On BNA-normality and solvability of finite groups

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Abstract – Let $G$ be a finite group. A subgroup $H$ of $G$ is called a BNA-subgroup if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$. In this paper, some interesting properties of BNA-subgroups are given and, as applications, the structure of the finite groups in which all minimal subgroups are BNA-subgroups have been characterized.


Keywords. BNA-subgroup, minimal subgroup, soluble group, fitting height, $p$-length.

1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert [11]. $G$ always denotes a finite group, $|G|$ is the order of $G$, $\pi(G)$ denotes the set of all primes dividing $|G|$, $G_p$ is a Sylow $p$-subgroup of $G$ for some $p \in \pi(G)$.

Recall that a subgroup $H$ of $G$ is said to be an abnormal subgroup if $x \in \langle H, H^x \rangle$ for all $x \in G$. There have been several researches on normal and abnormal subgroups. In 1974, A. Fattahi classified the finite groups with only normal and abnormal subgroups [7]. G. Ebert and S. Bauman in 1975 studied the finite groups whose subgroups are either subnormal or abnormal [6]. Cuccia and
Liotta in 1982 showed that if $G$ is a finite group and, for every minimal subgroup $X$ of $G$, either $C_G(X)$ is subnormal or abnormal, then $G$ is soluble [4]. G. J. Wood in [15] studied the finite soluble groups whose subgroups are pronormal. Recently, Liu and Li in [13] classified CLT-groups with normal or abnormal subgroups, etc.

The abnormality and the normality are two basic concepts in the theory of groups, which are dual concepts. Precisely speaking, $G$ has only one subgroup, itself, that is both normal and abnormal in $G$. Each maximal subgroup of $G$ is either normal or abnormal.

In general, for any group $G$ and any subgroup $H$ of $G$, the following inclusion holds:

$$H \leq \langle H, H^x \rangle \leq \langle H, x \rangle,$$

for any $x \in G$.

For brevity, we introduce the following definition.

**Definition 1.1.** Let $G$ be a group. A subgroup $H$ of $G$ is called a BNA-subgroup of $G$ if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$, $H$ is also said to be BNA-normal in $G$.

Obviously, BNA-subgroups are in between normal subgroups and abnormal subgroups, all normal subgroups and all abnormal subgroups of $G$ are BNA-subgroups of $G$. The following example shows that the concept of BNA-subgroups is meaningful and feasible.

**Example 1.1.** Let $G = S_4$, the symmetric group on 4 letters. Then

1. the Sylow 3-subgroup of order 3 is not a BNA-subgroup of $G$;
2. each cyclic subgroup of order 4 is a BNA-subgroup of $G$, but is neither normal nor abnormal in $G$.

In fact, let $P = \langle (123) \rangle$ and $x = (34)$. Obviously, $P^x \neq P$ and $\langle P, P^x \rangle = A_4$, but $x \notin A_4$. Hence, $P$ is not a BNA-subgroup of $G$.

We know that $S_4$ has exactly three cyclic subgroups of order 4. Suppose that $C$ is one of them. Then $\langle C, C^y \rangle = C$ or $G$ for any $y \in G$. Therefore $C$ is a BNA-subgroup of $G$ (note that $C$ is neither normal nor abnormal in $G$).

There has been an interest to investigate the structure of a group $G$ under the assumption that minimal subgroups of $G$ have some properties in $G$. Itô proved that if the center of a group $G$ of order odd contains all minimal subgroups, then $G$ is nilpotent. Later Buckley [3] proved that if $G$ is a group of order odd whose minimal subgroups are normal in $G$, then the group $G$ is supersoluble.

In this paper, as a generalization, we consider the finite groups all of whose minimal subgroups are BNA-subgroups. Our main result is as follows.
Main Theorem. Suppose that all minimal subgroups of $G$ are BNA-subgroups of $G$. Then the following statements hold:

1. $G$ is soluble;
2. $G = TH$, where $T$ is a Sylow 2-subgroup of $G$, $H$ is a Hall 2'-subgroup of $G$;
3. $G$ is $p$-supersoluble for each odd prime $p$ dividing $|G|$;
4. the derived subgroup $G' = T_1 \rtimes H_1$, where $T_1 \leq T$ and $H_1 \leq H$;
5. the Fitting height of $G$ is bounded by 4;
6. for each odd prime $p$ dividing $|G|$, the $p$-length of $G$ is 1, that is, $G = O_{p'pp'}(G)$.

2. Preliminaries

In this section, we state some lemmas which are useful for our main result.

