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Irreducible characters of finite simple groups constant at the p-singular elements

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Abstract – In representation theory of finite groups an important role is played by irreducible characters of p-defect 0, for a prime p dividing the group order. These are exactly those vanishing at the p-singular elements. In this paper we generalize this notion investigating the irreducible characters that are constant at the p-singular elements. We determine all such characters of non-zero defect for alternating, symmetric and sporadic simple groups.

We also classify the irreducible characters of quasi-simple groups of Lie type that are constant at the non-identity unipotent elements. In particular, we show that for groups of BN-pair rank greater than 2 the Steinberg and the trivial characters are the only characters in question. Additionally, we determine all irreducible characters whose degrees differ by 1 from the degree of the Steinberg character.

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1. Introduction

Local representation theory studies properties of group representations depending on a prime p dividing the order of a finite group G and the structure of a Sylow p-subgroup S of G. Denote by $\Sigma_p(G)$ the set of all p-singular elements of G, that is, those of order divisible by p. In this theory a prominent role is played by

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irreducible characters of defect 0. These are exactly those vanishing at $\Sigma_p(G)$. In this paper we study irreducible characters that are constant at $\Sigma_p(G)$. We call such characters *p-constant*.

Although p-constant characters are very natural as a generalization of those of defect 0, they do not seem to be discussed in the literature.

If G has a single conjugacy class of p-singular elements then every irreducible character of G is p-constant. Groups G with single class of non-trivial p-elements are studied in [10]. Also, the trivial character is p-constant. It is less obvious that for $p > 2$ non-exceptional characters in the principal block with cyclic defect group are p-constant (see Theorem 1.3 below). We mention [13] where the authors study irreducible characters whose values at the p-singular elements are roots of unity, mainly for p-solvable groups.

In this paper we focus mainly on quasi-simple groups and in view of Lemma 2.2 below, we can concentrate on simple groups. Our main result is that on classification of all p-constant irreducible characters for quasi-simple groups of Lie type with defining characteristic p. Following [1, 1.17], a finite group of Lie type is the group of the fixed points of a (non-necessarily standard) Frobenius map acting on a connected reductive group. (The simple group ${}^2F_4(2)'$ will be considered in Section 5 together with the sporadic groups). Note that among the quasi-simple groups of Lie type, only $\mathrm{SL}_2(q)$ with q even has a single class of non-identity p-elements. Recall that, for every quasi-simple group G of Lie type of characteristic p, the Steinberg character is the only irreducible character of G of p-defect 0.

Theorem 1.1. *Let G be a quasi-simple finite group of Lie type of characteristic p and let τ be an irreducible character of G. Then τ is p-constant if, and only if, one of the following holds*:

(1) *τ is the Steinberg character of G or $\tau = 1_G$*;

(2) *$G \in \{\mathrm{SL}_2(q), \mathrm{SL}_3(q), \mathrm{SU}_3(q), {}^2B_2(q^2), {}^2G_2(q^2)\}$ and $\tau(1) = |G|_p \pm 1$. More precisely, $\tau(1) \neq |G|_p - 1$ if $G \in \{\mathrm{SU}_3(q), {}^2B_2(q^2), {}^2G_2(q^2)\}$, and $\tau(1) \neq |G|_p + 1$ if $G = \mathrm{SL}_3(q)$.*

One can be interested with the other quasi-simple groups. We state the following.

Problem 1.2. *Let G be a finite quasi-simple group. Determine the irreducible characters τ of G for which there exists a constant $c \neq 0$ such that $\tau(g) = c$ for all $g \in \Sigma_p(G)$.*

Toward this problem, we have the following technical but useful observations. Recall that, when G has cyclic Sylow p-subgroups, $\operatorname{Irr} G$ consists of so called *exceptional* and *non-exceptional* characters, see [7, Chapter VII].

THEOREM 1.3. *Let G be a finite group with Sylow p-subgroup S, and let B be the principal p-block of G.*

(1) *If χ is an irreducible p-constant character of non-zero defect, then χ belongs to B.*

(2) *Assume further that the defect group S of B is cyclic and that B contains d ordinary exceptional characters. Let $\chi \neq 1_G$ be an irreducible character belonging to B. Then χ is p-constant if, and only if, one of the following occurs*:

 (a) $d = 1$;

 (b) $d > 1$, $p > 2$ *and χ is not exceptional.*

 In addition, if χ is p-constant then $\chi(g) = 1$ or -1 for $g \in \Sigma_p(G)$.

In fact, we have more precise information on χ in the case (b) of (2) above in terms of the Brauer tree of the principal block. This reduces Problem 1.2 to groups with non-cyclic Sylow p-subgroups. For alternating groups we have the following result:

THEOREM 1.4. *Let $G = \mathbb{A}_n$, $n > 4$, be an alternating group, and let p be a prime such that $n \geq 2p$. Let τ be a p-constant non-linear irreducible character of non-zero defect. Then one of the following holds*:

(1) $p > 2$, $n = 2p$ *and τ is an irreducible constituent of an irreducible character of $\mathbb{S}_n$ corresponding to the partition $(p, 1^p)$ or $(p, 2, 1^{p-2})$*;

(2) $p > 2$, $n = 2p + 1$ *and τ is an irreducible constituent of an irreducible character of $\mathbb{S}_n$ corresponding to the partition $(p+1, 1^p)$ or $(p+1, 2, 1^{p-2})$*;

(3) $p = 2$ *and* $(n, \chi(1)) \in \{(5, 3), (5, 5), (6, 9), (7, 15)\}$.

All these characters take value 1 *or* -1 *on* $\Sigma_p(G)$.

For finite simple groups we obtain the following result.

