Berezin transform and Stratonovich–Weyl correspondence for the multi-dimensional Jacobi group

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ABSTRACT – We study the Berezin transform and the Stratonovich–Weyl correspondence associated with a holomorphic representation of the multi-dimensional Jacobi group.


KEYWORDS. Berezin quantization, Berezin transform, quasi-Hermitian Lie group, unitary representation, holomorphic representation, reproducing kernel Hilbert space, multi-dimensional Jacobi group, Stratonovich–Weyl correspondence.

1. Introduction

This paper is part of a program to study Berezin transforms and Stratonovich–Weyl correspondences associated with holomorphic representations. The notion of Stratonovich–Weyl correspondence was introduced in [31] in order to extend the usual Weyl correspondence between functions on $\mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$ (see [1] and [21]) to the general setting of a Lie group acting on a homogeneous space. Stratonovich–Weyl correspondences were systematically studied by J. M. Gracia-Bondía, J. C. Várilly, and various co-workers, see in particular [23], [20], [18], and [22]. The following definition is taken from [22].

DEFINITION 1.1. Let $G$ be a Lie group and $\pi$ a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Let $M$ be a homogeneous $G$-space and $\mu$ a (suitably normalized) $G$-invariant measure on $M$. Then a Stratonovich–Weyl correspondence for the triple $(G, \pi, M)$ is an isomorphism $W$ from a vector space of operators on $\mathcal{H}$ to a vector space of (generalized) functions on $M$ satisfying the following properties:

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(1) \( W \) maps the identity operator of \( \mathcal{H} \) to the constant function \( 1 \);

(2) **Reality**: the function \( W(A^*) \) is the complex-conjugate of \( W(A) \);

(3) **Covariance**: we have \( W(\pi(g) A \pi(g)^{-1})(x) = W(A)(g^{-1} \cdot x) \);

(4) **Unitarity**: we have
\[
\int_M W(A)(x) W(B)(x) \, d\mu(x) = \text{Tr}(AB)
\]

In this context, \( M \) is generally a coadjoint orbit of \( G \) which is associated with \( \pi \) by the Kirillov–Kostant method of orbits [25]. For instance, consider the case when \( G \) is the \((2n + 1)\)-dimensional Heisenberg group \( H_n \). Each non-degenerate coadjoint orbit \( M \) of \( G \) is then diffeomorphic to \( \mathbb{R}^{2n} \) and is associated with a Schrödinger representation \( \pi \) of \( H_n \) on \( L^2(\mathbb{R}^n) \). In this case, the classical Weyl correspondence gives a Stratonovich–Weyl correspondence for the triple \((H_n, \pi, M)\), [21] and [22].

In the case when \( G \) is a quasi-Hermitian Lie group and \( \pi \) is a unitary representation of \( G \) (on a Hilbert space \( \mathcal{H} \)) which is holomorphically induced from a unitary character of a compactly embedded subgroup \( K \) of \( G \), we can apply an idea of [20] and we obtain a Stratonovich–Weyl correspondence by modifying suitably the Berezin correspondence \( S \) [14] (see also [2] and [3]).

More precisely, recall that \( S \) is an isomorphism from the Hilbert space of all Hilbert-Schmidt operators on \( \mathcal{H} \) (endowed with the Hilbert-Schmidt norm) onto a space of square integrable functions on a homogeneous complex domain [32]. The map \( S \) satisfies (1), (2), and (3) of Definition 1.1 but not (4). A Stratonovich–Weyl correspondence \( W \) is then obtained by taking the isometric part in the polar decomposition of \( S \), that is, \( W := (SS^*)^{-1/2}S \). Let us mention that \( B := SS^* \) is then the so-called Berezin transform which have been studied by many authors, see in particular [19], [27], [28], [32], and [33].

In [14], we considered the case when the Lie algebra \( \mathfrak{g} \) of \( G \) is reductive. In this case, we proved that \( B \) can be extended to a class of functions which contains \( S(d\pi(X_1X_2\cdots X_p)) \) for \( X_1, X_2, \ldots, X_p \in \mathfrak{g} \) and that the restrictions to each simple ideal of \( \mathfrak{g} \) of the mappings \( X \to S(d\pi(X)) \) and \( X \to W(d\pi(X)) \) are proportional (see also [12] and [13]).

The case when \( \mathfrak{g} \) is not reductive is more delicate. In [16] we investigated the case of the diamond group and, in [17], we studied \( B \) and \( W \) in the case of the Jacobi group.
The aim of the present paper is to generalize the results of [17] to the case of the multi-dimensional Jacobi group, which is technically more complicated. The multi-dimensional Jacobi group plays a central role in different areas of Mathematics and Physics and its holomorphic unitary representations were studied intensively, see [26], [9], [10], [4], and [6]. In particular, the metaplectic factorization should be used to reduce the study of the highest weight representations of a quasi-Hermitian Lie group to that of some generalized multi-dimensional Jacobi group [26]. Then the study of the case of the multi-dimensional Jacobi group can be considered as a first step towards the general case.

In this paper, we begin by some generalities on the multi-dimensional Jacobi group (Section 2) and its holomorphic representations (Section 3). Then we introduce the Berezin correspondence \( S \), the Berezin transform \( B \) and the Stratonovich–Weyl correspondence \( W \) (Section 4). In Section 5, we show that, under some technical assumptions, the Berezin transform of \( S(d\pi(X_1X_2\ldots X_p)) \) is well-defined for each \( X_1, X_2, \ldots, X_p \in g \). In Section 6, we identify a class of functions which is stable under \( B \) and contains \( S(d\pi(X)) \) for each \( X \in g \). We also give an expression of \( W(d\pi(X)) \) in terms of some integrals of Hua’s type (see [24]).

2. The multi-dimensional Jacobi group

The material of this section and of the following section is essentially taken from [21], Chapter 4, [26], Chapters VII and XII and [15].

Consider the symplectic form \( \omega \) on \( \mathbb{C}^n \times \mathbb{C}^n \) defined by

\[
\omega(((z, w), (z', w'))) = \frac{i}{2} \sum_{k=1}^{n} (z_k w_k' - z_k' w_k).
\]

for \( z, w, z', w' \in \mathbb{C}^n \). The \((2n + 1)\)-dimensional real Heisenberg group is

\[
H := \{(z, \bar{z}), c): z \in \mathbb{C}^n, c \in \mathbb{R}\}
\]

endowed with the multiplication

\[
((z, \bar{z}), c) \cdot ((z', \bar{z}'), c') = \left( (z + z', \bar{z} + \bar{z}'), c + c' + \frac{1}{2} \omega((z, \bar{z}), (z', \bar{z}')) \right).
\]

Then the complexification \( H^c \) of \( H \) is

\[
H^c := \{(z, w), c): z, w \in \mathbb{C}^n, c \in \mathbb{C}\}
\]

and the multiplication of \( H^c \) is obtained by replacing \((z, \bar{z})\) by \((z, w)\) and \((z', \bar{z}')\) by \((z', w')\) in the preceding equality. We denote by \( \mathfrak{h} \) and \( \mathfrak{h}^c \) the Lie algebras of \( H \) and \( H^c \).
Now consider the group $S := \text{Sp}(n, \mathbb{C}) \cap SU(n, n) \simeq \text{Sp}(n, \mathbb{R})$, see [26], p. 501, and [21], p. 175. Then $S$ consists of all matrices

$$h = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \quad P, Q \in M_n(\mathbb{C}), \quad PP^* - QQ^* = I_n, \quad PQ^t = QP^t$$

and $S^c = \text{Sp}(n, \mathbb{C})$.

The group $S$ acts on $H$ by

$$h \cdot ((z, \bar{z}), c) = (h(z, \bar{z}), c) = (Pz + Q\bar{z}, \bar{Q}z + \bar{P}\bar{z}, c)$$

where the elements of $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}^n$ are considered as column vectors. Then we can form the semi-direct product $G := H \times S$ called the multi-dimensional Jacobi group. The elements of $G$ can be written as $((z, \bar{z}), c, h)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $h \in S$. The multiplication of $G$ is thus given by

$$((z, \bar{z}), c, h)\cdot((z', \bar{z}'), c', h') = \left((z, \bar{z}) + h(z', \bar{z}'), c + c' + \frac{1}{2}\omega((z, \bar{z}), h(z', \bar{z}')), hh'\right).$$

The complexification $G^c$ of $G$ is then the semi-direct product

$$G^c = H^c \rtimes \text{Sp}(n, \mathbb{C})$$

whose elements can be written as $((z, w), c, h)$ where $z, w \in \mathbb{C}^n$, $c \in \mathbb{C}$, $h \in \text{Sp}(n, \mathbb{C})$ and the multiplication of $G^c$ is obtained by replacing $\bar{z}$ and $\bar{z}'$ by $w$ and $w'$ in the preceding formula.

