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ON OPERATORS FACTORIZABLE THROUGH $L_p$ SPACE

by

Stanislaw KWAPIEN

In this paper we give some necessary and sufficient conditions for an operator between Banach spaces be factorizable through $L_p$ space, also conditions for factorizability through a subspace, a quotient and a subspace of a quotient of $L_p$. Hence, we obtain characterizations of Banach spaces isomorphic with complemented subspaces, with subspaces, with quotients and with subspaces of quotients of $L_p$. These conditions are given in terms of $p$-absolutely summing and $p$-integral operators. We use the general theory of ideals of operators, necessary definitions and facts of the theory given in § I. For more detailed treatment the reader is referred to the paper [3], by A. Grothendieck, where it is exposed in frame of tensor product theory, and also to papers of A. Pietsch. We end the paper with some applications.

§ I. Normed ideals of operators.

In the sequel $L(E,F)$ will denote all bounded linear operators from Banach space $E$ into Banach space $F$ and $\|u\|$ the norm of an operator.

Let for each pair of Banach spaces $E$, $F$ be given a linear subspace $A(E,F)$ of $L(E,F)$ and $\alpha_{E,F}$ a norm on $A(E,F)$ such that

1. if $u \in A(E,F)$, $v \in L(X,E)$, $w \in L(E,Y)$ then $wuv \in A(X,Y)$
   and $\alpha_{X,Y}(wuv) \leq \alpha_{E,F}(u) \|w\| \|v\|$

2. if $u \in A(E,F)$ then $\alpha_{E,F}(u) \geq \|u\|$

3. if $u \notin L(E,F)$ is one dimensional then $u \notin A(E,F)$
   and $\alpha_{E,F}(u) = \|u\|$

Then we say that $|A, \alpha|$ is a normed linear ideal of operators.

In further we shall write $\alpha(u)$ instead of $\alpha_{E,F}(u)$.

A normed linear ideal $|A, \alpha|$ is defined to be maximal if it satisfies the following condition:

if for $u \in L(E,F)$ there exists a constant $M$ such that for each finite dimensional Banach spaces $X$, $Y$ and operators $v \in L(X,E)$, $w \in L(F,Y)$

it is $\alpha(wuv) \leq M \|w\| \|v\|$ then $u \in A(E,F)$ and $\alpha(u) \leq M$. 

We say that $u \in A''(E,F)$ if there exists a constant $M$ such that for each finite dimensional Banach spaces $X, Y$ and operators $v \in L(X,E)$, $w \in L(Y,F)$ and $z \in A(Y,X)$ there holds

$$\|\text{trace } (wuvz)\| \leq M \|v\| \|w\| a(z).$$

The least such constant $M$ is denoted by $\alpha^a(u)$.

It is easy to check that $[A', \alpha^a]$ is a maximal normed ideal of operators. We call it the dual ideal of $[A, \alpha]$. Moreover, given normed linear ideal $[A, \alpha]$ we define the following ideals:

right injective envelope of $[A, \alpha]$, denoted $\langle A, \alpha \rangle$, as follows $u \in \langle A \rangle(E,F)$ if for some Banach space $G$ and isometric embedding $i$ of $F$ into $G$ it is $iu \in A(E,G)$,

$$\alpha(u) = \inf \alpha(iu),$$

left injective envelope of $[A, \alpha]$, denoted by $\langle A \rangle$, as follows $u \in \langle A \rangle(E,F)$ if for some Banach space $H$ and normed surjection $j$ of $H$ onto $F$ there exists $v \in A(H,F)$ such that $ju = \text{tt} jv$, $i$ is the canonical injection of $F$ in $F''$ and $\text{tt}$ is the second adjoint of $j$,

right projective envelope of $[A, \alpha]$, denoted by $\langle A \rangle$, as follows $u \in \langle A \rangle(E,F)$ if for each Banach space $H$ and a normed surjection $j$ of $H$ onto $F$ there exists $v \in A(H,F)$ such that $ju = \text{vi}$, $j$ is the cannonical injection of $F$ in $F''$.

left projective envelope of $[A, \alpha]$, denoted by $\langle A \rangle$, as follows $u \in \langle A \rangle(E,F)$ if for each Banach space $G$ and isometric embedding $i$ of $E$ into $G$ there exists $v \in A(G,F)$ such that $ju = vi$, $i$ is the cannonical injection of $F$ in $F''$.