THEOREM 1.5. *Let G be a finite simple group, p be a prime dividing the order of G and τ be an irreducible character of G. Assume that $\tau(g) = c$ for all $g \in \Sigma_p(G)$. Then, one of the following holds*:

(1) $c \in \{-1, 0, 1\}$;

(2) $G = M_{22}$, $p = 3$, $c = -2$ *and* $\tau(1) = 385$;

(3) *G is a group of Lie type of characteristic $r \neq p$ with a non-cyclic Sylow p-subgroup.*

Note that case (3) requires further analysis. This case is not vacuous: for instance the group $\mathrm{PSL}_3(7)$ admits an irreducible 3-constant character which takes value 2 at the 3-singular elements. For sporadic groups see Section 5.

In [15] Seitz discussed a question on pairs of irreducible characters of classical groups whose degrees differ by 1. He suggested examples, currently known as irreducible Weil characters, and studied these examples in certain details. To our knowledge, no further discussion of this question is available in the literature (but the Weil characters themselves attracted a lot of attention and have many applications). As a part of our proof of Theorem 1.1 we classify all cases where one of the characters is the Steinberg character St_G of a finite group of Lie type G.

THEOREM 1.6. *Let G be a quasi-simple finite group of Lie type. Then G admits an irreducible character τ such that $\tau(1) = St_G(1) \pm 1$ if, and only if, one of the following holds*:

(1) $\tau(1) = St_G(1) + 1$ *and* $G \in \{\mathrm{SL}_2(q), \mathrm{SU}_3(q), {}^2B_2(q^2), {}^2G_2(q^2)\}$;

(2) $\tau(1) = St_G(1) - 1$ *and* $G \in \{\mathrm{SL}_2(q), \mathrm{SL}_3(q), \mathrm{Sp}_4(q), G_2(q)\}$.

In Section 2 we give some basic properties of p-constant characters and we recall some results of E. Dade in order to prove Theorem 1.3. In Section 3 we deal with symmetric and alternating groups. In Section 4 we consider p-constant characters for finite groups of Lie type in characteristic p and prove Theorems 1.1 and 1.6. In Section 5 we analyse the sporadic groups and finally in Section 6 we prove Theorem 1.5.

2. Blocks with cyclic defect group

We first make the following observation for an arbitrary finite group G. Let p be a prime dividing the order of G and let S be a Sylow p-subgroup of G. Let $\mathbb{Z}$ denote the set of rational integers.

LEMMA 2.1. *Let τ be a generalized character of a group G such that $\tau(u) = a$ for some complex number a and every $1 \neq u \in S$. Then $a \in \mathbb{Z}$.*

PROOF. Let $\lambda \neq 1_S$ be an arbitrary linear character of S. Then

$$(\tau_{|S}, \lambda) = \frac{\sum_{u\in S} \tau(u)\overline{\lambda(u)}}{|S|} = \frac{\tau(1) + \sum_{1\neq u\in S} a\overline{\lambda(u)}}{|S|}$$
$$= \frac{\tau(1) + a\cdot|S|\cdot(\lambda, 1_S) - a}{|S|} = \frac{\tau(1) - a}{|S|},$$

since $(\lambda, 1_S) = 0$. Hence, $a = \tau(1) - |S|\cdot(\tau_{|S}, \lambda) \in \mathbb{Z}$. $\square$

The following lemma reduces Problem 1.2 to groups with trivial center, in particular, we can ignore quasi-simple groups that are not simple.

LEMMA 2.2. *Let G be a finite group, p a prime and let χ be an irreducible character of G of non-zero p-defect. Suppose that χ is non-trivial and p-constant. Then one of the following holds*:

(1) *p does not divide $|Z(G)|$, $Z(G) \leq \mathrm{Ker}(\chi)$ and the corresponding character $\bar{\chi}$ of $G/Z(G)$ is an irreducible p-constant character*;

(2) *$p = 2$, $|G|_2 = 2$ and $G = \mathrm{Ker}(\chi) \times O_2(G)$.*

PROOF. Suppose that $Z(G)$ is not contained in $\mathrm{Ker}(\chi)$, and let $z \in Z(G)$ such that $\chi(z) \neq \chi(1)$. If $z \notin \Sigma_p(G)$, then $zg \in \Sigma_p(G)$ for every p-element $g \in G$. So $\chi(g) = \chi(zg) = \frac{\chi(z)}{\chi(1)}\chi(g)$. As χ is not of p-defect 0, $\chi(g) \neq 0$ for some p-element g, and hence $\chi(z) = \chi(1)$, which is a contradiction.

So, there is a p-element $z \in Z(G)$. By Lemma 2.1, $\chi(z) \in \mathbb{Z}$ and so $\chi(z) = -\chi(1)$, whence $p = 2$ and $\mathrm{Ker}(\chi)$ has odd order. Let $h \in G$. If h has odd order, then $zh \in \Sigma_2(G)$ and $-\chi(1) = \chi(zh) = \frac{\chi(z)}{\chi(1)}\chi(h) = -\chi(h)$, whence $h \in \mathrm{Ker}(\chi)$. It follows that $G/\mathrm{Ker}(\chi)$ is a 2-group. If $h \neq z^{-1}$ is a 2-element of G, then $\chi(h) = \chi(z) = -\chi(1)$ which implies $-\chi(1) = \chi(zh) = \chi(1)$, a contradiction. So z is the only 2-element in G. Therefore, $G = \mathrm{Ker}(\chi) \times \langle z\rangle$. $\square$

Prior proving Theorem 1.3 we recall certain facts from representation theory of groups with cyclic Sylow p-subgroups. For further details, see [5]. Let G be a finite group with cyclic Sylow p-subgroup S. Set $C = C_G(S)$, $N = N_G(S)$ and $n = |N : C|$. As $(n, p) = 1$, it follows that n divides $p - 1$.

