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ABSOLUTELY- p -SUMMING OPERATORS IN \mathfrak{L}_r -SPACES

by

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The purpose of this paper is to give a uniform presentation of all known results about absolutely- p -summing operators in \mathfrak{L}_r -spaces.

§ 1. ABSOLUTELY- p -SUMMING OPERATORS (cf. [11]).

Let E and F be Banach spaces. We denote by $\mathfrak{L}(E, F)$ the set of all bounded linear operators from E into F . An operator $T \in \mathfrak{L}(E, F)$ is called absolutely- p -summing ($1 \leq p < \infty$) if there exists a constant $\rho \geq 0$ such that for every finite set of elements $x_1, \dots, x_m \in E$ the inequality

$$\left\{ \sum_i \|T x_i\|^p \right\}^{1/p} \leq \rho \sup_{\|a\| \leq 1} \left\{ \sum_i |\langle x_i, a \rangle|^p \right\}^{1/p}$$

holds. The set $\mathfrak{P}_p(E, F)$ of all absolutely- p -summing operators $T \in \mathfrak{L}(E, F)$ is a Banach space with norm defined by

$$\pi_p(T) := \inf \rho.$$

It is convenient to put

$$\mathfrak{P}_\infty(E, F) := \mathfrak{L}(E, F) \quad \text{and} \quad \pi_\infty(T) := \|T\|$$

If $1 \leq p \leq q < \infty$ then

$$\mathfrak{P}_p(E, F) \subset \mathfrak{P}_q(E, F) \quad \text{and} \quad \pi_p(T) \geq \pi_q(T).$$

§ 2. THE \mathfrak{L}_r -SPACES (cf. [8]).

In the following let l_r^n be the Banach space of all n -dimensional real vectors $x = (\xi_i)$ with the norm

$$\|x\|_r := \left\{ \sum_i |\xi_i|^r \right\}^{1/r} \quad \text{for } 1 \leq r < \infty \quad \text{and} \quad \|x\|_\infty := \sup_i |\xi_i|.$$

A real Banach space E is called an \mathfrak{L}_r -space if for every finite set of elements $x_1, \dots, x_m \in E$ there exist operators $A \in \mathfrak{L}(E, l_r^n)$ and $X \in \mathfrak{L}(l_r^n, E)$ such that

$$\|x_i - X A x_i\| \leq 1 \quad \text{for } i=1, \dots, m \quad \text{and} \quad \|X\| \|A\| \leq c_E,$$

where the constant $c_E \geq 1$ depends only on E . We note that our definition is a slightly weaker than the definition of J. Lindenstrauss and A. Pełczyński.

All function spaces $L_r(S, \Sigma, \mu)$ are of type \mathfrak{L}_r . The operators A and X can be constructed as follows. Given $x_1, \dots, x_m \in L_r(S, \Sigma, \mu)$ we find step functions $x_1^0, \dots, x_m^0 \in L_r(S, \Sigma, \mu)$ such that $\|x_i - x_i^0\|_r \leq 1/2$. Then there exist disjoint subsets $S_1, \dots, S_n \in \Sigma$ with $\mu(S_i) > 0$ such that the step functions x_1^0, \dots, x_m^0 are linear combinations of the corresponding characteristic functions f_1, \dots, f_n . Now we define the operators A and X by

$$A x := (\mu(S_k))^{-1/r'} \int_S x(s) f_k(s) d\mu(s)$$

and

$$X(\xi_k) := \sum_k \xi_k \mu(S_k)^{-1/r} f_k.$$

Then we have

$$\|A\| = \|X\| = 1$$

and since $X A f_k = f_k$ the estimate

$$\|x_i - X A x_i\| \leq \|x_i - x_i^0\| + \|X A x_i^0 - X A x_i\| \leq 1$$

holds.

Now we show that it is possible to reduce the considerations of absolutely- p -summing operators in \mathfrak{L}_r -spaces to finite dimensional l_r^n -spaces.

Proposition : The following statements are equivalent :

(1) There exists a constant $c_{rs,pq} > 0$ such that

$$\pi_p(T) \leq c_{rs,pq} \pi_q(T) \quad \text{for all } T \in \mathcal{L}(l_r^n, l_s^n) \text{ and } n = 1, 2, \dots$$

(2) For every \mathcal{L}_r -space L_r and every \mathcal{L}_s -space L_s the inclusion

$$\mathcal{P}_q(L_r, L_s) \subset \mathcal{P}_p(L_r, L_s)$$

holds.

Proof : (1) \rightarrow (2) Let $T \in \mathcal{P}(E, F)$ and $x_1, \dots, x_m \in E$. Then for all $\epsilon > 0$ there exist $A \in \mathcal{L}(E, l_r^n)$, $X \in \mathcal{L}(l_r^n, E)$, $B \in \mathcal{L}(F, l_s^n)$, and $Y \in \mathcal{L}(l_s^n, F)$ such that

$$\|x_i - X A x_i\| \leq \epsilon, \quad \|T x_i - Y B T x_i\| \leq \epsilon, \quad \|X\| \|A\| \leq c_E, \quad \text{and } \|Y\| \|B\| \leq c_F.$$

Then

$$\begin{aligned} \|T x_i\| &\leq \|T x_i - Y B T x_i\| + \|Y B T x_i - Y B T X A x_i\| + \|Y B T X A x_i\| \\ &\leq \epsilon(1 + c_F \|T\|) + \|Y B T X A x_i\|, \end{aligned}$$

$$\left\{ \sum_i \|Y B T X A x_i\|^p \right\}^{1/p} \leq \pi_p(Y B T X A) \sup_{\|a\| \leq 1} \left\{ \sum_i |\langle x_i, a \rangle|^p \right\}^{1/p},$$

and

$$\begin{aligned} \pi_p(Y B T X A) &\leq \|Y\| \pi_p(B T X) \|A\| \leq c_{rs,pq} \|Y\| \pi_q(B T X) \|A\| \\ &\leq c_{rs,pq} \|Y\| \|B\| \pi_q(T) \|X\| \|A\| \leq c_{rs,pq} c_E c_F \pi_q(T). \end{aligned}$$

Consequently,

$$\begin{aligned} \left\{ \sum_i \|T x_i\|^p \right\}^{1/p} &\leq \\ &\leq \epsilon(1 + c_F \|T\|)^{1/p} + c_{rs,pq} c_E c_F \pi_q(T) \sup_{\|a\| \leq 1} \left\{ \sum_i |\langle x_i, a \rangle|^p \right\}^{1/p}. \end{aligned}$$

If $\varepsilon \rightarrow 0$, we obtain

$$\pi_p(T) \leq c_{rs,pq} c_E c_F \pi_q(T) \quad \text{and} \quad T \in P_p(E, F).$$

(2) \rightarrow (1). Since the sequence space l_r , resp. l_s , is of type \mathfrak{L}_r , resp. \mathfrak{L}_s , we have

$$P_q(l_r, l_s) \subset P_p(l_r, l_s).$$

Consequently, by the closed graph theorem there exists a constant $c_{rs,pq} > 0$ such that

$$\pi_p(T) \leq c_{rs,pq} \pi_q(T) \quad \text{for all} \quad T \in P_q(l_r, l_s).$$

Let us consider the operators

$$Q_n(\xi_1, \dots, \xi_n, \dots) := (\xi_1, \dots, \xi_n)$$

and

$$J_n(\xi_1, \dots, \xi_n) := (\xi_1, \dots, \xi_n, 0, \dots).$$

Then for every $T \in \mathfrak{L}(l_r^n, l_s^n)$ since $T = Q_n(J_n T Q_n)J_n$ we have

$$\pi_p(T) \leq \pi_p(J_n T Q_n) \leq c_{rs,pq} \pi_q(J_n T Q_n) \leq c_{rs,pq} \pi_q(T).$$

§ 3. HISTORICAL REMARKS.

The first result about absolutely- p -summing operators in \mathfrak{L}_r -spaces goes back to A. Grothendieck [5] who showed in 1956 that all bounded linear operators from an \mathfrak{L}_1 -space into an \mathfrak{L}_2 -space are absolutely-1-summing. A simplified proof of this important results was given by J. Lindenstrauss and A. Pełczyński [8].

