Berezin transform and Stratonovich–Weyl correspondence for the multi-dimensional Jacobi group
(addendum)

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ABSTRACT – We extend the results of [13] to the holomorphic representations of the non-scalar type of the multi-dimensional Jacobi group.


KEYWORDS. Berezin quantization, Berezin transform; quasi-Hermitian Lie group, multi-dimensional Jacobi group, unitary representation, holomorphic representation of the non-scalar type, Stratonovich–Weyl correspondence.

1. Introduction

We use the notation of [13], Section 2. In particular, we denote by $G$ the multi-dimensional Jacobi group and by $K$ the subgroup of $G$ consisting of all elements of the form $((0, 0), c, \left( \begin{array}{cc} P & 0 \\ 0 & \bar{P} \end{array} \right))$ where $c \in \mathbb{R}$ and $P \in U(n)$.

Recall that the unitary representations of $G$ considered in [13] are holomorphically induced from a unitary character of $K$. Here we consider, more generally, the unitary representations of $G$ which are holomorphically induced from a unitary representation $\rho$ of $K$, see [18], p. 515, and we extend the results of [13] to these representations. The main tool is then the generalized Berezin calculus for a reproducing kernel Hilbert space of vector-valued holomorphic functions, see [2], [17] and [12]. Most of the proofs are similar to those of [13] so we just sketch them briefly.

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2. Representations

Let $\gamma \in \mathbb{R}$ and let $\rho_0$ be a unitary irreducible representation of $U(n)$ on a (finite-dimensional) complex vector space $\mathcal{V}$. Let $\rho$ be the representation of $K$ on $\mathcal{V}$ defined by

$$\rho((0,0), c, \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}) = e^{i\gamma c} \rho_0(P).$$

We also denote by $\rho_0$ and $\rho$ the corresponding representations of $GL_n(\mathbb{C})$ and $K^c$.

Let $M_n(\mathbb{C}) = n^+ \oplus h_0 \oplus n^-$ be the usual triangular decomposition of $M_n(\mathbb{C})$. Then $\rho_0$ is associated with a dominant integral weight of the form

$$\lambda_{m_1,m_2,\ldots,m_n} : \text{Diag}(a_1, a_2, \ldots, a_n) \mapsto \sum_{i=1}^n m_ia_i$$

where $m_1 \geq m_2 \geq \cdots \geq m_n$ and $m_i \in \mathbb{Z}$ [15], p. 274. Let $m = \sum_{i=1}^n m_i$. Then we have $\rho_0(zI_n) = z^mI_\mathcal{V}$ for each $z \in \mathbb{C}^\times$.

Recall that for each $y \in \mathbb{C}^n$ and $Y \in M_n(\mathbb{C})$ such that $Y^t = Y$, we denote

$$a(y, Y) := ((y,0), 0, \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}) \in p^+$$

and

$$\mathcal{D} := \{a(y, Y) \in p^+ : I_n - Y\overline{Y} > 0\} \cong \mathbb{C}^n \times \mathcal{B}$$

where $\mathcal{B} := \{Y \in M_n(\mathbb{C}) : Y^t = Y, I_n - Y\overline{Y} > 0\}$.

Now we will apply the general considerations of [18] and [12] to the particular case of the multi-dimensional Jacobi group. Following [18], p. 497, we set $K(Z, W) := \rho(\kappa(\exp W^* \exp Z))^{-1}$ for $Z, W \in \mathcal{D}$ and $J(g, Z) := \rho(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$ and we introduce the Hilbert space $\mathcal{H}$ of all holomorphic functions on $\mathcal{D}$ with values in $\mathcal{V}$ such that

$$\|f\|^2_{\mathcal{H}} := \int_\mathcal{D} \langle K(Z, Z)^{-1} f(Z), f(Z) \rangle_{\mathcal{V}} d\mu(Z) < +\infty$$

where $\mu$ denotes the $G$-invariant measure on $\mathcal{D}$ defined in [13], Section 2.