Let B be a block of G having defect group S. By Brauer's first main theorem, there exists a unique block B_0 of N with the same defect group S such that

$B_0^G = B$. Let b_0 be a block of C such that $b_0^N = B_0$ (also S is the defect group of b_0). Let E be the subgroup of N fixing b_0 and $e = |E : C|$. Then E/C acts on S as a group of automorphisms and e divides $p-1$ (e is called the *inertia index* of B).

The set of non-trivial irreducible characters of S partitions into $(|S|-1)/e$ orbits under the action of E/C. Each of these orbits contains e elements. Let Λ be a complete set of representatives of these orbits. So $d = |\Lambda| = (|S|-1)/e$.

For a non-trivial character $\lambda \in \operatorname{Irr} S$ let η_λ denote the sum of all N/C-conjugates of λ. In particular, $\eta_\lambda(1) = n$ and $(\eta_\lambda, 1_S) = 0$. Note that if λ, μ are N-conjugate, then $\eta_\lambda = \eta_\mu$.

Lemma 2.3 ([5, Theorem 1 and Corollary 1.9]). *Under the previous hypothesis on $G, S, B, \ldots$, the block B contains e non-exceptional characters $\chi_1, \ldots, \chi_e$ and $d = |\Lambda|$ exceptional characters χ_λ $(\lambda \in \Lambda)$. Let $g \in S$ be of order $|S|$ and let ϕ be the unique irreducible Brauer character of C contained in b_0. When $|S| > p$, let $x \in S$ be of order p and $S_1 = \langle x \rangle$. Set $N_1 = N_G(S_1)$ and $C_1 = C_G(S_1)$.*

(1) *For any $j = 1, \ldots, e$ one has*

$$\chi_j(g) = \varepsilon_j \phi(1) \cdot |N : E| \quad \text{and} \quad \chi_j(x) = \varepsilon_j \gamma \phi_1(1) \cdot |N_1 : EC_1|,$$

for some $\phi_1 \in \operatorname{IBr} C_1$, $\varepsilon_j = \pm 1$ and $\gamma = \pm 1$ that do not depend on g and x.

(2) *For any $\lambda \in \Lambda$ one has*

$$\chi_\lambda(g) = \varepsilon_0 \phi(1) \eta_\lambda(g) \quad \text{and} \quad \chi_\lambda(x) = \varepsilon_0 \gamma \phi_1(1) \sum_{y \in N_1/C_1} \lambda^y(x),$$

where ϕ_1, γ are the same as in item (1) *and $\varepsilon_0 = \pm 1$ does not depend on g, x and λ.*

(3) *$\chi_\lambda(S) \subset \mathbb{Q}$ if, and only if, $n = p-1$ and $|\lambda(S)| = p$.*

(4) *Assume B is the principal block of G and $d > 1$. Then the trivial character 1_G is not exceptional, except possibly when $p = 2$ and G has a normal subgroup of index $|S|$.*

Proof. (3) and (4) are not stated in [5], so we provide a proof here, although they can be known to some experts.

(3) Let $S = \langle g \rangle$. By item (2), $\chi_\lambda(g) = \varepsilon_0 \phi(1) \eta_\lambda(g)$, where $\eta_\lambda(1) = n$. Let ρ be a representation of S with character η_λ. Let $|\lambda(S)| = p^a$ for some integer $a > 0$ (so $\lambda(g)$ is a primitive p^a-root of unity). Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of $\rho(g)$. As η_λ is the sum of all N-conjugates of λ and $1_S \neq \lambda \in \operatorname{Irr} S$, it follows that $\alpha_1, \ldots, \alpha_n$ are (distinct) roots of the polynomial $t(x) := (x^{p^a} - 1)/(x^{p^{a-1}} - 1)$,

which is irreducible over $\mathbb{Q}$. Let $f(x)$ be the characteristic polynomial of the matrix $\rho(g)$, so $\alpha_1, \dots, \alpha_n$ are also the roots of $f(x)$. Suppose that that $\chi_\lambda(h)$ is rational for every $h \in S$. Then so is $\eta_\lambda(h)$ and $\eta_\lambda(h) = \alpha_1^k + \dots + \alpha_n^k \in \mathbb{Q}$ for every integer k. It is well known that the coefficients of the characteristic polynomial of a square matrix M, say, are polynomials of the traces of M^i for various integers i. It follows that the coefficients of $f(x)$ are rational. This implies that the polynomial $t(x)$ is reducible over $\mathbb{Q}$, unless $t(x) = f(x)$. In the latter case all primitive p^a-roots are the roots of $f(x)$. Therefore, $n = p - 1$ and $a = 1$.

The converse is obvious.

(4) Suppose the contrary, that 1_G is an exceptional character (this belongs to the principal block). We first recall that the principal blocks of G and N correspond to each other under the Brauer correspondence [4, 61.16]. Furthermore, it follows from by [4, 61.7 and 61.11], that 1_N is the only irreducible Brauer character in the principal block of N. In particular, ϕ is the trivial Brauer character of C. The group E above is in fact the stabilizer in N of this character, and hence $E = N$, $e = n$ in this case.

By item (3), $n = p - 1$. Let $1_G = \chi_\lambda$ for some $\lambda \in \Lambda$ such that $|\lambda(S)| = p$. Therefore, $\eta_\lambda(g) = -1$. Since $d > 1$ we have $|S| > p$. Clearly, $|N_1/C_1| \leq p - 1$. As $N \subseteq N_1$ and $|N/C| = p - 1$, we have $|N_1/C_1| = p - 1$. By item (2) we get $1 = \chi_\lambda(x) = \pm\phi_1(1) \cdot |N_1 : C_1|$ for some $\phi_1 \in \mathrm{IBr}\, C_1$, whence $n = 1$ and $p = 2$.

The statement on the structure of G follows from the Burnside Normal Complement Theorem [8, Theorem 14.3.1]. □

Proof of Theorem 1.3. Clearly we may assume $\chi \neq 1_G$.