In 1967, A. Pełczyński [9] and A. Pietsch [11] proved that the absolutely- p -summing operators in Hilbert spaces coincide with the Hilbert-Schmidt operators. This proof used Chintchin's inequality for Rademacher functions. Finally, D.J.H. Garling [3] determined the exact value of the π_p -norm of diagonal operators in l_2 .

Important progress was made in 1969, when L. Schwartz [13], [14], [15] remarked, in his theory of p -radonifying operators, that it is possible to use in place of Rademacher functions general sequences of independent and equidistributed random variables. By his method S. Kwapien [7] and P. Saphar [12] proved the fundamental theorems on absolutely- p -summing operators in \mathfrak{L}_r -spaces.

§ 4. A PROBABILITY LEMMA.

For $1 \leq s \leq 2$ let μ_s be the probability measure on the real line which is uniquely determined by its characteristic function

$$e^{-|\alpha|^s} = \int_{\mathbb{R}} e^{i\alpha\beta} d\mu_s(\beta).$$

If $1 \leq s < 2$ and $1 \leq p < s$ or if $s = 2$ and $1 \leq p < \infty$ then the moments

$$c_{sp} := \left\{ \int_{\mathbb{R}} |\beta|^p d\mu_s(\beta) \right\}^{1/p} > 0$$

exist (cf. [4]).

Let μ_s^n be the n-dimensional product measure of μ_s then the following probability lemma holds. It was used in functional analysis at first by J. Bretagnolle, D. Dacunha-Castelle and J. D. Krivine [1].

Lemma : If $y \in \mathbb{R}^n$ then

$$\left\{ \int_{\mathbb{R}^n} |\langle y, b \rangle|^p d\mu_s^n(b) \right\}^{1/p} = c_{sp} \|y\|_s.$$

Proof : We consider on the probability space $[\mathbb{R}^n, \mu_s^n]$ the independant random variables

$$f_i(b) := \beta_i \quad \text{for } i = 1, \dots, n.$$

Then

$$\hat{f}_i(\alpha) = e^{-|\alpha|^s},$$

where \hat{f}_i is the characteristic function of f_i .

Consequently, the random variable

$$f(b) := \langle y, b \rangle = \sum_i \eta_i f_i(b)$$

has the characteristic function

$$\hat{f}(\alpha) = e^{-\|y\|_s^s |\alpha|^s}.$$

The same characteristic function corresponds to the random variable

$$\varphi(\beta) := \|y\|_s \beta$$

which is defined on the probability space $[\mathbb{R}, \mu_s]$.

Therefore, the two random variables f and φ are equidistributed and we have

$$\int_{\mathbb{R}^n} |\langle y, b \rangle|^p d\mu_s^n(b) = \int_{\mathbb{R}} |\beta|^p d\mu_s(\beta) \|y\|_s^p.$$

§ 5. ABSOLUTELY-p-SUMMING OPERATORS IN l_s^n -SPACES.

We begin the central part of this paper with some few lemmata.

Lemma 1 : Let $T \in \mathcal{L}(E, l_s^n)$. If $1 < s < 2$ and $1 \leq p < s$ or if $s = 2$ and $1 \leq p < \infty$ then

$$\pi_p(T) \leq c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \|T'b\|^p d\mu_s^n(b) \right\}^{1/p}.$$

Proof : It follows from the probability lemma that if $x_1, \dots, x_m \in E$ then

$$\begin{aligned} \left\{ \sum_i \|T x_i\|_s^p \right\}^{1/p} &= c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \sum_i |\langle T x_i, b \rangle|^p d\mu_s^n(b) \right\}^{1/p} \\ &\leq c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \|T'b\|^p d\mu_s^n(b) \right\}^{1/p} \sup_i \left\{ \sum |\langle x_i, a \rangle|^p \right\}^{1/p} \\ &\quad \|a\| \leq 1 \end{aligned}$$

Lemma 2 : Let $T \in \mathcal{L}(l_s^n, F)$. If $1 < s < 2$ and $1 \leq p < s$ or if $s = 2$ and $1 \leq p < \infty$ then

$$c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \|T x\|^p d\mu_s^n(x) \right\}^{1/p} \leq \pi_p(T).$$

Proof : The main theorem of absolutely-p-summing operators (cf. [8], [11]) implies that there exists a measure μ on the closed unit ball U_s^n of l_s^n such that

$$\|T x\| \leq \left\{ \int_{U_s^n} |\langle x, a \rangle|^p d\mu(a) \right\}^{1/p} \text{ for all } x \in E \text{ and } \mu(U_s^n)^{1/p} = \pi_p(T).$$

Therefore, it follows from the probability lemma that

$$\begin{aligned} \left\{ \int_{\mathbb{R}^n} \|T x\|^p d\mu_s^n(x) \right\}^{1/p} &\leq \left\{ \int_{\mathbb{R}^n} \int_{U_s^n} |\langle x, a \rangle|^p d\mu(a) d\mu_s^n(x) \right\}^{1/p} \\ &\leq \int_{U_s^n} c_{sp}^p \|a\|_s^p d\mu(a) \quad \text{ }^{1/p} \\ &\leq c_{sp} \pi_p(T). \end{aligned}$$

Now we obtain the following lemma, which was proved by S. Kwapień [7], immediately.

Lemma 3 : Let $T \in \mathcal{L}(E, l_s^n)$. If $1 < s < 2$ and $1 \leq p \leq q < s$ or if $s = 2$ and $1 \leq p \leq q < \infty$ then

$$\pi_p(T) \leq c_{sq} c_{sp}^{-1} \pi_q(T').$$

In particular,

$$\pi_p(T) \leq \pi_p(T').$$

Proof : Applying lemma 1 to $T \in \mathcal{L}(E, l_s^n)$ and lemma 2 to $T' \in \mathcal{L}(l_s^n, E')$ we obtain

$$\begin{aligned} \pi_p(T) &\leq c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \|T'b\|^p d\mu_s^n(b) \right\}^{1/p} \\ &\leq c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \|T'b\|^q d\mu_s^n(b) \right\}^{1/q} \leq c_{sq} c_{sp}^{-1} \pi_q(T'). \end{aligned}$$

Remark (of C. Sunyack) : Let $T \in \mathcal{L}(l_s^n, l_s^n)$. Then by lemma 3

$$\pi_p(T) = \pi_p(T'),$$

and from the inequality in the proof of lemma 3 we obtain the equality

$$\pi_p(T) = c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \|T'b\|^p d\mu_s^n(b) \right\}^{1/p} = c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \|Tx\|^p d\mu_s^n(x) \right\}^{1/p}.$$

In particular, if I_n is the identity operator of the Hilbert space l_2^n then (cf. [3])

$$\pi_p(I_n) = \left(\frac{\Gamma(\frac{n+p}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}) \Gamma(\frac{1+p}{2})} \right)^{1/p}$$

The next lemma was proved by J. S. Cohen [2] and P. Saphar [12].

Lemma 4 : Let $T \in \mathcal{L}(E, l_s^n)$. Then

$$\pi_s(T) \leq \pi_s(T').$$

Proof : If e_1, \dots, e_n are the usual unit vectors we have

$$\|Tx\|_s = \left\{ \sum_k |\langle Tx, e_k \rangle|^s \right\}^{1/s} = \left\{ \sum_k |\langle x, T'e_k \rangle|^s \right\}^{1/s}$$

and

$$\left\{ \sum_k \|T'e_k\|_s^s \right\}^{1/s} \leq \pi_s(T') \sup_{\|y\|_s \leq 1} \left\{ \sum_k |\langle y, e_k \rangle|^s \right\}^{1/s} = \pi_s(T').$$

Consequently, if $x_1, \dots, x_m \in E$ then

$$\begin{aligned} \left\{ \sum_i \|T x_i\|_s^s \right\}^{1/s} &= \left\{ \sum_{ik} |\langle x_i, T'e_k \rangle|^s \right\}^{1/s} \\ &\leq \left\{ \sum_k \|T'e_k\|_s^s \right\}^{1/s} \sup_{\|a\| \leq 1} \left\{ \sum_i |\langle x_i, a \rangle|^s \right\}^{1/s} \\ &\leq \pi_s(T') \sup_{\|a\| \leq 1} \left\{ \sum_i |\langle x_i, a \rangle|^s \right\}^{1/s}. \end{aligned}$$

The proofs of the following propositions are obtained by different combinations of lemma 3 and 4.