Let $\pi$ be the unitary representation of $G$ on $\mathcal{H}$ defined by

$$(\pi(g) f)(Z) = J(g^{-1}, Z)^{-1} f(g^{-1} \cdot Z).$$

In [12], we verified that $\pi$ is obtained by holomorphic induction from $\rho$, that is, $\pi$ corresponds to the natural action of $G$ on the square-integrable holomorphic sections of the Hilbert $G$-bundle $G \times_\rho \mathcal{V}$ over $G/K \cong \mathcal{D}$. Moreover, $\pi$ is irreducible since $\rho$ is irreducible, see [18], p. 515.
The evaluation maps $K_Z: \mathcal{H} \to \mathcal{V}$, $f \to f(Z)$ are continuous [18], p. 539. The vector coherent states of $\mathcal{H}$ are the maps $E_Z = K_Z^*: \mathcal{V} \to \mathcal{H}$ defined by $(f(Z), v) = (f, E_Z v)_{\mathcal{H}}$ for $f \in \mathcal{H}$ and $v \in \mathcal{V}$. We have the following result, see [18], p. 540.

**Proposition 2.1.** (1) There exists a constant $c_\rho > 0$ such that

$$E_Z^* E_W = c_\rho K(Z, W)$$

for each $Z, W \in \mathcal{D}$. (2) For $g \in G$ and $Z \in \mathcal{D}$,

$$E_{g \cdot Z} = \pi(g) E_Z J(g, Z)^*.$$ 

Now we give explicit formulas for $K$ and $J$ and we compute $c_\rho$ (see Lemma 3.1 in [13]).

**Proposition 2.2.** (1) Let $Z = a(y, Y) \in \mathcal{D}$ and $W = a(v, V) \in \mathcal{D}$. We set

$$E(y, v, Y, V) := 2y^t (I_n - \bar{V} Y)^{-1} \bar{v} + y^t (I_n - \bar{V} Y)^{-1} \bar{V} y + \bar{v}^t Y (I_n - \bar{V} Y)^{-1} \bar{v}.$$ 

Then,

$$K(Z, W) = \exp \left( \frac{\gamma}{4} E(y, v, Y, V) \right) \rho_0 (I_n - Y \bar{V}).$$

(2) $\mathcal{H} \neq (0)$ if and only if $\gamma > 0$ and $m + n + 1/2 < 0$. In this case,

$$c_\rho = \text{Dim}(\mathcal{V}) \left( \frac{\gamma}{2\pi} \right)^n J_n (-m - n - 3/2)^{-1}.$$ 

(3) For each $g = ((z_0, \bar{z}_0), c_0, (\frac{P}{\bar{Q} y})) \in G$ and each $Z = a(y, Y) \in \mathcal{D}$, we have

$$J(g, Z) = e^{i y c_0} \exp \left( \frac{\gamma}{4} (z_0^t \bar{z}_0 + 2\bar{z}_0^t P y + y^t P^t \bar{Q} y - (\bar{z}_0 + \bar{Q} y)^t (P Y + Q)(\bar{Q} y + \bar{P})^{-1} (\bar{z}_0 + \bar{Q} y) \right) \rho_0 ((\bar{Q} Y + \bar{P} y)^t)^{-1}.$$ 

**Proof.** (1) and (3) are simple calculations. To prove (2), we use the formula

$$(I_n - \bar{Y} Y)^{-1} = \text{Det}(I_n - \bar{Y} Y)^{-1} C(I_n - \bar{Y} Y)^t$$.
for each $Y \in \mathcal{B}$, where $C(A)$ denotes the cofactor matrix of a matrix $A$. From this formula, we deduce

$$\text{Tr} \, \rho_0 (I_n - \bar{Y} Y)^{-1} = \text{Det}(I_n - \bar{Y} Y)^{-m} \, \text{Tr} \, \rho_0 (C(I_n - \bar{Y} Y)^t)$$

and then we can prove (2) by following the same lines as in the proof of Proposition 3.2 of [13], using Theorem XII.5.6 of [18].

