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COMPLEX ANALYTIC GEOMETRY OF COMPLEX PARALLELIZABLE MANIFOLDS

Jörg Winkelmann

Abstract. — We investigate complex parallelizable manifolds, *i.e.*, complex manifolds arising as quotients of complex Lie groups by discrete subgroups. Special emphasis is put on quotients by discrete subgroups which are cocompact or at least of finite covolume.

These quotient manifolds are studied from a complex-analytic point of view. Topics considered include submanifolds, vector bundles, cohomology, deformations, maps and functions. Furthermore arithmeticity results for compact complex nilmanifolds are deduced.

An exposition of basic results on lattices in complex Lie groups is also included, in order to improve accessibility.

***Résumé* (Géométrie analytique complexe et variétés complexes parallélisables)**

On étudie les variétés complexes parallélisables, c'est-à-dire les variétés quotients des groupes de Lie complexes par des sous-groupes discrets. On s'intéresse tout particulièrement aux quotients par des sous-groupes discrets cocompacts ou de covolume fini.

Ces variétés quotients sont étudiées du point de vue de la géométrie analytique complexe. On traite notamment les sujets suivants : les sous-variétés, les fibrés vectoriels, la cohomologie, les déformations, les applications et les fonctions. De plus, on en déduit des résultats d'arithméticité pour des nil-variétés complexes compactes.

Pour faciliter la lecture du texte, on a inclus un exposé de résultats de base sur les réseaux dans les groupes de Lie complexes.

Contents

PREFACE

This book is concerned with quotients of complex Lie groups by discrete subgroups. Of particular interest are those quotients where the discrete subgroup is large in a certain sense, e.g. cocompact or of finite covolume. We are interested in the complex-analytic properties of these quotient manifolds. Typical questions include those on the existence of meromorphic, holomorphic or plurisubharmonic functions, existence and structure of complex subspaces and vector bundles, the rigidity of the complex structure and the nature of certain cohomology groups. Arithmeticity results for complex nilmanifolds are derived.

The core material for this book comes from the "Habilitationsschrift" of the author ([**153**]). However, there are a number of changes. We provide a more complete exposition of the basic techniques used in this area and include proofs for auxiliary results wherever possible without too much of an effort. In the Habilitationsschrift mainly only the quotients by *cocompact* discrete subgroups were considered. Although substantial additional work was required, most results of the Habilitationsschrift could be proved for lattices, *i.e.*, discrete subgroups with finite covolume. These have been included in the present work.

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CHAPTER 1

INTRODUCTION

1.1. Summary

This book is organized in the following way. In the first chapter we introduce some basic notions, provide some first examples of complex parallelizable manifolds and collect some general results on Haar measures and on subgroups of linear groups.

In the second chapter we discuss arithmetic groups. We cite arithmeticity and superrigidity results and a criterion for the cocompactness of arithmetic groups. Furthermore, some examples of arithmetic lattices are presented.

In the third chapter a number of foundational results on lattices in complex Lie groups are presented. Particular emphasis is given to density results and results implying that certain orbits are closed. Moreover, the Albanese torus and the algebraic reduction is discussed and it is shown that compact complex tori are the only quotients of complex Lie groups by lattices which admit a Kähler metric.

In the fourth chapter, complex analytic subspaces of quotients of complex Lie groups by lattices are studied. Normally, there are no hypersurfaces, but there always exists closed analytic subspaces of higher codimension if the group is non-commutative. We discuss the Kodaira-dimension of subspaces and show that in a certain way it closely reflects the behaviour of subspaces in tori.

In the fifth chapter, holomorphic mappings and automorphism groups of complex parallelizable manifolds are investigated. Often holomorphic mappings are automatically equivariant. This permits a complete description of self-maps in general and the automorphism group in particular.

Chapter 6 is concerned with basic facts on homogeneous vector bundles and conditions under which a homogeneous vector bundle must be flat.

In Chapter 7 we deduce classification results for flat vector bundles over parallelizable manifolds. In particular we introduce the notion of *essentially antiholomorphic* representations and prove that every flat vector bundle over a quotient of a semisimple complex Lie group without rank-1-factor by a lattice is induced by an essentially antiholomorphic representation. Furthermore, the topological structure of homogeneous vector bundles is investigated and it is deduced that every positive-dimensional compact complex manifold admits a non-trivial holomorphic vector bundle.

In Chapter 8 we study deformations of the complex structure of compact complex parallelizable manifolds and investigate the cohomology group $H^1(\cdot, \mathcal{O})$. Forthermore we use the Serre-construction to prove the existence of non-homogeneous vector bundles over certain quotients $SL_2(\mathbb{C})/\Gamma$.

Chapter 9 is devoted to the structure theory of complex nilmanifolds. We prove that they are always built from complex tori admitting a certain kind of complex multiplication – a result which can be regarded as a kind of arithmeticity result for complex nilmanifolds.

In the final chapter we discuss criteria for determining which complex linear algebraic groups satisfy the following statement: *If Γ is a Zariski dense discrete subgroup of G, then every Γ-invariant holomorphic function on G is constant.*

As an appendix, we provide a diagram displaying the dependency relations between different density properties for discrete subgroups of complex Lie groups.

Each chapter of this book starts with a survey of the results obtained in that chapter. Of course, these survey are more detailed than the present "tour d'horizon".

1.2. General references

General knowledge about complex manifolds, Lie groups and linear algebraic groups is presupposed.

As general references we suggest

- [**46, 50, 71**] for complex analytic spaces,
- [**48, 53**] for algebraic geometry,
- [**23, 20, 64, 115**] for linear algebraic groups,
- [**74**] for Lie groups,
- [**19, 65**] for arithmetic groups,
- [**92, 118, 123, 163**] for discrete subgroups in Lie groups,
- [**2, 61, 59, 77**] for complex transformation groups.
- [**24, 100**] for locally compact topological groups
- [**66**] for fibre bundles and
- [**111**] for number theory.

The note [**155**] contains a short survey of many of the results treated in this book. A number of results of the author described here have already been published elsewhere ([**152, 153, 154, 156, 157, 158, 159, 160, 161**]).

1.3. Parallelizable Manifolds

We will now introduce the notion of parallelizability.

DEFINITION 1.3.1. — A connected complex manifold X is called *parallelizable* if the (holomorphic) tangent bundle is holomorphically trivial.

It is called *group-theoretically parallelizable* if X is biholomorphic to a quotient G/Γ where G denotes a connected complex Lie group and Γ a discrete subgroup.

Using fundamental vector fields induced by the G-left action on G/Γ, it is evident that a group-theoretically parallelizable complex manifold is parallelizable. Conversely, every *compact* paralllelizable complex manifold is group-theoretically parallelizable.

THEOREM 1.3.2 (Wang [**149**]). — *Let X be a connected compact parallelizable complex manifold and $\mathrm{Aut}_{\mathcal{O}}(X)$ the group of holomorphic automorphisms of X equipped with the compact-open-topology. Let G denote the connected component of id_X in $\mathrm{Aut}_{\mathcal{O}}(X)$*

Then $X \simeq G/\Gamma$ where Γ is a discrete subgroup of G. In particular X is group-theoretically parallelizable.

Proof. — Since X is compact, every holomorphic function on X is constant. Hence the triviality of the tangent bundle T_X implies that $\Gamma(X, T_X)$ is a complex vectorspace of the same dimension as X. Recall that a vector field with compact support on a manifold is always integrable to a one-parameter group of automorphisms. It follows that $\Gamma(X, T_X)$ is the Lie algebra of a complex Lie group G acting on X. Moreover, since $\mathrm{Aut}_{\mathcal{O}}(X)$ is a complex Lie group ([**15**]), it follows that this Lie group G is the connected component of the whole automorphism group of X. The tangent space of an orbit of G at a point $p \in X$ is the vector subspace of T_pX spanned by the vector fields induced by the G-action. Hence every point of X is contained in an open G-orbit. Since X is connected, it follows that G acts transitively on X. Hence $X \simeq G/\Gamma$ (with Γ discrete, because $\dim(X) = \dim(G)$). □

If X is a compact complex manifold on which a solvable (resp. nilpotent) complex Lie group acts transitively, then X is called a (compact) complex *solvmanifold* (resp. *nilmanifold*). Every compact complex solvmanifold is parallelizable [**10**].

1.4. First examples

Here we present some elementary examples of compact complex parallelizable manifolds, *i.e.*, quotients of complex Lie groups by discrete cocompact subgroups. The simplest examples are naturally compact complex tori, *i.e.*, quotients $\mathbb{C}^N/\Pi$ of complex vector spaces by lattices. These are the only Kähler compact complex parallelizable manifolds ([**149**], see also thm. 3.14.1).

Another well-known example is the Iwasawa-manifold, which is obtained as $X = G_{\mathbb{C}}/G_{\mathbb{Z}\oplus i\mathbb{Z}}$ with

$$G_A = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in A \right\}$$

The Iwasawa manifold is a popular example of a non-Kähler compact complex manifold.

Note that $\mathbb{Z} \oplus i\mathbb{Z}$ is just the ring of algebraic integers in the number field $k = \mathbb{Q}[i]$. This suggests the following generalization:

Let k be a imaginary quadratic extension field of $\mathbb{Q}$, *i.e.*, $k = \mathbb{Q}[\sqrt{q}]$ for some negative rational number $q \in \mathbb{Q}^-$. Fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$ and let $A = \sigma(\mathcal{O}_k)$, where $\mathcal{O}_k$ denotes the ring of algebraic integers in k. Then $G_{\mathbb{C}}/G_A$ is a compact complex parallelizable manifold.

EXAMPLE 1.4.1 (Otte-Potters [**117**]). — Let k be a totally real number field of degree d, $\mathcal{O}_k$ its ring of algebraic integers and $\sigma_1, \dots, \sigma_d : k \to \mathbb{R} \subset \mathbb{C}$ be the d distinct embeddings of k into $\mathbb{R}$. Let $H = \mathbb{C}^* \ltimes_\rho \mathbb{C}$ be the semi-direct product given by $\rho : \mathbb{C}^* \simeq \mathrm{Aut}(\mathbb{C}, +)$, $G_1 = H^d$ and G be the subgroup of those $((\lambda_1, x_1), \dots, (\lambda_d, x_d))$ for which $\Pi_{i=1}^d \lambda_i = 1$. Now we obtain a discrete cocompact subgroup of G in the following way. We embedd $\mathcal{O}_k^* \ltimes (\mathcal{O}_k \oplus i\mathcal{O}_k)$ into H^d via $(\sigma_1, \dots, \sigma_d)$. The theorem of Dirichlet ensures that $\mathrm{rank}_{\mathbb{Z}} \mathcal{O}_k^* = d-1$, ensuring cocompactness of $\Gamma = \mathcal{O}_k^* \ltimes (\mathcal{O}_k \oplus i\mathcal{O}_k)$ in G.

EXAMPLE 1.4.2 (Nakamura [**110**]). — Let $A \in SL_2(\mathbb{Z})$ with $\mathrm{Trace}(A) \notin \{-2, 2\}$. Then A has a unique eigenvalue α with $|\alpha| > 1$. Now let E be an elliptic curve (*i.e.*, a one-dimensional torus) and X the quotient of $X_1 = \mathbb{C}^* \times E \times E$ by the $\mathbb{Z}$-action on X_1 generated by

$$(x, s, t) \mapsto (\alpha x, A \cdot \begin{pmatrix} s \\ t \end{pmatrix}).$$

Then X is a compact complex parallelizable manifold.

EXAMPLE 1.4.3. — Let $V = (\mathbb{C}^2, +)$ and Λ be the lattice in V given by $(\mathbb{Z} \oplus i\mathbb{Z})^2$. Let $A \in SL_2(\mathbb{Z}[i])$ with $|\mathrm{Trace}(A)| > 2$, e.g.

$$A = \begin{pmatrix} 4+i & 2-i \\ 1+i & 1 \end{pmatrix}$$

Let Γ be the subgroup of $SL_2(\mathbb{C}) \ltimes V$ generated by $(A, 0)$ and (I, λ) with $\lambda \in \Lambda$ and let G be its Zariski closure in $SL_2(\mathbb{C}) \ltimes V$. Observe that $|\mathrm{Trace}(A)| > 2$ implies that A is diagonalizable with one eigenvalue of absolute value larger than 1. Therefore $\{A^n : n \in \mathbb{Z}\}$ is discrete and contained in a commutative reductive subgroup of G. Compactness of both $\mathbb{C}^*/\mathbb{Z}$ and V/Λ implies that Γ is cocompact in G.

Already in these examples it is apparent that arithmetic methods play an important role in constructing lattices. We will discuss arithmetic groups in more detail in chapter 2.

1.5. Haar measure

A locally compact topological group is a group which is simultaneously a locally compact topological space with countable basis of topology such that the group structure is compatible with the topological structure. Every locally compact topological group G (thus in particular every Lie group) admits a left-invariant and a right-invariant regular Borel measure which are both unique up to multiplication by a positive real scalar, called the left- resp. right-invariant *Haar measure* [**24**]. Let μ_G denote a left-invariant Haar measure and for $g \in G$ let $R_g : G \to G$ denote the map given by $R_g(x) = xg^{-1}$. Let $R_g^* \mu_G$ be the measure given by

$$R_g^* \mu_G(S) = \mu_G(R_g(S)) = \mu_G(Sg^{-1}).$$

Then $R_g^* \mu_G$ is again a left-invariant Haar measure. Hence there exists a continuous group homomorphism $\Delta_G : G \to \mathbb{R}^+$ such that $R_g^* \mu_G = \Delta(g)\mu_G$ for all $g \in G$. This group homomorphism Δ_G, also called *modular function*, measures to which degree a left-invariant Haar measure on G fails to be right-invariant. Thanks to the Riesz representation theorem we may consider the corresponding linear functionals on the respective spaces of continuous functions with compact support instead of these regular Borel measures itself. Then $R_g^* \mu_G = \Delta(g)\mu_G$ translates into

$$\int_G f(xg)d\mu_G(x) = \Delta_G(g) \int_G f(x)d\mu_G(x).$$

Note that $\Delta_G \mu_G$ is a right-invariant Haar measure. Furthermore $\xi^* \mu_G = \Delta_G \cdot \mu_G$ where $\xi : G \to G$ denotes taking the inverse, *i.e.*, $\xi(g) = g^{-1}$. This translates into

$$\int_G \Delta_G(g) f(g) d\mu_G(g) = \int_G f(g^{-1}) d\mu_G(g).$$

For a discrete group Γ the counting measure is the Haar measure. Since the counting measure is evidently left- and right-invariant, $\Delta_\Gamma \equiv 1$ for every discrete group Γ.

For a real Lie group G the Haar measure can be realized by a left-invariant volume form induced by a non-zero element $\omega \in \bigwedge^{\dim G} (\mathcal{L}ie\, G^*)$ and Δ is given by $\Delta(g) = |\det \operatorname{Ad}(g)|$ where $\operatorname{Ad} : G \to GL(\mathcal{L}ie\, G)$ is the adjoint representation.

LEMMA 1.5.1. — *Let G be a locally compact topological group and H a closed subgroup. Then G/H admits a left invariant Borel measure if and only if $\Delta_G|_H = \Delta_H$.*

Proof. — Let ω be a left-invariant Borel measure on G/H. Then a left-invariant measure η on G is given by

$$\int_G f(g) d\eta(g) = \int_{G/H} \left(\int_H f(gh) d\mu_H(h) \right) d\omega(gH).$$

Now $R_h^* \eta = \Delta_H(h)\eta$ for $h \in H$ and $R_h^* \mu_G = \Delta_G(h)\mu_G$ for all $h \in G$. Unicity of the Haar measure implies that $\eta = C\mu_G$ for some constant $C > 0$. It follows that $\Delta_G|_H = \Delta_H$.

Now we have to show the converse. Thus let us assume $\Delta_G|_H = \Delta_H$. Let *pi* denote the natural projection $\pi : G \to G/H$ and let Σ denote the set of all continuous functions ζ on G such that

1. $\zeta(g) \geq 0$ for all $g \in G$,
2. $\operatorname{supp}(\zeta) \cap \pi^{-1}(K)$ is compact for every compact subset $K \subset G/H$.
3. $\int_H \zeta(gh) d\mu_H(h) = 1$ for all $g \in G$.

For every $\zeta \in \Sigma$ a Borel measure η on G/H can be defined by

$$\int_{G/H} f(gH) d\eta(gH) = \int_G f(gH)\zeta(g) d\mu_G(g)$$

We claim that $\Delta_G|_H = \Delta_H$ implies that this measure is independent of the choice of $\zeta \in \Sigma$.

Indeed, let $\zeta, \widetilde{\zeta}, \xi \in \Sigma$, $f_0 \in \mathcal{C}_c(G/H)$ and define $f = f_0 \circ \pi$.

Then $\int_H \zeta(gh) - \widetilde{\zeta}(gh) d\mu_H(h) = 0$ for all $g \in G$ and this implies

$$\begin{aligned}
0 &= \int_G \xi(g) f(g) \left[\int_H \zeta(gh) - \widetilde{\zeta}(gh) d\mu_H(h) \right] d\mu_G(g) \\
&= \int_H \int_G \xi(g) f(g) \left[\zeta(gh) - \widetilde{\zeta}(gh) \right] d\mu_G(g) d\mu_H(h) \\
&= \int_H \Delta_G(h) \int_G \xi(gh^{-1}) \underbrace{f(gh^{-1})}_{=f(g)} \left[\zeta(g) - \widetilde{\zeta}(g) \right] d\mu_G(g) d\mu_H(h) \\
&= \int_G f(g) \left[\zeta(g) - \widetilde{\zeta}(g) \right] \int_H \Delta_H(h) \frac{\Delta_G(h)}{\Delta_H(h)} \xi(gh^{-1}) d\mu_H(h) d\mu_G(g) \\
&= \int_G f(g) \left[\zeta(g) - \widetilde{\zeta}(g) \right] \underbrace{\int_H \xi(gh) d\mu_H(h)}_{=1} d\mu_G(g) \\
&= \int_G f(g)\zeta(g) d\mu_G(g) - \int_G f(g)\widetilde{\zeta}(g) d\mu_G(g)
\end{aligned}$$

Finally left-invariance of the Borel measure η follows from the fact that $\zeta \circ L_g \in \Sigma$ for all $g \in G$, $\zeta \in \Sigma$ and $L_g(x) = g \cdot x$. □

LEMMA 1.5.2. — *Let G be a locally compact topological group, $H \subset I \subset G$ closed subgroups and assume that G/H carries a G-left invariant probability measure μ. Then there is a G-left-invariant probability measure on G/I, too.*

Proof. — Consider the projection $\pi : G/H \to G/I$ and define the desired measure η on G/I by $\eta(S) = \mu(\pi^{-1}(S))$ for $S \subset G/I$. □

LEMMA 1.5.3. — *Let G be a locally compact topological group and Γ a discrete subgroup. Assume that there exists a G-invariant probability measure on G/Γ.*

Then $\Delta_G \equiv 1$.