(1) It follows from Lemma 2.1 that $\chi(S) \subset \mathbb{Z}$, so $\chi - a \cdot 1_G$ (where $a = \chi(s)$, $1 \neq s \in S$) is a non-zero generalized character vanishing at the p-singular elements. It follows from [7, Ch.IV, Lemma 3.14] that χ and 1_G belong to the same block. As 1_G is in the principal block, so is χ.

(2) By (1), 1_G and χ belong to the principal block, and, by assumption, $\chi \neq 1_G$. Consider the Brauer tree associated to the principal block. Recall that one node of the Brauer tree corresponds to the sum of all $d = |\Lambda|$ exceptional characters (denoted by χ_0), and the other nodes are in bijective correspondence with the e non-exceptional characters of the block.

(i) The theorem is true if both χ and 1_G are not exceptional or $d = 1$.

Let v, w be the nodes at the Brauer tree corresponding to the characters 1_G and χ, let $n_1 = v, n_2, \dots, n_k = w$ be the consequent nodes of the path connecting v and w, and let ψ_i be the ordinary character corresponding to n_i for $i = 1, \dots, k$

(one of the characters ψ_i coincides with χ_0, which is irreducible if and only if $d = 1$). By [7, Ch.VII, Lemma 2.15], $\psi_i + \psi_{i+1}$ is the character of a projective indecomposable module for $i = 1, \ldots, k-1$. Let $g \in \Sigma_p(G)$. Then $\psi_i(g) = -\psi_{i+1}(g)$ for every $i = 1, \ldots, k-1$. It follows that $\psi_i(g) = (-1)^{i+1}\psi_1(g)$. As $\psi_1 = 1_G$, we arrive at the case (2)(a). In addition, this proves the additional statement of the theorem in this case.

(ii) The theorem is true if $d > 1$ and χ or 1_G is exceptional.

Let $d > 1$. If 1_G is exceptional then, by Lemma 2.3(4), $n = e = 1$, $p = 2$ and $G/P \cong S$ for a normal subgroup P of G. We show that the same is true if 1_G is non-exceptional.

Let $\chi = \chi_\lambda$ for some $\lambda \in \Lambda$. By Lemma 2.3(3), $\chi_\lambda(S) \not\subset \mathbb{Q}$ unless $n = p-1$ and $|\lambda(S)| = p$. In this case $\eta_\lambda(g) = -1$ and $\chi(g) = \pm 1$. Let $x \in S$ be of order p and $S_1 = \langle x \rangle$. Set $N_1 = N_G(S_1)$ and $C_1 = C_G(S_1)$. As $\lambda(x) = 1$, by Lemma 2.3(2), we have $\chi_\lambda(x) = \pm\phi_1(1) \cdot |N_1 : C_1|$ for some $\phi_1 \in \mathrm{IBr}\, C_1$. As χ is p-constant, $\chi(g) = \chi(x)$, whence $N_1 = C_1$. It is well known $N \cap C_1 = C$. As $N \subseteq N_1$, we have $N = C$, and hence $n = 1$. Then $n = p - 1$ implies $p = 2$.

By the Burnside Normal Complement Theorem [8, Theorem 14.3.1], G has a normal 2-complement, that is, G has a normal subgroup P, say, of index $|S|$ as claimed.

Furthermore, χ belongs to the principal block B, so the irreducible constituents of $\chi|_P$ are in the principal block b of P, see [12, Theorem 9.2]. As P is a p'-group, 1_P is the only irreducible character in b. Therefore, $\chi|_P = \chi(1) \cdot 1_P$, and hence P is in the kernel of χ. So χ is linear. Since $d > 1$, necessarily $|S| > 2$. In this case, the hypothesis χ be p-constant leads to the contradiction $\chi = 1_G$.

To prove the converse, suppose that χ is non-exceptional. By Lemma 2.3(1), $\chi(S) \subset \mathbb{Q}$. By (i), we only have to deal with the case where 1_G is exceptional. Then by Lemma 2.3(4), $n = e = 1$ and $G/P \cong S$ for a normal subgroup P of G. Then we have seen in the previous paragraph that $\chi(P) = 1$. Now $\chi(S) \subset \mathbb{Q}$ implies $\chi^2 = 1_G$. Then $\chi(g) = \chi(g^2)$ leads to $\chi = 1_G$ which is a contradiction. □

It is well known that the defect group of the principal block of G coincides with a Sylow p-subgroup. Therefore, if a p-constant character belongs to a block with cyclic defect group then, by Theorem 1.3(1), the Sylow p-subgroups of G are cyclic.

3. Symmetric and alternating groups

We first consider the case where Sylow p-subgroups of $G = \mathbb{S}_n$ are cyclic, equivalently with $p \leq n < 2p$. By Theorem 1.3, a character $\tau \in \mathrm{Irr}\, G$ of non-zero defect is p-constant if, and only if, τ belongs to the principal block. Therefore, it suffices to determine the non-linear irreducible characters that are in the same block as 1_G. However, this is already known, see [9, 6.1.21]. Specifically, if χ_λ is the irreducible character of G corresponding to a partition λ of n, then χ_λ is in the principal block if and only if the p-core of λ is the same as that of the trivial partition (n). (See [9, p.76] for the notion of p-core.) If $n = p$ then the p-core of (n) is empty; this implies that λ is a hook. If $p < n < 2p$ then the p-core of (n) is $(n-p)$. It follows that λ is the partition associated to the diagram obtained from a hook diagram associated to a partition $\lambda' \neq (p)$ for $\mathbb{S}_p$ either by adding $(n-p)$ boxes to the second row, or by adding the additional row of $(n-p)$ boxes above the diagram of λ', provided this yields a proper diagram. In more accurate terms this is described in the following lemma.