Proposition 1 : Let $T \in \mathcal{L}(l_r^n, l_s^n)$. If $2 < r < \infty$, $1 \leq s < 2$, and $1 \leq p < r'$ then

$$\pi_1(T) \leq c_{r',p} c_{r',1}^{-1} \pi_p(T)$$

Proof : Applying lemma 3 to $T' \in \mathcal{L}(l_s^n, l_r^n)$ we obtain

$$\pi_1(T') \leq c_{r',p} c_{r',1}^{-1} \pi_p(T).$$

On the other hand by lemma 3 in the case $1 < s < 2$, and by lemma 4 in the case $s = 1$,

$$\pi_1(T) \leq \pi_1(T').$$

Theorem 1 (P. Saphar [12]) : Let $T \in \mathcal{L}(l_r^n, F)$. If $2 < r < \infty$ and $1 \leq p < r'$ then

$$\pi_1(T) \leq c_{r',p} c_{r',1}^{-1} \pi_p(T).$$

Proof : Without loss of generality we may assume that the Banach space F has the extension property. Consequently (cf. [10]), for all $\varepsilon > 0$ there exists a factorization

$$T : l_r^n \xrightarrow{A} l_\infty^m \xrightarrow{D} l_p^m \xrightarrow{Y} F$$

such that

$$\|A\| \leq 1, \|Y\| \leq 1, \text{ and } \pi_p(D) \leq \pi_p(T) + \varepsilon.$$

Now it follows by proposition 1 that

$$\begin{aligned}\pi_1(T) &\leq \|Y\| \pi_1(D A) \leq c_{r,p} c_{r,1}^{-1} \pi_p(D A) \\ &\leq c_{r,p} c_{r,1}^{-1} [\pi_p(T) + \varepsilon].\end{aligned}$$

Proposition 2 (S. Kwapien [7]) : Let $T \in \mathcal{L}(l_r^n, l_2^n)$. If $1 < r \leq \infty$ then

$$\pi_1(T) \leq c_{2r} c_{21}^{-1} \pi_{r'}(T).$$

Proof : Applying lemma 3 to $T \in \mathcal{L}(l_r^n, l_2^n)$ and lemma 4 to $T' \in \mathcal{L}(l_2^n, l_{r'}^n)$ we obtain

$$\pi_1(T) \leq c_{2r} c_{21}^{-1} \pi_{r'}(T') \quad \text{and} \quad \pi_{r'}(T') \leq \pi_{r'}(T).$$

The case $r = 1$, which is not dealt with in proposition 2, is identical with the fundamental theorem of A. Grothendieck [5].

Proposition 2^G : Let $T \in \mathcal{L}(l_1^n, l_2^n)$. Then

$$\pi_1(T) \leq c_G \|T\|.$$

Remark : If the constant c_G is the best possible then

$$\pi/2 \leq c_G \leq \sinh \pi/2.$$

Theorem 2 (S. Kwapien)[7]: Let $T \in \mathcal{L}(l_r^n, F)$. If $r = 1$, resp. $1 < r \leq 2$, then

$$\pi_1(T) \leq c_G \pi_2(T), \quad \text{resp.} \quad \pi_1(T) \leq c_{2r} c_{21}^{-1} \pi_2(T).$$

Proof : For all $\varepsilon > 0$ there exists a factorization

$$T : l_r^n \xrightarrow{A} l_\infty^m \xrightarrow{D} l_2^m \xrightarrow{Y} F$$

such that

$$\|A\| \leq 1, \|Y\| \leq 1, \text{ and } \pi_2(D) \leq \pi_2(T) + \varepsilon.$$

Now it follows by proposition 2 that

$$\begin{aligned}\pi_1(T) &\leq \|Y\| \pi_1(D A) \leq c_{2r} c_{21}^{-1} \pi_{r'}(D A) \\ &\leq c_{2r} c_{21}^{-1} \pi_2(D A) \leq c_{2r} c_{21}^{-1} [\pi_2(T) + \varepsilon].\end{aligned}$$

The proof in the case $r = 1$ is the same.

Proposition 3 : Let $T \in \mathfrak{L}(l_2^n, l_s^n)$. If $1 \leq s \leq p < \infty$ then

$$\pi_s(T) \leq c_{2p} c_{2s}^{-1} \pi_p(T).$$

Proof : Applying lemma 4 to $T \in \mathfrak{L}(l_2^n, l_s^n)$ and lemma 3 to $T' \in \mathfrak{L}(l_s^n, l_2^n)$ we obtain

$$\pi_s(T) \leq \pi_s(T') \quad \text{and} \quad \pi_s(T') \leq c_{2p} c_{2s}^{-1} \pi_p(T).$$

Theorem 3 (S. Kwapien [7]) : Let $T \in \mathfrak{L}(E, l_s^n)$. If $1 \leq s \leq 2$ and $2 \leq p < \infty$ then

$$\pi_2(T) \leq c_{2p} c_{2s}^{-1} \pi_p(T).$$

Proof : If $x_1, \dots, x_m \in E$ we define the operator $X \in \mathfrak{L}(l_2^m, F)$ by

$$X(\xi_i) = \sum_i \xi_i x_i.$$

Then

$$\|X\| = \sup_{\|a\| \leq 1} \left\{ \sum_i |\langle x_i, a \rangle|^2 \right\}^{1/2}.$$

Consequently, by proposition 3 we have

$$\begin{aligned} \left\{ \sum_i \|T x_i\|^2 \right\}^{1/2} &= \left\{ \sum_i \|T X e_i\|^2 \right\}^{1/2} \\ &\leq \pi_2(T X) \sup_{\|f\|_2 \leq 1} \left\{ \sum_i |\langle e_i, f \rangle|^2 \right\}^{1/2} \\ &\leq \pi_s(T X) \leq c_{2p} c_{2s}^{-1} \pi_p(T X) \\ &\leq c_{2p} c_{2s}^{-1} \pi_p(T) \sup_{\|a\| \leq 1} \left\{ \sum_i |\langle x_i, a \rangle|^2 \right\}^{1/2}. \end{aligned}$$

Because of symmetry it seems very probable that we have :

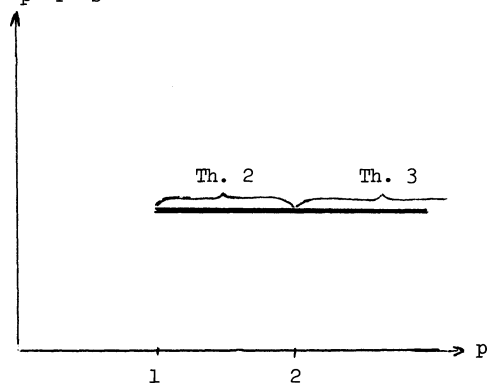
Theorem 4 (CONJECTURE) : Let $T \in \mathfrak{L}(E, l_s^n)$. If $2 < s < p \leq q < \infty$ then, with a constant $c_{s,pq} > 0$,

$$\pi_p(T) \leq c_{s,pq} \pi_q(T).$$

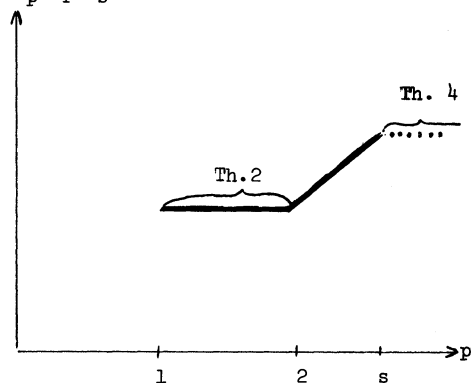
Finally, we illustrate the results in the following diagrams where the ordinate

is a symbolic measure of the largeness of $\rho_p(L_r, L_s)$.