Proof. — From lemma 1.5.1 we deduce that $\Delta_G|_\Gamma = \Delta_\Gamma \equiv 1$. Thus $\Gamma \subset \ker \Delta_G$. By lemma 1.5.2 above it follows that $G/\ker \Delta_G$ admits a G-invariant probability measure. Since $G/\ker \Delta_G$ embeds into $(\mathbb{R}, +)$, this can happen only if $G = \ker \Delta_G$, *i.e.*, $\Delta_G \equiv 1$. □

LEMMA 1.5.4. — *Let G be a locally compact topological group and Γ a discrete subgroup such that G/Γ is compact. Then G/Γ admits a G-left invariant probability measure.*

Proof. — Note that G is locally compact and that the projection $\pi : G \to G/\Gamma$ is an open map. Hence compactness of G/Γ implies that there exists a compact subset $F_0 \subset G$ with $\pi(F_0) = G/\Gamma$. Choose a measurable subset $F \subset F_0$ such that $\pi|_F : F \to G/\Gamma$ is bijective. Let $\widehat{\mu}_G = \Delta_G \mu_G$. Then $\widehat{\mu}_G$ is a right-invariant Haar measure. Compactness of F_0 implies that all the gF are measurable with $\widehat{\mu}_G(gF) < \infty$. Furthermore one may check easily that $\widehat{\mu}_G(gF) = \Delta_G(g)\widehat{\mu}_G(F)$. On the other hand, given $g \in G$ we may define sets $F_{g,\gamma}$ by $F_{g,\gamma} = F \cap \left(gF\gamma^{-1}\right)$. Since π projects both F and gF bijectively onto G/Γ, it is clear that $F = \dot{\cup}_{\gamma\in\Gamma} F_{g,\gamma}$ and $gF = \dot{\cup}_{\gamma\in\Gamma} F_{g,\gamma} \cdot \gamma$. It follows that

$$\Delta(g)\widehat{\mu}_G(F) = \widehat{\mu}_G(gF) = \sum_{\gamma\in\Gamma} \widehat{\mu}_G(F_{g,\gamma} \cdot \gamma) = \sum_{\gamma\in\Gamma} \widehat{\mu}_G(F_{g,\gamma}) = \widehat{\mu}_G(F).$$

Therefore $\Delta_G(g) = 1$ for all $g \in G$. Due to lemma 1.5.1 it follows that G/Γ admits an invariant Borel measure ω. Finally compactness of G/Γ implies $\omega(G/\Gamma) < \infty$. Thus $C\omega(G/\Gamma) = 1$ for a suitable choosen positive number $C \in \mathbb{R}^+$. □

DEFINITION 1.5.5. — A subgroup Γ in a locally compact topological group G is called a *lattice* if Γ is discrete and there exists a G-invariant probability measure on the quotient G/Γ.

Another name for lattices is *discrete subgroups of finite covolume.*

DEFINITION 1.5.6. — A subgroup Γ in a topological group G is called *cocompact* if the quotient G/Γ is compact.

A cocompact subgroup is necessarily closed. A discrete cocompact subgroup is a lattice by cor. 1.5.4 above.

Sometimes a discrete subgroup Γ of a locally compact group G is called *uniform* if it is cocompact.

1.6. Unimodular groups

DEFINITION 1.6.1. — A locally compact topological group G is called *unimodular* iff $\Delta \equiv 1$, *i.e.*, iff a left-invariant Haar measure is also right-invariant.

Using this terminology lemma 1.5.3 may be reformulated as follows: *If a locally compact topological group G contains a lattice Γ, then G must be unimodular.*

If G is a locally compact topological group such that every continuous group homomorphism from G to $\mathbb{R}$ is trivial, then G must be unimodular. This includes in particular compact topological groups and semisimple Lie groups and every locally compact topological group G with $G = [G, G]$.

Other examples of unimodular groups are nilpotent locally compact groups and discrete groups.

For Lie groups it is easy to see that nilpotency implies unimodularity: If G is a nilpotent Lie group, then $\mathrm{Ad}(g)$ is unipotent for every $g \in G$. Hence $|\det \mathrm{Ad}(g)| = 1$ for all $g \in G$.

On the other hand, using structure theory of semisimple groups, it is easy to see that parabolic subgroups of semisimple Lie groups are never unimodular.

1.7. Subgroups of linear groups

DEFINITION 1.7.1. — A group G is called *torsion-free* if it contains no elements of finite order, *i.e.*, $g^n \neq e$ for all $g \in G \setminus \{e\}$ and $n \in \mathbb{N}$.

Finitely generated linear groups are almost torsion-free.

PROPOSITION 1.7.2 (Selberg). — *Let k be a field of characteristic zero, $n \in \mathbb{N}$ and let Γ be a finitely generated subgroup of $GL(n,k)$.*

Then Γ admits a torsion-free subgroup Γ_0 of finite index.

Proof. — Let $a_1, \dots a_g$ denote a set of generators of Γ and let T denote the set of all torsion elements of $\Gamma \setminus \{e\}$. For each torsion element $t \in T$ we choose a root of unity $\omega_t \neq 1$ which is an eigenvalue of $t \in GL(n,k)$. Now GL_n^g contains a subvariety V defined over $\mathbb{Q}$ such that for every field $K \supset \mathbb{Q}$ there is a one-to-one correspondence between K-rational points in V and group homomorphisms $\rho : \Gamma \to GL_n(K)$ given by $(\rho(a_1), \dots, \rho(a_g)) \sim \rho$. There is a subvariety V_0 corresponding to those group homomorphism ρ such that ω_t is an eigenvalue of $\rho(t)$ for all $t \in T$. This subvariety is defined over $\overline{\mathbb{Q}}$ and non-empty. Hence there is a $\overline{\mathbb{Q}}$-rational point in V_0 and consequently there exists a group homomorphism $\tau : \Gamma \to GL_n(\overline{\mathbb{Q}})$ such that $\ker \tau \cap T = \varnothing$. Since Γ is finitely generated, $\tau(\Gamma)$ is actually contained in $GL_n(K)$ for some finitely generated field $K \subset \overline{\mathbb{Q}}$, *i.e.*, $\tau(\Gamma)$ is contained in $GL_n(K)$ for some number field K. Now for each $t \in T$ the eigenvalue ω_t of $\tau(t)$ is contained in an extension field K_t with $\deg(K_t/K) \leq n$. Recall that for a fixed number d there exist only finitely many roots of unity which are contained in a number field of degree $\leq d$. It follows that there exists a natural number M such that for every $t \in T$ the order of t divides M.

Let R denote the subring of K generated by all the matrix coefficients of all the generators $\tau(a_i)$ and $\tau(a_i^{-1})$ and let p be a prime which neither divides M nor is invertible in R. (Such a choice is possible, because R is a finitely generated ring.) Now consider the ring homomorphism $\psi_0 : R \to R/pR$ and the induced group homomorphism $\psi : GL_n(R) \to GL_n(R/pR)$. Since R is finitely generated as a ring, it follows that R/pR is a finite field. Thus the kernel $\ker(\psi \circ \tau)$ is a subgroup of finite index in Γ. We claim that this subgroup is torsion-free. Indeed, if $g \in \ker(\psi \circ \tau) \setminus \{e\}$, then $\tau(g) = I_n + pA$ for some $A \in M(n, R)$ and the calculation

$$(I_n + pA)^m = I_n + mpA \qquad \text{modulo } p^2$$

shows that $g^m \neq e$ unless p divides m. But p was choosen in such a way that it does not divide the order of any torsion element of Γ. Thus $\Gamma_0 = \ker(\psi \circ \tau)$ is torsion-free. □

COROLLARY 1.7.3. — *Let k be a field of characteristic zero, $n \in \mathbb{N}$ and $\Gamma \subset GL(n,k)$ be a subgroup such that every element of Γ is of finite order.*

Then Γ is locally finite, *i.e., every finitely generated subgroup of Γ is finite.*

DEFINITION 1.7.4. — A group Γ is called *residually finite*, if for every $\gamma \in \Gamma \setminus \{e\}$ there exists a group homomorphism ξ from Γ to a finite group F such that $\xi(\gamma) \neq e$.

PROPOSITION 1.7.5 (Malcev, [**87**]). — *Let k be a field, $n \in \mathbb{N}$ and let Γ be a finitely generated subgroup of $GL_n(k)$.*

Then Γ is residually finite.

Proof. — Let $a_1, \dots, a_g$ be a set of generators of Γ and R denote the subring of k generated by the a_i and a_i^{-1}. Then R is a finitely generated ring. For every prime ideal $p \subset R$ and every number $k \in \mathbb{N}$ the quotient ring R/p^k is a finite set. Moreover $\cap_k p^k = \{0\}$. Now consider the induced group homomorphisms $\xi_n : GL(n,R) \to GL(n, R/p^k)$. All the groups $GL(n, R/p^k)$ are finite and $\cap_k p^k = \{0\}$ implies $\cap_k \ker \xi_k = \{e\}$. Hence $GL(n,R)$ and thereby Γ are residually finite. □

We will also make use of the *Tits-alternative.*

THEOREM 1.7.6 (Tits). — *Let k be a field of characteristic zero, $n \in \mathbb{N}$ and Γ a subgroup of $GL(n,k)$.*

Then either Γ contains a non-commutative free subgroup or Γ contains a solvable subgroup of finite index.

For the proof see [**145**].

COROLLARY 1.7.7 (Zassenhaus). — *Let k be a field of characteristic zero, $n \in \mathbb{N}$ and Γ a subgroup of $GL(n,k)$.*

Then Γ is solvable if and only if every finitely generated subgroup of Γ is solvable.

One of the ingredients of Tits' proof is the subsequent theorem of Schur ([**131**], see also [**35**], p. 582).

THEOREM 1.7.8 (Schur). — *There exists a map $F : \mathbb{N} \to \mathbb{N}$ such that the following is true:*

Let k be a field of characteristic zero, $n \in \mathbb{N}$ and $\Gamma \subset GL(n,k)$ a subgroup such that every element γ of Γ is of finite order (the order possibly depending on γ).

Then there exists an abelian normal subgroup $\Gamma_0 \subset \Gamma$ of finite index such that $\#(\Gamma/\Gamma_0) \leq F(n)$.

This result of Schur contains an earlier theorem of Jordan ([**68**], see also [**16**]) as a special case.

COROLLARY 1.7.9 (Jordan's theorem). — *There exists a function $k : \mathbb{N} \to \mathbb{N}$ such that for every finite subgroup $F \subset GL_n(\mathbb{C})$ there exists a normal abelian subgroup $A \triangleleft F$ such that F/A has at most $k(n)$ elements.*

Instead of $GL(n,\mathbb{C})$ we may consider arbitrary real Lie groups with finitely many connected components.

COROLLARY 1.7.10. — *Let H be a (real) Lie group with finitely many connected components.*

Then there exists a number $k(H)$ such that for every finite subgroup $F \subset H$ there exists an normal abelian subgroup $A \triangleleft F$ such that F/A has at most $k(H)$ elements.

Proof. — Let K be a maximal compact subgroup of H. Then every finite subgroup of H is conjugate to a finite subgroup of K. Since K is compact, there exists a faithful representation $\rho : K \to GL_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Therefore the statement follows from cor. 1.7.9. □

LEMMA 1.7.11. — *Let G be a connected real Lie group, Γ a subgroup such that each element $\gamma \in \Gamma$ is of finite order.*

Then Γ is almost abelian and relatively compact in G.

Proof. — If G is abelian, then $G \simeq \mathbb{R}^k \times (S^1)^n$. In this case $\Gamma \subset (S^1)^n$ and the statement is immediate.

Now let us assume that G may be embedded into a complex linear algebraic group $\widetilde{G}$. Let H denote the (complex-algebraic) Zariski closure of Γ in $\widetilde{G}$. By the above mentioned theorem of Schur (thm. 1.7.8) Γ is almost abelian, hence H^0 is abelian. This completes the proof for this case, since we already discussed the situation where G is abelian.

Finally let us discuss the general case. By the above considerations $\mathrm{Ad}(\Gamma_0)$ is contained in an abelian connected compact subgroup K of $\mathrm{Ad}(G)$ for some subgroup Γ_0 of finite index in Γ. Now $N = (\mathrm{Ad})^{-1}(K)$ is a central extension $1 \to Z \to N \to K \to 1$. (where Z is the center of G). But complete reducibility of the representations of compact groups implies that this sequence splits on the Lie algebra level. Hence N is abelian and we can complete the proof as before. □

LEMMA 1.7.12. — *Let G be a connected complex linear algebraic group and Γ a Zariski dense subgroup.*

Then Γ contains a finitely generated subgroup Γ_0 such that the Zariski closure of Γ_0 contains the commutator group G' of G.

Proof. — Consider all finitely generated subgroups of Γ and their Zariski closure in G. There is one such group Γ_0 for which the dimension of the Zariski closure A is maximal. Clearly A must contain the connected component of the Zariski closure for any finitely generated subgroup of Γ. This implies that A^0 is normalized by Γ. Since Γ is Zariski dense, it follows that A^0 is normal in G. Furthermore maximality implies that the group Γ/A^0 contains no element of infinite order. Hence Γ/A^0 is almost abelian by lemma 1.7.8, which implies that G/A^0 is abelian. Thus $G' \subset A$. □

Caveat: There is no hope for $A = G$, even if Γ is discrete. For instance, let $\Lambda \subset SL(2,\mathbb{C})$ be a discrete free subgroup with infinitely many generators $\lambda_1, \lambda_2, \ldots$ and let Γ be the subgroup of $G = SL(2,\mathbb{C}) \times \mathbb{C}^*$ generated by $\gamma_n = (\lambda_n, e^{2\pi i/n})$ with $n = 1, 2, \ldots$.

1.8. Lattices are finitely generated

Let G be a Lie group. An arbitrary discrete subgroup is not necessarily finitely generated. For instance, the universal covering of $\mathbb{C} \setminus \mathbb{Z}$ is the unit disk and this fact implies that $\pi_1(\mathbb{C} \setminus \mathbb{Z})$ can be embedded into $PSL_2(\mathbb{R}) = \mathrm{Aut}(\Delta)$. However, for lattices the situation is better. If Γ is a discrete cocompact subgroup of a Lie group G, then the quotient G/Γ is a compact manifold. This implies that the fundamental group $\pi_1(X)$ is finitely presentable and in particular finitely generated. From this one easily deduces that Γ must be finitely presentable as well.

For non-cocompact lattices the argumentation is more complicated, however it is nevertheless true that they are finitely generated.

THEOREM 1.8.1 (Raghunathan, see **[123]**, thm. 6.18). — *Let G be a connected (real or complex) Lie group and Γ a lattice in G. Then Γ is finitely generated.*

1.9. Algebraicity of Lie groups

We will need some algebraicity results for Lie groups.

If G is a simply connected complex nilpotent Lie group, the exponential map is a biholomorphic map from $\mathcal{L}ie(G)$ to G. As a vector space $\mathcal{L}ie(G)$ carries naturally a structure as an algebraic variety. Via the exponential map this induces a natural structure of a unipotent algebraic group on G.

For a semisimple complex Lie group S let K be a maximal compact subgroup and consider the K-action on the vector space $\mathcal{O}(G)$ of holomorphic functions which is induced by the K-action on G by left multiplication. Then there is a canonical algebraic structure (independent of the choice of K) on G given in the following way: A holomorphic function f on G is algebraic if and only if the vector space spanned by the K-orbit $K(f)$ through f in $\mathcal{O}(G)$ is finite-dimensional.

In this way, both semisimple and simply connected nilpotent complex Lie groups are intrinsically algebraic.

PROPOSITION 1.9.1. — *Let G be a simply connected complex Lie group with $G = G'$. Then G carries the structure of a linear algebraic group. Moreover, for every complex linear algebraic group H every homomorphism of complex Lie groups $f : G \to H$ is already a morphism of algebraic groups.*

Proof. — Since G is simply connected, it is linear and algebraicity follows from $G = G'$. Let f be a homomorphism of complex Lie groups to some complex linear algebraic group H. Let $I = f(G)$. Then $I = I'$ and consequently I is algebraic and moreover the nilradical of I is unipotent. Now every homomorphism of complex Lie groups between unipotent groups is algebraic. Furthermore every homomorphism of complex Lie groups between semisimple groups is algebraic. Using Levi-decomposition this implies that f is algebraic. □

Recall that every continuous group homomorphism between Lie groups is real-analytic. Using this fact we can improve the algebraicity result via restriction of scalars.

PROPOSITION 1.9.2. — *Let G, H be complex linear algebraic groups and assume $G = G'$.*

Then every continuous group homomorphism $\rho : G \to H$ is already real-algebraic.

Using similar methods one may also prove an algebraicity result for commutator groups. First we develop an auxiliary lemma.

LEMMA 1.9.3. — *Let G be a connected algebraic group, $\Gamma \subset G$ a Zariski dense subgroup. Then the commutator group Γ' is Zariski dense in G'.*

Proof. — For $n \in \mathbb{N}$ consider the morphism of algebraic varieties $\zeta_n : (G \times G)^n \to G$ given by

$$\zeta_n : ((a_1, b_1), \dots, (a_n, b_n)) \mapsto (a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_n b_n a_n^{-1} b_n^{-1})$$

Let S_n denote the Zariski closure of the image. Then S_n is an increasing sequence of irreducible algebraic subvarieties of G'. This sequence necessarily becomes stationary. Therefore $S_N = G'$ for N sufficiently large. Finally observe that $\zeta_N((\Gamma \times \Gamma)^n)$ is Zariski dense in S_n and $\zeta_N((\Gamma \times \Gamma)^n) \subset \Gamma'$. □

PROPOSITION 1.9.4. — *Let G be a connected complex Lie subgroup of $GL_n(\mathbb{C})$ and $\bar{G}$ its Zariski closure. Then $G' = (\bar{G})'$.*

In particular G' is an algebraic subgroup.

Proof. — Let $G = S \cdot R$ be a Levi-Malcev decomposition, *i.e.*, S is a maximal connected semisimple Lie subgroup of G and R the radical of G (*i.e.*, the maximal connected normal solvable Lie subgroup). Then S is already algebraic. It follows that $\bar{G} = S \cdot \bar{R}$ where $\bar{R}$ denotes the Zariski closure of R. By standard structure theory of linear algebraic groups it follows that $(\bar{G})' = S \ltimes U$ for some unipotent group U. Since G' is a connected complex Lie subgroup of $(\bar{G})'$ and $S \subset G'$, it follows that $G' = S \ltimes V$ for some connected Lie subgroup V of U. But connected Lie subgroups of unipotent groups are algebraic. Hence G' is already an algebraic group. Since G' is Zariski dense in $(\bar{G})'$ by the above lemma 1.9.3, it follows that $G' = (\bar{G})'$. □

1.10. Tits-fibration

Let G be a connected complex Lie group and H a closed complex Lie subgroup. Let H^0 be the connected component of H and $N = N_G(H^0)$ its normalizer, *i.e.*

$$N_G(H^0) = \left\{g \in G : gH^0g^{-1} = H^0\right\}.$$

The projection map $\pi : G/H \to G/N$ is called the *Tits-fibration* of G/H. Its properties may be summarized as follows.