Proposition 3.4 of [13] can be generalized as follows.

**Proposition 2.3.** For $X \in \mathfrak{g}^c$, $f \in \mathfrak{h}$ and $Z \in \mathcal{D}$,

$$d \pi(X) f(Z) = d\rho(p_{t^c} (e^{-ad Z} X)) f(Z) - (df)_{Z} (p_{p^+} (e^{-ad Z} X)).$$

In particular,

1. if $X \in \mathfrak{p}^+$, then $d \pi(X) f(Z) = -(df)_Z (X)$;
2. if $X \in \mathfrak{t}^c$, then $d \pi(X) f(Z) = d\rho(X) f(Z) + (df)_Z ([Z, X])$;
3. if $X \in \mathfrak{p}^-$, then

$$d \pi(X) f(Z) = (d\rho \circ p_{t^c}) \left( - [Z, X] + \frac{1}{2} [Z, [Z, X]] \right) f(Z)$$

$$- (df_Z \circ p_{p^+}) \left( - [Z, X] + \frac{1}{2} [Z, [Z, X]] \right).$$

Let $(E_k)$ be a basis of $\mathfrak{p}^+$. Then, for each $f \in \mathfrak{h}$ and each $k$, we denote

$$(\partial_k f)(Z) = \frac{d}{dt} f(Z + tE_k)|_{t=0}.$$ 

From the preceding proposition we deduce the following result.

**Proposition 2.4.** For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, $d \pi(X_1 X_2 \cdots X_q)$ is a sum of terms of the form $P(Z) \partial_{k_1} \partial_{k_2} \cdots \partial_{k_r}$ where $r \leq q$ and $P(Z)$ is a polynomial of degree $\leq 2q$.

## 3. Berezin calculus

First we introduce the Berezin quantization map associated with $\rho_0$, see [3], [4], [1], [5], and [19].

Let $\tilde{\varphi}_0$ be the linear form on $M_n(\mathbb{C})$ defined by $\tilde{\varphi}_0 = -i \Lambda_{m_1,m_2,\ldots,m_n}$ on $\mathfrak{h}_0$ and $\tilde{\varphi}_0 = 0$ on $\mathfrak{n}^\pm$. We denote by $\varphi_0$ the restriction of $\tilde{\varphi}_0$ to $u(n)$. Then the orbit $o(\varphi_0)$ of $\varphi_0$ under the coadjoint action of $U(n)$ is then said to be associated with $\rho_0$. 

Note that the stabilizer of \( \varphi_0 \) for the coadjoint action of \( U(n) \) contains the torus
\[
H_0 := \{ \text{Diag}(ia_1, ia_2, \ldots, ia_n) : a_j \in \mathbb{R} \}.
\]
We say that such an element \( \varphi_0 \) is \textit{regular} if the stabilizer of \( \varphi_0 \) is equal to \( H_0 \), see [5]. Then we can verify that \( \varphi_0 \) is regular if and only if one has \( m_1 > m_2 > \cdots > m_n \). In the rest of the paper, we assume that \( \varphi_0 \) is regular.

Note also that a complex structure on \( o.'0/ \) is then defined by the diffeomorphism \( o.'0/ = U(n)/H_0/ \cong \text{GL}_n(\mathbb{C})/H_0^c N^- \) where \( N^- \) is the analytic subgroup of \( \text{GL}_n(\mathbb{C}) \) with Lie algebra \( n^- \).

Without loss of generality, we can assume that \( V \) is the space of holomorphic functions on \( o.'0/ \) as in [5]. For \( \varphi \in o.(\varphi_0) \) there exists a unique function \( e_\varphi \in V \) (called a coherent state) such that \( \langle a, e_\varphi \rangle_V \) for each \( a \in V \). The Berezin calculus on \( o.(\varphi_0) \) associates with each operator \( B \) on \( V \) the complex-valued function \( s(B) \) on \( o.(\varphi_0) \) defined by
\[
s(B)(\varphi) = \frac{\langle Be_\varphi, e_\varphi \rangle_V}{\langle e_\varphi, e_\varphi \rangle_V}
\]
which is called the symbol of \( B \). Then we have the following proposition, see [14], [1] and [5].

