3r}\). By the proof of Theorem 3.3 it follows that \( H \simeq \mathbb{Z}_p \times \mathbb{Z}_p \). We will show that there exists
an automorphism \( \varphi \) of \( F \) with \( \varphi(H) = R_1 \) and so \( F/H \cong F/R_1 \). Since \( F/H \) is of nilpotency class three, we have that \( \Gamma_3(F) \not\subseteq H \). As \( Z(F) \) is of rank three and \( H \cong \mathbb{Z}_p \times \mathbb{Z}_p \), we get \( |H \cap \Gamma_3(F)| = p \). We construct the automorphism \( \varphi \) in two steps. First we give an automorphism which maps \( \langle \chi \rangle \) with \( \chi \) and a subgroup \( \langle \psi \rangle \) of nilpotency class three, we have that \( \varphi(\chi) = \chi \) and \( \varphi(\psi) = \psi \). Therefore we may assume that \( \varphi(\chi) = \chi \) and \( \varphi(\psi) = \psi \). Since \( \varphi(S) = S \) and \( \varphi(\psi^2) = \psi^2 \), we get \( |\varphi(\chi) \langle \chi \rangle | = 2^4 \). Finally let \( \varphi(\chi) = \chi \) and \( \varphi(\psi) = \psi \). Then we have \( \varphi(\chi) = \chi \) and \( \varphi(\psi) = \psi \). In both cases we have now found an automorphism of \( F \) which maps \( H \) onto a subgroup \( H^* \) of \( Z(F) \) with

\[
H^* \cap \Gamma_3(F) = \langle v_2 \rangle.
\]

Therefore we may assume that \( H^* = \langle v_2, v^n_1 u^{np^{r-1}} \rangle \) with \( m, n \in \mathbb{Z} \) and \( n \not\equiv 0 \pmod{p} \). Since \( n \not\equiv 0 \pmod{p} \), we have

\[
H^* = \langle v_2, v^n_1 u^{p^{r-1}} \rangle
\]

with \( h \equiv mn^{-1} \pmod{p} \). First let \( h \not\equiv 0 \pmod{p} \). We consider the automorphism \( \gamma \) of \( F \) such that \( \gamma(x) = x^h \) and \( \gamma(y) = y \). We have \( \gamma(v_2) = v_2^h \in H^* \) and

\[
\gamma([x, y] [x, y]^{p^{r-1}}) = [x, y, x]^{h^2} [x, y]^{hp^{r-1}} = ([x, y, x]^{h^2} [x, y]^{hp^{r-1}}) = H^*.
\]

So \( \gamma(v_1 u^{p^{r-1}}) = (v_1^h u^{p^{r-1}})^h \) and \( R^v_1 = H^* \).

Finally let \( h \equiv 0 \pmod{p} \). So \( H^* = \langle v_2, u^{p^{r-1}} \rangle \). Since \( [x^{p^{r-1}}, y] = u^{p^{r-1}} \in H^* \), we have that \( x^{p^{r-1}} H^* \in Z(F/H^*) \). Similarly \( y^{p^{r-1}} H^* \in Z(F/H^*) \). But the images of \( x^{p^{r-1}} \) and \( y^{p^{r-1}} \) under the canonical epimorphism of \( F/H^* \) onto \( F/F' \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^r} \) are independent, and so the center of \( F/H^* \) is not cyclic. This case does not occur.

Let \( F/H \in \text{Min}(N_2) \) be of exponent \( 2^3 \) and maximal order \( 2^6 \). Then \( |H| = 2^4 \) and \( H \) must contain exactly one of the three subgroups \( \langle v_1 \rangle, \langle v_2 \rangle, \langle v_1 v_2 \rangle \) of \( \Gamma_3(F) \). The automorphism \( \alpha \) of \( F \), defined by \( \alpha(x) = y \) and \( \alpha(y) = x^{-1} y^{-1} \), is of order 3 and acts transitively on the non-identity elements of \( \Gamma_3(F) \). So without loss of generality we may assume \( H \cap \Gamma_3(F) = \langle v_2 \rangle \) and \( v_1 \not\in H \). Now consider the intersection of \( H \) with the subgroup \( E = \langle v_1, v_2, u^2 \rangle = \Omega_1(F') \). Since \( E/E \cap H \cong EH/H \leq Z(F/H) \) which is cyclic, we get \( |E \cap H| = 2^2 \).
The subgroups of $E$ of order $2^2$, that contain $v_2$ but not $v_1$ are precisely $L_1 = \langle v_2, u^2 \rangle$ and $L_2 = \langle v_2, u^2 v_1 \rangle$. If $L_2 \leq H$, then $v_1 L_2, x^2 L_2, u y^2 L_2 \in Z(F/L_2)$. So $Z(F/H)$ is not cyclic, because $Z(F/H) \cong Z((F/L_2)/(H/L_2))$ contains $Z(F/L_2)/(H/L_2)$ and $x^2 L_2, u y^2 L_2 \not\in H/L_2$ since $H \leq Z(F)$. Therefore $L_1 \leq H$ and $H/L_1$ is a subgroup of rank 2 of $Z(F/L_1)$ that does not contain $v_1 L_1$. Since $|Z(F)/L_1| = 2^3$, we get the following four subgroups:

$$\begin{align*}
H_1 &= \langle v_2, u^2, v_1 x^4, v_1 y^4 \rangle, & H_2 &= \langle v_2, u^2, x^4, v_1 y^4 \rangle, \\
H_3 &= \langle v_2, u^2, v_1 x^4, y^4 \rangle, & H_4 &= \langle v_2, u^2, x^4, y^4 \rangle.
\end{align*}$$

By a simple calculation, using the relations (3) and (4), we see that $F/H_1$ and $F/H_2$ have cyclic center, while the centers of the two remaining quotients are not cyclic. Finally, the theorem for the case $p = 2$ and $r = 3$ is proved by the automorphism $\beta$ defined by $\beta(x) = xy$, $\beta(y) = y$ that fixes $v_2$ and $u^2$ and maps $H_1$ onto $H_2$.

Let $F/H \in \text{Min}(N_2)$ be of exponent 4 and maximal order $2^5$. Then $|H| = 4$ and $F/H$ is nilpotent of class 3 with cyclic center (see Lemma 2.1). Since $\Gamma_3(F) = \langle v_1, v_2 \rangle \cong \mathbb{Z} \times \mathbb{Z}$, we must have $|H \cap \Gamma_3(F)| = 2$. As in the previous case, without loss we may assume $H \cap \Gamma_3(F) = \langle v_2 \rangle$. Let $L = \langle v_2 \rangle$. It is easy to see that $Z(F/L) = \langle v_1 L \rangle \times \langle y^2 L \rangle \cong \mathbb{Z} \times \mathbb{Z}$. Now $H/L \leq F/L$ and $|H/L| = 2$. If $v_1 L \in H/L$, then $\Gamma_3(F) = \langle v_1, v_2 \rangle \leq L$ and so $F/L$ would be of class two, a contradiction. Hence $v_1 \not\in H/L$, and hence either $H = \langle v_2, y^2 \rangle$ or $H = \langle v_2, v_1 y^2 \rangle$. But the automorphism $\gamma$ of $F$, defined by $\gamma(x) = x$ and $\gamma(y) = x^2 y$, centralizes $\Gamma_3(F)$ and maps $y^2$ to $v_1 y^2$. Therefore all the quotients $F/H \in \text{Min}(N_2)$ of order $2^5$ are isomorphic. 

**Remark 3.1.** In the case $p = 2$ and $r \geq 4$, there are non-isomorphic groups in $\text{Min}(N_2)$ of exponent $2^r$ and maximal order $2^{3(r-1)}$. In fact, the two quotients $F/R_2$ and $F/R_2^*$, where $R_2 = \langle v_2, v_1 u^{2^{r-2}}, v_1 x^{2^{r-1}}, v_1 y^{2^{r-1}} \rangle$ and $R_2^* = \langle v_2, v_1 u^{2^{r-2}}, x^{2^{r-1}}, y^{2^{r-1}} \rangle$, have cyclic center but one can check that the power $2^{r-1}$ of an element $g = u^k v_1^j v_2^m x^i y^j$ in $F$ is

$$g^{2^{r-1}} = (u^k x^i y^j)^{2^{r-1}} = (x^{2^{r-1}})^i (y^{2^{r-1}})^j (u^{-2^{r-2}(2^{r-1})})^i j;$$