THEOREM 1.10.1 (Tits [**143**]). — *Let G be a connected complex Lie group, H a closed complex Lie subgroup, $N = N_G(H^0)$ and $\pi : G/H \to G/N$ the natural projection.*

Then the base manifold G/N admits a G-equivariant embedding into a projective space $\mathbb{P}_M(\mathbb{C})$ and the fiber N/H is a parallelizable complex manifold.

If G/H is compact, then π is universal in the sense that for every connected complex Lie group I acting transitively on $X = G/H$, every $m \in \mathbb{N}$ and every I-equivariant holomorphic map $f : X \to \mathbb{P}_m(\mathbb{C})$ there exists a holomorphic map $F : G/N \to \mathbb{P}_m(\mathbb{C})$ such that $f = F \circ \pi$.

Proof. — The base manifold G/N can be embedded into a Grassmann manifold of vector subspaces of $\mathcal{L}ie(G)$ in the following way: The point gN is mapped to the point in the Grassmann manifold corresponding to the vector subspace $\mathrm{Ad}(g)\,(\mathcal{L}ie(H))$ of $\mathcal{L}ie(G)$. By use of Plücker coordinates this Grassmann manifold can be embedded equivariantly into some projective space $\mathbb{P}_M(\mathbb{C})$.

The fiber N/H is isomorphic to the quotient of the complex Lie group N/H^0 by the discrete group H/H^0 and therefore parallelizable.

Finally, we have to show the universality property. If G/H is compact, then both $Y = G/N$ and N/H are compact. Being a compact orbit of a subgroup of some $GL_M(\mathbb{C})$ acting linearly on some $\mathbb{P}_M(\mathbb{C})$, it is clear that Y is isomorphic to a quotient of a semisimple Lie group by a parabolic subgroup and therefore simply connected. Thus N/H must be connected. It follows that every holomorphic function on N/H is constant. For every fiber $F = \pi^{-1}(y)$ of $\pi : G/H \to G/N$ the map π induces a morphism $D\pi : TX|_F \to T_y(G/N)$. Considering this map, $\mathcal{O}(N/H) = \mathbb{C}$ implies that if a holomorphic vector field on X is tangent to F in one point, it must be tangent to F everywhere. Therefore every complex Lie group I acting transitively on X contains a complex Lie subgroup J stabilizing F and acting transitively on F. Parallelizability of N/H implies that the J-action on N/H is induced by a group homomorphism $\rho : J \to N/H^0$. Using this fact, the desired universality property is a direct consequence of lemma 3.4.3. □

1.11. Linearity of simply connected complex Lie groups

At many points in our work the following fact is useful. It is a consequence of Th. XVIII.4.7 in **[57]**.

PROPOSITION 1.11.1. — *Let G be a simply connected complex Lie group. Then there exists an injective homomorphism of complex Lie groups $i : G \to GL_n(\mathbb{C})$ for some $n \in \mathbb{N}$.*

Once one obtained an embedding into $GL_n(\mathbb{C})$ one can construct a better one.

LEMMA 1.11.2. — *Let G be a simply connected complex Lie group and $i : G \to GL(N, \mathbb{C})$ be an injective morphism of complex Lie groups.*

Then there exists a number $M > 0$ and an injective morphism of complex Lie groups $j : G \to GL(M, \mathbb{C})$ such that $j(G)$ is closed in $GL(M, \mathbb{C})$ and $j(N)$ is unipotent where N denotes the nilradical of G.

Proof. — Note that $G/G' \simeq (\mathbb{C}^d, +)$ for some $d \geq 0$ since G is simply connected. Let $\pi : G \to G/G'$ denote the natural projection and choose a morphism of complex Lie groups $\nu : G/G' \to GL(2d, \mathbb{C})$ which embedds G/G' as a closed unipotent subgroup. Now define j by $j = (i, \nu\circ\pi)$ and recall that $i(G')$ is an algebraic subgroup of $GL(N, \mathbb{C})$ (prop. 1.9.4). It follows that j has the desired properties. □

This has the following consequence.

COROLLARY 1.11.3. — *Let G be a simply connected complex Lie group.*

Then G is Stein as a complex manifold.

This is a special case of the results of Matsushima and Morimoto (**[96]**) who studied the question which complex Lie groups are Stein as complex manifolds.

1.12. Fundamental groups

We will now apply the above mentioned linearity result to parallizable manifolds. We start by the following observations.

REMARK 1.12.1. — Let G be a connected (complex) Lie group. The universal covering $\widetilde{G}$ of G admits a natural structure of a (complex) Lie group such that the projection $\pi : \widetilde{G} \to G$ is a homomorphism of (complex) Lie groups. Every action of G on some space X can be lifted to a $\widetilde{G}$-action on X (with $\ker \pi$ being contained in the ineffectivity). If Γ is a discrete subgroup of G, then $G/\Gamma \simeq \widetilde{G}/\pi^{-1}(\Gamma)$ and $X = G/\Gamma$ is a manifold with $\widetilde{G}$ as universal covering and $\pi^{-1}(\Gamma)$ as fundamental group.

PROPOSITION 1.12.2. — *Let X be a compact complex parallelizable manifold and $\widetilde{X}$ its universal covering.*

Then $\widetilde{X}$ is a Stein manifold.

Proof. — The complex manifold X may be realized as a quotient of a simply connected complex Lie group G by a discrete subgroup Γ (thm. 1.3.2) and a simply connected complex Lie group is necessarily a Stein manifold (cor. 1.11.3). □

LEMMA 1.12.3. — *Let G be a connected complex Lie group, Γ a discrete subgroup and $X = G/\Gamma$.*

Then the fundamental group $\pi_1(X)$ can be embedded into some $GL_n(\mathbb{C})$.

Proof. — This follows with the help of prop. 1.11.1. □

COROLLARY 1.12.4. — *Let G be a connected complex Lie group, Γ a finitely generated discrete subgroup and $X = G/\Gamma$.*

Then the fundamental group $\pi_1(X)$ is residually finite.

Proof. — Combine the preceding corollary with prop. 1.7.5. □

COROLLARY 1.12.5. — *Let G be a simply connected complex Lie group and Γ a lattice in G.*

Then $\Gamma \simeq \pi_1(G/\Gamma)$ is residually finite.

Proof. — This follows from thm. 1.8.1 in combination with prop. 1.7.2. □

CHAPTER 2

ARITHMETIC GROUPS

2.1. Survey

The single most important method of constructing lattices in Lie groups is to use arithmetic groups. In this chapter we first discuss lattices in nilpotent Lie groups, where arithmeticity statements take a particularly simple and complete form. Later we discuss arithmetic groups in arbitrary Lie groups. We quote the main arithmeticity results for lattices, in particular Margulis' superrigidity theorem. We also cite a criterion determining whether a given arithmetic group is a lattice or cocompact. These criteria are used to deduce some partial answers to the following question *"Which complex Lie groups do contain discrete cocompact subgroups ?"*

Most of the material in this chapter is well-known and covered in the standard literature on arithmetic groups (see e.g. [**19, 65, 92, 123**]).

2.2. Lattices in nilpotent Lie groups

For lattices in real simply connected nilpotent Lie groups there is a very complete theory going back to the work of Malcev [**89**]. There are many expositions of this theory, see e.g. [**123**].

THEOREM 2.2.1 (Malcev). — *Let G be a simply connected real nilpotent Lie group and Γ a discrete subgroup. Let V be the $\mathbb{Q}$-vector subspace of $\mathcal{L}ie(G)$ spanned by* $\exp^{-1}(\Gamma)$ *and let H denote the intersection of all connected Lie subgroups of G containing Γ. Then the following assertions are true:*

- *H/Γ is compact.*
- *V is a $\mathbb{Q}$-Lie algebra and the embedding of V into $\mathcal{L}ie(G)$ induces an isomorphism between $V \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathcal{L}ie(H)$.*

Conversely, let $\mathbf{n}$ be a nilpotent $\mathbb{Q}$-Lie algebra $\mathbf{n}$ with a $\mathbb{Q}$-vector space basis $\langle \alpha_1, \dots, \alpha_d \rangle$ and let N denote the simply connected real nilpotent Lie group corresponding to $\mathbf{n} \otimes_{\mathbb{Q}} \mathbb{R}$.

Then there exists a natural number M such that $\exp\left(\langle M\alpha_1, \dots, M\alpha_r \rangle_{\mathbb{Z}}\right)$ *is a discrete cocompact subgroup of N.*

Thus there is of one-to-one correspondence between (commensurability classes of) lattices in simply connected nilpotent Lie groups and $\mathbb{Q}$-structures of the corresponding Lie algebra.

COROLLARY 2.2.2. — *Let G be a simply connected real nilpotent Lie group with Lie algebra $\mathcal{L}ie(G)$. Then there exists a discrete cocompact subgroup $\Gamma \subset G$ if and only if there exists a nilpotent $\mathbb{Q}$-Lie algebra $\mathbf{n}$ such that $\mathcal{L}ie(G) \simeq \mathbf{n} \otimes_{\mathbb{Q}} \mathbb{R}$.*

COROLLARY 2.2.3. — *Let Γ be a discrete cocompact subgroup in a simply connected real nilpotent Lie group G. Let $G^{(k)}$ denote the derived series (i.e., $G^{(1)} = G$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$), G^k the (descending) central series (i.e., $G^1 = G$ and $G^{k+1} = [G, G^k]$) and C_k the ascending central series (i.e., $C_0 = \{e\}$ and C_{k+1}/C_k is the center of G/C_k). Then all the quotients $G^{(k)}/(G^{(k)} \cap \Gamma)$, $G^k/(G^k \cap \Gamma)$ and $C_k/(C_k \cap \Gamma)$ are compact for all k.*

Proof. — If $\mathcal{L}ie(G)$ can be defined over $\mathbb{Q}$, the same is true for all the Lie algebras of all the groups in the derived and descending resp. ascending central series. □

Since evidently there exist only countably many non-isomorphic $\mathbb{Q}$-Lie algebras, this correspondence can be used to prove that there exist real and complex nilpotent Lie groups without lattices.

PROPOSITION 2.2.4. — *There exists simply connected complex nilpotent Lie groups which do not admit any discrete cocompact subgroup.*

Proof. — It suffices to show that there exist uncountably many non-isomorphic real nilpotent Lie algebras admitting a complex structure. Let V, W be complex vector spaces of dimension n, m respectively. Let $\mathrm{Hom}^*(\wedge^2 V, W)$ denote the set of all alternating bilinear maps $\phi : V \times V \to W$ for which $\phi(V \times V)$ generates W as a vector space. Then the double coset space

$$S_{n,m} = GL_{\mathbb{R}}(V) \backslash \mathrm{Hom}^*_{\mathbb{C}}(\wedge^2 V, W) / GL_{\mathbb{R}}(W)$$

enumerates all equivalence classes of real nilpotent Lie algebras $\mathbf{n}$ which admit a complex structure and fulfill $\dim_{\mathbb{R}}(\mathbf{n}/\mathbf{n}') = 2n$, $\dim_{\mathbb{R}}(\mathbf{n}') = 2m$ and $[n, n'] = \{0\}$. Now $\dim_{\mathbb{R}} \mathrm{Hom}^*_{\mathbb{C}}(\wedge^2 V, W) = mn(n-1)$ while $\dim_{\mathbb{R}} GL_{\mathbb{R}}(V) \times GL_{\mathbb{R}}(W) = 4n^2 + 4m^2$. Hence for $mn(n-1) > 4n^2 + 4m^2$ the set $S_{n,m}$ can not be countable. □

2.3. Arithmetic subgroups

In this section we discuss general properties of arithmetic groups.

THEOREM 2.3.1 ([**21, 107**]). — *Let $G \subset GL_n$ be an algebraic subgroup defined over $\mathbb{Q}$, $\Gamma = G(\mathbb{Z}) = G \cap GL_n(\mathbb{Z})$ and let $\mathrm{X}_{\mathbb{Q}}(G)$ denote the group of $\mathbb{Q}$-characters, i.e., the group of all $\mathbb{Q}$-group morphisms from G to the multiplicative group G_m.*

Then Γ is a discrete subgroup of the real Lie group $G(\mathbb{R})$ with the following properties:

1. *The quotient manifold $G(\mathbb{R})/\Gamma$ has finite volume with respect to a G-left invariant Borel measure on $G(\mathbb{R})/\Gamma$ if and only if $\mathtt{X}_{\mathbb{Q}}(G) = \{1\}$.*
2. *The quotient manifold $G(\mathbb{R})/\Gamma$ is compact if and only if G is $\mathbb{Q}$-anisotropic.*

REMARK 2.3.2. — If we replace $\mathbb{Q}$ by an arbitrary number field K, the corresponding statements still hold with $\mathbb{Z}$ replaced by the ring $\mathcal{O}_K$ of algebraic integers of K and $\mathbb{R}$ being replaced by $\Pi_{\nu\in\mathcal{R}_\infty} K_v$ where $\mathcal{R}_\infty$ is the set of all archimedean valuations of K and K_v is the completion of K with respect to the valuation v.

However, this yields no new examples of lattices, since $G(\mathbb{Z}) \simeq H(\mathcal{O}_K)$ and $G(\mathbb{R}) \simeq \Pi_{\nu\in\mathcal{R}_\infty} H(K_v)$ for $G = R_{K/\mathbb{Q}}H$ if K is a number field and H a K-group (where $R_{K/\mathbb{Q}}$ denotes the restriction of scalars functor).

REMARK 2.3.3. — For a field K of characteristic zero and an algebraic K-group G the following conditions are equivalent:

1. G is K-anisotropic.
2. G does not contain any K-split torus.
3. $\mathtt{X}_K(G) = \{1\}$ and every unipotent element of $G(K)$ is contained in the unipotent radical $R_u(G)$ of G.

Since every complex semisimple Lie group can be defined over $\mathbb{Q}$ it is immediate that every complex semisimple Lie group contains a lattice, e.g. $SL_n(\mathbb{Z}[i])$ is a (non-cocompact) lattice in $SL_n(\mathbb{C})$. Constructing *cocompact* lattices is more intricate.

THEOREM 2.3.4 (Borel). — *Let G be a (real or complex) semisimple Lie group. Then there exists a discrete cocompact subgroup.*

Borel's proof (as given in **[18]**) for the existence of discrete cocompact subgroups in every semisimple (real) Lie group may be sketched as follows. Given a real simple Lie group G one can show that there exists a compact semisimple real Lie group G_u and an algebraic group H defined over $\mathbb{Q}$ such that H is simple over $\mathbb{Q}$ but isomorphic to $G \times G_u$ over $\mathbb{R}$. A faithful representation of H over $\mathbb{Q}$ yields a discrete arithmetic subgroup $\Gamma \subset H(\mathbb{Q}) \subset G\times G_u$. $\mathbb{Q}$-simplicity of H implies that $H(\mathbb{Q})\cap G$ is finite, hence $\pi_2 : H(\mathbb{Q}) \to G_u$ is almost injective. Therefore $H(\mathbb{Q})$ has no unipotent elements, *i.e.*, H is $\mathbb{Q}$-anisotropic. It follows that Γ is cocompact in $H(\mathbb{R})$, implying that $\pi_2(\Gamma)$ is a discrete cocompact subgroup of G. (For details see **[18]**, **[123]**.)

2.4. Arithmeticity of lattices

In many cases one can prove that lattices are necessarily arithmetic in a certain sense.

For instance, the theory of Malcev for lattices in nilpotent Lie groups implies the following.

PROPOSITION 2.4.1. — *Let G be a simply connected real nilpotent Lie group and Γ a lattice in G.*

Then there exists a unipotent algebraic group U defined over $\mathbb{Z}$ such that there exists an isomorphism of Lie groups $\phi : U(\mathbb{R}) \xrightarrow{\sim} G$ such that $\phi(U(\mathbb{Z}))$ is commensurable to Γ (i.e., the intersection $\phi(U(\mathbb{Z})) \cap \Gamma$ is a subgroup of finite index in both $\phi(U(\mathbb{Z}))$ and Γ).

For solvable groups one has the following result of Mostow.

THEOREM 2.4.2 (Mostow, see **[123]**, thm. 4.34). — *Let Γ be a lattice in a real simply connected solvable Lie group G.*

Then there exists an injective Lie group homomorphism $\rho : G \to GL_n(\mathbb{R})$ with $\rho(\Gamma) \subset GL_n(\mathbb{Z})$.

Later on arithmeticity results have been obtained for lattices in many classes of groups. For lattices in *complex* semisimple groups, this culminated in the following arithmeticity theorem of Margulis.

THEOREM 2.4.3 (Margulis, **[92]**). — *Let Γ be a lattice in a semisimple complex Lie group S. Assume that Γ is* irreducible, *i.e., there does not exist any normal Lie subgroup $H \subset S$ for which $H \cap \Gamma$ is a lattice in H. Assume furthermore that $S \not\simeq SL_2(\mathbb{C})$.*

Then Γ is arithmetic in the following sense: There exists a $\mathbb{Q}$-group H and a proper morphism of Lie groups $\tau : H(\mathbb{R}) \to S$ such that Γ is commensurable to $\tau(H(\mathbb{Z}))$.

Cocompact lattices in $SL_2(\mathbb{C})$ are not necessarily arithmetic. However, at least they fulfill the following weak "arithmeticity" condition (see **[123]**, thm. 7.67):

PROPOSITION 2.4.4. — *Let Γ be a discrete cocompact subgroup of $SL_2(\mathbb{C})$.*

Then there exists a number field K and an element $g \in SL_2(\mathbb{C})$ such that $g\Gamma g^{-1} \subset SL_2(K)$.

2.5. Superrigidity

We will need the following special case of Margulis' superrigidity theorem **[92]**.

THEOREM 2.5.1 (Superrigidity). — *Let S be a simply connected semisimple complex Lie group and Γ a lattice. Assume that no factor of S is isomorphic to $SL_2(\mathbb{C})$.*

Then there exists a simply connected compact real semisimple Lie group K, a subgroup $\Gamma_0 \subset \Gamma$ of finite index and a group homomorphism $\xi : \Gamma_0 \to K$ such that the following condition is fulfilled:

Given a real linear algebraic group H and a group homomorphism $\rho : \Gamma \to H$ there exists a subgroup $\Gamma_\rho \subset \Gamma_0$ of finite index and a real-algebraic group homomorphism $\widetilde{\rho} : S \times K \to H$ such that $\widetilde{\rho}|_{\Gamma_0}$ coincides with $\rho \circ (i, \xi) : \Gamma_0 \to H$.

Superrigidity is used in the proof of the arithmeticity results of Margulis stated in the preceding section. Margulis proved the above stated superrigidity result using methods from ergodic theory. Later, different proofs using harmonic maps were developed ([**34, 49, 70, 99**]).

We will also make use of the following related result.