We denote by $s, s^c, g$ and $g^c$ the Lie algebras of $S, S^c, G$ and $G^c$. The Lie brackets of $g^c$ are given by

$$[[((z, w), c, A), ((z', w'), c', A')]] = (A(z', w') - A'(z, w), \omega((z, w), (z', w'))), [A, A']).$$

Let $\theta$ denotes conjugation over the real form $g$ of $g^c$. For $X \in g^c$, we set $X^* = -\theta(X)$. We can easily verify that if $X = ((z, w), c, \begin{pmatrix} A & B \\ C & -A' \end{pmatrix}) \in g^c$ then we have

$$X^* = \left((-\bar{w}, -\bar{z}), -\bar{c}, \begin{pmatrix} \bar{A} & -\bar{B} \\ -\bar{C} & \bar{A} \end{pmatrix} \right).$$

Also, we denote by $g \rightarrow g^*$ the involutive anti-automorphism of $G^c$ which is obtained by exponentiating $X \rightarrow X^*$ to $G^c$.

Let $K$ be the subgroup of $G$ consisting of all elements $((0, 0), c, \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix})$ where $c \in \mathbb{R}$ and $P \in U(n)$. Then the Lie algebra $\mathfrak{k}$ of $K$ is a maximal compactly embedded subalgebra of $g$ and the subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ consisting of all elements
of the form \(((0,0), c, A)\) where \(A\) is diagonal is a compactly embedded Cartan subalgebra of \(g\) [26], p. 250. Following [26], p. 532, we set

\[
p^+ = \left\{ \left( (y, 0), 0, \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \right) : y \in \mathbb{C}^n, Y \in M_n(\mathbb{C}), Y^t = Y \right\}
\]

and

\[
p^- = \left\{ \left( (0, v), 0, \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix} \right) : v \in \mathbb{C}^n, V \in M_n(\mathbb{C}), V^t = V \right\}.
\]

Then we have the decomposition \(g^c = p^+ \oplus \mathfrak{t}^c \oplus p^-\).

Henceforth we denote by \(a(y, Y)\) the element \(\left( (y, 0), 0, \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \right)\) of \(p^+\). Also, we denote by \(p_{p^+}, p_{\mathfrak{t}^c}\) and \(p_{p^-}\) the projections of \(g^c\) onto \(p^+, \mathfrak{t}^c\) and \(p^-\) associated with the above direct decomposition.

Let \(P^+\) and \(P^-\) be the analytic subgroups of \(G^c\) with Lie algebras \(p^+\) and \(p^-\). Then we have

\[
P^+ = \left\{ \left( (y, 0), 0, \begin{pmatrix} I_n & Y \\ 0 & I_n \end{pmatrix} \right) : y \in \mathbb{C}^n, Y \in M_n(\mathbb{C}), Y^t = Y \right\}
\]

and

\[
P^- = \left\{ \left( (0, v), 0, \begin{pmatrix} I_n & 0 \\ V & I_n \end{pmatrix} \right) : v \in \mathbb{C}^n, V \in M_n(\mathbb{C}), V^t = V \right\}.
\]

In particular, we see that \(G\) is a group of the Harish-Chandra type [26], p. 507 (see also [30]), that is, the following properties are satisfied:

1. \(g^c = p^+ \oplus \mathfrak{t}^c \oplus p^-\) is a direct sum of vector spaces, \((p^+)^* = p^-\) and \([\mathfrak{t}^c, p^+] \subset p^-\);
2. the multiplication map \(P^+ K c P^- \to G^c, (z, k, y) \to zky\) is a biholomorphic diffeomorphism onto its open image;
3. \(G \subset P^+ K c P^-\) and \(G \cap K c P^- = K\).

We can easily verify that \(g = ((z_0, w_0), c_0, \left( \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)) \in G^c\) has a \(P^+ K c P^-\)-decomposition

\[
g = \left( (y, 0), 0, \begin{pmatrix} I_n & Y \\ 0 & I_n \end{pmatrix} \right) \cdot \left( (0, 0), c, \begin{pmatrix} P & 0 \\ 0 & (P^t)^{-1} \end{pmatrix} \right) \cdot \left( (0, v), 0, \begin{pmatrix} I_n & 0 \\ V & I_n \end{pmatrix} \right)
\]

if and only if \(\text{Det}(D) \neq 0\) and, in this case, we have \(y = z_0 - BD^{-1}w_0, Y = BD^{-1}, v = D^{-1}w_0, V = D^{-1}C, P = A - BD^{-1}C = (D^t)^{-1}\) and \(c = c_0 - (1/4)i(z_0 - BD^{-1}w_0)^t w_0\).

We denote by

\[
\zeta: P^+ K c P^- \to P^+, \quad \kappa: P^+ K c P^- \to K c, \quad \eta: P^+ K c P^- \to P^-
\]

the projections onto \(P^+, K c\) and \(P^-\)-components.
We can introduce an action (defined almost everywhere) of \( G^c \) on \( p^+ \) as follows. For \( Z \in p^+ \) and \( g \in G^c \) with \( g \exp Z \in P^+ K^c P^- \), we define the element \( g \cdot Z \) of \( p^+ \) by
\[
g \cdot Z := \log(\exp Z).
\]
From the above formula for the \( P^+ K^c P^- \)-decomposition, we deduce that the action of \( g = ((z_0, w_0), c_0, (A \frac{P}{Q}, B)) \in G^c \) on \( a(y, Y) \in p^+ \) is given by
\[
g \cdot a(y, Y) = a(y', Y')
\]
where \( Y' := (AY + B)(CY + D)^{-1} \) and
\[
y' := z_0 + Ay - (AY + B)(CY + D)^{-1}(w_0 + Cy).
\]
This implies that
\[
\mathcal{D} := G \cdot 0 = \{a(y, Y) \in p^+: I_n - Y \tilde{Y} > 0\} \cong \mathbb{C}^n \times \mathcal{B},
\]
where \( \mathcal{B} := \{Y \in M_n(\mathbb{C}) : Y' = Y, I_n - Y \tilde{Y} > 0\} \).

Now we introduce a useful section \( Z \to g_Z \) for the action of \( G \) on \( \mathcal{D} \). Let \( Z = a(y, Y) \in \mathcal{D} \). Define \( g_z := ((z_0, z_0), 0, (\frac{P}{Q}, \frac{Q}{P})) \in G \) as follows. We set
\[
z_0 = (I_n - Y \tilde{Y})^{-1}(y + Y \tilde{y}), \quad P = (I_n - Y \tilde{Y})^{-1/2}, \quad Q = (I_n - Y \tilde{Y})^{-1/2}Y.
\]
Then one has \( g_Z \cdot 0 = Z \).

From the above formula for the action of \( G \) on \( \mathcal{D} \), we can deduce the \( G \)-invariant measure \( \mu \) on \( \mathcal{D} \). Let \( \mu_L \) be the Lebesgue measure on \( \mathcal{D} \cong \mathbb{C}^n \times \mathcal{B} \). Thus, we easily obtain that \( d\mu(Z) = \text{Det}(I_n - Y \tilde{Y})^{-(n+2)} d\mu_L(y, Y) \), see for instance [5]. This result can be also deduced from the general formula for the invariant measure, see [26], p. 538.

In the rest of the paper, we fix the normalization of the Lebesgue measure as follows. For \( y \in \mathbb{C}^n \), write \( y = (a_1 + ib_1, a_2 + ib_2, \ldots, a_n + ib_n) \) with \( a_j, b_j \in \mathbb{R} \) for \( j = 1, 2, \ldots n \). Then we take the measure Lebesgue on \( \mathbb{C}^n \) to be \( dy := da_1 db_1 da_2 db_2 \ldots da_n db_n \). Similarly, writing \( Y \in \mathcal{B} \) as \( Y = (y_{kl}) \), we denote by \( dY \) the Lebesgue measure on \( \mathcal{B} \) defined by \( dY := \prod_{kl} dy_{kl} \). Thus we set \( d\mu_L(y, Y) := dy \) \( dY \).