so we have

$$g^{2^{r-1}} R_2 = v_1^{i+j+ij} R_2$$

and

$$g^{2^{r-1}} R_2^* = v_1^{ij} R_2^*.$$ 

It follows that the number of the elements of order $2^r$ is different in the two quotients and $F/R_2, F/R_2^*$ are not isomorphic.
Remark 3.2. The referee suggested to investigate the existence of groups in $\text{Min}(N_2)$ of exponent $p^r$ and order $p^k$ for all $k$ with $r + 2 \leq k < 3r$. He gave an example of minimal order $p^{r+2}$. Namely the group:

$$G_1 = \langle \tilde{x}, \tilde{y}, \tilde{u} \mid \tilde{x}^{p^r} = 1 = \tilde{y}^{p^r} = \tilde{u}^p, [\tilde{x}, \tilde{y}] = \tilde{u}, [\tilde{u}, \tilde{x}] = \tilde{x}^{p^{r-1}}, [\tilde{u}, \tilde{y}] = 1 \rangle.$$
We have $G_1 = F/L_1$ where $L_1 = \langle v_2, u^p, x^{p^{r-1}}v_1^{-1}, y^p \rangle$.

An other example of minimal order non-isomorphic to the previous one is given by

$$G_2 = \langle \tilde{x}, \tilde{y}, \tilde{u} \mid \tilde{x}^{p^r} = 1 = \tilde{y}^{p^r} = \tilde{u}^p, [\tilde{x}, \tilde{y}] = \tilde{u}, [\tilde{u}, \tilde{x}] = \tilde{y}^{p^{r-1}}, [\tilde{u}, \tilde{y}] = 1 \rangle;$$
in fact, $G_2$ has an abelian maximal subgroup $\langle \tilde{u}, \tilde{y} \rangle$, while $G_1$ has no abelian maximal subgroup. This is the quotient of $F$ by the subgroup:

$$L_2 = \langle v_2, u^p, x^p, y^{p^{r-1}}v_1^{-1} \rangle.$$ 

Other examples of order $p^{r+\frac{4r+1}{2}}$, with $r = 2h + 1$, are given by splitting meta-cyclic groups:

$$M_h = \langle \tilde{x}, \tilde{y}, \tilde{u} \mid \tilde{x}^{p^{2h+1}} = 1 = \tilde{x}^{p^{h+1}}, [\tilde{y}, \tilde{x}] = \tilde{y}^{p^h} \rangle.$$
These are the quotients of $F$ by the subgroups:

$$N_h = \langle v_2, uy^h, x^{p^{h+1}}, v_1y^{p^{2h}} \rangle.$$

The problem of the existence of groups in $\text{Min}(N_2)$ of order other than of the maximal one seems of non easy solution. We have to construct quotients $F/L$ of $F$ with cyclic center. Considering the automorphisms $\alpha$ and $\beta$ used in the proof of the Theorem 3.4, we can assume, W.L.O.G., that $L \geq H^* = \langle v_2, u^{p^{r-1}} \rangle$.

We prove that the orders of such quotients cannot be greater than $p^{2r+1}$. Since $Z(F/H^*) \cong \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $L$ has to contain a subgroup isomorphic to $\mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p$. In fact

$$F/L \cong (F/H^*)/(L/H^*)$$
and

$$Z(F/L) \geq (Z(F/H^*)(L/H^*))/(L/H^*);$$
since both

$$Z(F/H^*) = \langle u^{pH^*}, v_1H^*, x^{p^{r-1}}H^*, y^{p^{r-1}}H^* \rangle$$
and

$$(Z(F/H^*)(L/H^*))/(L/H^*)$$
has to be cyclic, it follows that $L/H^*$ has to contain a complement of $\langle v_1H^* \rangle$ in $Z(F/H^*)$. Thus $|L| \geq p^{r+1}$ and $|F/L| \leq p^{2r+1}$. 

4. \( \text{Min}(N_2) \)-groups with three generators

It follows from Lemma 2.3 that a group \( G \in \text{Min}(N_2) \), with three generators and exponent \( 3^r \) \((r \geq 1)\), belongs to the variety \( V \) of all 2-Engel groups of exponent \( 3^r \). So \( G \) is a quotient of \( Fr_3(V) \).

**Proposition 4.1.** Let \( F = Fr_3(V) \) be the relatively free group with free generators \( x, y, z \) in the variety \( V \).

a) \( |\Gamma_3(F)| = 3 \) and \( |F| = 3^{6r+1} \).

b) \( Z(F) \cong Z_{3^r-1} \times Z_{3^r-1} \times Z_{3^r-1} \times Z_3 \).

c) Every proper subgroup of \( F \) is nilpotent of class two.

d) \( F \) belongs to \( \text{Min}(N_2) \) if and only if \( r = 1 \).

e) Let \( F/H \) be a quotient of \( F \) of class three. Then \( F/H \in \text{Min}(N_2) \) if and only if \( Z(F/H) \) is cyclic.