THEOREM 2.5.2 ([**92**]). — *Let G be a simply connected complex Lie group without $SL_2(\mathbb{C})$-factors, Γ a lattice and Λ a normal subgroup of Γ.*

Then either Λ or Γ/Λ is finite.

COROLLARY 2.5.3. — *Under the assumptions of the theorem $\Gamma/[\Gamma,\Gamma]$ is finite and every group homomorphism from Γ to $(\mathbb{Z},+)$ is constant.*

2.6. Cohomology groups of arithmetic groups

PROPOSITION 2.6.1 (Margulis). — *Let S be a simply connected semisimple complex Lie group without SL_2-factor, let $\Gamma \subset S$ be a lattice and let ρ be a finite-dimensional representation of Γ (over $\mathbb{C}$, $\mathbb{R}$ or any field of characteristic zero).*

Then $H^1(\Gamma, \rho) = \{0\}$. Moreover, ρ is completely reducible.

Proof. — For the vanishing of $H^1(\Gamma, \rho)$ see [**92**]. Here we explain how the vanishing result implies the complete reducibility. Let

$$(*) \qquad 0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

be a short exact sequence of Γ-modules (over a field of characteristic zero). We have to show that it splits. Taking the tensor product with V_3^* yields another short exact sequence of G-modules:

$$0 \longrightarrow V_3^* \otimes V_1 \longrightarrow V_3^* \otimes V_2 \longrightarrow V_3^* \otimes V_3 \longrightarrow 0.$$

The vanishing of $H^1(\Gamma, \cdot)$ for every finite-dimensional representation implis that the associated sequence of invariant submodules is exact. In particular $(V_3^* \otimes V_2)^\Gamma \to (V_3^* \otimes V_3)^\Gamma$ is surjective. Hence the identity homomorphism $\mathrm{id}_{V_3} \in \mathrm{Hom}(V_3, V_3)$ lifts to an element in $\mathrm{Hom}(V_3, V_2)^G$, *i.e.*, $(*)$ splits as a sequence of G-modules. $\square$

There is another vanishing result going back to Každan. Každan introduced what is now called "property T" ([**72**]). By definition, a locally compact topological group has this property if the trivial representation is isolated in the space of all unitary representations. This property is equivalent to the vanishing of $H^1(\cdot, \rho)$ for every unitary representation ρ. If Γ is a lattice in a Lie group G, then Γ has property T if and only if G has property T. A simply connected semisimple complex Lie group has property T if and only if none of its simple factors is isomorphic to $SL_2(\mathbb{C})$.

See **[72]** and **[92]**, Ch. III for more information about property T.

2.7. Non-arithmetic lattices

$SL_2(\mathbb{C})$ is the only simply connected simple complex Lie group which admits non-arithmetic lattices, see **[86, 148]**.

Certain discrete cocompact subgroups in $SL_2(\mathbb{C})$ arise from differential geometry. There is a well-known link between three-dimensional real hyperbolic manifolds (hyperbolic in the sense of Riemannian geometry, *i.e.*, constant negative sectional curvature) and complex-parallelizable manifolds of the form $X \simeq SL_2(\mathbb{C})/\Gamma$. See **[142]** for more information about three-manifolds.

THEOREM 2.7.1. — *Let M be an oriented connected compact Riemannian manifold, $\dim_{\mathbb{R}}(M) = 3$, with constant negative sectional curvature.*

Then $\pi_1(M)$ may be embedded into $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$ as discrete cocompact torsion-free subgroup. The preimage $\tau^{-1}(\Gamma)$ in $SL_2(\mathbb{C})$ (where $\tau : SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ denotes the natural projection) is again a discrete cocompact group.

REMARK 2.7.2. — Evidently the preimage $\tau^{-1}(\Gamma)$ is not torsion-free, because $\ker\tau = \{\pm I\}$. However, every discrete cocompact subgroup in a linear Lie group contains a torsion-free subgroup of finite index (prop. 1.7.2).

In this way the above procedure yields discrete cocompact torsion-free subgroups in $SL_2(\mathbb{C})$.

Proof. — The universal covering $\widetilde{M}$ of M is isometrically diffeomorphic to the quotient $PSL_2(\mathbb{C})/K$ where K is a maximal compact subgroup of $PSL_2(\mathbb{C})$. Moreover $PSL_2(\mathbb{C})$ is the full group of orientation preserving isometries of $\widetilde{M}$. Thus the action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations induces an embedding $\pi_1(M) \hookrightarrow PSL_2(\mathbb{C})$. The image of this embedding is obviously discrete. The cocompactness follows from the compactness of M and K, because $\pi_1(M)\backslash PSL_2(\mathbb{C})/K \simeq M$. It is torsion-free, because no finite group can act freely on the real manifold $\widetilde{M} \simeq \mathbb{R}^3$. □

Such considerations produce the following result of Jørgensen, which is a key element in constructing a number of examples involving $SL_2(\mathbb{C})$. Similar examples were also constructed by Millson **[97]**.

PROPOSITION 2.7.3 (see **[69]**). — *There exists a discrete cocompact subgroup $\Gamma \subset SL_2(\mathbb{C})$ with a surjective group homomorphism $\rho : \Gamma \to \mathbb{Z}$.*[1]

[1] Jørgensen has proved more, namely that such a ρ can be choosen in such a way that $\ker\rho$ is finitely presentable.

Thurston even conjectured that every discrete cocompact subgroup of $SL_2(\mathbb{C})$ admits a subgroup of finite index Γ_0 such that $\mathrm{rank}_{\mathbb{Z}}(\Gamma_0/\Gamma_0') > 0$ (**[141]**).

2.8. Arithmeticity and complex structure

Given a simply connected simple complex Lie group G, the result of Borel (theorem 2.3.4, **[18]**) proves that there is a $\mathbb{Q}$-simple group H such that $H(\mathbb{Z})$ is cocompact in $H(\mathbb{R})$ and $H(\mathbb{R})$ is isomorphic as a real Lie group to a direct product of G and a compact real semisimple Lie group U. Naturally, one would like to know whether this compact real factor U can be avoided. The answer is the following.

PROPOSITION 2.8.1 (Otte, **[116]**). — *Let G be a semisimple $\mathbb{C}$-group. Then there exists a $\mathbb{Q}$-anisotropic $\mathbb{Q}$-simple $\mathbb{Q}$-group H with $G(\mathbb{C}) \simeq H(\mathbb{R})$ as topological groups if and only if $G \simeq G_1^d$ for some $d \in \mathbb{N}$ where G_1 is a simple $\mathbb{C}$-group of type A_n, i.e., isomorphic to some SL_{n+1}.*

Sketch of proof. — For a finite separable field extension K/k let $R_{K/k}$ denote the "restriction of scalars functors" as explained in **[19, 92]**. Then $G(\mathbb{C}) \simeq H(\mathbb{R})$ is equivalent to $R_{\mathbb{C}/\mathbb{R}}G \simeq H$ and it follows that $H \simeq R_{K/\mathbb{Q}}I$ for some totally imaginary extension field K of $\mathbb{Q}$ and a K-group I with $I \simeq G$ over $\mathbb{C}$. Hence the existence of such a $\mathbb{Q}$-anisotropic group H is equivalent to the existence of a K-anisotropic K-form of G for some totally imaginary extension field K. Now K-forms can be parametrized by Galois cohomology and results on Galois cohomology (see **[51, 133, 134, 31]**) imply that such a K-form exists if and only if G is of type A_n. □

This has consequences for the existence of cocompact lattices in "mixed" (*i.e.*, neither solvable nor semisimple) complex Lie groups.

PROPOSITION 2.8.2 (Otte-Potters **[117]**). — *Let G be a simply connected complex Lie group, R its radical, and $S = G/R$. Assume that one of the simple factors of S is not of type A_n* and *acts non-trivially on R by conjugation.*

Then G does not admit a discrete cocompact subgroup.

Sketch of the proof. — Assume that there exists a discrete cocompact subgroup Γ. Let $\pi : G \to S$ denote the natural projection. Then $\Lambda_1 = \pi(\Gamma)$ is a cocompact lattice in S (thm. 3.5.3). Let N denote the nilradical of G and N' its commutator group. Then S acts non-trivially on N/N' and $\Lambda_2 = (\Gamma \cap N)/(\Gamma \cap N')$ is a lattice in N/N' (cor. 2.2.3). Thus $\Lambda_1 \ltimes \Lambda_2$ is a cocompact lattice in $S \ltimes (N/N')$. Therefore there is no loss in generality in assuming that R is a vector group $(\mathbb{C}^d, +)$. If we forget the complex structure of R, then $\Lambda_1 \subset GL(2d, \mathbb{Z})$ after a suitable change of coordinates on R, since Λ_1 stabilizes the lattice Λ_2 in R. Thus $R_{\mathbb{C}/\mathbb{R}}S$ is defined over $\mathbb{Q}$ and by cocompactness of Λ_1 it follows that it is $\mathbb{Q}$-anisotropic. Hence the result by prop. 2.8.1. □

Conversely there exist such semi-direct products with $SL_n(\mathbb{C})$ containing cocompact lattices.

PROPOSITION 2.8.3. — *For every $n \geq 2$ there exists a complex vector space V and an irreducible representation $\rho : SL_n(\mathbb{C}) \to GL(V)$ such that $SL_n(\mathbb{C}) \ltimes (V, +)$ admits a discrete cocompact subgroup.*

Let $S = SL_2(\mathbb{C})$ and $\xi : S \to GL(W)$ be an irreducible representation.

Then $S \ltimes_\xi (W, +)$ contains a discrete cocompact subgroup if and only if $\dim(W)$ *is odd.*

Proof. — We will check later in detail that for every $n \geq 2$ and every imaginary quadratic number field K there exists a K-anisotropic K-form of SL_n (see §2.9).

Now let us discuss SL_2. We are looking for an imaginary quadratic number field K and K-form S of SL_2 such that there exists a absolutely irreducible K-representation of S. For $n = 2k+1$ an n-dimensional absolutely irreducible K-representation can be obtained as an irreducible component of $\otimes^k \mathrm{Ad}$ where Ad is the adjoint representation. Thus for every K-form of SL_2 and every odd number n there exists an absolutely irreducible n-dimensional K-representation. Conversely, assume that S is a K-form with an absolutely irreducible K-representation ρ of dimension $n = 2k$. Let τ be an absolutely irreducible K-representation of dimension $2k - 1$. Then $\rho \otimes \tau$ contains a two-dimensional representation, implying that there is a K-morphism from S to SL_2. It follows that S must be K-split as soon as it admits an even-dimensional absolutely irreducible representation. □

2.8.1. Example for SL_2. — We will now describe an elementary way to find a k-anisotropic k-form over a quadratic imaginary number field k for SL_2 ($\sim A_1$).

For a three-dimensional simple Lie algebra over a field k of characteristic zero there is an easy criterion to determine whether there exists an non-zero element v with $\mathrm{ad}(v)$ nilpotent: $\mathrm{ad}(v)$ is nilpotent if and only if $\mathrm{Kill}(v, v) = 0$. Thus a three-dimensional simple k-Lie algebra is k-anisotropic if and only if $\mathrm{Kill}(\cdot, \cdot)$ is a k-anisotropic quadratic form, *i.e.*, if and only if $\mathrm{Kill}(\cdot, \cdot)$ does not represent zero.

For every $\alpha, \beta \in k^*$ a simple three-dimensional k-Lie algebra with vector space basis A, B, C is given by

$$[A, B] = C, \quad [B, C] = \alpha A, \quad [C, A] = \beta B.$$

The Killing form is given by

$$\mathrm{Kill}(xA + yB + zC, xA + yB + zC) = -2(\beta x^2 + \alpha y^2 + \alpha\beta z^2).$$

Hence what we need is an imaginary quadratic field extension $k/\mathbb{Q}$ and numbers $\alpha, \beta \in k$ such that

$$(*) \qquad \alpha x^2 + \beta y^2 + \alpha\beta z^2 = 0$$

has no non-trivial solution.

For instance, let $k = \mathbb{Q}[i]$, $\alpha = i$, $\beta = 1 + 2i$. If $(*)$ has a non-trivial solution, then it has a non-trivial solution in $\mathcal{O}_k = \mathbb{Z}[i]$ with x, y, z coprime. But calculations modulo 4 show that $(*)$ implies that all x^2, y^2, z^2 equal $2i$ or 0 modulo 4. This implies that x, y, z are all divisible by $1 + i$, contrary to x, y, z coprime. Hence $(*)$ has no non-trivial solution.

2.8.2. Arithmetic subgroups of solvable groups. — In order to find discrete cocompact subgroups of solvable (non-nilpotent) linear algebraic groups one needs tori anisotropic over a number field.

A one-dimensional k-torus is a k-form of the multiplicative group G_m. Thus one-dimensional k-tori are classified by

$$H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Aut}(G_m)) \simeq \mathrm{Hom}(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2\mathbb{Z})$$

It follows that anisotropic one-dimensional k-tori correspond to quadratic extension fields of k. This correspondence can be made explicit in the following way: For every non-square $q \in k$ define

$$H_q = \left\{ \begin{pmatrix} x & y \\ qy & x \end{pmatrix} : x^2 - qy^2 = 1 \right\}$$

Then H_q is a one-dimensional k-torus, split over $k[\sqrt{q}]$, but not split over k.

More generally anisotropic tori can be constructed *by restriction of scalars.* Let H be an n-dimensional K-torus and K/k a separable field extension of degree d. Then $R_{K/k}H$ is a dn-dimensional k-torus with $\mathrm{rank}_k(R_{K/k}H) = \mathrm{rank}_K(H) \leq n$. Thus $R_{K/k}H$ contains a k-anisotropic subtorus of dimension at least $(d-1)n$.

The example of Otte and Potters (example 1.4.1) may be interpreted in this way. Namely, let $H = G_m \ltimes_\rho G_a$ where ρ denotes the natural action of the multiplicative group G_m on the additive group G_a, k a totally real number field of degree d and $K = k[i]$. Then $R_{K/\mathbb{Q}[i]} G_m \simeq G_m \times A$ for some $(d-1)$-dimensional $\mathbb{Q}[i]$-anisotropic $\mathbb{Q}[i]$-torus A and in this way $R_{K/\mathbb{Q}[i]} H$ contains a $\mathbb{Q}[i]$-subgroup $G = A \ltimes R_{K/\mathbb{Q}[i]} G_a \simeq A \ltimes G_a^d$. Now $G_1(\mathbb{Z}[i])$ is a discrete cocompact subgroup in $G_1(\mathbb{C})$ and this is precisely the example described in ex. 1.4.1.

2.9. Arithmetic groups arising from Division algebras

There is a standard method to construct cocompact arithmetic subgroups of $SL_n(\mathbb{C})$ using division algebras.

Let A be a finite-dimensional division algebra over a number field K. (This means A is an associative algebra with 1, not necessarily commutative, such that every non-zero element admits a multiplicative inverse. Sometimes division algebras are called skewfields.) The norm homomorphism $N_{A/K} : A \to K$ is defined as follows: $N_{A/K}(x) = \det(L_x)$ where L_x denotes left multiplication by an element $x \in A$, regarded as K-linear endomorphism of the K-vectorspace A. The norm is a polynomial function on the vector space A. Let $SL_1(A)$ denote the set of all elements x of A with $N_{A/K}(x) = 1$. Then we obtain a K-group G such that $G(K) \simeq SL_1(A)$ and $G \simeq SL_d$ over the algebraic closure of K (with $d^2 = \dim_K(A)$. Since A is a division algebra, $(1 - x)$ is nilpotent only if $1 = x$. Therefore $G(K)$ contains no unipotent elements except 1. Thus the associated arithmetic subgroup $G(\mathcal{O}_K)$ of $\Pi_{\nu\in\mathcal{R}_\infty} G(K_\nu)$ is cocompact.

We now want to verify that this method yields cocompact arithmetic subgroups of $SL_d(\mathbb{C})$ for every $d \in \mathbb{N}$. For this it suffices to construct a division algebra of dimension d^2 over a imaginary quadratic field extension of $\mathbb{Q}$ for every d. In fact, division algebras of arbitrary dimension d^2 exist over any number field. Just to be on the safe side, we prove this in detail.

PROPOSITION 2.9.1. — *Let k be a number field and $d \in \mathbb{N}$. Then there exists a division algebra A, defined over k, with $\dim_k(A) = d^2$ such that the center of A is isomorphic to k.*

Proof. — Every division algebra A over a number field k arises in the following way: There is a cyclic Galois extension l/k of degree d, a generator σ of the Galois group $\mathrm{Gal}(l/k)$ and an element $\lambda \notin N_{l/k}(l^*)$, $\lambda \neq 0$ such that the elements a of A are given by $a = \sum_{i=0\ldots d-1} a_i t^i$ with $a_i \in l$ and the multiplication on A is determined by $pt = t\sigma(p)$ for $p \in l$ and $t^d = \lambda$. (see e.g. **[130]**).

Hence the statement of the proposition is equivalent to the assertion that there exists a cyclic Galois extension l/k of degree d for every given number field k and $d \in \mathbb{N}\setminus\{1\}$ and that the norm endomorphism $N_{l/k} : l^* \to k^*$ for such a field extension is not surjective. These two assertions are shown below. □

LEMMA 2.9.2. — *Let $K/\mathbb{Q}$ be a finite extension, $d \in \mathbb{N}$, $d \geq 2$.*
Then there exists a cyclic field extension L/K with $d = [L : K]$.

Proof. — Let $n = [K : \mathbb{Q}]$. There is a prime p with $p \equiv 1 \mod dn$, because every arithmetic progression contains infinitely many primes. Let ζ_p be a primitive p-th root of unity and $F = K(\zeta_p)$. Since $\mathbb{Q}(\zeta_p) \subset K(\zeta_p)$ and $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$, it follows that $[F : \mathbb{Q}]$ is divisible by dn. Since $[K : \mathbb{Q}] = n$, it follows that $[F : K]$ is divisible by d. Now observe that $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is a cyclic Galois extension. This implies that $K(\zeta_p)/K$ is a Galois extension such that $\mathrm{Gal}(F/K)$ is isomorphic to a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Subgroups of cyclic groups are again cyclic, hence F/K is a cyclic extension. Since d divides $[F/K]$, it follows that there exists a cyclic extension L/K of degree d (with $L \subset F$). □

LEMMA 2.9.3. — *Let L/K be a finite abelian Galois extension of number fields. Then $N_{L/K} : L^* \to K^*$ is not surjective.*

Proof. — Let $P \in K[X]$ denote the corresponding irreducible polynomial. Using the Chebotarev Density Theorem (see e.g. **[111]**) it follows that there exists a completion

$K \hookrightarrow k$ with respect to some valuation v such that P is irreducible in $k[X]$. Hence v can be extended to L in such a way that the completion l of L is a non-trivial finite extension of k. Now l/k is abelian, because L/K is abelian. Hence we obtain from local class field theory that $N_{l/k}(l^*)$ is a non-trivial closed subgroup in k^*. Since K is dense in k, this implies that $N_{L/K} : L^* \to K^*$ is not surjective. □

As a consequence, we obtain the following existence result.