One can verify the following

[I.1] if $[A, \alpha]$ is maximal then each of the above defined ideals is maximal also,

[I.2] if $[A, \alpha]$ is maximal then $[(A'')', (\alpha')]$ is equal to $[A, \alpha]$,

[I.3] $[(A'')', (\alpha')]$ is equal to $[A', \alpha']$,

[I.4] $[(A'')', (\alpha')]$ is equal to $[A*, \alpha*]$.

Example I. Ideal of p-absolutely summing operators, $|||p, \pi_p|||$

$u \in \pi_p(E,F)$ if for some constant $M$ for each $x_1, \ldots, x_n \in E$ there holds

$$\sum_{i=1}^{n} \|u(x_i)\|^p \leq M \sup_{x' \in E'} \|x'\|^p \sum_{i=1}^{n} |\langle x_i, x' \rangle|^p,$$

$\pi_p(u)$ is the least such constant $M$. 

Example 2. Ideal of $p$-integral operators, $\mathcal{I}_p, \mathcal{I}_p$

$u \in \mathcal{I}_p(E,F)$ if there exists a probability measure space $(\Omega, \mathcal{F}, \mu)$ and operators $v \in L(E, L^p(\Omega, \mu))$ and $w \in L(L_p(\Omega, \mu), F')$ such that $wv = iu$, where $j$ is the canonical injection of $L^p(\Omega, \mu)$ into $L_p(\Omega, \mu)$ and $i$ the canonical injection of $F$ into $F'$.

$I_p(u)$ is defined as $\inf \|v\|_{L_p} \|w\|$, infimum is taken over all such probability measure spaces $(\Omega, \mathcal{F}, \mu)$ and operators $v$ and $w$.

It was proved by A. Pietsch that

$$I_p \setminus I_{\mathcal{I}_p} \text{ is equal to } \mathcal{I}_p, \mathcal{I}_p,$$

$$I_p, \mathcal{I}_p \text{ is equal to } L_{L_p, q}, \mathcal{I}_p \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$  

§ 2. Ideal of $L_p$ factorizable operators

By $L_p$ space we shall mean any Banach space isometric with the space $L_p(\Omega, \mu)$ for some measure space $(\Omega, \mathcal{F}, \mu)$.

We say that $u \in \mathcal{I}_p(E,F)$ if for some $L_p$ space there exist operators $v \in L(E, L_p)$ and $w \in L(L_p, F')$ such that $iu = wv$, $i$ is the canonical injection of $F$ into $F'$.

$\gamma_p(u)$ is defined as $\inf \|v\|_{L_p} \|w\|$, $v$ and $w$ are as in the definition of $\mathcal{I}_p(E,F)$.

Proposition I. Let $1 < p < \infty$. $\Gamma_p, \gamma_p$ is a maximal normed ideal of operators.

Proof. We shall make use of the following equality

$$\text{inf}_{t>0} (p^{-1} t^p a^p + q^{-1} t^q b^q)$$

which is valid for positive numbers $a, b$ and $q$ defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Let for $k = 1, 2 \ u_k \in \Gamma_p(E,F)$ and let $iu_k = v_k$, where $v_k \in L(E, L_p(\Omega, \mu_k))$, $w_k \in L(L_p(\Omega, \mu_k), F'))$ and $\|v_k\| \|w_k\| \leq \gamma_p(u_k) + \varepsilon$ (cf. the definition of $\Gamma_p, \gamma_p$).

Let $\Omega_0$ be the disjoint sum of $\Omega_1$ and $\Omega_2$ and let $u_1 = \frac{1}{2}(u_1 + u_2)$.

We define $v_0 \in L(E, L_p(\Omega_0, \mu_0))$ and $w_0 \in L(L_p(\Omega_0, \mu_0), F')$ as follows $v_0(x)$ is a function on $\Omega_0$ which coincides with $v_1(x)$ on $\Omega_1$ and with $v_2(x)$ on $\Omega_2$, $w_0(r) = w_1(r_1) + w_2(r_2)$, where $r_1 = r|\Omega_1$ and $r_2 = r|\Omega_2$.