Now we aim to compute the adjoint and coadjoint actions of \( G^c \). First, we compute the adjoint action of \( G^c \) as follows. Let \( g = (v_0, c_0, h_0) \in G^c \) where \( v_0 \in \mathbb{C}^{2n} \), \( c_0 \in \mathbb{C} \) and \( h_0 \in S^c = \text{Sp}(n, \mathbb{C}) \) and \( X = (w, c, U) \in g^c \) where \( w \in \mathbb{C}^{2n}, c \in \mathbb{C} \) and \( U \in s^c \). We set \( \exp(tX) = (w(t), c(t), \exp(tU)) \). Then, since
the derivatives of \( w(t) \) and \( c(t) \) at \( t = 0 \) are \( w \) and \( c \), we find that
\[
\text{Ad}(g)X = \left. \frac{d}{dt} (g \exp(tX)g^{-1}) \right|_{t=0} = \left( h_0w - (\text{Ad}(h_0)U)v_0, c + \omega(v_0, h_0w) - \frac{1}{2}\omega(v_0, (\text{Ad}(h_0)U)v_0), \text{Ad}(h_0)U \right).
\]

On the other hand, let us denote by \( \xi = (u, d, \varphi) \), where \( u \in \mathbb{C}^{2n}, d \in \mathbb{C} \) and \( \varphi \in (s^c)^* \), the element of \((g^c)^*\) defined by
\[
\langle \xi, (w, c, U) \rangle = \omega(u, w) + dc + \langle \varphi, U \rangle.
\]
Moreover, for \( u, v \in \mathbb{C}^{2n} \), we denote by \( v \times u \) the element of \((s^c)^*\) defined by
\[
\langle v \times u, U \rangle := \omega(u, UV) \text{ for } U \in s^c.
\]
Let \( \xi = (u, d, \varphi) \in (g^c)^* \) and \( g = (v_0, c_0, h_0) \in G^c \). Then, by using the relation \( \langle \text{Ad}^*(g)\xi, X \rangle = \langle \xi, \text{Ad}(g^{-1})X \rangle \) for \( X \in g^c \), we obtain
\[
\text{Ad}^*(g)\xi = \left( h_0u - dv_0, d, \text{Ad}^*(h_0)\varphi + v_0 \times (h_0u - \frac{d}{2}v_0) \right)
\]
By restriction, we also get the formula for the coadjoint action of \( G \). The following lemma will be needed later.

**Lemma 2.1** ([15]). The elements \( \xi_0 \) of \( g^* \) fixed by \( K \) are the elements of the form \((0, d, \varphi_\lambda)\) where \( d, \lambda \in \mathbb{R} \) and \( \varphi_\lambda \in s^* \) is defined by \( \langle \varphi_\lambda, \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \rangle = i\lambda \text{Tr}(A) \).

### 3. Holomorphic representations

The holomorphic representations of the multi-dimensional Jacobi group were studied by many authors, see in particular [26], [9], [10], [4], [5], and [6]. We follow here the general presentation of [26], Chapter XII (see also [14]).

Let \( \chi \) be a unitary character of \( K \). The extension of \( \chi \) to \( K^c \) is also denoted by \( \chi \). We set \( K_\chi(Z, W) := \chi(\kappa(\exp W^* \exp Z))^{-1} \) for \( Z, W \in \mathbb{D} \) and \( J_\chi(g, Z) := \chi(\kappa(g \exp Z)) \) for \( g \in G \) and \( Z \in \mathbb{D} \). We consider the Hilbert space \( \mathcal{H}_\chi \) of all holomorphic functions \( f \) on \( \mathbb{D} \) such that
\[
\|f\|_\chi^2 := \int_{\mathbb{D}} |f(Z)|^2 K_\chi(Z, Z)^{-1} c_\chi d\mu(Z) < +\infty
\]
where the constant \( c_\chi \) is defined by
\[
c_\chi^{-1} = \int_{\mathbb{D}} K_\chi(Z, Z)^{-1} d\mu(Z).
\]
We shall see that, under some hypothesis on $\chi$, $c_\chi$ is well-defined and $\mathcal{H}_\chi \neq (0)$. In that case, $\mathcal{H}_\chi$ contains the polynomials [26], p. 546. Moreover, the formula

$$\pi_\chi(g) f(Z) = J_\chi(g^{-1}, Z) f(g^{-1} \cdot Z)$$

defines a unitary representation of $G$ on $\mathcal{H}_\chi$ which is a highest weight representation [26], p. 540.

The space $\mathcal{H}_\chi$ is a reproducing kernel Hilbert space. More precisely, if we set $e_Z(W) := K_\chi(W, Z)$ then we have we have the reproducing property $f(Z) = \langle f, e_Z \rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$ [26], p. 540. Here $\langle \cdot, \cdot \rangle_\chi$ denotes the inner product on $\mathcal{H}_\chi$.

Here we fix $\chi$ as follows. Let $\chi \in \mathbb{R}$ and $m \in \mathbb{Z}$. Then, for each $k = ((0, 0), c, \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}) \in K$, we set $\chi(k) := e^{iyc}(\text{Det } P)^m$.

We need the following lemma.

**Lemma 3.1 ([24]).** Let $\lambda \in \mathbb{R}$. The integral

$$J_n(\lambda) := \int_{\mathbb{B}} \text{Det}(I_n - Y\overline{Y})^\lambda dY$$

is convergent if $\lambda > -1$ and in this case we have

$$J_n(\lambda) = \pi^{n(n+1)/2} \frac{\Gamma(2\lambda + 3)\Gamma(2\lambda + 5) \cdots \Gamma(2\lambda + 2n - 1)}{\Delta},$$

where

$$\Delta := (\lambda + 1)(\lambda + 2) \cdots (\lambda + n)\Gamma(2\lambda + n + 2)\Gamma(2\lambda + n + 3) \cdots \Gamma(2\lambda + 2n).$$

Then we have the following result.

**Proposition 3.2.** (1) Let $Z = a(y, Y) \in \mathcal{D}$ and $W = a(v, V) \in \mathcal{D}$. We set

$$E(y, v, Y, V) := 2y^t(I_n - \overline{V}Y)^{-1} \overline{v} + v^t(I_n - \overline{Y}Y)^{-1} \overline{V}y + \overline{v}^t Y(I_n - \overline{Y}Y)^{-1} \overline{v}.$$

Then we have

$$K_\chi(Z, W) = \text{Det}(I_n - Y\overline{Y})^m \exp\left(\frac{\sqrt{\gamma}}{4} E(y, v, Y, V)\right).$$

(2) We have $\mathcal{H}_\chi \neq (0)$ if and only if $\gamma > 0$ and $m + n + 1/2 < 0$. In this case, we also have $c_\chi^{-1} = (2\pi)^n \gamma^{-n} J_n(-m - n - 3/2)$. 
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(3) For each \( g = ((z_0, \bar{z}_0), c_0, (\frac{P}{Q}, \bar{P})) \in G \) and each \( Z = a(y, Y) \in D \), we have

\[
J(g, Z) = e^{i\nu c_0} \text{Det}(\bar{Q}Y + \bar{P})^{-m} \exp \left( \left( \frac{\nu}{4} \left( z_0^T \bar{z}_0 + 2 \bar{z}_0^T P y + y^T P^T \bar{Q} y \right) - (\bar{z}_0 + \bar{Q} y)^T (P Y + Q)(\bar{Q}Y + \bar{P})^{-1}(\bar{z}_0 + \bar{Q} y) \right) \right)
\]

Proof. We can verify (1) and (3) by computations based on the formula for \( \kappa \) given in Section 2. To prove (2), recall that, by [26], Theorem XII.5.6, we have \( \mathcal{H}_\chi \neq (0) \) if and only if

\[
I_\chi := \int_D K_\chi(Z, Z)^{-1} d\mu(Z) < \infty.
\]

Then we have to study the convergence of \( I_\chi \). By taking into account the expression of \( \mu \) given in Section 2, we get

\[
I_\chi = \int_D \exp \left( -\frac{\nu}{4} E(y, y, Y, Y) \right) \text{Det}(I_n - Y \bar{Y})^{-m-n-2} d\mu_L(y, Y)
\]

and, by making the change of variables \( y \to (I_n - Y \bar{Y})^{1/2} y \) whose Jacobian is \( \text{Det}(I_n - Y \bar{Y}) \), we find that

\[
I_\chi = \int_{\mathbb{R}^{n\times 2}} \text{Det}(I_n - Y \bar{Y})^{-m-n} \exp \left( -\frac{\nu}{4} (2y^T \bar{y} + y^T \bar{Y} y + \bar{y}^T Y \bar{y}) \right) dy dY.
\]

But by [21], p. 258, we have

\[
I_\chi = \left( \frac{2\pi}{\nu} \right)^n \int_{\mathbb{R}} \text{Det}(I_n - Y \bar{Y})^{-m-n-3/2} dY
\]

for \( \nu > 0 \). The result then follows from Lemma 3.1 \( \square \)

Note that we can deduce from (3) of Proposition 3.2 an explicit but rather complicated expression for \( \pi_\chi(g) \). Now we consider the derived representation \( d\pi_\chi \).