COROLLARY 2.9.4. — *For every $d, g \in \mathbb{N}$ $(d \geq 2)$ and every quadratic imaginary number field K there exists a K-simple K-group H such that $H(\mathcal{O}_K)$ is an irreducible discrete cocompact subgroup in $H(\mathbb{C}) \simeq (SL_d(\mathbb{C}))^g$.*

COROLLARY 2.9.5. — *For every $d, g \in \mathbb{N}$ $(d \geq 2)$ there exists a discrete cocompact subgroup in a non-trivial semi-direct product $G = (SL_d(\mathbb{C}))^g \ltimes (\mathbb{C}^{g(d^2-1)})$.*

Proof. — Take K and H as in the preceding corollary and consider the semi-direct product $G = H \ltimes_{\mathrm{Ad}} \mathcal{L}ie\, H$. Then $G(\mathcal{O}_K)$ is a discrete cocompact subgroup in $G(\mathbb{C})$. □

CHAPTER 3

CLOSED ORBITS AND DENSITY RESULTS

3.1. Survey

In this chapter we collect a number of basic facts on lattices and Lie groups which we will need in the sequel. Particular emphasis is given to density results for lattices and the question : *"Given a lattice Γ in a Lie group G, which Lie subgroups of G have a closed or compact orbit in G/Γ ?"*

The density results include generalizations of the Borel density theorem and generally state that, given a lattice Γ in a complex Lie group G, certain Γ-invariant mathematical objects are automatically G-invariant. For instance, this is true for vector subspaces of finite-dimensional representations of G as well as for plurisubharmonic functions on G.

The results on closedness of orbits include Mostow's theorem which for lattices in complex Lie groups implies that both the radical and the nilradical of the Lie group have closed orbits in the quotient of the Lie group by the lattice. A very basic closedness result concerns centralizers: If Γ is a discrete subgroup in a Lie group G and $S \subset \Gamma$ is a subset, then $C_G(S)\Gamma$ is closed in G. This is the first step toward a collection of results on closed orbits of non-normal abelian subgroups. The investigation of closed orbits culminates in theorem 3.10.1 which states that G is generated as a group by all connected complex commutative Lie subgroups H for which $H/(H \cap \Gamma)$ is compact. To our knowledge this is a new result. We conclude with the derivation of some basic complex-analytic properties of quotients $X = G/\Gamma$ of a complex Lie group G by a lattice Γ. For instance we show that there are no non-constant plurisubharmonic functions on such a quotient, that the algebraic reduction maps such a quotient onto an abelian variety and that such a quotient carries a Kähler metric if and only if it is a compact complex torus. Furthermore we introduce the Albanese torus and consider group actions on cohomology groups.

Most of the results of this chapter are well-known.

3.2. Centralizer Orbits

Given a (discrete) subgroup Γ in a Lie group G we are interested in subgroups $H \subset G$ such that the H-orbit $H/(H \cap \Gamma)$ through $e\Gamma$ in G/Γ is closed.

Using *centralizers* is a good starting point in constructing such subgroups.

LEMMA 3.2.1. — *Let G be a (real) Lie group and Γ a discrete subgroup. Let Δ be a subset of Γ and $C = C_G(\Delta) = \{g \in G : g\delta = \delta g \ \forall \delta \in \Delta\}$ its centralizer. Then $C \cdot \Gamma$ is closed.*

This is stated in **[123]**, **[136]** under the additional assumptions that G/Γ has a finite G-invariant measure and Δ is finite. However, the first of these two additional assumptions is simply superfluous, while the second can be circumvented.

Proof. Case Δ finite. — Let $c_n \in C$, $\gamma_n \in \Gamma$ with $\lim_n(c_n\gamma_n) = z$. We have to show $z \in C\Gamma$. For any $d \in \Delta$ we have

$$z^{-1}dz = \lim \gamma_n^{-1}c_n^{-1}dc_n\gamma_n = \lim \gamma_n^{-1}d\gamma_n.$$

Since $d \in \Gamma$ and Γ discrete, this implies that there is a number $N(d)$ (depending on $d \in \Delta$) such that $z^{-1}dz = \gamma_n^{-1}d\gamma$ for all $n > N(d)$. Let $N = \max_{d\in\Delta} N(d)$. Then $\gamma_N z^{-1}dz\gamma_N^{-1} = d$ for all $d \in \Delta$. Hence $z\gamma_N^{-1} \in C$ and thereby $z \in C\Gamma$.

General case. — Let $(\Delta_j)_{j\in J}$ denote the family of all finite subsets of Δ. All the $C_G(\Delta_j)\Gamma$ are closed, hence $H = \bigcap_j (C_G(\Delta_j)\Gamma)$ is closed. Now it is easy to show that there exists a finite subset $\Delta_0 \subset \Delta$ such that $\dim(H) = \dim(C_G(\Delta_0)\Gamma) = \dim(C_G(\Delta_0))$. It follows that $C_G(\Delta_j)^0 \supset C_G(\Delta_0)^0$ for all $j \in J$. Thereby $C_G(\Delta) \supset C_G(\Delta_0)^0 = H^0$. This implies

$$H^0 \subset \left(\bigcap_j C_G(\Delta_j)\right)\Gamma \subset \left(\bigcap_j C_G(\Delta_j)\Gamma\right) = H.$$

Since H/H^0 is discrete, it follows that $C_G(\Delta)\Gamma = \left(\bigcap C_G(\Delta_j)\right)\Gamma$ is closed in G. $\square$

Note. — For the case Δ finite it is enough to assume that G is a topological group with a topology fulfilling the first axiom of countability. However, in the proof of the general case we used the assumption that G is a Lie group.

3.2.1. Abelian subgroups. — Our goal here is to show that there are abelian subgroups with closed orbits. For this we recall the following fact.

LEMMA 3.2.2. — *Let G be a connected non-commutative complex Lie group, and Z_G its center. For $g \in G$ let $C(g)$ be the centralizer of g. Then*

$$(*) \qquad \dim C(g) \geq \dim Z_G + 1.$$

Proof. — Consider the map $\mu : G \times G \to G \times G$, $(g,h) \mapsto (g, ghg^{-1}h^{-1})$. Let C be the irreducible component of the preimage of $G \times \{e\}$ which contains $G \times \{e\}$, and $\widetilde{\mu} : C \to G \times \{e\} \simeq G$ the induced map. Note that $\widetilde{\mu}^{-1}(g)$ is a union of components of $C(g)$, among which is always $C(g)^0$. Now there is an open set of elements $g \in G$ which are in the image of exp and are not in Z_G. For such elements $(*)$ holds. The result then follows from the semi-continuity of the fiber-dimension of the holomorphic map $\widetilde{\mu}$. $\square$

LEMMA 3.2.3. — *Let G be a connected complex Lie group and Γ a discrete subgroup. Then there exists a positive-dimensional connected complex abelian subgroup H with $H \cdot \Gamma$ closed, i.e., the H-orbit through the neutral point $e\Gamma$ in G/Γ is closed.*

Proof. — First assume that Γ is not central in G. Then there exists an element $\gamma \in \Gamma$ which is not central in G, and $C(\gamma)$ is a Lie subgroup with a closed orbit and $0 < \dim C(\Gamma) < G$. Thus we may complete the proof using induction on $\dim G$.

Second, let us assume that Γ is contained in the center Z of G and that $\dim Z > 0$. Then $Z = Z\Gamma$ is closed and abelian.

Finally, we have to consider the case where the center Z of G is discrete and $\Gamma \subset Z$. Complex semisimple Lie groups have finite center, hence $R \cap \Gamma$ is of finite index in Γ. As usual, let R resp. N denote the radical resp. nil-radical of G. Let $K = \ker \rho$ where $\rho : R \to \operatorname{Aut}(N)$ is the natural group homomorphism given by conjugation. Then $\Gamma \cap R \subset K$ and the connected component of K equals the center of N which is positive-dimensional and commutative. Thus K^0 is commutative with $K^0\Gamma$ closed and $\dim(K^0) > 0$. □

3.3. Cartan subgroups

DEFINITION 3.3.1. — Let $\mathbf{g}$ be a Lie algebra. A subalgebra $\mathbf{h}$ is called *Cartan subalgebra* if the following conditions are fulfilled:

1. The Lie algebra $\mathbf{h}$ is nilpotent.
2. $\mathbf{h}$ equals its own normalizer, *i.e.*, $\mathbf{h}$ contains all $x \in \mathbf{g}$ for which $\operatorname{ad}(x)$ stabilizes $\mathbf{h}$.

Correspondingly, we call a connected Lie subgroup H of a Lie group G a *Cartan subgroup* if its Lie algebra $\mathcal{L}ie(H)$ is a Cartan subalgebra of $\mathcal{L}ie(G)$.[(1)]

If G is a linear algebraic group, then it contains a Zariski open subset Ω such that for every $g \in \Omega$ the connected component of the centralizer $C(g_{ss})$ of g_{ss} is a Cartan subgroup, where g_{ss} denotes the Jordan semisimple part of $g \in G$.

As a consequence we obtain the following result.

PROPOSITION 3.3.2 (Mostow, see **[105, 106]**). — *Let G be a complex linear algebraic group and Γ a discrete Zariski dense subgroup. Then there exists a Cartan subgroup H of G such that $H\Gamma$ is closed in G.*

Proof. — Since Γ is Zariski dense, it has non-empty intersection with the Zariski open subset Ω mentioned above. Thus the statement follows from lemma 3.2.1. □

Aided by the theorems 3.4.1 and 3.9.1 obtained later this has the following implication.

(1)Mostow used a slightly different notion in **[105]**. Our Cartan subgroups are the connected components of Cartan subgroups in the sense of Mostow.

COROLLARY 3.3.3. — *Let G be a complex semisimple linear algebraic group and Γ a lattice in G.*

Then there exists a maximal torus[(2)] $T \subset G$ *such that* $T/(T \cap \Gamma)$ *is compact.*

The next corollary will turn out to be useful in our examination of the cycle space in section §4.11.

COROLLARY 3.3.4. — *Let G be a connected complex Lie group, Γ a lattice and assume that G is not nilpotent. Then there exists a connected complex Lie subgroup $H \subset G$ such that $H/(H \cap \Gamma)$ is compact and H is not normal in G.*

Proof. — Since G is not nilpotent, a Cartan subgroup of G can not equal the whole group G. Condition *(ii)* of the definition 3.3.1 then implies that Cartan subgroups of G are non-normal Lie subgroups. □

3.4. Density results

For a discrete subgroup Γ of a locally compact group G the property of being a lattice can be interpreted as the property that G/Γ is "small" in a certain sense, *i.e.*, as the property that Γ is large in G in a certain way. This interpretation is a key idea in many results on lattices. This is particularly true for results stating that in many cases lattices in linear algebraic groups over locally compact fields are dense with respect to the Zariski topology. The first such result is due to A. Borel who proved in **[17]** that a lattice in a real semisimple group without compact factors must be Zariski dense. For lattices in complex Lie groups we have the following density result.

THEOREM 3.4.1 (Moskowitz). — *Let G be a connected complex Lie group, $\Gamma \subset G$ a lattice and $\rho : G \to GL(n, \mathbb{C})$ a holomorphic representation.*

Then the Zariski closures of $\rho(\Gamma)$ and $\rho(G)$ coincide.

This was proved by Moskowitz in 1978 (see **[102]**). Apparently independently, it was later reproved by Iwamoto (**[67]**). Here we will give a slightly different, more geometric proof.

LEMMA 3.4.2. — *Let G be a complex linear algebraic group, H an algebraic subgroup. There does not exist a G-left invariant probability measure on G/H unless* $\dim(G) = \dim(H)$.

Proof. — Assume the converse and let A be a connected one-dimensional algebraic subgroup of G which is not contained in H. Then A is isomorphic either to the multiplicative group $\mathbb{C}^*$ or to the additive group $\mathbb{C}$. The quotient $X = G/H$ may be embedded equivariantly into some projective space and by the Flag theorem X

(2) torus in the sense of linear algebraic groups

contains an open quasi-affine A-invariant subvariety Ω. We consider the A-action on $\mathbb{C}[\Omega]$. Every $f \in \mathbb{C}[\Omega]$ is contained in a finite-dimensional A-invariant vector subspace $\mathbb{C}[\Omega]$. If $A \simeq \mathbb{C}^*$, then $\mathbb{C}[\Omega]$ is generated as vector space by functions $f \in \mathbb{C}[\Omega]$ such that $f(\lambda \cdot x) = \lambda^n f(x)$ for some $n = n(f) \in \mathbb{Z}$. If $A \simeq \mathbb{C}$, then A acts on $\mathbb{C}[\Omega]$ as a unipotent group and we can find a finite-dimensional vector subspace V of $\mathbb{C}[\Omega]$ and an element $f_0 \in \mathbb{C}[\Omega]$ such that

- A acts trivially on V,
- A stabilizes $V \oplus \langle f_0 \rangle_{\mathbb{C}}$ and
- f_0 is not A-invariant.

It follows that $f_0(t \cdot x) = f_0(x) + tg(x)$ for some $g \in V$. In this case $f(x) = f_0(x)/g(x)$ defines an A-equivariant map from $\Omega_1 = \{x \in \Omega : g(x) \neq 0\}$ to $\mathbb{C}$.

Thus in any case there exists an open subset $\Omega_1 \subset X$ and an A-equivariant map from Ω_1 to $\mathbb{C}$ or $\mathbb{C}^*$. The existence of such an equivariant map implies that there is a non-empty open (in the euclidean topology) subset $W \subset \Omega$ and an element $a \in A$ such that all the $a^n(W)$ $(n \in \mathbb{Z})$ are disjoint. It follows that $\mu(W) = 0$ for every A-invariant probability measure μ on X. Thus X contains an open subset of measure zero. But the measure was assumed to be G-invariant and G acts transitively on X. This implies $\mu(X) = 0$ contradicting the assumption that μ is a probability measure. Hence there is no G-invariant probability measure on X. □

LEMMA 3.4.3. — *Let G be a connected complex Lie group and H a closed subgroup such that G/H admits a G-left invariant probability measure.*

Then there does not exist any non-constant G-equivariant map to a projective space.

Proof. — Let $\phi : X \to \mathbb{P}_N$ be an equivariant holomorphic map and $\rho : G \to GL(n, \mathbb{C})$ denote the corresponding homomorphism of complex Lie groups. Then $\rho(G')$ is algebraic (prop. 1.9.4) and has closed orbits in $\phi(X) = G/I$. Hence $G'I$ is closed in G. The invariant probability measure μ on G/H can be pushed down to an invariant probability measure on $G/G'I$. Since $G/G'I$ is an abelian group, this implies that $G/G'I$ is compact. On the other hand $G/G'I$ can be embedded in a linear algebraic group, hence compactness of $G/G'I$ implies $G = G'I$. In other words, G' acts transitively on $\phi(X) = G/I$. Since $\rho(G')$ is linear algebraic, we may now apply the preceding lemma and conclude that ϕ must be constant. □

COROLLARY 3.4.4. — *Let G be a connected complex Lie group, H a closed subgroup such that G/H admits a G-left invariant probability measure and $\rho : G \to GL(n, \mathbb{C})$ be a holomorphic group homomorphism.*

Then the Zariski closures of $\rho(G)$ and $\rho(H)$ in $GL(n, \mathbb{C})$ coincide.

Proof. — The quotient $\overline{\rho(G)}/\overline{\rho(\Gamma)}$ is a quotient of linear algebraic groups and therefore admits an equivariant embedding into a projective space. □

COROLLARY 3.4.5. — *Let G, H and $\rho : G \to GL(n,\mathbb{C})$ be as above and assume that $W \subset \mathbb{C}^n$ is a $\rho(H)$-stable vector subspace.*

Then W is $\rho(G)$-stable.

REMARK 3.4.6. — It is crucial that ρ is a *finite-dimensional* representation. For instance, consider $V = L^2(G/H)$ with $\rho : G \to GL(V)$ being the action induced by the G-left action on G/H and $W = \{f \in V : f(eH) = 0\}$. Then W is $\rho(\Gamma)$-stable, but not $\rho(G)$-stable.

COROLLARY 3.4.7. — *Let G be a connected complex Lie group, H a closed subgroup such that G/H admits a G-left invariant probability measure, I be a connected complex Lie subgroup of G and $N_G(I) = \{g \in G : gIg^{-1} = I\}$.*

If $H \subset N_G(I)$, then $N_G(I) = G$, i.e., I is normal in G.

Proof. — Apply the preceding corollary to the adjoint representation $\mathrm{Ad} : G \to GL(\mathcal{L}ie\, G)$. □

COROLLARY 3.4.8. — *Let G be a connected complex Lie group and H a closed subgroup such that G/H admits a G-left invariant probability measure.*

Then the connected component H^0 of H is normal in G and G/H is a complex parallelizable manifold.

Proof. — Evidently $H \subset N_G(H^0)$, hence we may invoke the preceding corollary with $I = H^0$. □

COROLLARY 3.4.9. — *Let G be a complex Lie group, Γ a lattice in G and $I \subset G$ a closed complex Lie subgroup with $\Gamma \subset I$.*

Then the connected component I^0 of I is normal in G and I/I^0 is a lattice in G/I.

Proof. — This is a corollary of the preceding results, because G/I admits a G-left invariant probabiliy measure due to lemma 1.5.2. □

We would like to remark that for semisimple groups there exists a stronger density result due to Borel which implies the following.

THEOREM 3.4.10 (Borel [**17**]). — *Let G be a simply connected semisimple complex Lie group, Γ a lattice, $\rho : G \to GL_n(\mathbb{C})$ a continuous (not necessarily holomorphic) group homomorphism. Then $\rho(\Gamma)$ and $\rho(G)$ have the same Zariski closure in $GL_n(\mathbb{C})$.*

This can be generalized to the following result.

PROPOSITION 3.4.11. — *Let G be a simply connected complex Lie group with $G = G'$, Γ a lattice, $\rho : G \to GL_n(\mathbb{C})$ a continuous (not necessarily holomorphic) group homomorphism.*

Then $\rho(\Gamma)$ and $\rho(G)$ have the same Zariski closure in $GL_n(\mathbb{C})$.

REMARK 3.4.12. — For $G' \neq G$ there often exists a surjective continuous group homomorphism $\rho : G \to S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$ with $\Gamma \subset \ker \rho$. Thus the assumption $G = G'$ is crucial.

Proof. — Let H denote the Zariski closure of $\rho(G)$ in $GL_n(\mathbb{C})$. Then $H = H'$. It follows that the radical R_H is unipotent. Now $\Gamma \cap R_G$ is cocompact in R_G (see thm. 3.5.3), R_G is nilpotent (because $G = G'$) and $\rho|_{R_G}$ is a (real) unipotent representation. By a result of Malcev ([**89**]) it follows that the Zariski closure of $\rho(\Gamma \cap R_G)$ coincides with the Zariski closure of $\rho(R_G)$. Consider the induced group homomorphism $G/R_G \to H/\overline{\rho(R_G)} = I$. Now I is a linear algebraic group and the image of G/R is Zariski dense in I. Thereby semisimplicity of G/R implies semisimplicity of I. This is equivalent to the property that $\rho(R_G)$ is Zariski dense in R_H. By this argumentation the problem is reduced to the semisimple case and we can invoke the preceding theorem in order to conclude that $\rho(G)$ and $\rho(\Gamma)$ have the same Zariski closure. □

Next present some applications of the density result.