Lemma 2.1. Let $H \leq K \leq G$ and $N \trianglelefteq G$. Suppose that $H$ is a BNA-subgroup of $G$. Then

1. $H$ is a BNA-subgroup of $K$;
2. $HN$ is a BNA-subgroup of $G$;
3. $HN/N$ is a BNA-subgroup of $G/N$;
4. Any maximal subgroup of $G$ is a BNA-subgroup of $G$.

Proof. (1) Let $x$ be an element of $K$ such that $H \neq H^x$. Then $x \in \langle H, H^x \rangle$ by Definition 1.1 and so $H$ is a BNA-subgroup of $K$.

(2) Let $x$ be an element of $G$. If $H = H^x$, then $(HN)^x = HN$. If $x \not\in N_G(H)$, then $x \in \langle H, H^x \rangle$. Since $\langle H, H^x \rangle \leq \langle HN, (HN)^x \rangle$, then $x \in \langle HN, (HN)^x \rangle$. Thus $HN$ is a BNA-subgroup of $G$.

(3) It is easy to see $N_{G/N}(HN/N) \geq N_G(H)N/N$. If $xN \not\in N_{G/N}(HN/N)$, then $x \not\in N_G(H)$. As $H$ is a BNA-subgroup of $G$, then $x \in \langle H, H^x \rangle$. Therefore $xN \in \langle HN/N, H^xN/N \rangle = \langle HN/N, (HN/N)^xN \rangle$. Definition 1.1 implies that $HN/N$ is a BNA-subgroup of $G/N$.

(4) Definition 1.1 implies (4). \hfill \Box

Lemma 2.2. Let $H \leq G$ and be a BNA-subgroup of $G$. Then

1. the normal closure $H^G = H$ or $H^G = G$;
2. if, in addition, $H$ is subnormal in $G$, then $H$ is normal in $G$. 


Proof. (1) If $H^G < G$, then there exists some element $x$ of $G$ such that $x \notin H^G$. Since $H^x \leq H^G$ for all $x \in G$, then $\langle H, H^x \rangle \leq H^G$ and it follows that $x$ is not in $\langle H, H^x \rangle$. Thus, $x \in N_G(H)$ by Definition 1.1. Consequently, $G = H^G \cup N_G(H)$, which implies that $N_G(H) = G$. Therefore $H^G = H$.

(2) It follows from (1).

Lemma 2.3. Let $H$ be a BNA-subgroup of $G$. Then

1. $N_G(H) \leq \langle H, H^x \rangle$ whenever $H^x \neq H$;
2. if $N_G(H) \leq K \leq G$, then $K$ is an abnormal subgroup of $G$.

Proof. (1) Suppose that $H^x \neq H$ for some $x \in G$. By Definition 1.1, we have $x \in \langle H, H^x \rangle$. Let $n$ be an arbitrary element of $N_G(H)$, then $H^{nx} = H^x \neq H$. Similarly, we can get that $nx \in \langle H, H^{nx} \rangle = \langle H, H^x \rangle$. This implies $n \in \langle H, H^x \rangle$ and so $N_G(H) \leq \langle H, H^x \rangle$, as desired.

(2) Let $x$ be an element of $G$, then we have $\langle H, H^x \rangle \leq \langle N_G(H), N_G(H^x) \rangle \leq \langle K, K^x \rangle$. If $x \in N_G(H)$, then $x \in \langle K, K^x \rangle$ holds obviously. If $x \notin N_G(H)$, Definition 1.1 implies that $x \in \langle H, H^x \rangle$. Therefore $x \in \langle K, K^x \rangle$ holds.

Lemma 2.4. Let $G$ be a finite non-soluble group all of whose proper subgroups are soluble. Then $G/\Phi(G)$ is a minimal simple group, where $\Phi(G)$ is the Frattini subgroup of $G$.

Proof. Let $M$ be an arbitrary normal subgroup of $G$ containing $\Phi(G)$. If $M \not\leq \Phi(G)$, then there exists a maximal subgroup $H$ of $G$ such that $G = MH$. By the hypotheses, $H$ is soluble and hence $G/M \cong H/M \cap H$ is soluble. Because $G$ is non-soluble, $M$ is not soluble. We thus deduce that $M = G$ and so we can get that $G/\Phi(G)$ is a minimal simple group.

Lemma 2.5 ([14]). Let $G$ be a minimal non-abelian simple group (a non-abelian simple group all of whose proper subgroups are soluble). Then $G$ is one of the following groups:

1. $\text{PSL}(3, 3)$;
2. the Suzuki group $\text{Sz}(2^r)$ where $r$ is an odd prime;
3. $\text{PSL}(2, p)$ where $p$ is a prime with $p > 3$ and $p^2 \neq 1 \mod (5)$;
4. $\text{PSL}(2, 2^r)$ where $r$ is a prime;
5. $\text{PSL}(2, 3^r)$ where $r$ is an odd prime.
Recall that a normal subgroup $H$ of $G$ is supersolvably embedded in $G$ provided that every chief factor of $G$ contained in $H$ is cyclic.