Here we use the following notation. If \( L \) is a Lie group and \( X \) is an element of the Lie algebra of \( L \) then we denote by \( X^+ \) the right invariant vector field on \( L \) generated by \( X \), that is, \( X^+(h) = \frac{d}{dt}(\exp t X)h|_{t=0} \) for \( h \in L \).

By differentiating the multiplication map from \( P^+ \times K^c \times P^- \) onto \( P^+ K^c P^- \), we can easily prove the following result.
LEMMA 3.3. Let $X \in \mathfrak{g}^c$ and $g = z k y$ where $z \in P^+, k \in K^c$ and $y \in P^-$. We have

1. \[ d\zeta_g(X^{+}(g)) = (\text{Ad}(z) p_{p^{+}}(\text{Ad}(z^{-1}) X))^{+}(z); \]
2. \[ d\kappa_g(X^{+}(g)) = (p_{\mathfrak{k}^{c}}(\text{Ad}(z^{-1}) X))^{+}(k); \]
3. \[ d\eta_g(X^{+}(g)) = (\text{Ad}(k^{-1}) p_{p^{-}}(\text{Ad}(z^{-1}) X))^{+}(y). \]

From this, we easily deduce the following proposition (see also [26], p. 515).

PROPOSITION 3.4. For $X \in \mathfrak{g}^c$, $f \in \mathcal{H}_X$ and $Z \in \mathcal{D}$, we have

\[ d\pi_X(X) f(Z) = d\chi(p_{\mathfrak{e}c} (e^{-ad Z} X)) f(Z) - (df)_Z (p_{p^{+}}(e^{-ad Z} X)). \]

In particular, we have

1. If $X \in p^{+}$ then $d\pi_X(X) f(Z) = -(df)_Z(X)$;
2. If $X \in \mathfrak{t}^{c}$ then $d\pi_X(X) f(Z) = d\chi(X) f(Z) + (df)_Z([Z, X])$;
3. If $X \in p^{-}$ then
   \[ d\pi_X(X) f(Z) = (d\chi \circ p_{\mathfrak{e}c})(- [Z, X] + \frac{1}{2} [Z, [Z, X]]) f(Z) \]
   \[ - (df_Z \circ p_{p^{+}})(- [Z, X] + \frac{1}{2} [Z, [Z, X]]) \].

Now we need to introduce some notation. As usual, we write $Z \in \mathcal{D}$ as $Z = a(y, Y)$ where $y = (y_j)_{1 \leq j \leq n} \in \mathbb{C}^n$ and $Y = (y_{kl})_{1 \leq k, l \leq n} \in \mathbb{B}$. Define

\[ \mathcal{I} := \{1, 2, \ldots, n\} \cup \{(k, l): 1 \leq k, l \leq n\} \]

and consider $i \in \mathcal{I}$. Then we define $\partial_i$ as follows. If $i \in \{1, 2, \ldots, n\}$ then $\partial_i$ is the partial derivative with respect to $y_i$ and if $i = (k, l)$ then $\partial_i$ is the partial derivative with respect to $y_{kl}$. Moreover, we say that $P(Z)$ is a polynomial of degree $\leq q$ if $P(a(y, Y))$ is a polynomial of degree $\leq q$ in the variables $y_j$ and $y_{kl}$.

From the preceding proposition we deduce the following result.

PROPOSITION 3.5. For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, $d\pi_X(X_1 X_2 \ldots X_q)$ is a sum of terms of the form $P(Z) \partial_{i_1} \partial_{i_2} \ldots \partial_{i_r}$ where $r \leq q$, $i_1, i_2, \ldots, i_r \in \mathcal{I}$ and $P(Z)$ is a polynomial of degree $\leq 2q$.

PROOF. By Proposition 3.4 we see that, for each $X \in \mathfrak{g}^c$, $d\pi_X(X)$ is of the form $P^0(Z) + \sum_i P^i(Z) \partial_i$ where $P^0(Z)$, $P^i(Z)$ are polynomials of degree $\leq 2$. The result then follows by induction on $q$. \qed
4. **Generalities on the Stratonovich–Weyl correspondence**

In this section, we review some general facts about the Berezin correspondence, the Berezin transform and the Stratonovich–Weyl correspondence.

First at all, recall that the Berezin correspondence on \( D \) is defined as follows. Consider an operator (not necessarily bounded) \( A \) on \( \mathcal{H} \) whose domain contains \( e_Z \) for each \( Z \in \mathbb{D} \). Then the Berezin symbol of \( A \) is the function \( S_x(A) \) defined on \( \mathbb{D} \) by

\[
S_x(A)(Z) := \frac{\langle A e_Z, e_Z \rangle_x}{\langle e_Z, e_Z \rangle_x}.
\]

We can verify that each operator is determined by its Berezin symbol and that if an operator \( A \) has adjoint \( A^* \) then we have \( S_x(A^*) = \overline{S_x(A)} \), see [7] and [8]. Moreover, for each operator \( A \) on \( \mathcal{H} \) whose domain contains the coherent states \( e_Z \) for each \( Z \in \mathbb{D} \) and each \( g \in G \), the domain of \( \pi_x(g^{-1})A\pi_x(g) \) also contains \( e_Z \) for each \( Z \in \mathbb{D} \) and we have

\[
S_x(\pi_x(g)^{-1}A\pi_x(g))(Z) = S_x(A)(g \cdot Z),
\]

that is, \( S_x \) is \( G \)-equivariant, see [14]. We have also the following result.

**Proposition 4.1** ([14]). (1) For \( g \in G \) and \( Z \in \mathbb{D} \), we have

\[
S_x(\pi_x(g))(Z) = x(\kappa(\exp Z^* g^{-1} \exp Z)^{-1} \kappa(\exp Z^* \exp Z)).
\]

(2) For \( X \in \mathfrak{g}^c \) and \( Z \in \mathbb{D} \), we have

\[
S_x(d \pi_x(X))(Z) = d_x(p_{\mathfrak{g}^c}(\text{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*) X).
\]

Let \( \xi \) be the linear form on \( \mathfrak{g}^c \) defined by \( \xi = -i d_x \) on \( \mathfrak{g}^c \) and \( \xi = 0 \) on \( \mathfrak{p}^\pm \). Then we have \( \xi(\mathfrak{g}) \subset \mathbb{R} \) and the restriction \( \xi_x \) of \( \xi \) to \( \mathfrak{g} \) is an element of \( \mathfrak{g}^* \). In the notation of Section 2 we have \( \xi_x = (0, \gamma, -m \varphi_0) \) where \( \varphi_0 \in \mathfrak{s}^* \) is defined by \( \langle \varphi_0, (\frac{P}{Q} \frac{\mathcal{O}}{\mathcal{P}}) \rangle = i \text{Tr}(P) \).

We denote by \( O(\xi_x) \) the orbit of \( \xi_x \) in \( \mathfrak{g}^* \) for the coadjoint action of \( G \). This orbit is said to be associated with \( \pi_x \) by the Kostant–Kirillov method of orbits, see [25] and [14]. Moreover, we have the following result.

**Proposition 4.2** ([14]). (1) For each \( Z \in \mathbb{D} \), let

\[
\Psi_x(Z) := \text{Ad}^*(\exp(-Z^*) \zeta(\exp Z^* \exp Z)) \xi_x.
\]

Then, for each \( X \in \mathfrak{g}^c \) and each \( Z \in \mathbb{D} \), we have

\[
S(d \pi_x(X))(Z) = i \langle \Psi_x(Z), X \rangle.
\]
(2) For each \( g \in G \) and each \( Z \in \mathbb{D} \), we have \( \Psi_\chi(g \cdot Z) = \text{Ad}^*(g) \Psi_\chi(Z) \).

(3) The map \( \Psi_\chi \) is a diffeomorphism from \( \mathbb{D} \) onto \( \partial(\xi_\chi) \).

In order to make the expression of \( \Psi_\chi \) more explicit, we introduce the following notation. For \( \varphi \in \sigma^* \), let \( \alpha(\varphi) \) be the unique element of \( \sigma \) such that \( \langle \varphi, X \rangle = \text{Tr}(\alpha(\varphi)X) \) for each \( X \in \sigma \). In particular, one has \( \alpha(\varphi_0) = \frac{1}{2} \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix} \). Moreover, for \( u = (x, \tilde{x}) \in \mathbb{C}^{2n} \) and \( u = (y, \tilde{y}) \in \mathbb{C}^{2n} \) we have
\[
\theta(v \times u) = \frac{1}{2} \begin{pmatrix} -iyx^t & iyx^t \\ -i\tilde{y}\tilde{x}^t & i\tilde{y}\tilde{x}^t \end{pmatrix}.
\]
Note also that \( \theta \) intertwines \( \text{Ad}^* \) and \( \text{Ad} \). Then we have the following result.