Simple computations show that $i(u_1 + u_2) = v_0 v_0$ and

$$\|v_0\| \leq \frac{1}{2} \|v_1\|^p + \frac{1}{2} \|v_2\|^p, \frac{1}{p}$$

$$\|w_0\| \leq (2^{\frac{q}{p}} \|v_1\|^q + 2^{\frac{q}{p}} \|v_2\|^q, \frac{1}{q}$$
Applying 2.1 we obtain
\[ ||v_0|| ||w_0|| \leq p^{-1}||v_0||^p + q^{-1}||w_0||^q.\]
Hence and by 2.2, 2.3
\[ ||v_0|| ||w_0|| \leq \frac{1}{2}||v_1||^{p-1} + \frac{1}{2}||v_2||^{q-1} + \frac{1}{2}||v_0||^p + \frac{1}{2}||w_0||^q.\]
But we can replace \( v_1 \) by \( t_1 v_1 \) and \( w_1 \) by \( t_1^{-1} w_1 \) and the same with \( v_2 \) and \( w_2 \).
Taking the infimum with respect to \( t_1, t_2 \) the right side of the above inequality is equal to \( ||v_1|| ||w_1|| + ||v_2|| ||w_2|| \).
This proves that \( u_1 + u_2 \in \tau_p(E,F) \) and \( \gamma_p(u_1 + u_2) \leq \gamma_p(u_1) + \gamma_p(u_2) \).
If \( u \in \tau_p(E,F) \) then \( tu \) also and \( \gamma_p(tu) = |t| \gamma_p(u) \). Thus \( \tau_p(E,F) \) is a linear space and \( \gamma_p \) a norm on it. Properties 1., 2., 3. are obvious.
The maximality of \( |\tau_p, \gamma_p| \) may be obtained by the methods from the theory of ultraproducts of Banach spaces, developed by J. Krivine and D. Dacunha-Castelle, cf. [1].

**Proposition 2.** Let \( 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1 \). Then \( u \in \tau_p^*(E,F) \) if and only if there exist Banach space \( G \) and operators \( v \in \Pi_q(E,G) \), \( w \in \Pi_p(F', G') \) such that \( u = vw \),

\( \gamma_p(u) = \inf \pi_q(v) \pi_p(w) \), infimum is taken over all such \( G, v \) and \( w \).

**Proof.** Suppose \( u \in \tau_p^*(E,F) \). By the definition for each \( h \in L(E,F), g \in L(F,F') \) and \( 1 \)-identity operator in \( 1^p \) there holds

\[ |\text{trace}(guhi)| \leq \gamma_p(u)||g|| ||h|| \gamma_p(i). \]
Since \( \gamma_p(i) = 1 \) this is equivalent to: for each \( x_1, \ldots, x_n \in E, y_1, \ldots, y_n \in F' \)

\[ \sum_{i=1}^n <u(x_i), y_i> \leq \gamma_p(u) \sup_{x' \in K_1} \left( \sum_{i=1}^n |<x_i, x_i'>| \right)^{\frac{1}{q}} \sup_{y_j K_2} \left( \sum_{i=1}^n |<y_i, y_i'>| \right)^{\frac{1}{p}}, \]
where \( K_1 \) and \( K_2 \) are unite disks in \( E' \) and \( F' \) correspondingly.

Applying 2.1 we get

\[ \sum_{i=1}^n <u(x_i), y_i> \leq \gamma_p(u) \sup_{x' \in K_1, y \in K_2} \left( \sum_{i=1}^n q^{-1}|<x_i, x_i'>| + p^{-1}<y_i, y_i'> \right)^{\frac{1}{p}}. \]

By the theorem on separations of cones in locally convex spaces it is equivalent to the existence of a probability measure \( \mu \) on \( K \) - the cartesian product of \( K_1 \) and \( K_2 \) such that for each \( x \in E \) and \( y' \in F' \)
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\[ |\langle u(x), y' \rangle| \leq \gamma_p^c(u) \left( \int |\langle x, x' \rangle|^{q} \mu(x') \right)^{1/p} + \frac{1}{p} \left( \int |\langle y, y' \rangle|^{p} \mu(y) \right)^{1/p}. \]