COROLLARY 3.4.13. — *Let G be a simply connected complex Lie group, Γ a discrete subgroup, and H a connected complex subgroup for which $H \cap \Gamma$ is a lattice in H. Then $C_G(H) = C_G(H \cap \Gamma)$.*

Proof. — We may assume that $G \subset GL_m(\mathbb{C})$ (prop. 1.11.1). Since conjugation is an algebraic operation on $\overline{G}$, it follows that $C_G(A) = C_G(\overline{A})$ for every subgroup $A \subset G$, where $\overline{A}$ denotes the closure in $GL_m(\mathbb{C})$ with respect to the Zariski topology. Hence $C_G(H) = C_G(\overline{H}) = C_G(\overline{H \cap \Gamma}) = C_G(\Gamma)$. □

COROLLARY 3.4.14. — *Let G be a simply connected complex Lie group, Γ a lattice in G and let Z denote the center of G.*

Then $Z\Gamma$ is closed in G.

Proof. — Applying the preceding corollary with $H = G$ yields $Z = C_G(\Gamma)$. Thus closedness of $Z\Gamma$ follows from lemma 3.2.1. □

REMARK 3.4.15. — A result obtained below (thm. 3.9.1) implies that moreover $Z\Gamma/\Gamma = Z/(Z \cap \Gamma)$ is compact.

COROLLARY 3.4.16. — *Let G be a linear complex Lie group and Γ a lattice in G.*

Then G is abelian, nilpotent or solvable if and only if Γ is abelian, nilpotent resp. solvable.

Proof. — This follows from the density result, because these properties may be defined through the vanishing of certain relations involving commutators. □

By passing to the universal covering one can deduce:

Let G be a connected complex Lie group and Γ a lattice in G. Then G is solvable or nilpotent if and only if Γ is solvable resp. nilpotent.

This follows, because the classes of solvable resp. nilpotent groups are stable under central extensions. (The universal covering of a connected Lie group is a central extension.)

However, commutativity is not preserved under central extensions.

EXAMPLE 3.4.17. — Let $\widetilde{G}$ the three-dimensional Heisenberg group, *i.e.*, $\widetilde{G} \simeq \mathbb{C}^3$ as manifold with group multiplication given by

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$

Let $C = \{(0,0,z) : z \in \mathbb{Z} + i\mathbb{Z}\}$ and $\widetilde{\Gamma} = \{(x, y, z) : x, y, z \in \mathbb{Z} + i\mathbb{Z}\}$. Then $\Gamma = \widetilde{\Gamma}/C$ is a discrete cocompact subgroup of $G = \widetilde{G}/C$, but Γ is commutative while G is not.

3.5. Mostow fibration

Given a Lie group G, the maximal connected normal solvable Lie subgroup is called the *radical* of G and the maximal connected normal nilpotent Lie subgroup is called the *nilradical* of G.

By a result of Mostow **[106]** both the radical and the nilradical of a complex Lie group G have closed orbits in any quotient of G by a lattice Γ. Here we will deduce this result from a theorem of Auslander.

THEOREM 3.5.1 (Auslander). — *Let G be a real Lie group, H a discrete subgroup and R a normal solvable Lie subgroup.*

Then the connected component $U = (\overline{RH})^0$ of the closure of RH in G is solvable.

COROLLARY 3.5.2. — *Let G be a linear algebraic group defined over k with $k = \mathbb{R}$ or $k = \mathbb{C}$, R its radical and $\Gamma \subset G(k)$ be a Zariski dense discrete subgroup.*

Then $R(k)\Gamma$ is a closed subgroup of $G(k)$.

THEOREM 3.5.3 (Mostow). — *Let G be a connected complex Lie group, Γ a lattice, R the radical and N the nilradical of G.*

Then both $N/(N \cap \Gamma)$ and $R/(R \cap \Gamma)$ are compact.

Now we show how to deduce the theorem of Mostow from Auslander's theorem.

Proof. — Let $U = (\overline{R\Gamma})^0$. Then U is solvable and normalized by Γ. Let $\mathcal{L}ie(U)$ be the corresponding real Lie subalgebra of $\mathcal{L}ie(G)$ and $\mathbf{a}$ be the smallest k-Lie subalgebra of $\mathcal{L}ie(G)$ containing $\mathcal{L}ie(U)$. Then $\mathbf{a}$ is a solvable Lie subalgebra of $\mathcal{L}ie(G)$. Γ normalizes U. It follows that $\mathrm{Ad}(\Gamma)$ stabilizes both $\mathcal{L}ie(U)$ and $\mathbf{a}$. Since the adjoint representation is algebraic, it follows that $\mathbf{a}$ is an ideal. By the maximality property of the radical this implies that $\mathbf{a} = \mathcal{L}ie(R)$. It follows that $R = U$ and consequently

$R\Gamma$ must be closed. Closedness of $R\Gamma$ implies compactness of $R/(R \cap \Gamma)$ thanks to cor. 3.6.3. Finally compactness of $R/(R \cap \Gamma)$ implies compactness of $N/(N \cap \Gamma)$ by another result of Mostow (see **[103]** or **[123]**, p. 46). $\square$

REMARK 3.5.4. — The result of the closedness of the R-orbits holds also for real Lie groups which do not contain any non-trivial normal compact semisimple Lie subgroup.

On the other hand, if $G \simeq K \times H$ with K compact and semisimple and Γ is a lattice in H, then for any group homomorphism $\rho : \Gamma \to K$ the group $\widetilde{\Gamma} = \{(\rho(\gamma), \gamma) : \gamma \in \Gamma\}$ is a lattice in G. If H has a non-trivial radical R and $\rho(\Gamma \cap R)$ is infinite, then R is also the radical of G and $\widetilde{\Gamma}$ is a lattice in G such that $R/(R \cap \widetilde{\Gamma})$ is not compact.

WARNING 3.5.5. — The proof of Mostow's theorem given in **[123]**, 8.28 is wrong, see **[162]** for details.

3.6. Cocompactness of lattices

3.6.1. Lattices in solvable Lie groups are cocompact. — Examples of non-cocompact lattices occur in the context of semisimple Lie groups. For instance, $SL(2, \mathbb{Z})$ is a non-cocompact lattice in $SL(2, \mathbb{R})$. However, a lattice in a solvable real or complex solvable Lie group is always cocompact. Here we will prove this only for lattices in *complex* Lie groups, for the general case see **[123]**, thm. 3.1.

First we would like to recall that a locally compact topological group has finite volume with respect to a left- or right invariant Haar measure if and only if it is compact. Hence lattices in commutative locally compact topological groups are cocompact.

We also need the following auxiliary lemmata.

LEMMA 3.6.1. — *Let G be a group, Γ a subgroup and A a normal subgroup of Γ. Then Γ normalizes the centralizer $C_G(A)$ and consequently $C_G(A)\Gamma$ is a subgroup of G.*

Proof. — Let $\gamma \in \Gamma$, $c \in C_G(A)$ and $a \in A$. Then

$$(\gamma c \gamma^{-1})a = \gamma c(\gamma^{-1} a \gamma)\gamma^{-1} = \gamma(\gamma^{-1} a \gamma)c\gamma^{-1} = a(\gamma c \gamma^{-1}).$$

$\square$

LEMMA 3.6.2. — *Let G be a simply connected solvable complex Lie group and $\Gamma \subset G$ a lattice. Assume that G is not commutative. Then there exists a closed complex Lie subgroup I with $\Gamma \subset I \subset G$ such that $0 < \dim I < \dim G$, i.e., there is a non-trivial holomorphic fibration $G/\Gamma \to G/I$.*

The assumption of Γ being a lattice is superfluous, cf. **[61]**, thm. 4.1.

Proof. — Let Σ denote the intersection $\cap_g g\Gamma g^{-1}$. This is the ineffectivity of the Γ-action on G/Γ. By replacing G and Γ by their respective quotients by Σ we may assume that Γ acts effectively on G/Γ, *i.e.*, that $\Sigma = \{e\}$, *i.e.*, that Γ does not contain any normal subgroup of G.

Since G is non-commutative, the density results imply that Γ is non-commutative (see cor. 3.4.16). Let N denote the nilradical of G. Since $[\Gamma, \Gamma] \subset [G, G] \subset N$, it follows that $\Gamma \cap N \neq \{e\}$. Consider the centralizer $C_G(\Gamma \cap N)$. It contains the center of N and therefore is positive-dimensional. On the other hand, since Γ contains no normal subgroup of G, it is clear that $C_G(\Gamma \cap N) \neq G$. Now $I = C_G(\Gamma \cap N)\Gamma$ is closed (lemma 3.2.1) in G and it is a subgroup by lemma 3.6.2 above. Thus $I = C_G(\Gamma \cap N)\Gamma$ is a subgroup with the desired properties. □

COROLLARY 3.6.3. — *Let G be a connected solvable complex Lie group and Γ a lattice.*

Then G/Γ is compact.

Proof. — The assertion is trivial for $\dim_{\mathbb{C}}(G) = 1$. For $\dim_{\mathbb{C}}(G) > 1$ lemma 3.6.2 above yields a fibration $G/\Gamma \to G/I$ such that both $I/\Gamma = I^0/(I^0 \cap \Gamma)$ and G/I are of lower dimension. Hence we may argue by induction on the dimension of G. □

REMARK 3.6.4. — It is easy to see that lattices in arbitrary nilpotent locally compact topological group are cocompact. However, we do not know whether lattices in arbitrary solvable locally compact topological groups are neceassarily cocompact.

3.6.2. A general cocompactness criterion

THEOREM 3.6.5 (Každan-Margulis, [**73**]). — *Let G be a semisimple real or complex linear algebraic group and Γ a lattice.*

Then G/Γ is compact if and only if Γ contains only semisimple elements.

This has first been proved for arithmetic lattices by Mostow and Tamagawa [**107**], later by Každan and Margulis for arbitrary lattices.

Together with thm. 3.5.3 this yields a complete criterion determining which lattices in complex Lie groups are cocompact.

COROLLARY 3.6.6. — *Let G be a connected complex Lie group, $\Gamma \subset G$ a lattice, R the radical of G and $\pi : G \to G/R$ the natural projection.*

Then Γ is cocompact in G if and only if none of the elements $\pi(\gamma)$ with $\gamma \in \Gamma \setminus (\Gamma \cap R)$ is unipotent in S.

For linear algebraic groups this may be reformulated in the following way, yielding a close parallel of the cocompactness criterion for arithmetic groups mentioned in thm. 2.3.1 (ii).

COROLLARY 3.6.7. — *Let G be a complex linear algebraic group and Γ a lattice. Then Γ is cocompact in G if and only if every unipotent element in Γ is already contained in the unipotent radical of G.*

EXAMPLE 3.6.8. — Evidently $SL(n, \mathbb{Z} + i\mathbb{Z})$ contains unipotent elements, hence it can not be cocompact in $SL(n, \mathbb{C})$.

3.7. Density results II

Semisimple complex Lie groups are intrinsically algebraic. In this context it is reasonable to assume that many analytic objects on a semisimple complex Lie group are already constant, if they are invariant under a Zariski dense subgroup. The first result in this spirit is due to Barth and Otte who proved that, given a reductive complex Lie group G and a discrete Zariski dense subgroup Γ, every Γ-invariant holomorphic function on G is constant [**11**]. Akhiezer proved that every meromorphic function on a complex semisimple Lie group S is constant as soon as it is invariant under a Zariski dense subgroup Γ provided $\Gamma \subset S(\mathbb{Q})$ [**1**]. In general, *i.e.*, without the assumption $\Gamma \subset S(\mathbb{Q})$, this was proved by Huckleberry and Margulis [**58**]. Later K. Oeljeklaus and Berteloot showed that for an infinite discrete subgroup Γ in a semisimple complex Lie group S there does not exists any strictly plurisubharmonic Γ-invariant function on S [**12**]. Using similar methods we will here give a proof for the fact that there are no Γ-invariant plurisubharmonic functions on a complex Lie group if Γ is a lattice.

THEOREM 3.7.1. — *Let G be a connected complex Lie group and Γ a lattice. Then every holomorphic or plurisubharmonic function on G/Γ is constant.*

Proof. — Let R denote the radical of G and consider $\pi : G/\Gamma \to G/R\Gamma$. Recall that $R\Gamma$ is closed and that $R/(R \cap \Gamma)$ is compact (thm. 3.5.3). By the maximum principle for plurisubharmonic functions every plurisubharmonic function is constant on the compact complex space $R/(R \cap \Gamma)$. Hence every plurisubharmonic function on G/Γ is a pull-back of a plurisubharmonic function on $G/R\Gamma$ and the theorem follows from the proposition below. □

PROPOSITION 3.7.2. — *Let S be a semisimple complex linear algebraic group and Γ a Zariski dense discrete subgroup. Then every holomorphic or plurisubharmonic function on S/Γ is constant.*

Proof. — We start be deriving an auxiliary claim.

CLAIM 3.7.3. — Let $H \simeq (\mathbb{C}^*)^r$ and $h \in H$ be an element such that $\Lambda = \{h^n : n \in \mathbb{Z}\}$ is an infinite discrete subgroup.

Then there exists a connected complex Lie subgroup $A \subset H$ with $\dim A > 0$ such that every Λ-invariant plurisubharmonicfunction on H is already A-invariant.

To prove this claim, consider $\exp : \mathcal{L}ie(H) \to H$. Let $V_{\mathbb{Z}} = \exp^{-1}(\{h^n : n \in \mathbb{Z}\})$ and let $V_{\mathbb{R}}$ denote the $\mathbb{R}$-vector space generated by $V_{\mathbb{Z}}$. Then $\dim_{\mathbb{R}}(V_{\mathbb{R}}) = r + 1$. Hence $W = V_{\mathbb{R}} \cap iV_{\mathbb{R}}$ is a one-dimensional complex vector space. Define $A = \exp(W)$.

By construction the image of $\tau(A)$ of A under the projection $\tau : H \to H/\Lambda$ is relatively compact. It follows that for every plurisubharmonic function f on H/Λ the restriction of f to $\tau(A)$ must be bounded. Since every bounded subharmonic function on $\mathbb{C}$ is constant, it follows that every Λ-invariant plurisubharmonic function on H is A-invariant. This proves the claim.

Now let I denote the set of all $g \in S$ such that $f(g) = f(e)$ for every Γ-right invariant plurisubharmonic function on S. Then I is a closed subgroup of S. Let I^0 denote its connected component and let J denote the maximal connected complex Lie subgroup of I^0 (*i.e.*, $\mathcal{L}ie(J) = \mathcal{L}ie(I) \cap i\,\mathcal{L}ie(I)$). Clearly J is invariant under conjugation by elements of Γ. The normalizer of a connected complex Lie subgroup J of S equals $\{g \in S : \mathrm{Ad}(g)(\mathcal{L}ie\, J) \subset \mathcal{L}ie\, J\}$ and therefore is an algebraic subgroup of S. Since Γ is Zariski dense in S, it follows that J is normal in S.

Since S is semisimple and may be assumed to be simply connected, we may assume that $S = S_0 \times J$ for some semisimple complex Lie group S_0. Let $\pi : S \to S_0 = S/J$ denote the natural projection.

Assume $\dim S_0 > 0$. Due to lemma 1.7.12 and prop. 1.7.2 the group $\pi(\Gamma)$ contains a subgroup Λ which is torsion-free and still Zariski dense in S_0. Since the set of all semisimple elements of S_0 contains a Zariski open subset of S_0, it follows that Λ contains a semisimple element of infinite order. But now the claim implies that there is a Lie subgroup $A \subset S_0$ with $\dim(A) > 0$ such that every Λ-invariant plurisubharmonic function on S_0 is already A-invariant.

This is a contradiction, since (by construction of J) it is clear that $\pi(\Gamma)$-invariant plurisubharmonic functions separate $\pi(\Gamma)$-orbits in S_0.

Thus $\dim(S_0) > 0$ is impossible, *i.e.*, $S = J$, *i.e.*, every Γ-invariant plurisubharmonic function on S is constant. □

REMARK 3.7.4. — Later, in chapter 10, we will prove another result in this area: *If G is a linear algebraic group with $G = G'$ such that the generic element in G is semisimple and Γ is a discrete Zariski dense subgroup, then every plurisubharmonic function on G/Γ is constant.* (see prop. 10.7.1).

Finally we state a consequence of the non-existence of non-constant holomorphic functions.

PROPOSITION 3.7.5. — *Let G be a linear complex Lie group and Γ a discrete subgroup such that every holomorphic function on G/Γ is constant. Let ϕ be an holomorphic Lie group automorphism of G such that $\phi|_\Gamma = \mathrm{id}_\Gamma$.*

Then $\phi = \mathrm{id}_G$.

Proof. — Consider the map $\zeta : G \to G$ given by $\zeta(g) = \phi(g)g^{-1}$. Note that $\phi|_\Gamma = \mathrm{id}_\Gamma$ implies $\zeta(g\gamma) = \zeta(g)$ for all $g \in G$ and $\gamma \in \Gamma$. Therefore ζ induces a holomorphic map from G/Γ to G which is constant by the assumptions of the proposition. Hence $\zeta \equiv \zeta(e) = e$, *i.e.*, $\phi = \mathrm{id}_G$. □

3.8. Closed orbits of Abelian subgroups

THEOREM 3.8.1. — *Let G be a complex linear algebraic group, Γ a Zariski dense discrete subgroup and G_0 the algebraic subgroup of G generated (as algebraic subgroup) by all connected abelian linear algebraic subgroups $A \subset G$ with $A\Gamma$ being closed in G.*

Then $G = G_0$.

The first step in the proof of the theorem are the lemmata below.

LEMMA 3.8.2. — *Let G be a Lie group, $I \subset H$ closed subgroups of G and Γ a discrete subgroup of G such that $H\Gamma$ is closed and $H \cap \Gamma \subset I$. Then $I\Gamma$ is closed, too.*

Proof. — Let $x \in \overline{I\Gamma}$. Then $x = \lim \alpha_n \gamma_n$ for some $\alpha_n \in I$, $\gamma_n \in \Gamma$. Since $H\Gamma$ is closed, $x = h\gamma$ for some $h \in H$, $\gamma \in \Gamma$. Then $h = \lim \alpha_n(\gamma_n\gamma^{-1})$. Closedness of $H\Gamma$ in G implies that $H\Gamma$ is a submanifold, therefore H is open in $H\Gamma$. Thus $\alpha_n(\gamma_n\gamma^{-1}) \in H$ for n sufficiently large. In this case $(\gamma_n\gamma^{-1}) \in I$, because $\alpha_n \in I \subset H$ and $H \cap \Gamma \subset I$. It follows that $\alpha_n\gamma_n\gamma^{-1} \in I$ for n sufficiently large. Hence $h \in I$ and $x \in I\Gamma$. □

LEMMA 3.8.3. — *Let G be a complex linear algebraic group, Γ a discrete subgroup and $\gamma \in \Gamma$.*

Then there exists an abelian algebraic subgroup $A \subset G$ such that $A\Gamma$ is closed in G and $\gamma \in A$.