**Lemma 2.6 ([1]).** Let $H$ be a normal subgroup of $G$. Suppose that all subgroup of prime order and cyclic subgroups of order 4 (if any) of $H$ are normal in $G$. Then $H$ is supersolvably embedded in $G$.

**Lemma 2.7 ([2] or [11], $P_{719}$).** If the normal subgroup $H$ of $G$ (not necessarily soluble) is supersolvably embedded in $G$, then $G/C_{G}(H)$ is supersoluble.

**Lemma 2.8.** Let $G$ be a $p$-soluble group for some odd prime $p \in \pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. If every subgroup of $P$ of order $p$ is normal in $N_{G}(P)$, then $G$ is $p$-supersoluble.

**Proof.** This is a special case of Theorem 1.1 of [12].

**Lemma 2.9 ([11]).** If $G$ is a $p$-supersoluble group, then $G'$ is $p$-nilpotent. If $G$ is a supersoluble group, then $G'$ is nilpotent.

3. **Proof of main theorem**

The proof of the main theorem will be finished by showing the following theorem.

**Theorem 3.1.** Suppose that all minimal subgroups of $G$ are BNA-subgroups of $G$. Then $G$ is soluble.

In fact, we can show the following stronger result:

**Theorem 3.2.** Suppose that all minimal subgroups of $G$ of order odd are BNA-subgroups of $G$. Then $G$ is soluble.

**Proof.** Assume that the theorem is false and let $G$ be a counterexample of minimal order. It follows from Lemma 2.1 that the condition is inherited by subgroups of $G$, so every proper subgroup of $G$ is soluble by the choice of $G$. It follows by Lemma 2.4 that $G/\Phi(G)$ is a minimal simple group and so $G$ is one of groups of Lemma 2.5.

(1) All minimal subgroups of $\Phi(G)$ of order odd are in $Z(G)$.

Suppose that some minimal subgroup $X$ of $\Phi(G)$ of order odd is not in $Z(G)$. Then $C_{G}(X) < G$ and so $C_{G}(X)$ is soluble. Furthermore, $X$ is subnormal in $G$ and $X$ is also a BNA-subgroup of $G$ by the hypotheses of the theorem, it follows by Lemma 2.2 (2) that $X$ is normal in $G$. So $C_{G}(X)$ is normal in $G$ and hence $G/C_{G}(X)$ is cyclic. Thus we can get that $G$ is soluble, a contradiction.
(2) Let $H$ be a Hall $2'$-subgroup of $\Phi(G)$, then $H \leq Z(G)$.

By (1), all minimal subgroups of $H$ are in $Z(G)$, so by Lemma 2.6, $H$ is supersolvably embedded in $G$. It follows from Lemma 2.7 that $G/C_G(H)$ is supersoluble. If $C_G(H) < G$, then $C_G(H)$ is soluble and so $G$ is soluble, a contradiction. Thus we have that $C_G(H) = G$, and so $H \leq Z(G)$.

(3) $\Phi(G)$ is a group of order odd and $\Phi(G) \leq Z(G)$.

Let $Z$ be a Sylow 2-subgroup of $\Phi(G)$. It follows from Lemma 2.1 that $G/Z$ satisfies the hypotheses of the theorem. If $Z \neq 1$, then $G/Z$ is soluble and so $G$ is soluble by the choice of $G$, a contradiction. Thus $Z = 1$, that is, $\Phi(G)$ is a $2'$-group and so $\Phi(G) \leq Z(G)$.

(4) $\Phi(G) = 1$, that is, $G$ is a minimal simple group.

Since $Z(G)/\Phi(G) \leq G/\Phi(G)$ and $G/\Phi(G)$ is a minimal simple groups, $\Phi(G) = Z(G)$. So $G$ is a quasisimple group with the center of order odd. We claim that $\Phi(G) = 1$, that is, $Z(G) = 1$. It will suffice to show that the Schur multiplier of each of the minimal simple groups is a 2-group. Indeed, this is true by checking the list on the Schur multipliers of the known simple groups ([9], P302).

(5) $G$ can not be $\text{PSL}(2, p)$, $\text{PSL}(2, 3^r)$ or $\text{PSL}(3, 3)$.