Lemma 3.1. *Let p be a prime such that $2 \leq p \leq n < 2p$. A non-linear irreducible character χ_λ of $\mathbb{S}_n$ of non-zero defect is p-constant if, and only if, n and λ satisfy one of the following conditions*:

(i) $n = p \geq 3$ *and* $\lambda = (b, 1^{p-b})$ *with* $2 \leq b \leq p-1$;

(ii) $n = p+1 \geq 4$ *and* $\lambda = (b+1, 2, 1^{p-b-2})$ *with* $1 \leq b \leq p-2$;

(iii) $n = p+r \geq 5$, $r \geq 2$, *and λ is one of the following partitions*:

$$(p-a, r+1, 1^{a-1}) \;\; (1 \leq a \leq p-r-1); \qquad (r, b, 1^{p-b}) \;\; (1 \leq b \leq r).$$

Note that if $n = p, p+1$ then G has a single block of non-zero defect [3, 86.10]. We consider now the alternating groups.

Proposition 3.2. *Let p be a prime such that $2 < p \leq n < 2p$. A non-linear irreducible character τ of $G = \mathbb{A}_n$ of non-zero defect is p-constant if, and only if, τ is a constituent of $\chi_\lambda|_G$, where $\chi_\lambda \in \operatorname{Irr} \mathbb{S}_n$ and one of the following holds*:

(i) $n = p \geq 5$ *and* $\lambda = (b, 1^{p-b})$ *with* $2 \leq b \leq p-1$ *and* $b \neq \frac{p+1}{2}$;

(ii) $n = p+1 \geq 6$ *and* $\lambda = (b+1, 2, 1^{p-b-2})$ *with* $1 \leq b \leq p-2$ *and* $b \neq \frac{p-1}{2}$;

(iii) $n = p+r$, $r > 2$, *and λ is one of the following partitions*:

$$(p-a, r+1, 1^{a-1}) \;\; (1 \leq a \leq p-r-1); \qquad (r, b, 1^{p-b}) \;\; (1 \leq b \leq r).$$

Proof. By [9, Theorem 6.1.46], the characters of the principal p-block of $\mathbb{A}_n$ are constituents of the characters χ_λ, where λ is one of the partitions described in Lemma 3.1. Denote by λ^T the partition associated with the diagram transpose to that of λ. If $\lambda \neq \lambda^T$, the restriction $\tau = \chi_\lambda|_G$ is irreducible and so τ is

p-constant. Consider now the case $\lambda = \lambda^T$. This happens only for $n = p$ when $\lambda = (\frac{p+1}{2}, 1^{\frac{p-1}{2}})$ and for $n = p+1$ when $\lambda = (\frac{p+1}{2}, 2, 1^{\frac{p-3}{2}})$. In these cases the group $\mathbb{A}_n$ has two conjugacy classes σ_+, σ_- of elements of order p. For the previous values of λ, the character χ_λ splits on $\mathbb{A}_n$ as two characters τ_1, τ_2. By [9, Theorem 2.5.13] we have $\tau_i(\sigma_\pm) = \frac{(-1)^{(p-1)/2} \pm \sqrt{p(-1)^{(p-1)/2}}}{2}$, whence these τ_i's are not p-constant. □

Now, suppose that $n \geq 2p$. From the proof of Propositions 4.2 and 4.3 of [11] we can deduce the following result.

Lemma 3.3. *Let χ_λ be the non-linear irreducible character of $H = \mathbb{S}_n$ associated to the partition λ. Let τ_λ be an irreducible character of $G = \mathbb{A}_n$ which is a constituent of $\chi_\lambda|_G$. Let $p > 2$ be a prime such that $n \geq 2p$. Then $\chi_\lambda(h) = 0$ for some $h \in \Sigma_p(H)$, unless possibly when λ is conjugate to one of the following partitions*:

(i) $2p \leq n = 2p + r \leq 3p - 1$ *and* $\lambda = (p + r, r + 1, 1^{p-r-1})$;

(ii) $n = 2p$, $\lambda = (p, 2, 1^{p-2})$;

(iii) $n = 2p + 1$, $\lambda = (p + 1, 1^p)$.

Similarly, $\tau_\lambda(g) = 0$ for some $g \in \Sigma_p(G)$, unless possibly when λ is one of the partitions of items (i) *to* (iii).

Let $p = 2$. If $n > 11$ then every non-linear irreducible character of $\mathbb{S}_n$ and every non-linear irreducible character of $\mathbb{A}_n$ vanishes at some 2-singular element.

To deal with the missing cases of the previous Lemma, we look at the character table of $G = \mathbb{S}_n, \mathbb{A}_n$, when $n \leq 11$, obtaining the following irreducible characters τ of G that do not vanish at $\Sigma_2(G)$:

$$\mathbb{S}_n\colon (n, \tau(1)) \in \{(4,3), (5,5)\};$$

$$\begin{aligned}\mathbb{A}_n\colon (n, \tau(1)) \in \{&(4,3), (5,3), (5,5), (6,5), (6,9), (7,15), (7,21), (7,35),\\ &(10,315), (11,165)\}.\end{aligned}$$

However, among these characters, only those described in Theorem 1.4(3) and the irreducible character of degree 3 of $\mathbb{A}_4$ are 2-constant.

As an application of Murnaghan-Nakayama formula (e.g., see [9, 2.4.7]) we prove the following.

Proposition 3.4. *Let p be a prime such that $n \geq 2p \geq 4$. Then the non-linear p-constant irreducible characters of $\mathbb{S}_n$ are all of p-defect* 0.

Proof. Let, as before, χ_λ be the irreducible character of $\mathbb{S}_n$ associated to the partition λ of n. Assume that χ_λ is of non-zero defect. When $p = 2$, it suffices to look at the character tables for the cases $n \leq 11$ as done before. So, suppose $p > 2$. By Lemma 3.3 we are left to consider the case $n = 2p + r$ $(0 \leq r < p)$.