**Proof.** a) Note that \( F/F' \) is a 3-generated group of exponent \( 3^r \), so \( |F/F'| \leq 3^{3r} \). Similarly, we have \( |F'/\Gamma_3(F)| \leq 3^{3r} \). Now we show that \( |\Gamma_3(F)| = 3 \). In fact, \( \Gamma_3(F) \) is generated by the basic commutators of weight three and, as \( F \) is 2-Engel, they are all equal to 1, except at most \([y, x, z]\) and \([z, x, y]\) (see, for example [2, p. 54]). Moreover, in a 2-Engel group \( G \), for all \( x_1, x_2, x_3 \in G \) the following conditions hold:

i) \([x_1, x_3, x_2] = [x_1, x_2, x_3]^{-1} \),

ii) \([x_1^{-1}, x_2] = [x_1, x_2^{-1}] = [x_1, x_2]^{-1} \)

(see (2) and (3) in the proof of Satz 6.5 in [1, p. 288]).

So we get

\[
[z, x, y] = [[x, z]^{-1}, y] \quad \text{by ii)} \\
\quad = [x, z, y]^{-1} \quad \text{by i)} \\
\quad = [x, y, z] = [[y, x]^{-1}, z] \quad \text{by ii)} \\
\quad = [y, x, z]^{-1}.
\]

Hence \( \Gamma_3(F) = \langle [x, y, z] \rangle \) is cyclic of order 3 (see [4, p. 358]) and \( |F| \leq 3^{3r+1} \).

We now construct a group \( F_0 \), belonging to the variety \( V \), which has order \( 3^{3r+1} \). Then it follows that \( F_0 \cong F \) and \( |F| = 3^{3r+1} \).

Let \( A \) be the abelian group of exponent \( 3^r \) defined by

\[
A = \langle z \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \times \langle v_3 \rangle \cong Z_{3^r} \times Z_{3^r} \times Z_{3^r} \times Z_3
\]

and let \( Q \) be the group of exponent \( 3^r \) and of nilpotency class 2 defined by

\[
Q = \langle x, y \mid x^{3^r} = y^{3^r} = 1, u = [x, y], u^{3^r} = 1, [u, x] = [u, y] = 1 \rangle.
\]
Let $F_0 = [A]Q$ be the semidirect product of $A$ and $Q$ with the action of $Q$ on $A$ defined by

$$
\begin{align*}
  z^x &= zv_2^{-1}, & v_1^x &= v_1v_3, & v_2^x &= v_2, & v_3^x &= v_3, \\
  z^y &= zv_1, & v_1^y &= v_1, & v_2^y &= v_2v_3, & v_3^y &= v_3.
\end{align*}
$$

Since

$$u = [x, y], \quad v_1 = [z, y], \quad v_2 = [x, z], \quad v_3 = [v_1, x] = [v_2, y]$$

we obtain that $F_0 = \langle x, y, z \rangle$ and we have $|F_0| = |A||Q| = 3^{3r+1}3^r = 36r+1$.

Also we have

$$[z, u] = v_3, \quad [u, v_1] = [u, v_2] = [u, v_3] = 1.$$ 

So $F_0' = \langle u, v_1, v_2, v_3 \rangle$ and $\Gamma_3(F_0) = \langle v_3 \rangle$ is of order 3. Therefore $F_0$ is nilpotent of class 3.

To prove a) we only need to show that the group $F_0$ we have constructed belongs to the variety $V$. In other words, we have to show that $F_0$ is a 2-Engel group of exponent $3^r$. Since the right 2-Engel elements form a subgroup of a group (see [3]), it is sufficient to check that the generators $x, y, z$ of $F_0$ are right 2-Engel elements. In fact, by the definition of $F_0$, it is easy to see that the basic commutators of weight three on the generators, are the following:

$$[x, y, y] = [x, y, x] = [z, x, x] = [z, y, z] = [z, y, y] = [z, x, z] = 1$$

$$[x, y, z] = v_3^{-1}, \quad [z, y, x] = v_3.$$

We observe that $v_3 \in Z(F_0)$ by (11) and (12). Then it follows that $F_0$ is nilpotent of class 3 and $\Gamma_3(F_0) = \langle v_3 \rangle$ is of order 3.