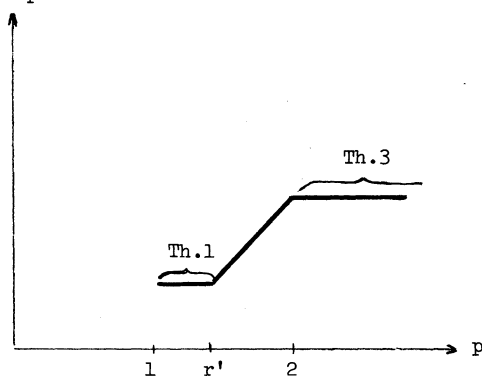
$\rho_p(L_r, L_s), 1 \leq r \leq 2, 1 \leq s \leq 2$



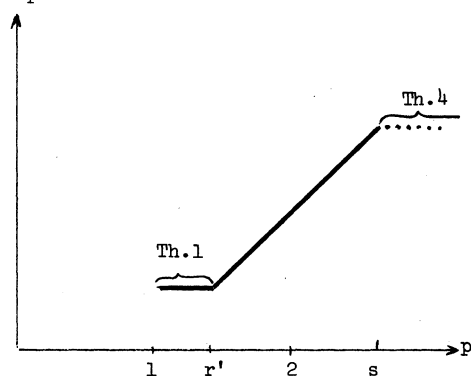
$\rho_p(L_r, L_s), 1 \leq r \leq 2, 2 \leq s \leq \infty$



$\rho_p(L_r, L_s), 2 \leq r \leq \infty, 1 \leq s \leq 2$



$\rho_p(L_r, L_s), 2 \leq r \leq \infty, 2 \leq s \leq \infty$



Remarks :

- (1) If the spaces L_r and L_s are infinite dimensional then $\rho_p(L_r, L_s)$ is strictly increasing
 - (2) $\rho_p(L_r, L_s)$ depends continuously on p if and only if it is constant since B. Maurey proved, assuming approximations property, the following results.
- If $\rho_p(E, F) = \bigcap_{\varepsilon > 0} \rho_{p+\varepsilon}(E, F)$ then $\rho_p(E, F) = \rho_{p+\varepsilon}(E, F)$ for $0 < \varepsilon < \varepsilon_0$.

If $\rho_p(E, F) = \bigcup_{\varepsilon > 0} \rho_{p-\varepsilon}(E, F)$ then $\rho_p(E, F) = \rho_{p-\varepsilon}(E, F)$ for $0 < \varepsilon < \varepsilon_0$:

§ 6. THE v_p -NORM (cf. [10], [23], [24]).

In the following let us assume that at least one of the Banach spaces E and F has finite dimension. Then every operator $T \in \mathcal{L}(E, F)$ can be represented in the form

$$T = \sum_i \langle x_i, a_i \rangle y_i \quad \text{for all } x \in E$$

with $a_1, \dots, a_n \in E'$ and $y_1, \dots, y_n \in F$. Now the v_p -norm is defined by

$$v_p(T) := \inf \left[\left\{ \sum_i \|a_i\|^p \right\}^{1/p} \sup_{\|b\| \leq 1} \left\{ \sum_i |\langle y_i, b \rangle|^{p'} \right\}^{1/p'} \right],$$

$1 < p < \infty$, where the infimum is taken over all possible representations. In the case $p = 1$ and $p = \infty$ we put

$$v_1(T) := \inf \left[\sum_i \|a_i\| \|y_i\| \right]$$

and

$$v_\infty(T) := \inf \left[\sup_i \|a_i\| \sup_{\|b\| \leq 1} \sum_i |\langle y_i, b \rangle| \right].$$

It follows from the well-known relations

$$\pi_p(T) = \sup \{ |\text{trace}(ST)| : S \in \mathcal{L}(F, E), v_p(S) \leq 1 \} \quad \text{for all } T \in \mathcal{L}(E, F)$$

and

$$v_{p'}(S) = \sup \{ |\text{trace}(ST)| : T \in \mathcal{L}(E, F), \pi_p(T) \leq 1 \} \quad \text{for all } S \in \mathcal{L}(F, E)$$

that the inequalities

$$\pi_p(T) \leq c \pi_q(T) \quad \text{for all } T \in \mathcal{L}(E, F)$$

and

$$v_{q'}(S) \leq c v_{p'}(S) \quad \text{for all } S \in \mathcal{L}(F, E)$$

are equivalent.

We have

$$\pi_p(T) \leq v_p(T) \quad \text{for all } T \in \mathcal{L}(E, F),$$

and in the case $p = 2$,

$$\pi_2(T) = v_2(T) \quad \text{for all } T \in \mathcal{L}(E, F).$$

If at least one of the Banach spaces E and F has the extension property then also the equation

$$\pi_p(T) = v_p(T) \quad \text{for all } T \in \mathcal{L}(E, F)$$

is valid. On the other side A. Pełczyński [21] has shown that there exists no constant $c > 0$ such that for every bounded linear operator T between arbitrary finite dimensional Banach spaces the inequality

$$v_p(T) \leq c \pi_p(T)$$

holds.

Problem. : If $1 \leq r, s \leq \infty$ and $1 < p < \infty$, does there exist a constant $c_{rsp} > 0$ such that

$$v_p(T) \leq c_{rsp} \pi_p(T) \quad \text{for all } T \in \mathcal{L}(l_r^n, l_s^n) ?$$

Now we prove further results by duality.

Theorem 1* : Let $T \in \mathcal{L}(E, l_s^n)$. If $2 < s < p \leq \infty$ then

$$v_p(T) \leq c_{s,p}, c_{s,1}^{-1} v_\infty(T).$$

Proof : If $2 < s < \infty$ and $1 \leq p' < s'$ then by theorem 1 we have

$$\pi_1(S) \leq c_{s,p}, c_{s,1}^{-1} \pi_{p'}(S) \quad \text{for all } S \in \mathcal{L}(l_s^n, E).$$

Consequently, there holds the dual inequality

$$v_p(T) \leq c_{s,p}, c_{s,1}^{-1} v_\infty(T) \quad \text{for all } T \in \mathcal{L}(E, l_s^n).$$

Theorem 2* : Let $T \in \mathcal{L}(E, l_s^n)$. If $s = 1$, resp. $1 < s \leq 2$, then

$$v_2(T) \leq c_G v_\infty(T), \text{ resp. } v_2(T) \leq c_{2s}, c_{21}^{-1} v_\infty(T)$$

Theorem 3* : Let $T \in \mathcal{L}(l_r^n, F)$. If $1 \leq r \leq 2$ and $1 < p \leq 2$ then

$$v_p(T) \leq c_{2p}, c_{21}^{-1} v_2(T)$$

Theorem 4* (CONJECTURE): Let $T \in \mathfrak{L}(l_r^n, F)$. If $2 < r < \infty$ and $1 < p < q < r'$ then, with a constant $c_{r,pq} > 0$,

$$v_p(T) \leq c_{r,pq} v_q(T)$$

Finally, we formulate some spacial cases of theorem 1* and 2*.

Proposition 4 (S. Kwapien [7]): Let $T \in \mathfrak{L}(l_\infty^n, l_s^n)$. If $2 < s < p < \infty$ then

$$v_p(T) \leq c_{s,p}, c_{s,1}^{-1} \|T\|$$

Proposition 5 (J. Lindenstrauss and A. Pełczyński [8]): Let $T \in \mathfrak{L}(l_\infty^n, l_s^n)$. If $s = 1$, resp. $1 < s \leq 2$, then

$$v_2(T) \leq c_G \|T\|, \text{ resp. } v_2(T) \leq c_{2s}, c_{21}^{-1} \|T\|.$$

Proof: The results follow from the fact that

$$v_\infty(T) = \|T\| \quad \text{for all } T \in \mathfrak{L}(l_\infty^n, F)$$

Remark: It is easy to prove the following stronger form of

Lemma 4: Let $T \in \mathfrak{L}(E, l_s^n)$. Then

$$v_s(T) \leq \pi_s(T).$$

One can obtain further results by using this inequality.

§ 7. IDENTITY OPERATORS IN l_r^n -SPACES.

Let I_n be the identity operator from l_r^n into l_s^n . We define the limit order $\lambda_I(r, s, \pi_p)$ to be the infimum of all real numbers λ for which there exists a constant $c_{rs,p} > 0$ such that the inequality

$$\pi_p(I_n : l_r^n \rightarrow l_s^n) \leq c_{rs,p} n^\lambda$$

for all $n = 1, 2, \dots$ holds. The limit order $\lambda_I(r, s, v_p)$ is defined in the same way.