First, take $\lambda = (p + r, r + 1, 1^{p-r-1})$ with $0 \leq r < p$. We apply Murnaghan-Nakayama formula to permutations σ whose cyclic decomposition is of type $(2p)(r)$ or $(p + r)(r)$, obtaining $\chi_\lambda((2p)(r)) = (-1)^{r+1}$ and $\chi_\lambda((p + r)(p)) = (-1)^r$. When $\lambda = (p + r, r + 1, 1^{p-r-1})^T$, we obtain $\chi_\lambda((2p)(r)) = -1$ and $\chi_\lambda((p + r)(p)) = +1$. This means that these characters χ_λ are not constant on $\Sigma_p(G)$. Now, if $n = 2p$ and $\lambda = (p, 2, 1^{p-2})$, then $\chi_\lambda((2p)) = 0$. Finally, if $n = 2p + 1$ and $\lambda = (p + 1, 1^p)$, then $\chi_\lambda((2p)(1)) = 0$. $\square$

Proof of Theorem 1.4. If $n \geq 2p > 4$, by Lemma 3.3 we are reduced to the following cases:

(a) $n = 2p$ and $\lambda = (p, 2, 1^{p-2}), (p, 1^p)$;

(b) $n = 2p + 1$ and $\lambda = (p + 1, 1^p), (p + 1, 2, 1^{p-2})$;

(c) $n = 2p + r$, $r > 1$, and $\lambda = (p + r, r + 1, 1^{p-r-1})$.

Actually, we can exclude case (c). Using Murnaghan-Nakayama formula we obtain $\chi_\lambda((p + r - 1)(p)(1)) = (-1)^r$ and $\chi_\lambda((2p)(r - 1)(1)) = (-1)^{r+1}$.

The case $p = 2$ follows from Lemma 3.3 and previous direct computations for $n \leq 11$. $\square$

4. Groups of Lie type

Following [1, 1.17] we use the term "a group of Lie type" to refer to groups of shape $\mathbf{G}^F$, where $\mathbf{G}$ is a connected reductive algebraic group in defining characteristic p with an algebraic group endomorphism $F: \mathbf{G} \to \mathbf{G}$ such that the subgroup $\mathbf{G}^F := \{g \in \mathbf{G}: F(g) = g\}$ is finite. Such an endomorphism is called a *Frobenius map* (F is not necessarily the standard Frobenius map). In what follows $\mathbf{G}$ is assumed to be simple, not necessarily simply connected.

In Lemma 4.1 below the term "regular character" is used as in the Deligne–Lusztig theory. More precisely, a regular character is defined to be a constituent of a Gelfand–Graev character [6, 14.39], where the latter is the induced character λ^G when λ is a linear character of a Sylow p-subgroup U satisfying a certain non-degeneracy condition. Every group of Lie type has at least one Gelfand–Graev character. In addition, every Gelfand–Graev character is multiplicity free and does not have 1_G as a constituent.

Lemma 4.1. *Let* $\mathbf{G}$ *be a connected reductive group defined over a field of characteristic* p*,* F *a Frobenius endomorphism and* $G = \mathbf{G}^F$*. Let* U *be a Sylow* p*-subgroup of* G *and let* τ *be an irreducible character of* G *such that* $\tau(u) = a \neq 0$ *for all* $1 \neq u \in U$*. Then either* $\tau(1) = 1$ *or* τ *is regular,* $a = \pm 1$ *and* $\tau(1) = a + |U|$*.*

Proof. By Lemma 2.1, $a \in \mathbb{Z}$ and so $\chi = \tau - a \cdot 1_G$ is a Syl_p-vanishing generalized character of G (i.e. vanishing on $U \setminus \{1\}$). If $\chi(1) = 0$ then $\tau(1) = a$, and hence $U \leq \mathrm{Ker}(\tau)$. It follows that the normal subgroup X of G generated by the unipotent elements is contained in $\mathrm{Ker}(\tau)$. It is well known that G/X is abelian, and hence $\tau(1) = 1$.

Suppose $\chi(1) \neq 0$. Then $\chi(1)$ is a multiple of $|U|$ (cf. [14, Lemma 2.4]). Observe that for any linear character λ of U we have

$$(\chi, \lambda^G) = (\chi_{|U}, \lambda) = \frac{\chi(1)}{|U|} = (\tau, \lambda^G) - a \cdot (1_G, \lambda^G).$$

In particular, considering $\lambda \neq 1_U$ non-degenerate, τ is a regular character of G. As λ^G is multiplicity free [1, Theorem 8.1.3], we have $(\tau, \lambda^G) = 1$, whence $\chi(1) = |U|$. Furthermore, $\chi(1) = |U|$ implies that $(\chi, \lambda^G) = 1$ for any linear character λ of U. In particular, taking $\lambda = 1_U$ we obtain $1 = (\tau, 1_U^G) - a$, whence $(\tau, 1_U^G) = a + 1 \geq 0$ and $a \geq -1$. We show that $a = \pm 1$.

We first consider the case where $Z(\mathbf{G})$ is connected. Since τ is constant on $U \setminus \{1\}$, the average value of τ on any set of regular unipotent elements of G coincides with its value a. Hence, by [1, Theorem 8.3.3(i)], we have $a = \pm 1$ (as $a \neq 0$).