Replacing $x$ by $tx$ and $y'$ by $t^{-1}y'$ and taking infimum we have by 2.1

\[ |\langle u(x), y' \rangle| \leq \gamma_p^c(u) \left( \int |\langle x, x' \rangle|^{q} \mu(x') \right)^{1/q} \left( \int |\langle y, y' \rangle|^{p} \mu(y) \right)^{1/p}. \]

Let $v \in L(E, L_q(K, \mu))$ be defined by $v(x)(x'x') = \langle x, x' \rangle$ on $K$,

similarly $w \in L(F', L_p(K, \mu))$ is defined by $w(y')(x'y') = \langle y, y' \rangle$ on $K$.

Let $G$ denote the closure of $v(E)$ in $L_q(K, \mu)$ and $H$ the closure of $w(F')$ in $L_p(K, \mu)$.

By Pietsch theorem 1.5 $v \in \Pi_q(E, G)$, $w \in \Pi_p(F', H)$ and $\pi_q(v)$, $\pi_p(w) \leq 1$.

The inequality 2.1 implies the existence of an operator $z \in L(G, H')$ such that

$\|z\| \leq \gamma_p^c(u)$ and $t_w z = i u$, $i$ being the canonical injection of $F$ into $F'$.

The image of $G$ by $t_w z$ is in $F$, so let $v = t_w z$ be considered as a member of $L(G, F)$. Then

$\pi_p(t_w) \leq \pi_p(w) \|z\| \leq \gamma_p^c(u)$.

Thus, $G$, $v$ and $w$ satisfy the required conditions of Proposition 2, moreover

$\pi_q(v) \pi_p(t_w) \leq \gamma_p^c(u)$. This proves the necessity.

Now assume $u = wv$, where $v \in \Pi_q(E, G)$ and $t_w \in \Pi_p(F', G')$.

Let $X$ and $Y$ be finite dimensional Banach spaces, $h \in L(X, E)$, $g \in L(F, Y)$ and $z \in \Pi_p(Y, X)$. We have to prove

$|\text{trace}(zguh)| \leq \pi_q(v) \pi_p(t_w) \gamma_p(z) \|g\| \|h\|.$

Let $z = z_1 z_2$, $z_1 \in L(L_p, X)$, $z_2 \in L(Y, L_p)$ and $\|z_1\| \|z_2\| \leq \gamma_p(z) + \varepsilon$.

Then $vzh_1 \in \Pi_q(L_p, G)$ and $t(z_2 gw) \in \Pi_p(L'_p, G')$. It was proved by A. Persson [8],

that if $r \in \Pi_p(G'_p, G')$ then $r \in I_p(G'_p, L'_p)$ and $\pi_p(r) \leq \pi_p(t_r)$. Applying this

we obtain that

$z_2 gw \in I_p(G'_p, L'_p)$ and $\pi_p(z_2 gw) \leq \pi_p(t(z_2 gw))$.

Since $|I_p, i_p|$ is the dual ideal of $|\Pi_q, \pi_q|$ we have

$|\text{trace}(z_2 gwvzh)| \leq i_p(z_2 gw) \pi_q(vzh) \pi_p(tw g z_2) \pi_q(vzh)$. Hence
Because \( \|z_1\| \|z_2\| \leq \gamma_p(z) + \varepsilon \) and \( \varepsilon \) is arbitrary small this ends the proof.

**Corollary 1.** \( u \in L(E,F) \) is factorizable through \( L_p \) space (i.e. \( u \in \Gamma_p(E,F) \)) if and only if for each Banach space \( G \) and \( v \in \Pi_q(F,G) \) it is \( t(vu) \in I_q(G',E') \).

**Proof.** Let \( u \in \Gamma_p(E,F) \) and \( v \in \Pi_q(F,G) \). By Proposition 2 if \( t^w \in \Pi_p(E',G') \) then \( wv \in \Pi_p(F,E) \). From this we deduce that \( t(vu) \in \Pi_p(G',E') \) and hence \( t(vu) \in I_q(G',E') \).