Moreover, since $A$ is abelian, the relations (11), (12) yield

$$\begin{align*}
  [x^\alpha, z^\alpha] &= v_2^{\alpha\alpha}, & [v_1^b, x^\alpha] &= v_3^{b\alpha}, & [z^a, y^\beta] &= v_1^{a\beta} \\
  [v_2^c, y^\beta] &= v_3^{c\beta}, & [u^y, z^a] &= v_3^{-a\gamma}
\end{align*}$$

where $a, b, c, \alpha, \beta, \gamma$ belong to $\mathbb{Z}_{3^r}$. Using the above relations, we can directly check that for all $g \in F_0$ we have

$$[x, g, g] = [y, g, g] = [z, g, g] = 1.$$ 

Write $g = vw$ with $v \in A$ and $w \in Q$. Since $Q$ is of class 2 and $A$ is abelian, we have $[x, w, w] = [x, v, v] = 1$. So

$$[x, g, g] = [x, v, w][x, w, v].$$
Letting $w = y^j s$, where $s \in \langle x, u \rangle$, and $v = z^j \widehat{v}$, where $\widehat{v} \in \langle v_1, v_2, v_3 \rangle$, the relations displayed in (11), (12), and (13) yield

$$[x, v, w] = [x, z^j, w] = [v^j_2, y^i] = v^{ij}_3$$

and

$$[x, w, v] = [x, y^i, v] = [u^i, z^j] = v^{-ij}_3.$$  

So $[x, g, g] = 1$.

The proof that $y$ is right 2-Engel is analogous.

For $z$ we observe that, since $A$ is abelian and $[z, Q]$ is contained in $A$, we have

$$[z, v, v] = [z, w, v] = z, w, v = 1.$$  

Moreover, letting $w = x^h y^i u^k$, by relations (11), (12), and (13) we have

$$[z, w] = [z, x^h y^i u^k] = [z, y^i][z, x^h]c, \quad c \in Z(F_0).$$

It follows that

$$[z, w, w] = [v^i_1 v^j_{2h}, x^h y^i] = v^{hi}_3 v^{-hi}_3 = 1.$$  

It remains to check that the exponent of $F_0$ is $3^r$. By the Hall-Petrescu identity (see [1, p. 317]) we have

$$g^{3^r} = (wv)^{3^r} = v^{3^r} w^{3^r} c_1^{(3^r)} c_2^{(3^r)},$$

where $c_1 \in F_0'$ and $c_2 \in \gamma_3(F_0) = \langle v_3 \rangle$. Since $Q, A, F_0'$ are of exponent $3^r$ and $|\Gamma_3(F_0)| = 3$, we have $(wv)^{3^r} = 1$.

We can now identify $F$ with $F_0$.

b) By the relations (11) and (12) we have that $u^3, v_1^3, v_2^3, v_3 \in Z(F)$.

Conversely, computing the commutators between an element $g = vw = z^a v^b_1 v^c_2 v^d_3 x^\alpha y^\beta u^\gamma$ and the generators $x, y, z$ of $F$, we obtain

\begin{align*}
\text{(14)} & \quad [x, vw] = [x, w][x, v][x, v, w] = u^\beta v^a_2 v^b_3 v^c_1 x^\alpha y^\beta u^\gamma, \\
\text{(15)} & \quad [y, vw] = [y, w][y, v][y, v, w] = u^{-\alpha} v^a_1 v^c_3 v^d_1 v^b_3 x^\alpha, \\
\text{(16)} & \quad [z, vw] = [z, w][z, x^\alpha y^\beta u^\gamma] = [z, u^\gamma][z, y^\beta] x^\alpha[z, x^\alpha] = v^3_1 v^2_2 v^3_3 u^3 y^1.
\end{align*}