Historical remark: The π_p - and v_p -norm of the identity operator from l_r^n into itself was determined or estimated by D.J.H. Garling and Y. Gordon (cf. [16], [17], [18]). In the cases v_∞ and π_1 the first result was proved by B. Grünbaum [19] and D. Rutovitz [22]. A. Tong [26] has given necessary and sufficient conditions for a diagonal operator from l_r into l_s to be nuclear (cf. also L. Schwartz [25]).

Lemma 5 : If $\alpha + \beta \leq 1$,

$$\lambda_I(r, s, \pi_p) \leq \alpha \quad \text{and} \quad \lambda_I(s, r, v_p) \leq \beta$$

then

$$\lambda_I(r, s, \pi_p) = \alpha \quad \text{and} \quad \lambda_I(s, r, v_p) = \beta$$

Proof : Since

$$n = \text{trace } (I_n) \leq \pi_p(I_n : l_r^n \rightarrow l_s^n) v_p(I_n : l_s^n \rightarrow l_r^n)$$

we have

$$1 \leq \lambda_I(r, s, \pi_p) + \lambda_I(s, r, v_p) \leq \alpha + \beta = 1 .$$

Consequently, identity holds.

Lemma 6 :

$$\lambda_I(r, s, \|\cdot\|) \leq \begin{cases} 1/s - 1/r & \text{if } r \geq s \\ 0 & \text{if } r \leq s \end{cases} .$$

Proof : The result follows from the well-known inequality

$$\|I_n : l_r^n \rightarrow l_s^n\| \leq \begin{cases} 1/s - 1/r & \text{if } r \geq s \\ 1 & \text{if } r \leq s \end{cases} .$$

Lemma 7 :

$$\lambda_I(1, \infty, v_1) \leq 0 .$$

Proof : If $e = (\varepsilon_i)$ ranges over the set of all n -dimensional vectors with $\varepsilon_i = \pm 1$ then the identity operator I_n has the representation

$$I_n x = 2^{-n} \sum_e \langle x, e \rangle e \quad \text{for all } x \in l_1^n .$$

Consequently,

$$v_1(I_n : l_1^n \rightarrow l_\infty^n) \leq 1 .$$

Lemma 8 : If $1 < p \leq \infty$ then

$$\lambda_I(1, 2, v_p) \leq 0 .$$

Proof : We represent the identity operator I_n in the form

$$I_n x = 2^{-n} \sum_e \langle x, e \rangle e \quad \text{for all } x \in l_1^n.$$

Then

$$\left\{ \sum \|e\|_\infty^p \right\}^{1/p} = 2^{n/p}.$$

On the other hand, it follows from Littlewood's inequality (cf. [20]) that

$$\sup_e \left\{ \sum |\langle e, b \rangle|^{p'} \right\}^{1/p'} \leq 2^{n/p'} c_p, \quad \|b\|_2 \leq 1.$$

Therefore,

$$v_p(I_n : l_1^n \rightarrow l_2^n) \leq c_p.$$

Lemma 9 :

$$\lambda_I(1, 2, \pi_1) \leq 0.$$

Proof : From Littlewood's inequality we have

$$\|x\|_2 \leq c_L 2^{-n} \sum_e |\langle x, e \rangle|.$$

Consequently, if $x_1, \dots, x_m \in l_1^n$

$$\sum_i \|x_i\|_2 \leq c_L \sup_a \sum |\langle x_i, a \rangle|, \quad \|a\|_\infty \leq 1$$

and therefore,

$$\pi_1(I_n : l_1^n \rightarrow l_2^n) \leq c_L.$$

Remark : Lemma 9 follows also from proposition 2^G.

Lemma 10 :

$$\lambda_I(\infty, p, v_p) \leq 1/p$$

Proof : If e_1, \dots, e_n are the usual unit vectors we can represent the identity operator I_n in the form

$$I_n x = \sum_i \langle x, e_i \rangle e_i \quad \text{for all } x \in l_\infty^n.$$

Since

$$\{\sum_i \|e_i\|_1^p\}^{1/p} = n^{1/p} \quad \text{and} \quad \sup_{\|a\|_{p'} \leq 1} \{\sum_i |\langle e_i, a \rangle|^{p'}\}^{1/p'} = 1$$

we obtain

$$v_p(I_n : l^n \rightarrow l_p^n) \leq n^{1/p}.$$

Lemma 11 : If $1 \leq s < 2$ then

$$\lambda_I(s', s, \pi_1) \leq 1/s$$

Proof : In the case $s = 1$ the result follows from lemma 10. Now we assume $1 < s < 2$. Then there exists ϵ with $0 < \epsilon < s-1$. By lemma 3 and 10 we obtain

$$\begin{aligned} \pi_1(I_n : l_{s'}^n \rightarrow l_s^n) &\leq c_{s \ s-\epsilon} c_{s1}^{-1} \pi_{s-\epsilon}(I_n : l_{s'}^n \rightarrow l_s^n) \\ &\leq c_{s \ s-\epsilon} c_{s1}^{-1} \|I_n : l_{s'}^n \rightarrow l_\infty^n\| \pi_{s-\epsilon}(I_n : l_\infty^n \rightarrow l_{s-\epsilon}^n) \|I_n : l_{s-\epsilon}^n \rightarrow l_s^n\| \\ &\leq c_{s \ s-\epsilon} c_{s1}^{-1} n^{1/(s-\epsilon)}. \end{aligned}$$

Consequently

$$\lambda_I(s', s, \pi_1) \leq 1/(s-\epsilon)$$

The result follows since ϵ can be made as small as we please.

Remark : It should be possible to determine the exact asymptotic behaviour of $\pi_p(I_n : l_{s'}^n \rightarrow l_s^n)$ as n tends to infinity by using the relation

$$\pi_p(I_n : l_{s'}^n \rightarrow l_s^n) = c_{sp}^{-1} \left\{ \int_{R^n} \|x\|_s^p d\mu_s^n(x) \right\}^{1/p}, \quad 1 \leq p < s.$$

The limit orders $\lambda_I(r, s, \|\cdot\|)$ and $\lambda_I(s, r, v_1)$

By lemma 6 we have

$$(1) \quad \lambda_I(r, s, \|\cdot\|) \leq \begin{cases} 1/s - 1/r & \text{if } r \geq s \\ 0 & \text{if } r \leq s \end{cases}.$$

On the other hand it follows from lemma 6, 7 and 10 that

$$\lambda_I(s, r, v_1) \leq \lambda_I(s, 1, \|\cdot\|) + \lambda_I(1, \infty, v_1) + \lambda_I(\infty, r, \|\cdot\|) \leq 1/s' + 1/r$$

and

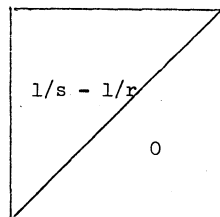
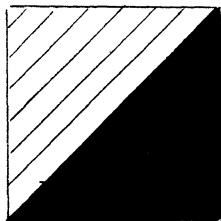
$$\lambda_I(s, r, v_1) \leq \lambda_I(s, \infty, \|\cdot\|) + \lambda_I(\infty, 1, v_1) + \lambda_I(1, r, \|\cdot\|) \leq 1.$$

In each case, choosing the best result we obtain

$$(1^*) \quad \lambda_I(s, r, v_1) \leq \begin{cases} 1/s' + 1/r & \text{if } r \geq s \\ 1 & \text{if } r \leq s \end{cases}.$$

Finally, lemma 5 implies that identity holds in (1) and (1*). In what follows we illustrate our results with pairs of diagrams in the unit square with coordinates $1/r$ and $1/s$. In the left hand diagram we plot the level curves of $\lambda_I(r, s, \pi_p)$. In the right hand diagrams we indicate the algebraic expression for $\lambda_I(r, s, \pi_p)$.