Conversely, if \( u \) satisfies the condition of Corollary then \( u \) belongs to the dual ideal of \( \{\Gamma_p, \gamma_p\} \). In view of the maximality of \( \{\Gamma_p, \gamma_p\} \), by 1.2, \( u \) is its member.

**Corollary 2.** Let \( 1 \leq p \leq \infty \). \( E \) is isomorphic with a complemented subspace of \( L_p \) if and only if for each Banach space \( G \) and \( v \in \Pi_q(E,G) \) it is \( t^v \in \Gamma_p(G',E') \).

**Proof.** By Corollary 1 we obtain that the identity operator in \( E \) belongs to \( \Gamma_p(E,E) \). This implies that \( E \) is reflexive and \( E \) isomorphic with a complemented subspace of \( L_p \).

§ 3. Some related ideals.

By \( S_p \) space, resp. \( Q_p \) space, resp. \( SQ_p \) space, we shall mean any Banach space isometric with a subspace of \( L_p \), resp. with a quotient of \( L_p \), resp. with a subspace of a quotient of \( L_p \).

We say that Banach space is of \( S_p \) type, resp. \( Q_p \) type, resp. \( SQ_p \) type, if it is isomorphic with \( S_p \) space, resp. \( Q_p \) space, resp. \( SQ_p \) space.

One can easy verify the following properties

\[ u \in \Gamma_p(E,F) \] if and only if for some \( S_p \) space there exist \( v \in L(E,S_p) \) and \( w \in L(S_p,F) \) such that \( u = vw \). Moreover

\[ \gamma_p(u) = \inf \|v\| \|w\|, \] infimum is taken over all such \( S_p \) spaces, \( v \) and \( w \).
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$|\Gamma_p^L, \gamma_p^L|$ is denoted by $|\Gamma_p^L, \sigma_p^L|$.

3.2. $u \in \Gamma_p^L(E,F)$ if and only if for some $Q_p$ space there exist $v \in L(E, Q_p)$ and $w \in L(Q_p, F''_p)$ such that $iu = wv$. Moreover

$\gamma_p^L(u) = \inf \|v\| \|w\|$, infimum is taken over all such $Q_p$ spaces, $v$ and $w$.

The ideal $|\Gamma_p^L, \gamma_p^L|$ is denoted by $|\Theta_p^L, \tau_p^L|$.

3.3. $u \in \Gamma_p^L(E,F)$ if and only if for some $SQ_p$ space there exist $v \in L(E, SQ_p)$ and $w \in L(SQ_p, F)$ such that $u = wv$. Moreover

$\gamma_p^L(u) = \inf \|v\| \|w\|$, inf is taken over all such $SQ_p$ spaces, $v$ and $w$.

The ideal $|\Gamma_p^L, \gamma_p^L|$ is denoted by $|\Xi_p^L, \sigma_p^L|$.

Taking into account the properties 1.3 - 1.6 and Proposition 2 we get

Proposition 3. $u \in \Gamma_p^H(E,F)$ if and only if there exist Banach space $G$ and operators $v \in I_p(E,G)$ and $w \in \Pi_p(F'_p, G')$ such that $iu = wv$, $i is the canonical injection of $F$ in $F''$. $\gamma_p^H(u) = \inf \|v\| \|w\|$, infimum is taken over all such $G$, $v$ and $w$.

Similar arguments to those used in the proofs of Corollaries 1,2 give

Corollary 3. $u \in \Gamma_p^H(E,F)$, i.e. $u$ is factorizable through $S_p$ space, if and only if for each Banach space $G$ and $v \in I_q(F,G)$ it is $t(vu) \in I_q(G', E')$.

Corollary 4. Let $1 \leq p \leq \infty$. $E$ is of $S_p$ type if and only if for each Banach space $G$ and operator $v \in I_q(E,G)$ it is $t v \in I_q(G', E')$.

The dual results to these are the following

Proposition 4. $u \in \Theta_p^H(E,F)$ if and only if there exist Banach space $G$ and operators $v \in \Pi_p(E,G)$ and $w \in I_p(F', G')$ such that $u = wv$,$\gamma_p^H(u) = \inf \|v\| \|w\|$, infimum is taken over all such $G$, $v$ and $w$.