It follows that $g \in Z(F)$ only if $a \equiv \beta \equiv \alpha \equiv 0 \pmod{3^r}$ and $b \equiv c \equiv \gamma \equiv 0 \pmod{3}$. So the elements of $Z(G)$ have the following form

$$g = v^3_1 v^3_2 v^3_3 u^3 v^1_1.$$
where \( b_1, c_1, \gamma_1 \in \mathbb{Z}_{3^{r-1}} \) and \( d \in \mathbb{Z}_3 \). Thus

\[
Z(F) = \langle v_1^3 \rangle \times \langle v_2^3 \rangle \times \langle v_3 \rangle \times \langle u^3 \rangle.
\]

c) It is sufficient to show that every maximal subgroup \( M \) of \( F \) is of class two. As \( F/\Phi(F) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \), we have \( M = \langle \Phi(F), x_1, x_2 \rangle \) for some \( x_1, x_2 \in M \). We want to show that \( M' = \langle [x_1, x_2], [x_i, F'], [x_i, F^3], \Phi(F)' \rangle \) \( (i = 1, 2) \) is contained in \( Z(M) \). In fact, \( F \) is nilpotent of class 3, so \( [x_i, F'] \leq Z(F) \). We observe that \( Z(F) = \langle v_3, F'^3 \rangle \) is contained in \( M \) and then \( Z(F) \leq Z(M) \). Therefore \( [x_i, F'] \leq Z(M) \). Since the identity \( [g_1, g_2^n] = [g_1, g_2]^n \) holds in the 2-Engel group \( F \), for all \( n \in \mathbb{Z} \) and \( g_1, g_2 \in F \), we have \( [x_i, F^3] = [x_i, F]^{3} \leq F'^3 \leq Z(M) \). In the same way we see that \( \Phi(F)' \leq Z(M) \). Finally \( [x_1, x_2] \in Z(M) \) because \( [x_1, x_2, x_1] = [x_1, x_2, x_2] = 1 \) holds in the 2-Engel group \( F \).

d) Suppose \( r > 1 \), then \( v_1^3, v_2^3, u^3 \) belong to \( Z(F) \) (see b)). So \( Z(F) \) is not cyclic, contradicting Lemma 2.1.

Conversely, let \( r = 1 \), then \( Z(F) = \langle v_3 \rangle \) is cyclic of order three. So, by Lemma 2.1 and c), \( F \) belongs to \( \text{Min}(N_2) \).

e) Let \( L = F/H \) be a quotient of \( F \) of class precisely three. If \( M/H \) is a maximal subgroup of \( L \), then \( M \) is a maximal subgroup of \( F \) and, by c), it is nilpotent of class two. Since \( \Gamma_3(L) = \Gamma_3(F)H/H \) is cyclic of order 3, by Lemma 2.1, \( L \in \text{Min}(N_2) \) if and only if \( Z(L) \) is cyclic.

**Proposition 4.2.** a) Let \( G \) be a 3-generated group in \( \text{Min}(N_2) \) with \( \exp(G) = 9 \). Then \( |G| \leq 3^7 \).

b) There are at least two non-isomorphic groups in \( \text{Min}(N_2) \) of exponent 9 and order 3.

**Proof.** a) Using the same notation as in the previous theorem, \( G \) has to be isomorphic to a quotient \( F/H \) of the relatively free group \( F \) with \( \exp(F) = 3^2 \). Since \( F/H \) has to be nilpotent of class 3, we have \( v_3 \notin H \). As \( Z(F/H) \) must be cyclic and \( Z(F) \) is elementary abelian of rank 4, then \( H \) must contain a subgroup \( K \) of \( Z(F) \) which is of rank 3 and \( v_3 \notin H \). Now \( Z(F) \) contains 40 subgroups of index 3. Among these, 13 contain \( v_3 \). So there are 27 subgroups of \( Z(F) \) which do not contain \( \langle v_3 \rangle \). The subgroup \( K_1 = \langle v_1^3, v_2^3, u^3 \rangle = (F')^3 \) is characteristic in \( F \) and the other 26 form a single orbit under the automorphism \( \varphi \) of \( F \) defined by

\[
x^\varphi = y, \quad y^\varphi = z, \quad z^\varphi = x^{-1}y.
\]
In fact consider the subgroup $K_2 = \langle v_1^3, v_2^3 v_3^{-1}, u^3 \rangle$ of $Z(F)$. We observe that $v_1\phi = v_1^{-1} v_2^{-1}$, $v_2\phi = u$, $u\phi = v_1^{-1}$. A straightforward calculation shows that $K_{2}^{\phi^3} = \langle v_1^3, v_2^3 v_3, u^3 \rangle \neq K_2$. As $\phi$ is an automorphism of order 26, the orbit of $K_2$ has length 26. So we may assume that $H$ contains one of the two subgroups $K_i$, $i = 1, 2$. Consider $F/K_1$. A generic element of $K_1$ can be written in the form