**Proposition 3.1.**

1. The map \( B \mapsto s(B) \) is injective.
2. For each operator \( B \) on \( V \), we have \( s(B^*) = s(B) \).
3. For each \( \varphi \in o.(\varphi_0) \), \( k \in U(n) \) and \( B \in \text{End}(V) \), we have
\[
s(B)(\text{Ad}^*(k)\varphi) = s(\rho_0(k)^{-1}B\rho_0(k))(\varphi).
\]
4. For each \( A \in u(n) \) and \( \varphi \in o.(\varphi_0) \), we have \( s(d\rho_0(A))(\varphi) = i\langle \varphi, A \rangle \).

We also need the following result, see [10] and [12].

**Proposition 3.2.** Let \( Z \in \mathfrak{d} \). There exists a unique element \( k_Z \) in \( K^c \) such that \( k_Z^* = k_Z \) and \( k_Z^2 = \kappa(\exp Z^* \exp Z)^{-1} \). Each \( g \in G \) such that \( g \cdot 0 = Z \) is then of the form \( g = \exp(-Z^*) \zeta(\exp Z^* \exp Z)k_Z^{-1}h \) where \( h \in K \). Consequently, the map \( Z \rightarrow g_Z := \exp(-Z^*) \zeta(\exp Z^* \exp Z)k_Z^{-1} \) is a section for the action of \( G \) on \( \mathfrak{d} \).

More explicitly, for each \( Z = a(y, Y) \in \mathfrak{d} \), we have
\[
k_Z = \begin{pmatrix} (0, 0), \frac{i}{8}E(y, y, Y, Y), \left( (I_n - Y \bar{Y})^{1/2} 0 \
0 (I_n - Y \bar{Y})^{-1/2} \right) \end{pmatrix}
\]
and
\[ g_Z = \left( (-\bar{w}, -w), \frac{i}{8}(y^t(I_n - \bar{Y}Y)^{-1}\bar{Y}y - \bar{y}^tY(I_n - \bar{Y}Y)^{-1}\bar{y}), M(Y) \right) \]
where \( w := -(I_n - \bar{Y}Y)^{-1}(\bar{y} + \bar{Y}y) \) and
\[
M(Y) := \begin{pmatrix}
\frac{1}{2}(I_n - Y\bar{Y})^{-1/2} & \frac{1}{2}(I_n - \bar{Y}Y)^{-1/2} \\
\frac{1}{2}(I_n - Y\bar{Y})^{-1/2} & \frac{1}{2}(I_n - \bar{Y}Y)^{-1/2}
\end{pmatrix}.
\]

Now, following [17], [2], [12], we define the pre-symbol \( S_0(A) \) of an operator \( A \) by
\[
S_0(A)(Z) = c_\rho^{-1}\rho(k_Z^{-1})E_Z^A E_Z \rho(k_Z^{-1})^* \]
and the Berezin symbol \( S(A) \) of \( A \) is defined as the complex-valued function on \( D \times o(\varphi_0) \) given by
\[
S(A)(Z, \varphi) = s(S_0(A)(Z))(\varphi).
\]
For each \( g \in G \) and \( Z \in D \), let \( k(g, Z) := g_Z^{-1}g^{-1}g_Z \cdot Z \in K \). Then we can write
\[
k(g, Z) = \begin{pmatrix}
(0, 0), c(g, Z), \left( \frac{P(g, Z)}{0} \right)
\end{pmatrix},
\]
where \( c(g, Z) \in \mathbb{R} \) and \( P(g, Z) \in U(n) \).

We have the following properties of \( S \), see [12].