Next, suppose that $Z(\mathbf{G})$ is not connected. Note that $\mathbf{G}$ can be embedded in a reductive group $\widehat{\mathbf{G}}$ with connected center such that the derived groups $\widehat{\mathbf{G}}'$ and $\mathbf{G}'$ coincide. Moreover, each Frobenius endomorphism of $\mathbf{G}$ extends to that of $\widehat{\mathbf{G}}$ [6, pp. 139–140]. We keep F to denote the extended endomorphism of $\widehat{\mathbf{G}}$. Then $G = \mathbf{G}^F \leq \widehat{G} = \widehat{\mathbf{G}}^F$; moreover G is a normal subgroup of $\widehat{G}$ with abelian quotient (*loc.cit.*). Let σ be an irreducible constituent of $\tau^{\widehat{G}}$. By Clifford's theorem, $\sigma_{|G} = e \sum_i^t \tau_i$, where $\{\tau_1 = \tau, \tau_2, \dots, \tau_t\}$ are the distinct conjugates of τ and $e = (\sigma, \tau^{\widehat{G}})$. Since τ is constant on the set of the non-trivial unipotent elements of G, so are all the τ_i's, and moreover, $\tau_i(u) = a$ for every $1 \neq u \in U$. This means that also σ is constant on $U \setminus \{1\}$, and, in addition, $\sigma(u) = et \cdot \tau(u) \neq 0$ is an integer. By the above, $\sigma(u) = \pm 1$, whence $et = 1$ and so $a = \tau(u) = \pm 1$. □

In the proof of the following two lemmas, we will make use of the Zsigmondy primes. Here, we briefly recall their definition. Let a, n be two positive integers. If $a \geq 2$, $n \geq 3$ and $(a, n) \neq (2, 6)$, then there exists a prime, denoted here by $\zeta_n(a)$, dividing $a^n - 1$ and coprime to $a^i - 1$ for every $1 \leq i < n$. This prime, not necessarily unique, is called a *Zsigmondy prime* (or a *primitive prime divisor* of $a^n - 1$). Observe that if $\zeta_n(a)$ divides $a^k - 1$, then n divides k.

LEMMA 4.2. *Let* $\mathbf{G}$ *be a simple connected reductive group and let* $G = \mathbf{G}^F$ *be the corresponding finite group. Then* $|G|_p - 1$ *divides* $|G|$ *if, and only if,* $G \in \{A_1(q), A_2(q), A_3(2), B_2(q), C_2(q), G_2(q)\}$.

PROOF. First, consider the groups of type ${}^2B_2(q^2)$ and ${}^2G_2(q^2)$. If $G = {}^2B_2(q^2)$, where $q^2 = 2^{2n+1}$, then $|G|_2 - 1 = q^4 - 1$ does not divide $|G| = q^4(q^2 - 1)(q^4 + 1)$, as $\gcd(q^2 + 1, q^4 + 1) = 1$. If $G = {}^2G_2(q^2)$, where $q^2 = 3^{2n+1}$, then $|G|_3 - 1 = q^6 - 1$ does not divide $|G| = q^6(q^2 - 1)(q^6 + 1)$, as $\gcd(q^6 - 1, q^6 + 1) = 2$.

Now, let $|G|_p = q^m$, with $G \notin \{{}^2B_2(q^2), {}^2G_2(q^2)\}$. We start our analysis with the cases where the existence of a Zsigmondy prime $\zeta_m(q)$ is not guaranteed, i.e. $m \leq 2$ or $(m, q) = (6, 2)$. If $m \leq 2$, then $G = A_1(q)$. In this case, $|G|_p - 1 = q - 1$ divides $|G| = q(q^2 - 1)$. If $(m, q) = (6, 2)$ then G is one of the following groups: $A_3(2)$, ${}^2A_3(2)$, $G_2(2)$. In this case, we can directly check when $|G|_p - 1$ divides $|G|$. This happens only when $G = A_3(2), G_2(2)$.

Hence, we may assume $m \geq 3$ and $(m, q) \neq (6, 2)$. Under this assumption, a Zsigmondy prime $\zeta_m(q)$ exists, and we check when this prime divides $|G|$. We show that this happens only for the groups of rank 2 in the statement.

If G is of type $A_n(q)$, then $m = \frac{n(n+1)}{2} \geq 3$. Suppose that $\zeta_m(q)$ divides $|G|$. Then $\frac{n(n+1)}{2} \leq n + 1$, whence $n^2 - n - 2 \leq 0$ and so $n = 2$. In this case, $|A_2(q)|_p - 1 = q^3 - 1$ divides $|G|$. If G is of type ${}^2A_n(q)$, then $m = \frac{n(n+1)}{2} \geq 3$. Suppose that $\zeta_m(q)$ divides $|G|$. Then $\frac{n(n+1)}{2} \leq 2(n + 1)$, whence $n = 3, 4$. For $n = 3$, $|{}^2A_3(q)|_p - 1 = q^3 - 1$ does not divide $|G| = q^3(q^3 + 1)(q^2 - 1)$ since $\gcd(q^3 - 1, q^3 + 1) \leq 2$. For $n = 4$, $|{}^2A_4(q)|_p - 1 = q^6 - 1$ and $|G| = q^6(q^4 - 1)(q^3 + 1)(q^2 - 1)$. Notice that $\zeta_3(q)$ does not divide $(q^4 - 1)(q^2 - 1)$, whence $|G|_p - 1$ does not divide $|G|$.

If G is of type $B_n(q)$ or $C_n(q)$, then $m = n^2 \geq 3$. The condition $\zeta_m(q)$ divides $|G|$ implies $n^2 \leq 2n$ and so $n = 2$. On the other hand, if $n = 2$ then $|G|_p - 1 = q^4 - 1$ divides $|G| = q^4(q^4 - 1)(q^2 - 1)$. If G is of type $D_n(q)$, then $m = n^2 - n \geq 3$. The condition $\zeta_m(q)$ divides $|G|$ implies $n^2 - n \leq 2n - 2$ and so $n = 1, 2$. If G is of type ${}^2D_n(q)$, then $m = n^2 - n \geq 3$. If $\zeta_m(q)$ divides $|G|$ then $n^2 - n \leq 2n$ and so $n = 3$. In this case, $|G|_p - 1 = q^6 - 1$ and

$|G| = q^6(q^4-1)(q^2-1)(q^3+1)$. However, the prime $\zeta_3(q)$ divides $|G|_p - 1$, but does not divide $|G|$.