Indeed, each of $\text{PSL}(2, p)$, $\text{PSL}(2, 3^r)$ and $\text{PSL}(3, 3)$ contains a subgroup which is isomorphic to $A_4$, the alternating group of degree 4. Let $P$ be a Sylow 3-subgroup of $A_4$. It follows from Lemma 2.3 that $N_G(P) \leq \langle P, P^x \rangle$ for all $x \in G$ such that $x \notin N_G(P)$. In particular, let $x$ be an arbitrary element of $A_4$ of order 2, then we have that $N_G(P) \leq A_4$. So $N_G(P) = P \leq C_G(P)$. Therefore $P$ has to be a Sylow 3-subgroup of $G$. By Burnside theorem, we can get that $G$ is 3-nilpotent, hence $G$ would not be a non abelian simple group, a contradiction. Therefore we conclude that $G$ is not any one of $\text{PSL}(2, p)$, $\text{PSL}(2, 3^r)$ or $\text{PSL}(3, 3)$.

(6) $G$ can not be $\text{PSL}(2, 2^r)$ or $S_z(2^r)$.

Suppose that $G \cong \text{PSL}(2, 2^r)$ or $S_z(2^r)$. By ([8], P466), we know that $G$ is a Zassenhaus group of odd degree and the stabilizer $M = [T]H$ of a point is a Frobenius group with kernel $T$ and with a complement $H$. For $\text{PSL}(2, 2^r)$, the kernel $T$ is an elementary 2-group of order $2^r$ and $H$ is cyclic of order $2^r - 1$. For $S_z(2^r)$, $T$ is a special 2-group of order $2^{2r}$ and $H$ is cyclic of order $2^r - 1$. Let $Q$ be a Sylow $q$-subgroup of $H$ and $L$ a minimal subgroup of $Q$ for some prime $q$ dividing $|H|$. It follows from Lemma 2.3 that $N_G(L) \leq \langle L, L^x \rangle \leq M$ for any 2-element $x$ of $M$. As $M$ is a Frobenius group, then $N_G(L)$ is a $2'$-subgroup of $M$ and so $N_G(L) \leq H$ is cyclic. Thus $N_G(L) = C_G(L) = H$. 
Since $C_G(Q) \leq N_G(Q) \leq N_G(L) = C_G(L) = H$, then $Q$ is also a Sylow $q$-subgroup of $G$ and so $C_G(Q) = N_G(Q) = H$. By Burnside theorem, $G$ is $q$-nilpotent, hence $G$ would not be a non abelian simple group, a contradiction. Therefore $G$ can not be any one of $\text{PSL}(2, 2^r)$ or $S_2(2^r)$ as well.

The proof of the theorem is now complete. \qed

**Definition 3.1** ([5]). Let $G$ be a finite group. The Fitting series

$$F_n(G), \ n = 0, 1, 2, \ldots$$

is defined inductively by $F_0(G) = 1$, $F_n(G)$ is the inverse image in $G$ of $F(G/F_{n-1}(G))$, for $n \geq 1$.

Evidently each $F_n(G)$ is a characteristic subgroup of $G$. If $G$ is solvable, then there is some integer $h \geq 0$ such that $F_h(G) = G$. We call the least such integer $h$ the Fitting height of $G$ and denote it by $h(G)$.

**Definition 3.2.** ([10]) Let $G$ be a finite $p$-soluble group for some prime $p$. Define the upper $p$-series

$$1 = P_0 \leq N_0 < P_1 < N_1 < P_2 < \cdots < P_l \leq N_l = G$$

inductively by the rule that $N_k/P_k$ is the greatest normal $p'$-subgroup of $G/P_k$, and $P_{k+1}/N_k$ the greatest normal $p$-subgroup of $G/N_k$. The number $l$, which is the least integer such that $N_l = G$, is called the $p$-length of $G$, and we denote it by $l_p$, or, if necessary, $l_p(G)$.

Recall that a finite $p$-group is said to be a $PN$-group if its subgroups of order $p$ are normal.

**Lemma 3.1** ([12], Lemma 1.4). Let $G$ be a finite $p$-soluble group for an odd prime $p$. If a Sylow $p$-subgroup of $G$ is a $PN$-group, then the $p$-length $l_p(G) \leq 1$.

Now we state and show the main theorem of this paper.