$$v_1^{3l}v_2^{3m}u^3n \quad \text{with} \quad l, m, n \in \{0, 1, 2\}.$$ 

From the relations (14), (15), and (16) we get that an element

$$gK_1 = z^a v_1^b v_2^c v_3^d x^\alpha y^\beta u^\gamma K_1$$

of $F/K_1$ belongs to $Z(F/K_1)$ if and only if

$$v_2^a v_3^{\beta-b} u^\beta = v_1^{3l_1} v_2^{3m_1} u^{3n_1},$$

$$v_1^{-a} v_3^{-c-\alpha} u^{-\alpha} = v_1^{3l_2} v_2^{3m_2} u^{3n_2},$$

$$v_1^{\beta} v_2^{-a} v_3^{y-\alpha\beta} = v_1^{3l_3} v_2^{3m_3} u^{3n_3},$$

for some $l_i, m_i, n_i \in \{0, 1, 2\}$; $i = 1, 2, 3$. It follows $a \equiv \beta \equiv b \equiv c \equiv \alpha \equiv \gamma \equiv 0 \pmod{3}$. Let $a = 3a_1$, $b = 3b_1$, $c = 3c_1$, $\alpha = 3\alpha_1$ and $\gamma = 3\gamma_1$. Then

$$gK_1 = z^{3a_1} v_1^{3b_1} v_2^{3c_1} v_3^{d} x^{3\alpha_1} y^{3\beta_1} u^{3\gamma_1} K_1 = z^{3a_1} v_3^{d} x^{3\alpha_1} y^{3\beta_1} K_1.$$ 

In a similar way we see that $gK_2 \in Z(F/K_2)$ if and only if $\beta \equiv c \equiv \alpha \equiv 0 \pmod{3}$ and $a \equiv 3b \pmod{9} \alpha \equiv 3\gamma \pmod{9}$. If $\beta = 3\beta_1$, $c = 2c_1$, and $\alpha = 3\alpha_1$, we have that

$$gK_2 = z^{3b} v_1^{b} v_2^{3c_1} v_3^{d} x^{3\gamma} y^{3\beta_1} u^\gamma K_2 = (z^{3} v_1)^b v_3^{d+c_1} (x^{3} u)^\gamma y^{3\beta_1} K_2.$$ 

Therefore $Z(F/K_1)$ and $Z(F/K_2)$ are abelian groups which can be represented as direct product

$$Z(F/K_1) = \langle z^3 K_1 \rangle \times \langle v_3 K_1 \rangle \times \langle x^3 K_1 \rangle \times \langle y^3 K_1 \rangle$$

and

$$Z(F/K_2) = \langle z^3 v_1^b K_2 \rangle \times \langle v_3 K_2 \rangle \times \langle x^3 u K_2 \rangle \times \langle y^3 K_2 \rangle$$

In order that a quotient $(F/K_1)/(H/K_i)$, $(i = 1, 2)$ of $F/K_i$ would be nilpotent of class 3 with cyclic center, we need that $v_3 K_i \neq Z(F/K_i)$ and that $H/K_i$ would contains a subgroup of rank 3 of $Z(F/K_i)$. So the order of a group $F/H \in \text{Min}(N_2)$ is at most $3^7$. 

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b) Consider the subgroups

\[ H_1 = \langle v_1^3, v_2^3, u^3, x^3, y^3, z^3 v_3^{-1} \rangle \quad \text{and} \quad H_2 = \langle v_1^3, v_2^3 v_3^{-1}, u^3, x^3 u, y^3, z^3 v_1 \rangle \]

which contain \( K_1 \) and \( K_2 \), respectively. By the same argument used above to determine the center of \( F/K_1 \), one can check easily that the center of \( Z(F/H_1) \) is cyclic. If \( g = z^a v_1^b v_2^c x^d y^e u^f \) is, as before, a generic element of \( F \), we have

\[ g^3 = z^{3a} v_1^{3(b-a\beta)} v_2^{3(c+a\alpha)} x^{3\alpha} y^{3\beta} u^{3(y-\alpha\beta)}. \]

Using this relation we see that the exponent of \( F/H_1 \) is 9. Moreover we see that the \( \mathcal{U}_1(F/H_1) = \langle v_3 H_1 \rangle \) while \( \mathcal{U}_1(F/H_2) = \langle v_1 H_2, v_3 H_2, u H_2 \rangle \) which is not cyclic.

References


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