Similarly, if G is of type $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, $E_8(q)$ or $F_4(q)$, then it is straightforward to see that $|G|_p - 1$ does not divide $|G|$. If G is of type ${}^2F_4(q^2)$, where $q^2 = 2^{2n+1}$, then $\zeta_{6n+3}(2)$ divides $|G|_2 - 1$ but does not divide $|G|$. If G is of type ${}^3D_4(q)$, then $m = 12$. In this case, $|G| = q^{12}\frac{(q^6-1)(q^{12}-1)}{(q^2+1)}$. Suppose that $|G|_p - 1 = q^{12} - 1$ divides $|G|$, then we obtain that $q^2 + 1$ divides $q^6 - 1$, i.e. $\zeta_4(q)$ divides $q^6 - 1$, which does not happen, since 4 does not divide 6. Finally, if G is of type $G_2(q)$ then $|G|_p - 1$ divides $|G|$. □

With the same techniques used in the proof of the previous lemma, we can also prove the following one.

Lemma 4.3. *Let* $\mathbf{G}$ *be a simple connected reductive group and let* $G = \mathbf{G}^F$ *be the corresponding finite group. Then* $|G|_p + 1$ *divides* $|G|$ *if, and only if,* $G \in \{A_1(q), {}^2A_2(q), {}^2B_2(q^2), {}^2G_2(q^2)\}$.

As previously remarked, if G is quasi-simple then the Steinberg character is the only irreducible character of p-defect 0. So we can now prove Theorem 1.1.

Proof of Theorem 1.1. Assume that $\tau(1) > 1$ and $\tau(u) = a \neq 0$. By Lemma 4.1, $a = \pm 1$ and τ is a regular character of degree $|G|_p \pm 1$. If $a = -1$, by Lemma 4.2 we are reduced to consider the following groups: $\mathrm{SL}_2(q), \mathrm{SL}_3(q), \mathrm{SL}_4(2), \mathrm{Sp}_4(q)$ and $G_2(q)$. Since $\chi = \tau + 1_G$ is a proper Syl_p-vanishing character, we may use [14]. If $a = 1$, by Lemma 4.3, it suffices to consider the following groups: $\mathrm{SL}_2(q), \mathrm{SU}_3(q), {}^2B_2(q^2)$ and ${}^2G_2(q^2)$. For all these groups the result follows by analysis of the character tables. □

Proof of Theorem 1.6. As $\tau(1) = |G|_p \pm 1$ and $\tau(1)$ divides $|G|$, it follows by Lemmas 4.2 and 4.3 that G must be one of the following groups: $\mathrm{SL}_2(q), \mathrm{SL}_3(q)$, $\mathrm{SL}_4(2), \mathrm{SU}_3(q), \mathrm{Sp}_4(q), {}^2B_2(q^2), G_2(q), {}^2G_2(q^2)$. So, it suffices to inspect the character tables of these groups to identify the irreducible characters with the degrees in question. □

5. Sporadic groups

The answer to Problem 1.2 for quasi-simple sporadic groups can be obtained directly from their character tables. We describe here the most interesting properties.

Proposition 5.1. *Let G be a finite quasi-simple sporadic group and let p be a prime dividing $|G|$. Let $\Delta_p(G)$ be the set of the non-linear irreducible characters of G whose p-defect is not 0 and which are constant on $\Sigma_p(G)$.*

(1) *For every G there exists a prime p such that the set $\Delta_p(G)$ is not empty.*

(2) *There exists exactly one prime p such that $\Delta_p(G) \neq \emptyset$ if, and only if, $p = 7$ and $G = J_2, 2.J_2$.*

(3) *The set $\Delta_2(G)$ is not empty if, and only if, $G = J_1$. In this case $\Delta_2(J_1) = \{\tau\}$ where $\tau(1) = 209$.*

(4) *G has a character $\tau \in \Delta_p(G)$ such that $|\tau(g)| \neq 1$ ($g \in \Sigma_p(G)$) if, and only if, $p = 3$, $G \in \{M_{22}, 2.M_{22}, 4.M_{22}\}$ and $\tau(1) = 385$. In these cases $\tau(g) = -2$.*

Finally we consider the simple group $G = {}^2F_4(2)'$, which admits the following p-constant irreducible characters (in the notation of [2]):

(1) $p = 3$: χ_8 of degree 325;

(2) $p = 5$: χ_9 of degree 351 and χ_{12}, χ_{13} of degree 624;

(3) $p = 13$: χ_4, χ_5 of degree 27, χ_7 of degree 300, χ_{15} of degree 675 and χ_{20} of degree 1728.

In all these cases, the characters take value ± 1 on $\Sigma_p(G)$.

6. Proof of Theorem 1.5

Let G be a finite simple group and let $\tau \in \mathrm{Irr}(G)$ such that $\tau(g) = c$ for all $g \in \Sigma_p(G)$, where p is a prime dividing $|G|$. By Lemma 2.1, $c \in \mathbb{Z}$ and $c = 0$ precisely when τ is of p-defect 0. Also, by Theorem 1.3(2) $c = \pm 1$ when G has a cyclic Sylow p-subgroup. So, assume that τ is not of p-defect 0 and that the Sylow p-subgroups of G are not cyclic. If G is an alternating group, then by Proposition 1.4 it follows that $c = \pm 1$. If G is a sporadic group, from Proposition 5.1 we get that either $c = \pm 1$ or $G = M_{22}$, $c = -2$, $p = 3$ and $\tau(1) = 385$. Finally, for groups of Lie type of characteristic p, the result follows from Lemma 4.1. This proves Theorem 1.5.

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