**Proposition 3.3.**
1. Each operator \( A \) is determined by \( S(A) \).
2. For each operator \( A \), we have \( S(A^*) = \overline{S(A)} \).
3. We have \( S(I_{3\ell}) = 1 \).
4. For each operator \( A \), \( g \in G \), \( Z \in D \) and \( \varphi \in o(\varphi_0) \), we have
\[
S(A)(g \cdot Z, \varphi) = S(\pi(g)^{-1}A\pi(g))(Z, \Ad^*(P(g, Z))\varphi).
\]

Now, we give some formulas for the Berezin pre-symbol of \( \pi(g) \) for \( g \in G \) and for the Berezin symbol of \( d\pi(X) \) for \( X \in \mathfrak{g}^c \). For \( \varphi \in u(n)^* \), we denote by \( \varphi^s \) the linear form on \( s \) defined by
\[
\left< \varphi^s, \left( \frac{P}{Q} \right) \right> = \langle \varphi, P \rangle
\]
and by \( \varphi^e \) the linear form on \( \mathfrak{g} \) defined by
\[
\left< \varphi^e, \left( (z, \bar{z}), c, \left( \frac{P}{Q} \right) \right) \right> = \langle \varphi, P \rangle + \gamma c.
\]
We also denote by \( \varphi^s \) and \( \varphi^e \) the extensions of \( \varphi^s \) and \( \varphi^e \) to \( s^c \) and \( \mathfrak{g}^c \).
Proposition 3.4 ([12]). (1) For \( g \in G \) and \( Z \in \mathcal{D} \), we have
\[
S_0(\pi(g))(Z) = \rho \left( k_Z^{-1} \kappa(\exp Z^* g^{-1} \exp Z)^{-1}(k_Z^{-1})^* \right).
\]
(2) For each \( X \in \mathfrak{g} \), \( Z \in \mathcal{D} \) and \( \varphi \in o(\varphi_0) \), we have
\[
S(d \pi(X))(Z, \varphi) = i \langle \text{Ad}^*(g_Z)\varphi, X \rangle.
\]
Recall that \( \xi_0 \in \mathfrak{g}^* \) is said to be regular if the stabilizer \( G(\xi_0) \) of \( \xi_0 \) for the coadjoint action is connected and if the Hermitian form \( (Z, W) \mapsto \langle \xi_0, [Z, W^*] \rangle \) is not isotropic [12].

Lemma 3.5. The linear form \( \varphi_0^e \) is regular if and only if we have \( m_j > 0 \) for each \( j \) or \( m_j < 0 \) for each \( j \).

Proof. On the one hand, by using the formula for the coadjoint action of \( G \) given in [13], Section 2, we can verify that \( G(\varphi_0^e) \) consists of all matrices of the form \( (0, 0), c, \left( \begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix} \right) \) where \( c \in \mathbb{R} \) and \( P \in U(n) \) is such that \( \text{Ad}^*(P)\varphi_0 = \varphi_0 \). Since \( \varphi_0 \) is assumed to be regular as an element of \( u(n)^* \), we get \( P \in H_0. \) Hence \( G(\varphi_0^e) \cong \mathbb{R} \times H_0 \) is connected.

On the other hand, for each \( Z = a(y, Y) \in \mathcal{D} \), we have
\[
\langle \varphi_0^e, [Z, Z^*] \rangle = -\langle \varphi_0, Y \bar{Y} \rangle - \frac{i}{2} \gamma |y|^2
= - \sum m_j |Y_j|^2 - \frac{i}{2} \gamma |y|^2
\]
where \( Y_1, Y_2, \ldots, Y_n \) denote the columns of \( Y \). The result hence follows. \( \square \)

Let us assume that \( \varphi_0^e \) is regular and denote by \( \mathcal{O}(\varphi_0^e) \) the orbit of \( \varphi_0^e \) for the coadjoint action of \( G \). Then we have the following proposition, see [12].