**Proposition 4.3 ([15]).** The map \( \psi_\chi : \mathbb{D} \to \partial(\xi_\chi) \) is given by
\[
\psi_\chi(a(y, Y)) = (-d(y_1, \tilde{y}_1), \gamma, \varphi(y, Y))
\]
where \( y_1 = (I_n - Y\tilde{Y})^{-1}(y + Y\tilde{y}) \) and
\[
\varphi(y, Y) := -m \text{Ad}^* \left( (I_n - Y\tilde{Y})^{-1/2} (I_n - Y\tilde{Y})^{-1/2}Y \right) \varphi_0
\]
\[
- \frac{\gamma}{2} (y_1, \tilde{y}_1) \times (y_1, \tilde{y}_1).
\]
Moreover, we have
\[
\alpha(\varphi(y, Y)) = -\frac{\gamma}{4} \begin{pmatrix} -iy_1\tilde{y}_1^t & iy_1\tilde{y}_1^t \\ -i\tilde{y}_1\tilde{y}_1^t & i\tilde{y}_1\tilde{y}_1^t \end{pmatrix} - \frac{m}{2} i \begin{pmatrix} A(Y) & B(Y) \\ -B(Y) & -A(Y) \end{pmatrix}.
\]
where
\[
A(Y) := (I_n + Y\tilde{Y})(I_n - Y\tilde{Y})^{-1/2}(I_n - Y\tilde{Y})^{-1/2};
\]
\[
B(Y) := -2Y(I_n - Y\tilde{Y})^{-1/2}(I_n - Y\tilde{Y})^{-1/2}.
\]

Now we recall briefly the construction of the Stratonovich–Weyl correspondence [20], [13], and [14]. Denote by \( \mathcal{L}_2(\mathcal{H}_\chi) \) the space of all Hilbert-Schmidt operators on \( \mathcal{H}_\chi \) and by \( \mu_\chi \) the \( G \)-invariant measure on \( \mathbb{D} \) defined by \( d\mu_\chi(Z) = e_\chi d\mu(Z) \). Then the map \( S_\chi \) is a bounded operator from \( \mathcal{L}_2(\mathcal{H}_\chi) \) into \( L^2(\mathbb{D}, \mu_\chi) \) which is one-to-one and has dense range [29], [32]. Moreover, the Berezin transform is the operator on \( L^2(\mathbb{D}, \mu_\chi) \) defined by \( B_\chi := S_\chi S_{\chi}^* \). We can easily verify that we have the following integral formula for \( B_\chi \):
\[
B_\chi F(Z) = \int_{\mathbb{D}} F(W) \frac{|\langle e_Z, e_W \rangle_\chi|^2_\chi}{\langle e_Z, e_Z \rangle_\chi \langle e_W, e_W \rangle_\chi} d\mu_\chi(W)
\]
(see [7], [32], and [33] for instance).
Let $\rho$ be the left-regular representation of $G$ on $L^2(\mathcal{D}, \mu_\chi)$. As a consequence of the equivariance property for $S_\chi$, we see that $B_\chi$ commute with $\rho(g)$ for each $g \in G$.

Consider the polar decomposition of $S_\chi$:

$$S_\chi = (S_\chi S_\chi^*)^{1/2} W_\chi = B_\chi^{1/2} W_\chi,$$

where $W_\chi := B_\chi^{-1/2} S_\chi$ is a unitary operator from $\mathcal{L}_2(\mathcal{H}_\chi)$ onto $L^2(\mathcal{D}, \mu_\chi)$. Note that, by (2) of Proposition 4.2, the measure $\mu_0 := (\Psi_{\chi}^{-1})^*(\mu_\chi)$ is a $G$-invariant measure on $\mathcal{O}(\xi_\chi)$. The following proposition is then immediate.

**Proposition 4.4.** 1) The map $W_\chi : \mathcal{L}_2(\mathcal{H}_\chi) \to L^2(\mathcal{D}, \mu_\chi)$ is a Stratonovich–Weyl correspondence for the triple $(G, \pi_\chi, \mathcal{D})$, that is, we have

1) $W_\chi(A^*) = \overline{W_\chi(A)}$;

2) $W_\chi(\pi_\chi(g) A \pi_\chi(g)^{-1})(Z) = W_\chi(A)(g^{-1} \cdot Z)$;

3) $W_\chi$ is unitary.

2) Similarly, the map $W_\chi : \mathcal{L}_2(\mathcal{H}_\chi) \to L^2(\mathcal{O}(\xi_\chi), \mu_0)$ defined by

$$W_\chi(A) = W_\chi(A) \circ \Psi_{\chi}^{-1}$$

is a Stratonovich–Weyl correspondence for the triple $(G, \pi_\chi, \mathcal{O}(\xi_\chi))$.

Note that we have relaxed here (1) of Definition 1.1 which is not adapted to the present setting since $I$ is not Hilbert-Schmidt. However, this requirement should be hold in some generalize sense, see for instance [22].

**5. Extension of the Berezin transform**

The aim of this section is to extend the Berezin transform to a class of functions which contains $S_\chi(d \pi_\chi(X))$ for each $X \in \mathfrak{g}^c$, in order to define and study $W_\chi(d \pi_\chi(X))$. This question was already investigated in [14] in the case of a reductive Lie group and in [17] in the case of the Jacobi group.

For $Z, W \in \mathcal{D}$, we set $l_Z(W) := \log(\exp Z^* \exp W) \in \mathfrak{p}^-$. 

**Lemma 5.1.** (1) For each $Z, W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have

$$\frac{d}{dt} e_Z(W + tV) \bigg|_{t=0} = -e_Z(W)(d_\chi \circ p_v)(\{l_Z(W), V\} + \frac{1}{2}[l_Z(W), [l_Z(W), V]])$$.
(2) For each $Z, W \in \mathbb{D}$ and $V \in p^+$, we have
\[
\frac{d}{dt} l_Z(W + tV) \bigg|_{t=0} = p_p \left( [l_Z(W), V] + \frac{1}{2} [l_Z(W), [l_Z(W), V]] \right).
\]

(3) For each $i_1, i_2, \ldots, i_q \in I$ and $Z \in \mathbb{D}$, the function $(\partial_{i_1} \partial_{i_2} \ldots \partial_{i_q} e_Z)(W)$ is of the form $e_Z(W) Q(l_Z(W))$ where $Q$ is a polynomial on $p^-$ of degree $\leq 2q$.

(4) For each $X_1, X_2, \ldots, X_q \in g^c$, the function $S_X(d \pi_X(X_1 X_2 \ldots X_q))(Z)$ is a sum of terms of the form $P(Z) Q(l_Z(Z))$ where $P$ and $Q$ are polynomials of degree $\leq 2q$.

**Proof.** The proof of this lemma is similar to those of Lemma 6.2 of [14] and Lemma 5.2 of [17]. Note that the proof of (1) is essentially based on Lemma 3.3, that (3) is a consequence of (1) and (2) and, finally, that (4) follows from (3) and Proposition 3.5.

We can then establish the main result of this section.

**Proposition 5.2.** If $q < (1/4)(-m - 2n)$ then for each $X_1, X_2, \ldots, X_q \in g^c$, the Berezin transform of $S_X(d \pi_X(X_1 X_2 \ldots X_q))$ is well-defined.