Corollary 5. $u \in \Theta_p^H(E,F)$, i.e. $u$ is factorizable through $Q_p$ space, if and only if for each Banach space $G$ and $v \in \Pi_q(F,G)$ it is $t(vu) \in \Pi_q(G', E')$.
Corollary 6. Let \( 1 \leq p \leq \infty \). \( E \) is of \( Q_p \) type if and only if for each Banach space \( G \) and operator \( v \in \Pi_p(E,G) \) it is \( t^v \in \Pi_q(G',E') \).

Now, combining the above results and again the properties 1.3 - 1.6, we arrive at

Proposition 5. \( u \in \mathcal{P}^2_p(E,F) \) if and only if there exist Banach space \( G \) and operators \( v \in I_q(E,G) \) and \( t^w \in I_p(F',G') \) such that \( uv = wv \) is the canonical injection of \( F \) into \( F' \).

\[ s^p_p(u) = \inf_{q} t_{p}(q(v) t_{p}(w)), \text{ infimum is taken over all such } G, v \text{ and } w. \]

Corollary 7. \( u \in \mathcal{F}_p(E,F) \), i.e. \( u \) is factorizable through \( SQ_p \) space, if and only if for each Banach space \( G \) and \( v \in I_q(F,G) \) it is \( t^v \in \Pi_q(G',E') \).

Corollary 8. \( E \) is of \( SQ_p \) type if and only if for each Banach space \( G \) and an operator \( v \in I_q(E,G) \) it is \( t^v \in \Pi_q(G',E') \).


The following result is an answer to Problem 6 of [7]

Theorem 1. Let \( 1 \leq s \leq p \leq r \leq \infty \) and let \( u \in L(L_s,L_r) \), then \( u \) is factorizable through \( L_p \) space.

Proof. By Corollary 2 it is enough to prove that \( t^v \in I_q(G',L'_r) \) whenever \( v \in \Pi_q(L_s,G) \). If \( v \in \Pi_q(L_s,G) \) then \( v \in \Pi_q(L_s,G') \), because \( q \leq s' \), where \( s' \) is defined by the equality \( \frac{1}{s'} + \frac{1}{s} = 1 \). By A. Persson theorem \( t^v \in I_{s'}(G',L'_s) \) and hence \( t^v \in I_{s'}(G',L'_r) \). But for \( s,p < r \leq 2 \), \( I_{s'}(F,L'_r) \) is equal to \( I_{s'}(F,L'_r) \) for each Banach space \( F \).

This is obtained from the dual equality \( \Pi_q(L'_r,F) = \Pi_q(L'_s,F) \) for \( s,p < r \leq 2 \), which is an easy consequence of Theorem 4 of [5], also cf. [10].

This proves the theorem in the case of \( s,p < r \leq 2 \). The case \( 2 \leq s,p \leq r \) is obtained by considering the adjoint operator \( t^u \). The remaining case may be also derived from Corollary 2. Since this case was proved by J. Lindenstrauss and A. Pelczynski we omit it, cf. [7].

If \((\Omega, \mathcal{M}, \mu)\) is a measure space and \( E \) is Banach space then by \( L_p(E, \Omega, \mu) \), briefly \( L_p(E) \), we denote Banach space of all measurable vector valued in \( E \) functions on \( \Omega \) which are strongly \( p \)-integrable.
Theorem 2. \( E \) is of \( SQ_p \) type if and only if for each operator \( u \in L_p(L_p) \) there corresponds an operator \( U \in L_p(L(E), L_p(E)) \) such that
\[
<U(f), x'> = u(<f, x'>) \quad \text{for each } x' \in E' \text{ and } f \in L_p(E).
\]

Proof. Let us observe that Theorem holds for \( E = L_p \) and that if it holds for any Banach space then for its subspaces and quotients also. These two observations prove the necessity, since \( SQ_p \) space is a subspace of a quotient of \( L_p \) space.