Proposition 3.6. The map \( \Psi: \mathcal{D} \times o(\varphi_0) \to \mathfrak{g}^* \) defined by
\[
\Psi(Z, \varphi) = \text{Ad}^*(g_Z)\varphi
\]
is a diffeomorphism form \( \mathcal{D} \times o(\varphi_0) \) onto \( \mathcal{O}(\varphi_0^e) \) such that
\[
\Psi(g \cdot Z, \varphi) = \text{Ad}^*(g) \Psi(Z, \text{Ad}^*(P(g, Z))(\varphi))
\]
for each \( g \in G, Z \in \mathcal{D} \) and \( \varphi \in o(\varphi_0) \).

More precisely, with the notation of [13], we have
\[
\Psi(Z, \varphi) = \left( \gamma(\bar{w}, w), \gamma, \text{Ad}^*(M(Y))\varphi - \frac{\gamma}{2}(\bar{w}, w) \times (\bar{w}, w) \right)
\]
where \( Z = a(y, Y) \in \mathcal{D} \) and \( w := -(I_n - \bar{Y} Y)^{-1}(\bar{y} + \bar{Y} y) \).
4. Berezin transform and Stratonovich–Weyl correspondence

Here we introduce the Berezin transform associated with $S$ and the corresponding Stratonovich–Weyl correspondence, following [12].

We fix a $K$-invariant measure $\nu$ on $o(\varphi_0)$ normalized as in [5], Section 2. Then the measure $\tilde{\mu} := \mu \otimes \nu$ on $D \times o(\varphi_0)$ is invariant under the action of $G$ on $D \times o(\varphi_0)$ given by $g \cdot (Z, \varphi) := (g \cdot Z, \text{Ad}^*(P(g, Z))^{-1}\varphi)$ and the measure $\mu_{\varphi_0} := (\Psi^{-1})^*(\tilde{\mu})$ is a $G$-invariant measure on $\varphi_0$.

We denote by $\mathcal{L}_2(\mathcal{H})$ the space of Hilbert-Schmidt operators on $\mathcal{H}$ endowed with the Hilbert-Schmidt norm. We also endow $\text{End}(V)$ with the Hilbert-Schmidt norm. We denote by $L^2(D \times o(\varphi_0))$ (respectively $L^2(\mathcal{H})$, $L^2(o(\varphi_0))$, $L^2(\varphi_0)$) the space of functions on $D \times o(\varphi_0)$ (respectively $D$, $o(\varphi_0)$, $\varphi_0$) which are square-integrable with respect to the measure $\tilde{\mu}$ (respectively $\mu, \nu, \mu_{\varphi_0}$). Then we have the following result, see for instance [6].

**Proposition 4.1.** The Berezin transform $b := ss^*$ is given by

$$b(a)(\psi) = \int_{o(\varphi_0)} a(\varphi) \frac{|\langle e_{\psi}, e_\varphi \rangle \nu|^2}{\langle e_{\varphi}, e_\varphi \rangle \nu \langle e_{\psi}, e_\psi \rangle \nu} d\nu(\varphi)$$

for each $a \in L^2(o(\varphi_0))$

Similarly, we have the following proposition.

**Proposition 4.2.** The Berezin transform $B := SS^*$ is a bounded operator of $L^2(D \times o(\varphi_0))$ and that, for each $f \in L^2(D \times o(\varphi_0))$, we have the following integral formula

$$B(f)(Z, \psi) = \int_{D \times o(\varphi_0)} k(Z, W, \psi, \varphi) f(W, \varphi) d\mu(W) d\nu(\varphi)$$

where

$$k(Z, W, \psi, \varphi) := \frac{|\langle \rho(g^{-1}g W), e_\varphi \rangle \nu|^2}{\langle e_{\varphi}, e_\varphi \rangle \nu \langle e_{\psi}, e_\psi \rangle \nu}.$$

Consider the left-regular representation $\tau$ of $G$ on $L^2(D \times o(\varphi_0))$ defined by

$$(\tau(g)(f))(Z, \varphi) = f(g^{-1} \cdot (Z, \varphi)).$$

Clearly, $\tau$ is unitary. Moreover, since $S$ is $G$-equivariant, we immediately verify that for each $f \in L^2(D \times o(\varphi_0))$ and each $g \in G$, we have $B(\tau(g)f) = \tau(g)(B(f))$. 