**Proof.** First, we fix $Z \in \mathbb{D}$ and we make the change of variables $W \to g_Z \cdot W$ in (4.1). Then we obtain
\[
(B_X F)(Z) = \int_{\mathbb{D}} F(g_Z \cdot W) \langle e_W, e_W \rangle_X^{-1} d\mu_X(W).
\]

We take $F = S_X(d \pi_X(X_1 X_2 \ldots X_q)$ and we set
\[
Y_k := \text{Ad}(g_Z^{-1}) X_k
\]
for $k = 1, 2, \ldots, q$. Then, by $G$-invariance of $S_X$, we have
\[
F(g_Z \cdot W) = S_X(n \pi_X(Y_1 Y_2 \ldots Y_q))(W)
\]
for each $W \in \mathbb{D}$. Recall that, by the preceding lemma, the function
\[
S_X(d \pi_X(Y_1 Y_2 \ldots Y_q))(W)
\]
is a sum of terms of the form $P(W) Q(l_W(W))$ where $P$ and $Q$ are polynomials of degree $\leq 2q$. Then we have to prove that, for each $q < (1/4)(-m - 2n)$ and each polynomials $P$ and $Q$ of degree $\leq 2q$, the integral
\[
I := \int_{\mathbb{D}} P(W) Q(l_W(W)) \langle e_W, e_W \rangle_X^{-1} d\mu_X(W)
\]
is convergent.
First, we note that if \( W = a(y, Y) \) then

\[
l_W(W) = \left( (0, -(I_n - \overline{Y} Y)^{-1}(\overline{y} + \overline{Y} y)), 0, \left( -(I_n - \overline{Y} Y)^{-1} \overline{Y} \right) \right).
\]

Thus we have

\[
I = c_x \int_{\mathcal{D}} P(y, Y) Q(-(I_n - \overline{Y} Y)^{-1}(\overline{y} + \overline{Y} y), -(I_n - \overline{Y} Y)^{-1} \overline{Y}) \exp \left( -\frac{y^t}{4} (2y^t (I_n - \overline{Y} Y)^{-1} \overline{y} \\
+ y^t (I_n - \overline{Y} Y)^{-1} \overline{Y} y + \overline{y}^t Y (I_n - \overline{Y} Y)^{-1} \overline{y}) \right) \det(I_n - Y \overline{Y})^{-m-n-2} d\mu_L(y, Y).
\]

As in the proof of Proposition 3.2, we make the change of variables

\[
y \mapsto (I_n - Y \overline{Y})^{1/2} y
\]

and we find that

\[
I = c_x \int_{\mathcal{D}} P((I_n - Y \overline{Y})^{1/2} y, Y) Q(-(I_n - \overline{Y} Y)^{-1/2}(\overline{y} + \overline{Y} y), -(I_n - \overline{Y} Y)^{-1} \overline{Y}) \exp \left( -\frac{y^t}{4} (2y^t \overline{y} + y^t \overline{Y} y + \overline{y}^t Y \overline{y}) \right) \det(I_n - Y \overline{Y})^{-m-n-1} d\mu_L(y, Y).
\]

Now we make the following remarks.

(1) Since \( P \) is a polynomial of degree \( \leq 2q \) and \( B \) is bounded, there exists a constant \( C_0 > 0 \) such that

\[
|P((I_n - Y \overline{Y})^{1/2} y, Y)| \leq C_0 \sum_{r \leq 2q} |y|^r
\]

for each \((y, Y) \in \mathbb{C}^n \times B\).

(2) By using the classical formula for the inverse of a matrix, for each \( Y \in B \) we have

\[
(I_n - \overline{Y} Y)^{-1} = \det(I_n - \overline{Y} Y)^{-1} C(I_n - \overline{Y} Y)^t
\]

where \( C(A) \) denotes the cofactor matrix of a matrix \( A \). From this we deduce that there exists a constant \( C'_0 > 0 \) such that

\[
|Q(-(I_n - \overline{Y} Y)^{-1/2}(\overline{y} + \overline{Y} y), -(I_n - \overline{Y} Y)^{-1} \overline{Y})| \leq C'_0 \det(I_n - Y \overline{Y})^{-2q} \sum_{r \leq 2q} |y|^r
\]

for each \((y, Y) \in \mathbb{C}^n \times B\).
(3) For each \((y, Y) \in \mathbb{C}^n \times \mathcal{B}\), we have
\[
2y^t \bar{y} + y^t \bar{Y} y + \bar{y}^t Y \bar{y} = 2(y^t y + \text{Re}(y^t \bar{Y} y)) \geq 2(1 - \|Y\|)|y|^2.
\]

Here \(\| \cdot \|\) denotes the operator norm corresponding to the Hermitian norm on \(\mathbb{C}^n\).

By using these remarks, we can reduce the study of the convergence of \(I\) to that of the integral
\[
I' := \int_{\mathcal{D}} \det(I_n - Y \bar{Y})^{-2q-m-n-1} |y|^{4q} e^{-(y/2)|y|^2(1-\|Y\|)} d\mu_L(y, Y).
\]

We set
\[
I(Y) := \int_{\mathbb{C}^n} |y|^{4q} e^{-\frac{y^t y}{2}(1-\|Y\|)} dy
\]
and, passing to spherical coordinates, we see that there exists some constants \(C, C' > 0\) such that, for each \(Y \in \mathcal{B}\), we have
\[
I(Y) = C \int_{0}^{+\infty} \chi^{4q+2n-1} e^{-(y/2)(1-\|Y\|)x^2} dx = C'(1 - \|Y\|)^{-2q-n}.
\]

Then we have to study the integral
\[
I' := \int_{\mathcal{B}} \det(I_n - Y \bar{Y})^{-2q-m-n-1} (1 - \|Y\|)^{-2q-n} dY.
\]

Now denote by \(\lambda_s(Y \bar{Y})\) the maximum of the eigenvalues of \(Y \bar{Y}\) and recall that \(\|Y\|^2 = \lambda_s(Y \bar{Y})\). Then we have
\[
\det(I_n - Y \bar{Y}) \leq 1 - \lambda_s(Y \bar{Y}) = 1 - \|Y\|^2 \leq 2(1 - \|Y\|)
\]
for each \(Y \in \mathcal{B}\). Thus we obtain
\[
\det(I_n - Y \bar{Y})^{-2q-m-n-1} (1 - \|Y\|)^{-2q-n} \leq 2^{2q+n} \det(I_n - Y \bar{Y})^{-4q-m-2n-1}
\]
for each \(Y \in \mathcal{B}\). But by Lemma 3.1, we see that \(J_n(-4q-m-2n-1)\) hence \(I''\) converges if \(q < \frac{1}{4}(-m-2n)\). This ends the proof.

\[
6. \text{Stratonovich–Weyl symbols of derived representation operators}
\]

Here we assume that \(-m > 2n + 4\). Then, by Proposition 5.2, \(B_\chi(S_\chi(d\pi_\chi(X)))\) is well-defined for each \(X \in \mathfrak{g}^c\). We aim to define also \(W_\chi(d\pi_\chi(X))\) for \(X \in \mathfrak{g}^c\). To this goal, we first introduce a space of functions on \(\mathcal{D}\) which is stable under \(B_\chi\) and contains \(S_\chi(d\pi_\chi(X))\) for each \(X \in \mathfrak{g}^c\).
Recall that, by Proposition 4.2 we have $S_x(d \pi_x(X))(Z) = i \xi(\text{Ad}(g_Z^{-1})X)$ for each $X \in g^c$ and $Z \in D$. This leads us to introduce the vector space $S$ generated by the functions $Z \rightarrow \phi_0(\text{Ad}(g_Z^{-1})X)$ where $X \in g^c$ and $\phi_0$ is an element of $(g^c)^*$ which is $\text{Ad}^*(K)$-invariant. Such elements $\phi_0$ were determined in [15], see Lemma 2.1 above. The following proposition is analogous to Proposition 6.2 of [17].

**Proposition 6.1.** Let $\phi: D \times g^c \rightarrow \mathbb{C}$ be a function such that

(i) for each $Z \in D$, the map $X \rightarrow \phi(Z, X)$ is linear;

(ii) for each $X \in g^c$, $g \in G$ and $Z \in D$, we have $\phi(g \cdot Z, X) = \phi(Z, \text{Ad}(g^{-1})X)$.

Then

1. the element $\phi_0$ of $(g^c)^*$ defined by $\phi_0(X) := \phi(0, X)$ is fixed by $K$;
2. for each $X \in g^c$ and $Z \in D$, we have

$$\phi(Z, X) = \phi_0(\text{Ad}(g_Z^{-1})X)$$

and

$$\phi(Z, X) = \phi_0(\text{Ad}(\xi(\exp Z^* \exp Z)^{-1} \exp Z^*)X) = (\phi_0 \circ p_{\mathfrak{c}})(\text{Ad}(\xi(\exp Z^* \exp Z)^{-1} \exp Z^*)X);$$

3. for each $X \in g^c$, the function $\psi: D \times g^c \rightarrow \mathbb{C}$ given by

$$\psi(\cdot, X) = B_x(\phi(\cdot, X))$$

is well-defined and satisfies (i) and (ii);

4. the vector space $S$ is generated by all the functions $Z \rightarrow \phi(Z, X)$ for $\phi$ as above and $X \in g^c$. Moreover, $S$ is stable under $B_x$.

**Proof.** (1) By (ii), for each $k \in K$ and $X \in g^c$, we have

$$(\text{Ad}^*(k)\phi_0)(X) = \phi_0(\text{Ad}(k^{-1})X) = \phi(0, \text{Ad}(k^{-1})X) = \phi(k \cdot 0, X) = \phi(0, X) = \phi_0(X).$$

Then $\phi_0$ is fixed by $K$.