Let \( p \neq 1, \infty \). By Corollary 8 it is enough to prove that if \( G \) is Banach space and \( v \in I_q((E,G)) \) then \( ^t v \in I_q(G', E') \). By Theorem 1 of [5] \( E' \)-separable \( ^t v \in I_q(G', E') \) if and only if for each \( w \in L(G, L_q) \) the operator \( wv \) is \( q \)-decomposable, cf. [5]. Let \( iv = v_2jv_1 \), where \( v_1 \in L(E, L_\infty) \), \( v_2 \in L(L_q, G') \) and \( j \) is the canonical injection of \( L_\infty \) into \( L_q \), be a factorization of \( q \)-integral operator, cf. § 1. Let \( w \in L(G, L_q) \) and let us denote by \( \bar{w} \) the canonical extension of \( w \) to an element of \( L(G', L_q) \). The operator \( jv_1 \) may be represented in the form \( <\cdot, f'> \) for some fixed \( f' \in L_q(E') \), i.e. \( jv_1(x) = <x, f'> \). Now, let \( U \in L_p(E), L_p(E) \) denote the operator corresponding to the operator \( ^t (\bar{w}v_2) \in L(L_p(L_p)) \), according to the assumption of Theorem. Then \( ^t U \in L(L(E'), L_q(E')) \) and it is seen that \( wv = \bar{w}v_2jv_1 \) is represented by \( <\cdot, ^t U(f')> \) and this denotes that \( wv \) is \( q \)-decomposable operator. This ends the proof. for \( p \neq 1, \infty \).

The case of \( p = 1, \infty \) is much more simpler, and we omit it. Let us observe that in this case each Banach space is of \( SQ_p \) type.

The case when \( E' \) is not separable follows from the fact that if each adjoint separable quotient of \( E \) is of \( SQ_p \) type then \( E \) is of \( SQ_p \) type.

Remark 1. All the propositions and corollaries of § 3 remain true if we replace everywhere in their formulations "Banach space 6" by "\( L_q \) space", resp. by "\( L_\infty \) space". We do not know if it is true with Proposition 2, cf. Problem 1.

If we replace "Banach space 6" by "\( L_q \) space" in Corollary 4 then it becomes a characterization of subspaces of \( L_q \), given independently by J. Holub, cf. [4].

Remark 2. In this paper we started with the ideal \( |\Gamma_p, \gamma_p| \) and then using the transformations of ideals defined in §1 some related ideals were introduced, cf. §3. It is possible to give a full list of ideals which may be obtained in this way. There is only finite number of them. In the case of \( p = 1,2, \infty \) it was done by A. Grothendieck, cf. [3].

Remark 3. Another version of Theorem 2 is the following
Theorem 2'. \( E \) is of SQ\(_p\) type if and only if there exists a constant \( M \) such that for each matrix \( (a_{i,j}) \) defining an operator \( u \in L(l_p, l_p) \) and each sequence \( (x_i) \) of elements from \( E \) there holds
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}^2 x_i^2 \leq M \| u \| \sum_{i=1}^{\infty} x_i^2.
\]

Remark 4. Theoreme 2' is especially interesting in the case of \( p = 2 \). Because spaces \( S_2 \), \( Q_2 \) and \( SQ_2 \) are Hilbert spaces we obtain a characterization of Banach spaces isomorphic with Hilbert space.

For \( p = 2 \) Corollary 4 coincides with a theorem proved by J. Cohen [2] and S. Kwapien [6].

Problem 1. Let \( 1 < p < \infty \). Is it true that Banach space of \( S_p \) type as well as of \( Q_p \) type is isomorphic with a complemented subspace of \( L_p \)?

Problem 2. Is the space \( L_2(L_p) \) of \( SQ \) type for \( s < r < 2 \) or \( 2 < r < s \)?

Problem 3. Let \( 1 < p < \infty \), and let \( u \in \Gamma_p(E,F) \), i.e. \( u = wv \) where \( v \in L(E,L_p) \), \( w \in L(L_p,F') \) and \( i \) is the canonical injection of \( F \) into \( F'' \).

Can \( u \) be represented in the form \( u = w' v' \), where \( v' \in L(E,L_p) \) and \( w' \in L(L_p,F) \)?

REFERENCES


Operators factorizable through $L_p$-spaces


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