$$\underline{\lambda_I(r, s, \|\cdot\|)}$$



The limit orders $\lambda_I(r, s, \pi_2)$ and $\lambda_I(s, r, v_2)$

By lemmas 6, 9 and 10 we have

$$\begin{aligned} \lambda_I(r, s, \pi_2) &\leq \lambda_I(r, \infty, \|\cdot\|) + \lambda_I(\infty, 2, \pi_2) + \lambda_I(2, s, \|\cdot\|) \\ &\leq 0 + 1/2 + \begin{cases} 1/s - 1/2 & \text{if } 1 \leq s \leq 2 \\ 0 & \text{if } 2 \leq s \leq \infty \end{cases} \end{aligned}$$

and

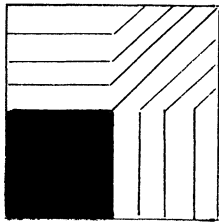
$$\begin{aligned} \lambda_I((r,s,\pi_2) &\leq \lambda_I(r,1, \|\cdot\|) + \lambda_I(1,2,\pi_2) + \lambda_I(2,s, \|\cdot\|) \\ &\leq 1/r' + 0 + \begin{cases} 1/s - 1/2 & \text{if } 1 \leq s \leq 2 \\ 0 & \text{if } 2 \leq s \leq \infty \end{cases} \end{aligned}$$

Consequently,

$$(2) \qquad \lambda_I(r,s,\pi_2) \leq \begin{cases} 1/r' + 1/s - 1/2 & \text{if } 1 \leq r \leq 2, \ 1 \leq s \leq 2, \\ 1/s & \text{if } 2 \leq r \leq \infty, \ 1 \leq s \leq 2, \\ 1/r' & \text{if } 1 \leq r \leq 2, \ 2 \leq s \leq \infty, \\ 1/2 & \text{if } 2 \leq r \leq \infty, \ 2 \leq s \leq \infty. \end{cases}$$

Since $\lambda_I(s,r,v_2) = \lambda_I(s,r,\pi_2)$ it follows from lemma 5 that identity holds in (2).

$$\lambda_I(r,s,\pi_2)$$



$\frac{1}{s}$	$\frac{1}{r'} + \frac{1}{s} - \frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{r'}$

The limit orders $\lambda_I(r,s,\pi_p)$ and $\lambda_I(s,r,v_{p'})$ with $1 \leq p < 2$

Since by theorem 2 and 2* for $1 \leq r \leq 2$ we have

$$\lambda_I(r,s,\pi_p) = \lambda_I(r,s,\pi_2) \quad \text{and} \quad \lambda_I(s,r,v_{p'}) = \lambda_I(s,r,v_2)$$

in the following we need only consider the case $2 < r \leq \infty$.

By lemma 6 and 11 we obtain

$$\lambda_I(r, s, \pi_p) \leq \lambda_I(r, s', \|\cdot\|) + \lambda_I(s', s, \pi_p) \leq 1/s \quad \text{if } r \leq s' \text{ and } 1 \leq s \leq 2,$$

and

$$\lambda_I(r, s, \pi_p) \leq \lambda_I(r, r', \pi_p) + \lambda_I(r', s, \|\cdot\|) \leq 1/r' \quad \text{if } r' \leq s \text{ and } 1 \leq r' \leq 2.$$

On the other hand it follows from lemma 6 and 10 that

$$\begin{aligned} \lambda_I(r, s, \pi_p) &\leq \lambda_I(r, \infty, \|\cdot\|) + \lambda_I(\infty, p, \pi_p) + \lambda_I(p, s, \|\cdot\|) \\ &\leq 0 + 1/p + \begin{cases} 1/s - 1/p & \text{if } p \geq s, \\ 0 & \text{if } p \leq s. \end{cases} \end{aligned}$$

In each case, choosing the best result we obtain

$$(3) \quad \lambda_I(r, s, \pi_p) \leq \begin{cases} 1/s & \text{if } p' \leq r \leq \infty, \quad -1 \leq s \leq p, \\ 1/p & \text{if } p' \leq r \leq \infty, \quad p \leq s \leq \infty, \\ 1/s & \text{if } 2 \leq r \leq p', \quad 1 \leq s \leq r', \\ 1/r' & \text{if } 2 \leq r \leq p', \quad r' \leq s \leq \infty. \end{cases}$$

By lemma 6 and 11

$$\begin{aligned} \lambda_I(s, r, v_{p'}) &\leq \lambda_I(s, \infty, \|\cdot\|) + \lambda_I(\infty, p', v_{p'}) + \lambda_I(p', r, \|\cdot\|) \\ &\leq 0 + 1/p' + \begin{cases} 1/r - 1/p' & \text{if } p' \geq r, \\ 0 & \text{if } p' \leq r. \end{cases} \end{aligned}$$

Moreover,

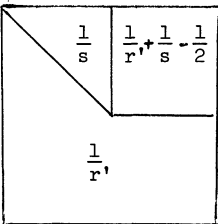
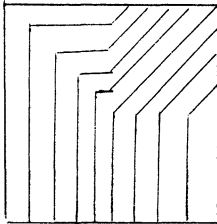
$$\lambda_I(s, r, v_{p'}) \leq \lambda_I(s, r, v_2) = 1/s' \quad \text{if } 1 \leq s \leq 2.$$

Consequently,

$$(3^*) \quad \lambda_I(s, r, v_{p'}) \leq \begin{cases} 1/s' & \text{if } p' \leq r \leq \infty, \quad 1 \leq s \leq p, \\ 1/p' & \text{if } p' \leq r \leq \infty, \quad p \leq s \leq \infty, \\ 1/s' & \text{if } 2 \leq r \leq p', \quad 1 \leq s \leq r', \\ 1/r & \text{if } 2 \leq r \leq p', \quad r' \leq s \leq \infty. \end{cases}$$

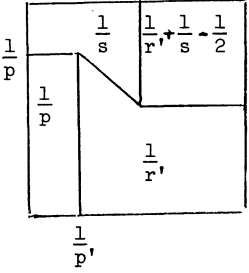
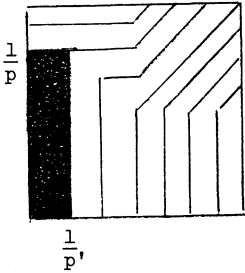
Finally, lemma 5 implies that identity holds in (3) and (3*).

$$\lambda_I(r,s,\pi_1)$$



$$\lambda_I(r,s,\pi_1)$$

$$1 < p < 2$$



The limit orders $\lambda_I(r,s,\pi_p)$ and $\lambda_I(r,s,v_p)$ with $2 < p < \infty$

Since by theorem 3 and 3* for $1 \leq s \leq 2$ we have

$$\lambda_I(r,s,\pi_p) = \lambda_I(r,s,\pi_2) \text{ and } \lambda_I(s,r,v_p) = \lambda_I(s,r,v_2)$$

in the following we need only consider the case $2 < s \leq \infty$.

Since

$$\lambda_I(r,s,\pi_p) \leq \lambda_I(r,s,v_p)$$

by (3*) we obtain

$$(4) \quad \lambda_I(r,s,\pi_p) \leq \begin{cases} 1/r' & \text{if } 1 \leq r \leq p', \quad p \leq s \leq \infty, \\ 1/p & \text{if } p' \leq r \leq \infty, \quad p \leq s \leq \infty, \\ 1/r' & \text{if } 1 \leq r \leq s', \quad 2 \leq s \leq p, \\ 1/s & \text{if } s' \leq r \leq \infty, \quad 2 \leq s \leq p. \end{cases}$$

It follows from lemma 6 and 10 that

$$\begin{aligned} \lambda_I(s, r, v_{p'}) &\leq \lambda_I(s, \infty, \|\cdot\|) + \lambda_I(\infty, p', v_{p'}) + \lambda_I(p', r, \|\cdot\|) \\ &\leq 0 + 1/p' + \begin{cases} 1/r - 1/p' & \text{if } p' \geq r, \\ 0 & \text{if } p' \leq r. \end{cases} \end{aligned}$$

On the other hand lemma 6 and 8 imply that

$$\begin{aligned} \lambda_I(s, r, v_{p'}) &\leq \lambda_I(s, 1, \|\cdot\|) + \lambda_I(1, 2, v_{p'}) + \lambda_I(2, r, \|\cdot\|) \\ &\leq 1/s' + 0 + 0 \quad \text{if } 2 \leq r. \end{aligned}$$

In each case, choosing the best result we obtain

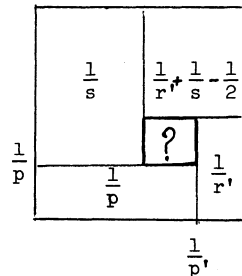
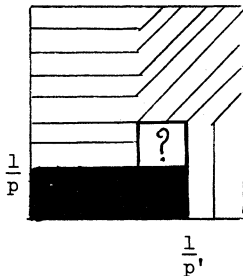
$$(4^*) \quad \lambda_I(s, r, v_{p'}) \leq \begin{cases} 1/r & \text{if } 1 \leq r \leq p', \quad p \leq s \leq \infty, \\ 1/p' & \text{if } p' \leq r \leq \infty, \quad p \leq s \leq \infty, \\ 1/r & \text{if } 1 \leq r \leq p', \quad 2 \leq s \leq p, \\ 1/s' & \text{if } 2 \leq r \leq \infty, \quad 2 \leq s \leq p. \end{cases}$$

Because the square

$$Q_{I,p} := \{(1/r, 1/s) : p' < r < 2, \quad 2 < s < p\}$$

does not appear in (4*), we have the open problem whether identity holds for all r and s in (4).

$$\frac{\lambda_I(r, s, \pi_p)}{2 < p < \infty}$$



§ 8. LITTLEWOOD OPERATORS IN l_r^n -SPACES.