(2) The first assertion follows from (ii). To prove the second assertion, recall that by [15], there exists $k_Z \in K$ such that $g_Z = \exp(-Z^*)\xi(\exp Z^* \exp Z)k_Z^{-1}$. Then we have

$$\phi(Z, X) = \phi_0(\text{Ad}(k_Z\xi(\exp Z^* \exp Z)^{-1} \exp Z^*)X) = \phi_0(\text{Ad}(\xi(\exp Z^* \exp Z)^{-1} \exp Z^*)X)$$
and, noting that $\phi_0|_{p^\pm} = 0$ by Lemma 2.1, we can conclude that

$$\phi(Z, X) = (\phi_0 \circ p_{\nu^r})(\text{Ad}(\xi \exp Z^* \exp Z)^{-1} \exp Z^*)X).$$

(3) By using the same arguments as in the proof of Proposition 5.2, we can verify that, for each $X \in g^c$, the Berezin transform of $\phi(\cdot, X)$ is well-defined. The second assertion follows from the fact that $B_X$ commutes to the $\rho(g), g \in G$.

(4) This follows from the preceding statements. $\square$

Now we need the following lemmas.

**Lemma 6.2.** For each $Y \in \mathcal{B}$, we have

$$I_1(Y) := \int_{\mathbb{C}^n} y^t \tilde{y} \exp \left( -\frac{\gamma}{4} (2y^t \tilde{y} + y^t \bar{Y} y + \tilde{y}^t Y \tilde{y}) \right) dy$$

$$= \frac{2}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n \text{Det}(I_n - Y \bar{Y})^{-1/2} \text{Tr}((I_n - Y \bar{Y})^{-1})$$

$$I_2(Y) := \int_{\mathbb{C}^n} y^t \bar{Y} y \exp \left( -\frac{\gamma}{4} (2y^t \tilde{y} + y^t \bar{Y} y + \tilde{y}^t Y \tilde{y}) \right) dy$$

$$= \frac{2}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n \text{Det}(I_n - Y \bar{Y})^{-1/2} (n - \text{Tr}((I_n - Y \bar{Y})^{-1})).$$

**Proof.** For $s \in [0, 1]$ and $Y \in \mathcal{B}$, let us introduce

$$J_s(Y) = \int_{\mathbb{C}^n} \exp \left( -\frac{\gamma}{4} (2y^t \tilde{y} + sy^t \bar{Y} y + s \tilde{y}^t Y \tilde{y}) \right) dy.$$ 

By [21], p. 258, we have

$$J_s(Y) = \left( \frac{2\pi}{\gamma} \right)^n \text{Det}(I_n - s^2 Y \bar{Y})^{-1/2}.$$ 

Then, by computing the derivative of $J_s(Y)$ at $s = 1$, we get

$$\int_{\mathbb{C}^n} (y^t \bar{Y} y + \tilde{y}^t Y \tilde{y}) \exp \left( -\frac{\gamma}{4} (2y^t \tilde{y} + y^t \bar{Y} y + \tilde{y}^t Y \tilde{y}) \right) dy$$

$$= -\frac{4}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n \text{Det}(I_n - Y \bar{Y})^{-1/2} \text{Tr}((I_n - Y \bar{Y})^{-1} Y \bar{Y}).$$

Thus we have

$$\text{(6.1)} \quad I_2(Y) + \overline{I_2(Y)} = -\frac{4}{\gamma} \left( \frac{2\pi}{\gamma} \right)^n \text{Det}(I_n - Y \bar{Y})^{-1/2} \text{Tr}(-I_n + (I_n - Y \bar{Y})^{-1}).$$
On the other hand, by integrating by parts, we get
\[ J_1(Y) = -\int_{C^n} y_k \frac{\partial}{\partial k} \left( \exp \left( -\frac{\sqrt{2}}{4} (2y^t \tilde{y} + y^t \tilde{Y} y + \tilde{y}^t Y \tilde{y}) \right) \right) dy \]
\[ = \frac{\sqrt{2}}{4} \int_{C^n} y_k (2\tilde{y}^t_k + 2 e_k^t \tilde{Y} y) \exp \left( -\frac{\sqrt{2}}{4} (2y^t \tilde{y} + y^t \tilde{Y} y + \tilde{y}^t Y \tilde{y}) \right) dy \]
for each \( k = 1, 2, \ldots, k \). By summing up over \( k \), we obtain
\[ (6.2) \quad n J_1(Y) = \frac{\sqrt{2}}{2} (I_1(Y) + I_2(Y)). \]
This last equation implies that \( I_2(Y) \) is real since \( J_1(Y) \) and \( I_1(Y) \) are real. Consequently, \( (6.1) \) gives the desired value for \( I_2(Y) \) hence \( (6.2) \) provides the desired value for \( I_1(Y) \).

The following lemma gives a useful expression for \( K_X(Z, Z) \) which will be used in the proof of Proposition 6.4.

**Lemma 6.3.** For each \( Z = a(y, Y) \), let \( z_0 := (I_n - Y \tilde{Y})^{-1}(y + Y \tilde{y}) \). Then we have
\[ K_X(Z, Z) = \exp \left( \frac{\sqrt{2}}{4} (2z_0^t \tilde{z}_0 - z_0^t \tilde{Y} z_0 - \tilde{z}_0^t Y \tilde{z}_0) \right) \text{Det}(I_n - Y \tilde{Y})^m. \]

**Proof.** The result follows from Proposition 3.2 by a routine computation. Alternatively, by [14], Lemma 4.1, we have
\[ \langle e_Z, e_Z \rangle_X = \langle e_{g_Z \cdot 0}, e_{g_Z \cdot 0} \rangle_X = \langle j(g_Z, 0) \pi(g_Z)e_0, j(g_Z, 0) \pi(g_Z)e_0 \rangle_X \]
\[ = |j(g_Z, 0)|^2 = |\chi(\kappa(g_Z))|^2 \]
and, by taking into account the expressions of \( \chi \) and \( g_Z \), we then recover the desired formula for \( K_X(Z, Z) \).

Let us introduce the following integral of Hua’s type:
\[ K_n(\lambda) := \int_{B} \text{Tr}((I_n - Y \tilde{Y})^{-1}) \text{Det}(I_n - Y \tilde{Y})^\lambda dy. \]
Since the maximum of the eigenvalues of \( (I_n - Y \tilde{Y})^{-1} \) is \( (1 - \lambda_s(Y \tilde{Y}))^{-1} \), we have
\[ \text{Tr}((I_n - Y \tilde{Y})^{-1}) \leq n (1 - \lambda_s(Y \tilde{Y}))^{-1} \leq n \text{Det}(I_n - Y \tilde{Y})^{-1} \]
and then we see that \( K_n(\lambda) \) converges for \( \lambda > 2 \) since \( J_n(\lambda) \) converges for \( \lambda > 1 \), see Lemma 3.1.

Also, we denote by \( \phi^1 \) and \( \phi^2 \) the elements of \( S \) defined by \( \phi^1_0 = (0, 1, 0) \) and \( \phi^2_0 = (0, 0, \varphi_0) \). We are now in position to establish the following proposition.
Proposition 6.4. Let
\[ \mu_n := \frac{c_X}{n^\gamma} \left( \frac{2\pi}{\gamma} \right)^n K_n \left( -m - n - \frac{3}{2} \right); \]
\[ \nu_n := -1 + \frac{2c_X}{n^\gamma} \left( \frac{2\pi}{\gamma} \right)^n K_n \left( -m - n - \frac{3}{2} \right). \]

Let \( \phi \in S \) defined by \( \phi_0 = (0, d, \lambda \varphi_0) \) with \( d, \lambda \in \mathbb{C} \). Let \( \psi \in S \) such that \( \psi(\cdot, X) = B_X(\phi(\cdot, X)) \) for each \( X \in \mathfrak{g}^\mathfrak{c} \). Then we have \( \psi_0 = (0, d, d\mu_n + \lambda \nu_n) \).

Proof. We will use the formula
\[ \psi_0(X) = \int_D \phi_0(\text{Ad}(g_Z^{-1})X))K_X(Z, Z)^{-1}c_Xd\mu(Z) \]
in order to compute the Berezin transforms \( \psi^1(\cdot, X) \) and \( \psi^2(\cdot, X) \) of \( \phi^1(\cdot, X) \) and \( \phi^2(\cdot, X) \).