**Theorem 3.3.** Suppose that all minimal subgroups of $G$ are BNA-subgroups of $G$. Then the following statements hold:

1. $G$ is soluble;
2. $G = TH$, where $T$ is a Sylow $2$-subgroup of $G$, $H$ is a Hall $2'$-subgroup of $G$;
3. $G$ is $p$-supersoluble for each odd prime $p$ dividing $|G|$;
(4) the derived subgroup \( G' = T_1 \rtimes H_1 \) where \( T_1 \leq T \) and \( H_1 \leq H \);
(5) the Fitting height of \( G \) is bounded by 4;
(6) for each odd prime \( p \) dividing \( |G| \), the \( p \)-length of \( G \) is 1, that is,
\[
G = O_{p | p'}(G).
\]

**Proof.** (1) is Theorem 3.1.

(2) Let \( T \) be a Sylow 2-subgroup of \( G \). Applying a well known theorem of P. Hall, we have that \( G \) possesses a Hall \( 2' \)-subgroup \( H \). Therefore (2) holds.

(3) For any odd prime \( p \) dividing \( |G| \), let \( P \) be a Sylow \( p \)-subgroup of \( G \). By Lemma 2.1, each subgroup \( X \) of \( P \) of order \( p \) is a BNA-subgroup of \( N_G(P) \) and \( X \) is subnormal in \( N_G(P) \). It follows from Lemma 2.2 that \( X \) is normal in \( N_G(P) \). By (1), \( G \) is soluble, of course, is \( p \)-soluble. Thus we can apply Lemma 2.8 to see that \( G \) is \( p \)-supersoluble.

(4) Clearly, \( G \) is \( p \)-supersoluble for any odd primes \( p \) dividing \( |G| \) by (3). Thus \( G' \) is \( p \)-nilpotent by Lemma 2.9 for all odd primes \( p \) of \( \pi(G) \). Let \( N(p) \) be the normal \( p \)-complement of \( G' \) for each odd prime \( p \) and set
\[
T_0 = \bigcap_p N(p).
\]
As \( N(p) \) is a \( p' \)-group for each odd prime \( p \), it is easy to see that \( T_0 \) must be a 2-group. Since \( N(p) \) contains all Sylow 2-subgroups of \( G' \) for each odd prime \( p \) dividing \( |G| \), \( T_0 \) is a Sylow 2-subgroup of \( G' \) and is normal in \( G \). So \( T_0 \leq T \).

As \( G \) is soluble, of course, \( G' \) is also soluble, so there is a Hall \( 2' \)-subgroup \( H_0 \) of \( G' \). Then \( H_0 \leq H^x \) for some \( x \in G \). Thus \( G' = T_0 \rtimes H_0 = T_0 \rtimes H_0^{x^{-1}} \). Consequently, \( H_0^{x^{-1}} \) is supersoluble and (4) holds.

(5) We can get that \( G' = T_1 \rtimes H_1 \) where \( T_1 \leq T \) and \( H_1 \leq H \) by (4). Set \( F_1 = T_1, F_2 = F_1(H_1)' \) and \( F_3 = G' \). We can get that \([G'/T_1, G'/T_1] = [T_1H_1/T_1, T_1H_1/T_1] = [H_1T_1, H_1T_1]T_1/T_1\). Since \( T_1 \leq G \), it follows from Lemma 1.10 of Chapter 3 in [11] that \([H_1T_1, H_1T_1] = [H_1, T_1](T_1)'(H_1)'\). By Lemma 1.6 of Chapter 3 in [11], we have that \([H_1, T_1] \leq T_1 \). Therefore \( G/T_1 \cong [G'/T_1, G'/T_1] = T_1(H_1)'/T_1 \), so we have \( F_2 \leq G \). Thus \( F_1, F_2 \) and \( F_3 \) are all normal in \( G \) and we have a chain of normal subgroups
\[
1 = F_0 \leq F_1 \leq F_2 \leq F_3 \leq F_4 = G.
\]
It follows by Lemma 2.9 that \((H_1)' \) is nilpotent. Now, it is easy to check that \( F_i/F_{i-1} \) is nilpotent, \( i = 1, 2, 3, 4 \). Therefore we conclude that the Fitting height of \( G \) is at most 4 by Definition 3.1.
(6) For any odd prime $p$ dividing $|G|$, let $P$ be a Sylow $p$-subgroup of $G$. By Lemma 2.1, each subgroup $X$ of $P$ of order $p$ is a BNA-subgroup of $P$. As $X$ is subnormal in $P$, it follows by Lemma 2.2 that $X$ is normal in $P$, so $P$ is a $PN$-group. $G$ is soluble, and of course, $p$-soluble. We apply Lemma 3.1 to see that $l_p(G) \leq 1$. If $l_p(G) = 0$, then $G$ is a $p'$-group, a contradiction. Therefore $l_p(G) = 1$, and so $G = O_{p'pp'}(G)$.

The proof of the theorem is now complete. □

**Question.** Are there finite groups $G$ that satisfy the condition of Theorem 3.3 with $l_2(G) \geq 2$?

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