In the following n ranges over the set of all natural numbers $n = 2^k$ with $k = 1, 2, \dots$. The symmetric Littlewood operators $A_n = (\alpha_{ik}^{(n)})$ are defined inductively by (cf. [20])

$$A_2 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \dots, A_{2n} := \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix}, \dots$$

Then

$$A_n^2 = n I_n \quad \text{and} \quad \alpha_{ik}^{(n)} = \pm 1.$$

The limit orders $\lambda_A(r, s, \pi_p)$ and $\lambda_A(r, s, v_p)$ are introduced in the same way as in the case of identity operators.

Lemma 12 : If $\alpha + \beta \leq 2$,

$$\lambda_A(r, s, \pi_p) \leq \alpha \quad \text{and} \quad \lambda_A(s, r, v_p) \leq \beta$$

then

$$\lambda_A(r, s, \pi_p) = \alpha \quad \text{and} \quad \lambda_A(s, r, v_p) = \beta.$$

Proof : Since

$$n^2 = \text{trace} (n I_n) \leq \pi_p(A_n : l_r^n \rightarrow l_s^n) v_p(A_n : l_s^n \rightarrow l_r^n)$$

we have

$$2 \leq \lambda_A(r, s, \pi_p) + \lambda_A(s, r, v_p) \leq \alpha + \beta \leq 2.$$

Consequently, identity holds.

Lemma 13 : If $2 \leq s \leq \infty$ then

$$\lambda_A(r, s, \|\cdot\|) \leq 1/s.$$

Proof : Since the operator $n^{-1/2} A_n$ is unitary we have

$$\|A_n : l_2^n \rightarrow l_2^n\| \leq n^{1/2}.$$

On the other hand, because $|\alpha_{ik}^{(n)}| = 1$, it follows that

$$\|A_n : l_1^n \rightarrow l_\infty^n\| \leq 1.$$

Finally, if $2 \leq s \leq \infty$, the M. Riesz' convexity theorem implies

$$\|A_n : l_{s'}^n \rightarrow l_s^n\| \leq n^{1/s}$$

Lemma 14 :

$$\lambda_A(1, 2, v_1) \leq 1/2.$$

Proof : The result follows from

$$\begin{aligned} v_1(A_n : l_1^n \rightarrow l_\infty^n) &= \pi_1(A_n : l_1^n \rightarrow l_\infty^n) \\ &= \pi_1(I_n : l_1^n \rightarrow l_2^n) \|A_n : l_2^n \rightarrow l_2^n\| \|I_n : l_2^n \rightarrow l_\infty^n\| \\ &\leq c_L n^{1/2}. \end{aligned}$$

The limit orders $\lambda_A(r, s, \|\cdot\|)$ and $\lambda_A(s, r, v_1)$

By lemma 6 and 13 we have

$$\begin{aligned} \lambda_A(r, s, \|\cdot\|) &\leq \lambda_I(r, 2, \|\cdot\|) + \lambda_A(2, 2, \|\cdot\|) + \lambda_I(2, s, \|\cdot\|) \\ &\leq \begin{cases} (1/2 - 1/r) + 1/2 + (1/s - 1/2) & \text{if } r \geq 2, 2 \geq s, \\ 0 + 1/2 + (1/s - 1/2) & \text{if } r \leq 2, 2 \geq s, \\ (1/2 - 1/r) + 1/2 + 0 & \text{if } r \geq 2, 2 \leq s. \end{cases} \end{aligned}$$

On the other hand we obtain

$$\begin{aligned} \lambda_A(r, s, \|\cdot\|) &\leq \lambda_I(r, s', \|\cdot\|) + \lambda_A(s', s, \|\cdot\|) \\ &\leq 0 + 1/s \quad \text{if } r \leq s' \quad \text{and } 2 \leq s, \end{aligned}$$

and

$$\begin{aligned} \lambda_A(r, s, \|\cdot\|) &\leq \lambda_A(r, r', \|\cdot\|) + \lambda_I(r', s, \|\cdot\|) \\ &\leq 1/r' + 0 \quad \text{if } r \leq 2 \quad \text{and } r' \leq s. \end{aligned}$$

Summarizing the results we have

$$(5) \quad \lambda_A(r, s, \|\cdot\|) \leq \begin{cases} 1/r' + 1/s - 1/2 & \text{if } 2 \leq r \leq \infty, 1 \leq s \leq 2, \\ 1/s & \text{if } 1 \leq r \leq 2, 1 \leq s \leq r', \\ 1/r' & \text{if } s' \leq r \leq \infty, 2 \leq s \leq \infty. \end{cases}$$

With the known values of $\lambda_I(s, r, v_1)$ we obtain

$$\lambda_A(s, r, v_1) \leq \lambda_I(s, 1, v_1) + \lambda_A(1, r, \|\cdot\|) \leq 1 + 1/r,$$

and

$$\lambda_A(s, r, v_1) \leq \lambda_A(s, \infty, \|\cdot\|) + \lambda_I(\infty, r, v_1) \leq 1/s' + 1.$$

On the other hand it follows from lemma 14 that

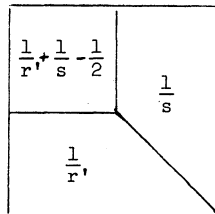
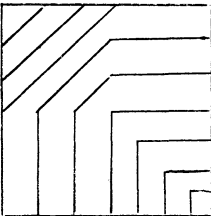
$$\begin{aligned} \lambda_A(s, r, v_1) &\leq \lambda_I(s, 1, \|\cdot\|) + \lambda_A(1, \infty, v_1) + \lambda_I(\infty, r, \|\cdot\|) \\ &\leq 1/s' + 1/2 + 1/r. \end{aligned}$$

In each case, choosing the best result we obtain,

$$(5^*) \quad \lambda_A(s, r, v_1) \leq \begin{cases} 1/r + 1/s' + 1/2 & \text{if } 2 \leq r \leq \infty, 1 \leq s \leq 2, \\ 1/s' + 1 & \text{if } 1 \leq r \leq 2, 1 \leq s \leq r', \\ 1/r + 1 & \text{if } s' \leq r \leq \infty, 2 \leq s \leq \infty. \end{cases}$$

Finally, lemma 12 implies that identity holds in (5) and (5*).

$$\underline{\lambda_A(r, s, \|\cdot\|)}$$



The limit orders $\lambda_A(r, s, \pi_2)$ and $\lambda_A(s, r, v_2)$

Since

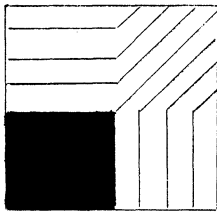
$$\lambda_A(r, s, \pi_2) \leq \lambda_I(r, 2, \pi_2) + \lambda_A(2, s, \|\cdot\|)$$

we obtain, using the known values of $\lambda_I(r, 2, \pi_1)$ and $\lambda_A(2, s, \|\cdot\|)$,

$$(6) \quad \lambda_A(r, s, \pi_2) \leq \begin{cases} 1/r' + 1/s & \text{if } 1 \leq r \leq 2, & 1 \leq s \leq 2, \\ 1/2 + 1/s & \text{if } 2 \leq r \leq \infty, & 1 \leq s \leq 2, \\ 1/r' + 1/2 & \text{if } 1 \leq r \leq 2, & 2 \leq s \leq \infty, \\ 1/2 + 1/2 & \text{if } 2 \leq r \leq \infty, & 2 \leq s \leq \infty. \end{cases}$$

Finally, it follows from lemma 12 and $\lambda_A(s, r, v_2) = \lambda_A(s, r, \pi_2)$ that identity holds in (6)

$$\underline{\lambda_A(r, s, \pi_2)}$$



$\frac{1}{2} + \frac{1}{s}$	$\frac{1}{r'} + \frac{1}{s}$
1	$\frac{1}{r'} + \frac{1}{2}$

The limit orders $\lambda_A(r, s, \pi_p)$ and $\lambda_A(s, r, v_p)$ with $1 \leq p < 2$

Since by theorem 2 and 2* for $1 \leq r \leq 2$ we have

$$\lambda_A(r, s, \pi_p) = \lambda_A(r, s, \pi_2) \quad \text{and} \quad \lambda_A(s, r, v_p) = \lambda_A(s, r, v_2)$$

in the following we need only consider the case $2 < r \leq \infty$.