Proof. — We will construct two sequences of algebraic subgroup C_n, D_n. We start by choosing the centralizer $C_G(\gamma) = \{g \in G : g\gamma = \gamma g\}$ as C_0. For given C_n $(n \geq 0)$ we define D_n as the Zariski closure of $C_n \cap \Gamma$ in G. If D_n is non-abelian, then $C_n \cap \Gamma$ is non-abelian and we choose an element $\gamma_n \in C_n \cap \Gamma$ which is not central in $C_n \cap \Gamma$ and let $C_{n+1} = D_n \cap C_G(\gamma_n)$. If D_n is abelian, we set $A = D_n$.

Note that by construction γ_n is central in C_{n+1}. This implies that the sequences of subgroups $C_n \cap \Gamma$ and D_n are strictly decreasing. Since the D_n are algebraic subgroups, it follows that the sequence must terminate, *i.e.*, there must be a number n such that D_n is abelian.

Furthermore by construction it is clear that γ is contained in all the C_n and D_n.

Finally we have to show that $C_n\Gamma$ and $D_n\Gamma$ are closed in G. This follows from the centralizer lemma (lemma 3.2.1) combined with lemma 3.8.2 above. □

COROLLARY 3.8.4. — *Let G be a complex linear algebraic group, Γ a discrete subgroup and $\gamma \in \Gamma$.*

Then there exists a connected abelian algebraic subgroup $A \subset G$ and a number $n \in \mathbb{N}$ such that $A\Gamma$ is closed in G and $\gamma^n \in A$.

Proof of the theorem. — It is clear that Γ normalizes G_0, since closedness of $A\Gamma$ implies closedness of $\gamma A \gamma^{-1}\Gamma = \gamma A\Gamma$ for any $A \subset G$, $\gamma \in \Gamma$. Since Γ is Zariski dense in G, it follows that G_0 is normal in G. Thus the quotient G/G_0 is an algebraic group and the projection $\pi : G \to G/G_0$ is a morphism of algebraic groups. Cor. 3.8.4

above implies that for every $\gamma \in \Gamma$ there is a number n such that $\gamma^n \in G_0$. Hence $\Lambda = \pi(\Gamma)$ containing only torsion elements. Let K denote the closure of Λ in G/G_0 with respect to the euclidean topology. Lemma 1.7.11 implies that K is compact with commutative connected component. Since $\pi(\Gamma)$ is Zariski dense, it follows that G/G_0 is a commutative reductive group, *i.e.*, a torus in sense of theory of algebraic groups. By general theory of linear algebraic groups it follows that there exists a commutative reductive group $T \subset G$ such that $\pi|_T : T \to G/G_0$ is a finite surjective morphism. Now $\pi(\Gamma)$ being relatively compact in G/G_0 implies that $T\Gamma$ is closed in G. But this implies $T \subset G_0$, since T is connected and commutative. Hence T and therefore G/G_0 must be trivial, *i.e.*, $G = G_0$. □

3.9. Closed orbits in quotients of finite volume

We will show that closed orbits of abelian subgroups in finite volume spaces are necessarily compact.

THEOREM 3.9.1. — *Let G be a locally compact topological group, H an abelian closed subgroup and Γ a lattice in G.*

Assume that $H\Gamma$ is closed in G.

Then $H/(H \cap \Gamma) \simeq H\Gamma/\Gamma$ is compact.

We are convinced that this is just a special case of a more general fact.

CONJECTURE 3.9.2. — *Let G be a locally compact topological group, H a closed subgroup and Γ a lattice.*

Then $H\Gamma$ is closed in G if and only if $H \cap \Gamma$ is a lattice in H.

For a cocompact lattice Γ this is an easy exercise and for abelian H it is a consequence of the theorem above. Raghunathan proved the implication "$H \cap \Gamma$ being a lattice in H implies $H\Gamma$ closed" in general and the converse for normal H (see [**123**], Thm. 1.13).

To prepare the proof of the theorem we need the following auxiliary lemma.

LEMMA 3.9.3. — *Let A be a non-compact locally compact topological group and $W_0 \subset A$ be a relatively compact open subset. Then there exists an infinite subset $\Lambda_0 \subset A$ such that $\lambda W \cap \eta W = \varnothing$ for all $\lambda, \eta \in \Lambda_0$ with $\lambda \neq \eta$.*

Proof. — Choose a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in A recursively in such a way that

$$\lambda_n \notin \cup_{k<n} \lambda_k W_0 (W_0^{-1})$$

This is possible since the set on the right hand is relatively compact in A. Finally set $\Lambda_0 = \{\lambda_n : n \in \mathbb{N}\}$. □

LEMMA 3.9.4. — *Let G be locally compact group, H a closed subgroup and Γ a discrete subgroup of G such that $H\Gamma$ is a closed subset of G.*

Then the quotient $H\backslash H\Gamma$ is discrete.

Proof. — The quotient $Q = H\backslash H\Gamma$ is a closed countable subset of $H\backslash G$. Every closed countable subset of a locally compact space contains an isolated point. (This follows from the Baire Category Theorem.)[3] Since Q is a Γ-orbit, this implies that Q is discrete. □

Now we are in a position to prove the theorem.

Proof of the theorem. — Let W be a relatively compact open neighbourhood of e in G. Since H is abelian, $H\cap\Gamma$ is normal in H. Thus $A = H/(H\cap\Gamma)$ is a locally compact topological group. Assume that A is non-compact. By the lemma we may choose an infinite subset $\Lambda \subset H$ such that $\lambda(W\cap H)(H\cap\Gamma) \cap \eta(W\cap H)(H\cap\Gamma) = \varnothing$ for all $\lambda, \eta \in \Lambda$ with $\lambda \neq \eta$. Next we choose a decreasing sequence of open sets $W_n \subset W$ such that $\cap_n \overline{W_n} = \{e\}$. By assumption the quotient G/Γ has finite volume. Every W_n is open and therefore has positive measure. Since Λ is infinite, it follows that there exists a divergent sequences $\alpha_n, \beta_n \in \Lambda$ with $\alpha_n \neq \beta_n$ such that $\alpha_n W_n \Gamma \cap \beta_n W_n \Gamma \neq \varnothing$. It follows that there are sequences $w_n, v_n \in W_n$ and $\gamma_n \in \Gamma$ such that $\alpha_n v_n = \beta_n w_n \gamma_n$ for all n. Now $\cap_n \overline{W_n} = \{e\}$ implies that $\lim v_n = e$ and $\lim w_n = e$. It follows that $\lim \alpha_n^{-1}\beta_n\gamma_n = e$. Observe that $\alpha_n^{-1}\beta_n\gamma_n \in H\Gamma$. Discreteness of $H\backslash H\Gamma$ implies that $\gamma_n \in H$ for n sufficiently large. Then $\gamma_n \in H\cap\Gamma$. With H being abelian and $\lim \alpha_n^{-1}\beta_n\gamma_n = e$ this implies that

$$\alpha_n(W\cap H)(H\cap\Gamma) \cap \beta_n(W\cap H)(H\cap\Gamma).$$

But this contradicts the fact that $\alpha_n \neq \beta_n$ and $\alpha_n, \beta_n \in \Lambda$. Hence it is not possible that $H/(H\cap\Gamma)$ is non-compact. □

3.10. Subgroups with a bounded orbit

Given a Lie group G and a lattice Γ in G we will show that there are many Lie subgroups H of G such that the H-orbit through $e\Gamma$ is relatively compact in G/Γ.

THEOREM 3.10.1. — *Let G be a connected complex Lie group, Γ a lattice and G_0 the subgroup of G generated by all connected commutative complex Lie subgroups A of G with $A/(A\cap\Gamma)$ compact.*

Then $G_0 = G$.

(3) In our notation, locally compact topological groups are always assumed to have a countable basis of topology and therefore are metrizable as topological spaces.

Proof. — If G is commutative, then G/Γ is a topological group with finite volume and therefore compact.

Now let us consider the case where G is not abelian, but solvable. Here we argue by induction over $\dim(G)$. Thus let G, Γ and G_0 be as above and assume that the proposition is valid for all solvable complex Lie groups of lower dimension. Let $\gamma \in \Gamma \setminus Z$ (where Z denotes the center of G) and let $C^0(\gamma)$ denote the connected component of the centralizer of γ in G. Then $0 < \dim C^0(\gamma) < \dim G$. Moreover $C^0(\gamma)\Gamma$ is closed in G. Since G is solvable and Γ a lattice, the quotient G/Γ must be compact (Cor. 3.6.3) and therefore closedness of $C^0(\gamma)\Gamma$ implies that $C^0(\gamma) \cap \Gamma$ is a lattice in $C^0(\gamma)$. Hence $C^0(\gamma) \subset G_0$ by the induction hypothesis. This is equivalent to the assertion that $V^{\mathrm{Ad}(\gamma)} \subset \mathcal{L}ie(G_0)$ where $V = \mathcal{L}ie\, G$. Let $H \subset GL(V)$ denote the Zariski closure of $\mathrm{Ad}(G)$. Then H is also the Zariski closure of $\mathrm{Ad}(\Gamma)$. Let k denote the generic dimension of $V^h = \{v : hv = v\}$ for $h \in H$ and Ω the set of all $h \in H$ for which $\dim V^h = k$. Then Ω is a Zariski open subset of H. Now $V^h \subset \mathcal{L}ie(G_0)$ for all $h \in \Gamma \cap \Omega$ and therefore for all $h \in \Omega$. It follows that $v \in \mathcal{L}ie\, G_0$ for all $v \in \mathrm{Ad}^{-1}(\Omega)$. Since $\mathrm{Ad}^{-1}(\Omega)$ is a non-empty open subset of $\mathcal{L}ie\, G$, this implies that $G = G_0$.

Finally we consider the general case. Let R denote the radical of G and consider the projection $\pi : G/ \to G/R = S$. The quotient group S is semisimple and therefore linear algebraic. The image $\pi(\Gamma)$ is a lattice in S. Let $A \subset S$ be an abelian subgroup with closed orbit in $Y = S/\pi(\Gamma)$. By thm. 3.9.1 such an orbit is compact. Hence $\pi^{-1}(A)$ is a solvable subgroup of G with compact orbit in G/Γ. Thus it follows from the proof for the solvable case that $\pi^{-1}(A)$ is generated by connected complex commutative subgroups with compact orbits in G/Γ.

Now let H be the subgroup of S generated by all those A. It is clear that H is arcwise connected and by thm. 3.8.1 it is also clear that H is Zariski dense in S. But arcwise-connected subgroups of Lie groups are necessarily Lie subgroups ([**43**]) and for a Zariski dense Lie subgroup H of a linear algebraic group S one has $[S, S] \subset H \subset S$ (prop. 1.9.4). Thus $S = [S, S]$ implies $H = S$. □

We will now deduce another variant of this theme, this time strengthening the assumption on the subgroups (requiring unipotency) while relaxing the assumption on the orbits (only relatively-compact instead of compact).

PROPOSITION 3.10.2. — *Let G be a simply connected complex Lie group and Γ a lattice.*

Let G_1 denote the subgroup of G generated by all unipotent subgroups $U \subset G'$ for which there exists a compact subset $F \subset G$ such that $U \subset F\Gamma'$.

Then $G_1 = G'$.

REMARK 3.10.3. — For a simply connected complex Lie group G the commutator group G' carries a unique structure of a complex linear algebraic group. Hence it makes sense to speak about unipotent subgroups of G'.

Proof. — First we note that we may assume $R \cap G' = \{e\}$, because $R \cap G'$ is a normal unipotent subgroup with compact orbits (prop. 3.11.2). This assumption implies that R is central and $G = S \times R$ with S semisimple. Let $\pi : G \to G/R \simeq S$ be the natural projection.

CLAIM 3.10.4. — Let H be a one-dimensional unipotent subgroup of S such that $H \cap \pi(\Gamma) \neq \{e\}$.

Then $H \subset G_1$.

Proof. — Let $u \in S$, $r \in R$ such that $ur \in \Gamma$ and $\pi(u) \in H \setminus \{e\}$. Let C denote the connected component of the centralizer of u in S. Clearly, H is contained in the center of C. Now $C \cdot R$ is the centralizer of ur in G. Hence the CR-orbit through $e\Gamma$ is closed (lemma 3.2.1). Let N denote the nilradical of C. Then NR is the nilradical of CR. It follows that the NR-orbit through $e\Gamma$ is compact (thm. 3.5.3). Now H is central in C, hence $H \subset N$ and consequently $H \subset G_1$. □

Next let V denote the subgroup of G/R generated by all one-dimensional unipotent subgroups having non-trivial intersection with $\pi(\Gamma)$. Then V is normalized by $\pi(\Gamma)$. The Borel Density theorem (thm. 3.4.10) thus implies that V is a normal subgroup of S. It follows that $S = S_0 \times V$ for some semisimple complex Lie subgroup $S_0 \subset S$.

CLAIM 3.10.5. — The S_0-orbit through $e\Gamma$ in G/Γ is relatively compact.

Proof. — By a criterion of Kazdan-Margulis (see **[123]**, Cor. 11.12) we have to show that given a sequence $s_n \in S_0$ there is no sequence $\gamma_n \in \Gamma \setminus \{e\}$ such that $\lim s_n \gamma_n s_n^{-1} = e$. Assume there are such sequences. Let $u_n \in S$ and $r_n \in R$ such that $u_n r_n = \gamma_n$. Then $\lim s_n u_n s_n^{-1} = e$ (because R is central). Since $\pi(\Gamma)$ is a lattice in S, it follows that u_n is unipotent for n sufficiently large (**[123]**, Cor. 11.18). But in this case $u_n \in V$ by construction and this implies that $s_n u_n s_n^{-1} = u_n$ for all n. Contradiction! □

Finally note that S_0 is a semisimple complex Lie group and therefore generated by its unipotent subgroups. □

3.11. Orbits of the commutator group

For a group G we always denote the commutator group by G'. This is the subgroup generated by all the commutators $ghg^{-1}h^{-1}$ with $g, h \in G$. For a connected complex Lie group G this is a connected closed complex Lie subgroup, namely the connected Lie subgroup corresponding to the commutator algebra of the Lie algebra of G. For most (though not all) parallelizable manifolds the orbits of the commutator group are closed.

LEMMA 3.11.1. — *Let G be a complex Lie group, Γ a discrete subgroup and A a normal simply connected commutative complex Lie subgroup of G. Assume that $A/(A\cap\Gamma)$ is compact and that the Zariski closures of* $\mathrm{Ad}(\Gamma)$ *and* $\mathrm{Ad}(G)$ *in* $GL(\mathcal{L}ie\, G)$ *coincide.*

Then $[\Gamma, A\cap\Gamma]$ is a lattice of $[G,A]$ and a subgroup of finite index in $[G,A]\cap\Gamma$.

Proof. — The group A is a complex vector space and conjugation in the group G induces a representation $\rho : G/A \to GL(A)$. This representation is naturally isomorphic to the restriction of the adjoint representation Ad to $\mathcal{L}ie\, A$.

For $g \in G$ define $\zeta_g \in \mathrm{End}(A)$ by $\zeta_g = \rho(g) - \mathrm{id}_A$. Let $\Lambda = \oplus_{\gamma\in\Gamma}\zeta_\gamma(A\cap\Gamma)$. Clearly, $[\Gamma, A\cap\Gamma]$ is the subgroup generated by Λ. Since $A\cap\Gamma$ is cocompact in A, $\langle A\cap\Gamma\rangle_{\mathbb{R}} = \langle A\cap\Gamma\rangle_{\mathbb{C}}$. This implies that $\langle\Lambda\rangle_{\mathbb{R}} = \langle\Lambda\rangle_{\mathbb{C}}$ and consequently that $\langle\Lambda\rangle_{\mathbb{Z}}$ is a lattice in the complex subspace $V = \langle\Lambda\rangle_{\mathbb{C}}$. We claim that $[A,G] = V$. Indeed, $\mathrm{Ad}(\gamma)(\mathcal{L}ie\, A) \subset \mathcal{L}ie\, V$ for all $\gamma\in\Gamma$ implies that $\mathrm{Ad}(g)(\mathcal{L}ie\, A)\subset\mathcal{L}ie\, V$ for all $g\in G$, because we assumed that the Zariski closures of $\mathrm{Ad}(\Gamma)$ and $\mathrm{Ad}(G)$ coincide. □

PROPOSITION 3.11.2. — *Let G be a simply connected complex Lie group, R its radical and Γ a lattice.*

Then $(G'\cap R)\Gamma$ is closed in G, $R\cap\Gamma'$ is a subgroup of finite index in $G'\cap R\cap\Gamma$ and a cocompact lattice in $G'\cap R$.

Proof. — First recall that a normal Lie subgroup in a simply connected Lie group is also simply connected.

Let N denote the nilradical and N^k the central series of N. By the theorems of Mostow (thm. 3.5.3) and Malcev (cor. 2.2.3) all of the groups N and N^k ($k\in\mathbb{N}$) have compact orbits in G/Γ. Furthermore N^k/N^{k+1} is a simply connected commutative normal Lie subgroup of G/N^{k+1} for all k.

Thus we may apply the preceding lemma repeatedly and obtain that $G'\cap N\cap\Gamma'$ is of finite index in $G'\cap N\cap\Gamma$ and a lattice in $N\cap G'$.

This is the statement of the proposition, since $G'\cap R\subset N$. □

COROLLARY 3.11.3 (Barth-Otte, [**10**]). — *Let G be a solvable complex Lie group and Γ a lattice.*

Then $G'\Gamma$ is closed in G.

THEOREM 3.11.4. — *Let G be a simply connected complex Lie group, R its radical and Γ a lattice. Assume that $\Gamma'/(R\cap\Gamma')$ is a subgroup of finite index in $\Gamma/(R\cap\Gamma)$.*

Then $G'\Gamma$ is closed in G, $G'\cap\Gamma$ is a lattice in G' and Γ' is a subgroup of finite index in $G'\cap\Gamma$.

COROLLARY 3.11.5. — *Let G be a simply connected complex Lie group, R its radical and Γ a lattice. Assume that no simple factor of G/R is isomorphic to $SL_2(\mathbb{C})$.*

Then $G'\Gamma$ is closed in G, $G' \cap \Gamma$ is a lattice in G' and Γ' is a subgroup of finite index in $G' \cap \Gamma$. Furthermore $(\Gamma \cap G')/\Gamma'$ is a finite group isomorphic to the torsion part of the first homology group $H_1(X, \mathbb{Z})$.