We write \( \psi_0^1 = (0, d_1, \lambda_1 \varphi_0) \) with \( d_1, \lambda_1 \in \mathbb{C} \). Let \( H_1 := ((0, 0), 1, 0) \). Then we have \( \text{Ad}(g_Z^{-1})H_1 = H_1 \) hence \( \phi_0^1(\text{Ad}(g_Z^{-1})H_1) = 1 \) for each \( Z \in D \). This gives
\[ \psi_0^1(H_1) = \int_D K_X(Z, Z)^{-1}c_Xd\mu(Z) = 1. \]

On the other hand, we also have \( \psi_0^1(H_1) = d_1 \). Then we find \( d_1 = 1 \).

Now, let \( H_2 := ((0, 0), 0, (I_n^0 0 \ 0 \ 0 - I_n)) \). Then, for each \( Z \in D \), we have
\[ \text{Ad}(g_Z^{-1})H_2 = \begin{pmatrix} \star & i z_0^t \bar{z}_0 & (I_n - Y \bar{Y})^{-1}(I_n + Y \bar{Y}) & \star \\ \star & \star \end{pmatrix} \]
where, as usual, \( z_0 = (I_n - Y \bar{Y})^{-1}(y + Y \bar{y}) \). Consequently, we have
\[ \phi_0^1(\text{Ad}(g_Z^{-1})H_2) = \frac{i}{2} z_0^t \bar{z}_0. \]

Thus, by Lemma 6.3, we get
\[ \psi_0^1(H_2) = \frac{ic_X}{2} \int_D z_0^t \bar{z}_0 \exp \left( -\frac{\gamma}{4} (2z_0^t \bar{z}_0 - z_0^t \bar{Y} z_0 - \bar{z}_0^t Y \bar{z}_0) \right) \]
\[ \text{Det}(I_n - Y \bar{Y})^{-m-n-2} dy \ dY \]
and we make the change of variables
\[ y = z_0 - Y \bar{z}_0 \]
whose Jacobian is $\text{Det}(I_n - Y \bar{Y})$. Hence, by using Lemma 6.2, we obtain

$$\psi_0^1(H_2) = \frac{i c_\chi}{\gamma} \left(\frac{2\pi}{\gamma}\right)^n K_n \left(-m - n - \frac{3}{2}\right).$$

On the other hand, it is clear that $\psi_0^1(H_2) = i \lambda_1 n$. Finally, we find that

$$\lambda_1 = \frac{c_\chi}{n\gamma} \left(\frac{2\pi}{\gamma}\right)^n K_n \left(-m - n - \frac{3}{2}\right) = \mu_n.$$

Similarly, we write $\psi_0^2 = (0, d_2, \lambda_2 \varphi_0)$. Since we have $\phi_0^2(\text{Ad}(g_Z^{-1})H_1) = 0$ for each $Z \in \mathcal{D}$, we first obtain $d_2 = \psi_0^2(H_1) = 0$. Moreover, for each $Z = a(y, Y) \in \mathcal{D}$, we also have

$$\phi_0^2(\text{Ad}(g_Z^{-1})H_2) = i \text{Tr}(I_n - Y \bar{Y})^{-1}(I_n + Y \bar{Y})$$

$$= i(-n + 2 \text{Tr}((I_n - Y \bar{Y})^{-1})).$$

Then, changing variables $y \rightarrow (I_n - Y \bar{Y})^{1/2} y$, we get

$$\psi_0^2(\text{Ad}(g_Z^{-1})H_2) = -in + 2i c_\chi \int_{\mathbb{B} \times \mathbb{C}^n} \exp \left(-\frac{\gamma}{4}(2y^t \bar{y} + y^t \bar{Y} y + \bar{y}^t Y \bar{y})\right)$$

$$\text{Tr}((I_n - Y \bar{Y})^{-1}) \text{Det}(I_n - Y \bar{Y})^{-m-n-1} dy \, dY.$$

Thus, by using [21], p. 248, we obtain

$$\psi_0^2(\text{Ad}(g_Z^{-1})H_2) = -in + 2i c_\chi \left(\frac{2\pi}{\gamma}\right)^n K_n \left(-m - n - \frac{3}{2}\right).$$

Also, we have $\psi_0^2(\text{Ad}(g_Z^{-1})H_2) = i \lambda_2 n$. This gives

$$\lambda_2 = -1 + \frac{2c_\chi}{n} \left(\frac{2\pi}{\gamma}\right)^n K_n \left(-m - n - \frac{3}{2}\right) = \nu_n.$$

This finishes the proof. $\square$

Recall that $c_\chi$ can be expressed in terms of the Hua’s integral $J_n(-m-n-3/2)$ which can be explicitly computed, see Proposition 3.2 and Lemma 3.1. However, it seems difficult to compute $K_n(-m-n-3/2)$ similarly.

Now we give the matrix of $B_\chi$ in a suitable basis of $\mathcal{S}$. First, we consider the basis of $g^c$ consisting of the elements

$$X_i = ((e_i, 0), 0, 0),$$

$$Y_j = ((0, e_j), 0, 0),$$
\[
F_{ij} = \begin{pmatrix} (0, 0), 0, \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix} \end{pmatrix},
\]
\[
G_{ij} = \begin{pmatrix} (0, 0), 0, \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix} \end{pmatrix},
\]
\[
H_1 = ((0, 0), 1, 0),
\]
\[
A_{ij} = \begin{pmatrix} (0, 0), 0, \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij} \end{pmatrix} \end{pmatrix}.
\]
for \(i, j = 1, 2, \ldots, n\), \(E_{ij}\) denoting the \(n \times n\) complex matrix whose \(ij\)-th entry is 1 and all of whose other entries are 0.

Note that \(\phi^2(\cdot, X_i) = \phi^2(\cdot, Y_j) = \phi^2(\cdot, H_1) = 0\). Then, from the preceding proposition, we easily deduce the following result.

**Corollary 6.5.** The functions \(\phi^1(\cdot, X_i), \phi^1(\cdot, Y_j), \phi^1(\cdot, H_1), \phi^1(\cdot, F_{ij}), \phi^1(\cdot, G_{ij}), \phi^1(\cdot, A_{ij}), \phi^2(\cdot, F_{ij}), \phi^2(\cdot, G_{ij})\) and \(\phi^2(\cdot, A_{ij})\) form a basis for \(S\) in which \(B_X\) has matrix
\[
\begin{pmatrix}
I_{2n+1} & O & O \\
O & I_{3n^2} & O \\
O & \mu_n I_{3n^2} & \nu_n I_{3n^2}
\end{pmatrix}.
\]

Recall that for each \(X \in \mathfrak{g}^c\), we have \(S_X(d\pi_X(X)) \in S\). Consequently, we see that \(W_X(d\pi_X(X)) = B_X^{-1/2}(S_X(d\pi_X(X)))\) is well-defined. Moreover, we have the following proposition.

**Proposition 6.6.** For each \(X \in \text{Span}_C\{H_1, X_i, Y_j, 1 \leq i, j \leq n\}\), we have
\(W_X(d\pi_X(X)) = S_X(d\pi_X(X)).\) For each \(X \in \text{Span}_C\{F_{ij}, G_{ij}, A_{ij}, 1 \leq i, j \leq n\}\), we have
\[
W_X(d\pi_X(X)) = S_X(d\pi_X(X)) + i(1 - v_n^{-1/2})\left(\frac{\gamma \mu_n}{1 - v_n} + m\right)\phi^2(\cdot, X).
\]

**Proof.** For each \(X \in \mathfrak{g}^c\) we have
\[
S_X(d\pi_X(X)) = d\chi(\text{Ad}(g_Z^{-1})X) = i\gamma \phi^1(\cdot, X) - i m \phi^2(\cdot, X).
\]
Now, by using the preceding corollary, we see that the matrix of \(B_X^{-1/2}\) with respect to the above basis of \(S\) is
\[
\begin{pmatrix}
I_{2n+1} & O & O \\
O & I_{3n^2} & O \\
O & \frac{\mu_n v_n^{-1/2}}{1 + v_n^{1/2}} I_3 & v_n^{-1/2} I_{3n^2}
\end{pmatrix}.
\]
This implies that for \( X \in \{ H_1, X_i, Y_j, 1 \leq i, j \leq n \} \), we have \( W_X(d\pi_X(x)) = S_X(d\pi_X(x)) \) and, for \( X \in \{ F_{ij}, G_{ij}, A_{ij}, 1 \leq i, j \leq n \} \), we have

\[
W_X(d\pi_X(x)) = i\gamma \left( \phi^1(\cdot, X) - \frac{\mu_n v_n^{-1/2}}{1 + v_n^{1/2}} \phi^2(\cdot, X) \right) - imv_n^{-1/2} \phi^2(\cdot, X).
\]

Hence the result follows. \( \square \)

**References**


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