Since

$$\lambda_A(r, s, \pi_p) \leq \lambda_I(r, p, \pi_p) + \lambda_A(p, s, \|\cdot\|)$$

we obtain, using the known values of $\lambda_I(r, p, \pi_p)$ and $\lambda_A(p, s, \|\cdot\|)$,

$$\lambda_A(r, s, \pi_p) \leq \frac{1}{p} + \begin{cases} 1/p' & \text{if } s \geq p', \\ 1/s & \text{if } s \leq p'. \end{cases}$$

On the other hand it follows from

$$\lambda_A(r, s, \pi_p) \leq \lambda_I(r, r', \pi_p) + \lambda_A(r', s, \|\cdot\|)$$

that

$$\lambda_A(r, s, \pi_p) \leq \frac{1}{r'} + \begin{cases} 1/r & \text{if } s \geq r, \\ 1/s & \text{if } s \leq r. \end{cases}$$

In each case, choosing the best result we obtain

$$(7) \quad \lambda_A(r, s, \pi_p) \leq \begin{cases} 1 & \text{if } p' \leq r \leq \infty, \quad p' \leq s \leq \infty, \\ 1/p + 1/s & \text{if } p' \leq r \leq \infty, \quad 1 \leq s \leq p', \\ 1 & \text{if } 2 \leq r \leq p', \quad r \leq s \leq \infty, \\ 1/r' + 1/s & \text{if } 2 \leq r \leq p', \quad 1 \leq s \leq r. \end{cases}$$

Moreover,

$$\begin{aligned} \lambda_A(s, r, v_{p'}) &\leq \lambda_A(s, \infty, \|\cdot\|) + \lambda_I(\infty, r, v_{p'}) \\ &\leq 1/s + \begin{cases} 1/r & \text{if } r \leq p', \\ 1/p' & \text{if } r \geq p', \end{cases} \end{aligned}$$

and

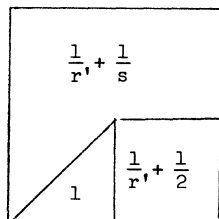
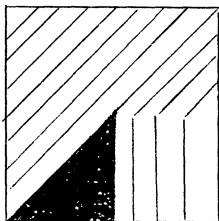
$$\begin{aligned} \lambda_A(s, r, v_{p'}) &\leq \lambda_A(s, 2, v_{p'}) + \lambda_I(2, r, \|\cdot\|) \\ &\leq \lambda_A(s, 2, v_2) \leq 1 \quad \text{if } 2 \leq s \leq \infty. \end{aligned}$$

Consequently,

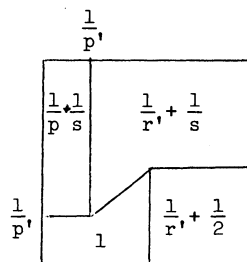
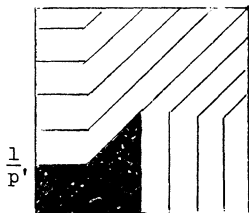
$$(7^*) \quad \lambda_A(s, r, v_{p'}) \leq \begin{cases} 1 & \text{if } p' \leq r \leq \infty, \quad p' \leq s \leq \infty, \\ 1/p' + 1/s' & \text{if } p' \leq r \leq \infty, \quad 1 \leq s \leq p', \\ 1 & \text{if } 2 \leq r \leq p', \quad r \leq s \leq \infty, \\ 1/r + 1/s' & \text{if } 2 \leq r \leq p', \quad 1 \leq s \leq r. \end{cases}$$

Finally, lemma 12 implies that identity holds in (7) and (7*).

$$\underline{\lambda_A(r, s, \pi_1)}$$



$$\frac{\lambda_A(r, s, \pi_p)}{1 < p < 2}$$



The limit orders $\lambda_A(r, s, \pi_p)$ and $\lambda_A(s, r, \nu_p)$ with $2 < p < \infty$

Since by theorem 3 and 3* for $1 \leq s \leq 2$ we have

$$\lambda_A(r, s, \pi_p) = \lambda_A(r, s, \pi_2) \quad \text{and} \quad \lambda_A(s, r, \nu_p) = \lambda_A(s, r, \nu_2)$$

in the following we need only consider the case $2 < s \leq \infty$.

From (7*) and

$$\lambda_A(r, s, \pi_p) \leq \lambda_A(r, s, \nu_p)$$

we obtain

$$(8) \quad \lambda_A(r, s, \pi_p) \leq \begin{cases} 1/r' + 1/s & \text{if } 1 \leq r \leq s, & 2 \leq s \leq p, \\ 1 & \text{if } s \leq r \leq \infty, & 2 \leq s \leq p, \\ 1/r' + 1/p & \text{if } 1 \leq r \leq p, & p \leq s \leq \infty, \\ 1 & \text{if } p \leq r \leq \infty, & p \leq s \leq \infty. \end{cases}$$

On the other hand, we have

$$\begin{aligned} \lambda_A(s,r,v_p) &\leq \lambda_I(s,p',v_{p'}) + \lambda_A(p',r, \|\cdot\|) \\ &\leq 1/p' + \begin{cases} 1/p & \text{if } p \leq r, \\ 1/r & \text{if } p \geq r, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \lambda_A(s,r,v_p) &\leq \lambda_I(s,2, \|\cdot\|) + \lambda_A(2,2,v_p) + \lambda_I(2,r, \|\cdot\|) \\ &\leq (1/2-1/s) + 1 + (1/r-1/2) \quad \text{if } 1 \leq r \leq 2 \text{ and } 2 \leq s \leq \infty. \end{aligned}$$

In each case, choosing the best result we obtain

$$(8^*) \quad \lambda_A(s,r,v_p) \leq \begin{cases} 1/r + 1/s' & \text{if } 1 \leq r \leq 2, \quad 2 \leq s \leq p, \\ 1 & \text{if } p \leq r \leq \infty, \quad 2 \leq s \leq p, \\ 1/r + 1/p' & \text{if } 1 \leq r \leq p, \quad p \leq s \leq \infty, \\ 1 & \text{if } p \leq r \leq \infty, \quad p \leq s \leq \infty. \end{cases}$$

Because the square

$$Q_{A,p} := \{ (1/r', 1/s) : 2 < r < p, \quad 2 < s < p \}$$

does not appear in (8*), we have the open problem wether identity holds for all r and s in (8).

$$\frac{\lambda_A(r,s,\pi_p)}{2 < p < \infty}$$



Final remark (Cf. end of § 5)

Let L_r and L_s be infinite dimensional. Then $\rho_p(L_r, L_s)$ is strictly increasing

- 1) if $2 \leq r \leq \infty$, $1 \leq s \leq 2$, and $r' \leq p \leq 2$ since $\lambda_A(r, s, \pi_p) = 1/p + 1/s$,
- 2) if $1 \leq r \leq 2$, $2 \leq s \leq \infty$, and $2 \leq p \leq s$ since $\lambda_A(r, s, \pi_p) = 1/p + 1/r'$,
- 3) if $2 \leq r \leq \infty$, $2 \leq s \leq \infty$, and $r' \leq p \leq s$ since $\lambda_I(r, s, \pi_p) = 1/p$.

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