Now, we introduce the polar decomposition of $S: \mathcal{L}_2(\mathcal{H}) \to L^2(\mathfrak{D} \times o(\varphi_0))$. We can write $S = (SS^*)^{1/2}W = B^{1/2}W$ where $W := B^{-1/2}S$ is a unitary operator from $\mathcal{L}_2(\mathcal{H})$ to $L^2(\mathfrak{D} \times o(\varphi_0))$. Then we have the following proposition, see [12]. The main point is that $W$ is $G$-equivariant since $S$ (hence $B$) is $G$-equivariant.

**Proposition 4.3.** (1) $W: \mathcal{L}_2(\mathcal{H}) \to L^2(\mathfrak{D} \times o(\varphi_0))$ is a Stratonovich–Weyl correspondence for the triple $(G, \pi, \mathfrak{D} \times o(\varphi_0))$.

(2) The map $W$ from $\mathcal{L}_2(\mathcal{H})$ to $L^2(\mathfrak{D}(\varphi_0^c))$ defined by $W(f) = W(f \circ \Psi)$ is a Stratonovich–Weyl correspondence for the triple $(G, \pi, \mathfrak{D}(\varphi_0^c))$.

5. Extension of the Berezin transform

Here we generalize Proposition 5.2 of [13], that is, we extend $B$ to a class of functions which contains $S(d\pi(X))$ for $X \in \mathfrak{g}^c$, in particular in order to define $W(d\pi(X))$.

For $Z, W \in \mathfrak{D}$, we set $l_Z(W) := \log \eta(\exp Z^* \exp W) \in \mathfrak{p}^\perp$. We need the following lemma which is the direct generalization of Lemma 5.1 of [13].

**Lemma 5.1.** (1) For each $Z, W \in \mathfrak{D}$, $V \in \mathfrak{p}^+$ and $v \in \mathcal{V}$,

\[
\frac{d}{dt} (E_Z v)(W + tV)\big|_{t=0} = -c_\rho(d\rho \circ p_{\mathcal{V}}) \left( [l_Z(W), V] + \frac{1}{2} [l_Z(W), [l_Z(W), V]] \right)(E_Z v)(W).
\]

(2) For $Z, W \in \mathfrak{D}$ and $V \in \mathfrak{p}^+$,

\[
\frac{d}{dt} l_Z(W + tV)\big|_{t=0} = p_{\mathfrak{p}^\perp} \left( [l_Z(W), V] + \frac{1}{2} [l_Z(W), [l_Z(W), V]] \right).
\]

(3) The function $(\partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} E_Z v)(W)$ is of the form $Q(l_Z(W))(E_Z v)(W)$ where $Q$ is a polynomial of degree $\leq 2q$ with values in $\text{End}(\mathcal{V})$.

(4) For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, the function $S_0(d\pi((X_1 X_2 \cdots X_q))$ is a sum of terms of the form $\rho(k_Z)^{-1} P(Z)Q(l_Z(Z))\rho(k_Z)$ where $P$ and $Q$ are polynomials of degree $\leq 2q$ with values in $\text{End}(\mathcal{V})$.

By combining the arguments of the proof of Proposition 6.5 in [8] with those of the proof of Proposition 5.2 in [13], we then obtain the following result. Recall that $m := \sum_i m_i$. 

Proposition 5.2. If \( q < \frac{1}{4}(-m - 2n) \) then for each \( X_1, X_2, \ldots, X_q \in \mathfrak{g}^c \), the Berezin transform of \( S(d\pi(X_1X_2\cdots X_q)) \) is well-defined.

We have then generalized Proposition 5.2 of [13]. However, it seems difficult to obtain here an explicit expression for \( W(d\pi(X)) \), \( X \in \mathfrak{g} \), as in [13], Section 6.

References


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