EXAMPLE 3.11.6

1. The exclusion of $SL_2(\mathbb{C})$-factors is necessary. Recall that there exists a discrete cocompact subgroup $\Lambda \subset SL_2(\mathbb{C})$ with a surjective group homomorphism $\tau : \Lambda \to \mathbb{Z}$ (2.7.3). Let $\Gamma \simeq \Lambda \times \mathbb{Z}^2$ be embedded in $G = SL_2(\mathbb{C}) \times \mathbb{C}$ by $(\lambda, m, n) \mapsto (\lambda, m + in + \sqrt{2}\tau(\lambda))$. Then the orbits of $G' = SL_2(\mathbb{C})$ in G/Γ are not closed.
2. Let
$$G = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{C} \right\},$$
$m \in \mathbb{N}$ and $\Gamma \subset G$ the set of those elements $g = (x, y, z) \in G$ for which $x, y, mz \in \mathbb{Z} + i\mathbb{Z}$. Then Γ is discrete, cocompact and Γ' is a subgroup of index m^2 in $G' \cap \Gamma$.
3. For *real* nilpotent Lie groups the similar statement is true; this is a result of Malcev. However, for arbitrary real solvable Lie groups no such result holds. For instance, let $V = \mathbb{R}^3$ and $\Gamma \subset V$ a lattice such that $\Gamma \cap W = \{0\}$ where $W = \{(x, y, 0) : x, y \in \mathbb{R}\}$. Let $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ act on V by
$$(x, y, z) \longmapsto (\cos\theta x + \sin\theta y, \cos\theta y - \sin\theta x, z).$$
Then $\{e\} \times \Gamma$ is discrete, cocompact in $S^1 \ltimes (V, +)$ and the G'-orbits are not closed.
4. It is crucial that Γ is supposed to be a lattice, even for complex nilpotent Lie groups. See **[112]** for an example of a nilpotent complex Lie group G and a discrete subgroup Γ such that every holomorphic function on G/Γ is constant and the G'-orbits in G/Γ are not closed.

3.11.1. Derived and central series. — Recursive application of the theorem implies the following.

COROLLARY 3.11.7. — *Let G and Γ be as in theorem 3.11.4 above. Let G^k denote the k-th term of the derived series of G. Then $G^k\Gamma$ is closed in G, $G^k \cap \Gamma$ is a lattice in G^k and Γ^k is a subgroup of finite index in $G^k \cap \Gamma$.*

Proof. — By the above proposition it suffices to consider the case where $R \cap G' = \{e\}$. Then R is central in G. Since G is simply connected, this implies $G = R \times S$ for a semisimple complex Lie group S. Let $\pi : G \to S$ denote the natural projection. Then $\pi(\Gamma)$ is a lattice in S. A theorem of Margulis (see 2.5.2) implies that $\pi(\Gamma')$ is also a lattice in S. It follows that $(\Gamma \cap R) \cdot (\Gamma \cap S) \supset (\Gamma \cap R) \cdot \Gamma'$ is already a lattice in G. In particular, Γ' is a lattice in S and hence of finite index in $S \cap \Gamma = G' \cap \Gamma$. □

PROPOSITION 3.11.8. — *Let G be a simply connected complex Lie group, Γ a lattice. Let $Z = C_1$ denote the center of G and define C_k recursively such that C_k/C_{k+1} is the center of G/C_{k+1}.*

Then all the orbits of all the subgroups C_k are closed.

Proof. — $Z\Gamma$ is closed in G by cor. 3.4.14. Hence $G/Z\Gamma = (G/Z)/(\Gamma/Z \cap \Gamma)$ is again a quotient of a simply connected complex Lie group (viz. G/Z^0) by a lattice (viz. $\Gamma/(Z^0 \cap \Gamma)$). Thus closedness of $C_k\Gamma$ in G can be proved by induction over k. □

By a result of Barth and Otte (see **[10]**) the assumption of Γ being a lattice may be weakened for nilpotent G.

PROPOSITION 3.11.9. — *Let G be a complex nilpotent Lie group, Γ a discrete subgroup such that every holomorphic function on G/Γ is constant. Let C_k be defined as above. Then all the C_k-orbits in G/Γ are closed.*

3.11.2. Invariant plurisubharmonic functions. — Although in general $G' \cap \Gamma$ is not a lattice in G', it is still large in a certain sense.

PROPOSITION 3.11.10. — *Let G be a complex Lie group, Γ a lattice in G.*

Then every Γ'-invariant plurisubharmonic function on G' is constant.

Proof. — Since $(R \cap G')/(R \cap \Gamma')$ is compact, it is clear that Γ'-invariant plurisubharmonic functions are constant along the $R \cap G'$-orbits. Thus we may assume that $R \cap G' = \{e\}$. Then G is a direct product of an abelian complex Lie group R and a semisimple complex Lie group S. Now S is a linear algebraic group and thm. 3.4.1 implies that $\tau(\Gamma)$ is Zariski dense in S where $\tau : S \times S \to S$ is the projection onto the second factor. From lemma 1.9.3 it follows that $\tau(\Gamma')$ is Zariski dense in $S = S'$. With $\Gamma' = \{e\} \times (S \cap \Gamma)'$ this in turn implies that $\Gamma \cap S$ is Zariski dense in S. Thus the proof is completed by prop. 3.7.2. □

3.12. On the number of compact orbits

The main goal of this section is to show that there is no non-trivial *continuous* family of compact orbits in G/Γ, where G is a *complex* Lie group and Γ is a discrete subgroup. In the case of *real* Lie groups such families can arise.

EXAMPLE 3.12.1. — Let $E = \mathrm{SO}_2 \ltimes \mathbb{R}^2$ be the group of orientation preserving Euclidean motions. Recall that $\mathrm{SO}_2 \cong S^1$ so that the universal cover G is a semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$. Let Γ be the center of G, *i.e.*, a copy of $\mathbb{Z}$ in the first factor which realizes the universal covering of S^1. Let H be the first factor itself. Note that the H-orbit in $G/\Gamma = E$ is just the first factor in E. Conjugating H yields the *continuous* family of semi-simple factors of E. Of course all such conjugates contain Γ.

The above phenomenon does not occur in the case of complex Lie groups.

PROPOSITION 3.12.2. — *Let G be a complex Lie group and Γ a discrete subgroup. Then there exist only countably many connected complex Lie subgroups H in G with $H/H \cap \Gamma$ compact.*

Proof. — First note that it is enough to handle the case where G is simply connected. Further recall that, since the fundamental group of a connected compact manifold is finitely generated, if the orbit $H/H \cap \Gamma$ is compact, then $H \cap \Gamma$ is finitely generated. Hence it is enough to prove the following statement:

> *Let G be a simply connected complex Lie group and Γ a finitely-generated discrete subgroup. Then there exist at most countably many connected complex subgroups H in G with $\Gamma \subset H$ and H/Γ compact.*

Since G is simply connected, we may replace it by its Zariski hull in some linear realization, *i.e.*, we may assume that G is a linear algebraic group. Now H/Γ being compact implies $A := \overline{\Gamma} = \overline{H}$ (thm. 3.4.1). Furthermore $\overline{H}' = H' = A'$ (prop. 1.9.4). Thus it suffices to consider only the groups H with $H' = A'$ and H/Γ compact. In particular $H \supset B := \overline{A'\Gamma}^{ana}$, where $\overline{}^{ana}$ denotes the closure in the complex analytic Zariski topology. It is therefore sufficient to count the complex groups $\widetilde{H}$ in the abelian group $\widetilde{A} = A/B^0$ which contain the discrete group $\widetilde{\Gamma} := B/B^0$ such that $\widetilde{H}/\widetilde{\Gamma}$ is compact. Thus it only remains to prove the proposition in the abelian case.

Now assume that G is abelian and simply connected. Then any connected Lie subgroup H with H/Γ compact is just the linear span of $H \cap \Gamma$ and is therefore determined by $H \cap \Gamma$ which is a finitely generated subgroup of Γ. Of course Γ has only countably many such subgroups. □

3.13. On the number of parallelizable manifolds

In a certain sense, only few Lie groups admit lattices. In particular, one can prove the following ([**157**]).

PROPOSITION 3.13.1. — *Let $k = \mathbb{R}$ or $k = \mathbb{C}$. Then up to isomorphism as k-Lie groups there are only countably many simply connected k-Lie groups admitting lattices.*

As mentioned above, for every semisimple complex Lie group there exists a discrete cocompact subgroup. Here we want to show that up to conjugacy there are only countably many such subgroups. This is equivalent to the assertion below.

PROPOSITION 3.13.2. — *There exist only countably many compact complex parallelisable manifolds $X = G/\Gamma$ with G semisimple (up to biholomorphic equivalence).*

Proof. — We have to consider compact quotients of semisimple Lie groups S by discrete subgroups Γ. As usual, let us call S/Γ irreducible, if there does not exist any almost direct product decomposition $S = S_1 \cdot S_2$ with $S_1 \cap S_2$ and $\Gamma/(S_1 \cap \Gamma) \cdot (S_2 \cap \Gamma)$ finite. Clearly it suffices to consider irreducible S/Γ. Recall that by standard theory of linear algebraic groups there exists only countably many non-isomorphic complex semisimple Lie groups. Thus it is sufficient to consider a fixed semisimple complex Lie group S which we may take to be simply connected. As a fundamental group of a compact manifold $\Gamma \simeq \pi_1(S/\Gamma)$ is finitely presentable ([**123**], Thm. 6.16). Up to isomorphism as abstract groups there exist only countably many finitely presentable groups. Thus we may restrict our attention to irreducible lattices in a fixed simply connected complex semisimple Lie group S which are isomorphic (as abstract groups) to a fixed finitely presentable group Γ_0. Any such lattice may be regarded as image of an injective group homomorphism $\rho : \Gamma_0 \to S$. Thus the set of all irreducible lattices corresponds to a subset R of $\mathrm{Hom}(\Gamma_0, S)$. As outlined in [**151**], the set $\mathrm{Hom}(\Gamma_0, S)$ may be endowed with the structure of an affine variety. There is a natural S-action by conjugation on $\mathrm{Hom}(\Gamma_0, S)$. Now local rigidity results ([**123**], Th. 7.66 & 6.7, [**151**]) imply that R is a union of open S-orbits in the variety $\mathrm{Hom}(\Gamma_0, S)$. Therefore R is just a union of finitely many S-orbits. However conjugate homomorphisms $\Gamma_0 \to S$ define isomorphic quotients. Thus the proof of countability is completed. □

3.14. Kähler parallelizable manifolds

THEOREM 3.14.1. — *Let G be a connected complex Lie group and Γ a lattice. Assume that $X = G/\Gamma$ carries a (not necessarily G-invariant) Kähler form.*

Then X is a compact complex torus.

For compact quotients this result is due to Wang [**149**].

Proof for the compact case. — For arbitrary one-forms ω and vector fields X, Y one has the relation

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

Holomorphic one-forms on X are in one-to-one correspondence with linear forms on the Lie algebra $\mathcal{L}ie(G)$. The above equation implies that such a holomorphic one-form is closed if and only if it vanishes on the commutator algebra of $\mathcal{L}ie(G)$. Hence G/Γ admits non-closed holomorphic one forms if G is non-commutative. However, on a compact Kähler manifold every holomorphic differential form is closed. Hence G/Γ compact Kähler implies that G is commutative, *i.e.*, X is a compact complex torus. □

Proof of the general case. — Assume that $X = G/\Gamma$ is Kähler. First we note that this implies that every connected complex Lie subgroup $H \subset G$ with compact orbit in X must be commutative. Let R denote the radical of G and C denote the centralizer of R in G. Since the R-orbits are compact, it follows that R is commutative, *i.e.*, $R \subset C$. Now consider the Mostow-fibration (thm. 3.5.3) $p : X = G/\Gamma \to G/R\Gamma = Y$. Let $\pi : G \to G/R = S$ denote the natural projection. If A is an abelian subgroup of S with compact orbit in Y, then $\pi^{-1}(A)$ has a compact orbit in G/Γ and therefore $\pi^{-1}(A)$ must be abelian, *i.e.*, $\pi^{-1}A \subset C$. Since $S = G/R$ is generated by such A (thm. 3.10.1), it follows that R is central in G. Then $p : X \to Y$ is a torus principal bundle. We may replace the Kähler form ω on X by its average over the torus principal right action. Contraction with respect to the fundamental vector fields of the torus action and their complex conjugates then yields a Kähler form on Y. But for an infinite discrete subgroup in a semisimple complex Lie group there are no invariant Kähler forms (This is a result of F. Berteloot and K. Oeljeklaus, see [**12**]). Hence Y must be point, which implies that X is a compact complex torus. □

3.15. Algebraic reduction

Given an irreducible compact complex space X the field of meromorphic functions $\mathcal{M}(X)$ is a finitely generated extension field of $\mathbb{C}$ by a theorem of Siegel. This implies that there exists a dominant meromorphic map f from X onto a projective variety V such that $f^* : \mathcal{M}(V) \to \mathcal{M}(X)$ is an isomorphism of fields. Such a meromorphic map f is called *algebraic reduction* of X. It is well-defined up to bimeromorphic equivalence. For more information on the algebraic reduction see [**125, 146**].

For homogeneous complex manifolds (not necessarily compact) an *algebraic reduction* exists in the following sense:

THEOREM 3.15.1 (see [**62**]). — *Let X be a complex homogeneous manifold. Then there exists a surjective $\mathrm{Aut}_{\mathcal{O}}(X)$-equivariant holomorphic map f onto a meromorphically separable homogeneous complex manifold Y such that $\mathcal{M}(X) = f^*\mathcal{M}(Y)$.*

Here being meromorphically separable can be defined in the following way:

DEFINITION 3.15.2. — A complex space Y is called *meromorphically separable* if, given two points x and y on Y there always exists a meromorphic function f on Y such that neither x nor y is contained in the indeterminacy set of f and $f(x) \neq f(y)$.

Meromorphically separable complex parallelizable manifolds are necessarily abelian varieties.

LEMMA 3.15.3. — *Let H be a connected complex Lie group and Λ a lattice. Assume that $Y = H/\Lambda$ is meromorphically separable.*

Then H is commutative and Y a complex abelian variety.

Proof. — Recall that the radical R of H has compact orbits in $Y = H/\Lambda$ (thm. 3.5.3). Thus we obtain a holomorphic fiber bundle $\tau : Y \to Z = H/R\Lambda$ with compact fibers. Using meromorphic separability of Y, for any point $y \in Y$ and any natural number k

with $0 \leq k \leq \dim Y$ we can find a closed analytic subset $W \subset Y$ of pure dimension k containing y (simply by taking irreducible components of intersections of zero divisors of meromorphic functions on Y). Since the projection map $\tau : Y \to Z$ is a proper holomorphic map, this implies the existence of analytic hypersurfaces in W unless W is trivial (*i.e.*, a point).

Recall that a theorem of Huckleberry and Margulis **[58]** states a quotient of a complex semisimple Lie group by a Zariski dense subgroup never contains analytic hypersurfaces. Hence in our situation W must be a point, *i.e.*, H is solvable. It follows that H/Λ is compact. Finally observe that a meromorphically separable compact homogeneous manifold is automatically projective and in particular Kähler ([**45**]). But a parallelizable compact complex manifold is Kähler if and only if it is a compact complex torus ([**149**] or thm. 3.14.1). □

Using this lemma, we can describe the algebraic reduction of complex parallelizable manifolds in the following way:

THEOREM 3.15.4. — *Let G be a connected complex Lie group and Γ a lattice. Then there exists an abelian variety A and a closed complex Lie subgroup $H \subset G$ with $\Gamma \subset H$ such that $G/H \simeq A$ as complex manifolds and such that the natural projection $\pi : X \to G/H = A$ induces an isomorphism between the function fields of A and X.*

3.16. The Albanese torus

For a connected compact complex manifold X there exists a torus $\mathrm{Alb}(X)$, called the *Albanese-torus* and a holomorphic map $\mathrm{Alb}_X : X \to \mathrm{Alb}(X)$, called the *Albanese-map* of X with the following universality property:

For every compact complex torus T and every holomorphic map $f : X \to T$ there exists an affin-linear map $F : \mathrm{Alb}(X) \to T$ such that $f = F \circ \mathrm{Alb}_X$.

For a compact Kähler manifold X the Albanese torus can be constructed as $\mathrm{Alb}(X) = \Omega^1(X)^*/\tau H_1(X,\mathbb{Z})$ where $\tau : H_1(X,\mathbb{Z}) \to \Omega^1(X)^*$ is the natural map given by

$$\tau(\gamma) : \omega \mapsto \int_\gamma \omega.$$

The Albanese map is given by

$$\mathrm{Alb}_X(x) : \omega \mapsto \int_{x_0}^{x} \omega$$

where x_0 is an arbitarily choosen fixed base point. This construction is possible, because for Kähler manifolds every holomorphic one-form is closed and $\tau H_1(X,\mathbb{Z})$ is a lattice in $\Omega^1(X)^*$ (both properties follow from Hodge theory).

We will now demonstrate the existence of an Albanese torus for quotients of complex Lie groups by lattices.

PROPOSITION 3.16.1. — *Let G be a complex Lie group and Γ a closed complex Lie subgroup. Assume that every holomorphic function on G/Γ is constant.*

Let I denote the smallest closed complex Lie subgroup of G containing $G'\Gamma$.

Then G/I is a commutative complex Lie group and the natural projection $\pi : G/\Gamma \to G/I$ has the following properties.

1. *For every connected complex Lie group T and every holomorphic map $f : G/\Gamma \to T$ there exists an element $a \in T$ and a holomorphic Lie group homomorphism $F : G/I \to T$ such that $f(x) = a \cdot F \circ \pi(x)$ for all $x \in X$.*
2. *If $I = G'\Gamma$, then every closed holomorphic 1-form on G/Γ is a pull-back from G/I.*
3. *If Γ is a lattice in G, then G/I is compact.*
4. *If Γ is a lattice in G, then every meromorphic function on G/Γ is a pull-back from G/I.*

Proof

1. Let Ad denote the adjoint representation of T and Z the connected component of the center of T. Since holomorphic functions on $GL(\mathcal{L}ie\,T)$ separate points, it is clear that $\mathrm{Ad} \circ f$ is constant for every holomorphic map $f : G/\Gamma \to T$. Thus there is no loss in generality in assuming $Z = T$.

   The cotangent bundle $T^*(Z)$ is spanned by invariant closed holomorphic 1-forms ω_i. Now

   $$0 = df^*\omega_i(X,Y) = f^*\omega_i([X,Y]) + X(f^*\omega_i)Y - Y(f^*\omega_i X) = f^*\omega_i([X,Y])$$

   for every holomorphic map $f : G/\Gamma \to T$, every closed holomorphic 1-form ω_i on Z and every holomorphic vector fields X, Y on $X = G/\Gamma$ (Note that $(f^*\omega_i Y)$ is a holomorphic function on G/Γ and therefore constant.)

   It follows that the G'-orbits are contained in the fibers of f. By construction of I it follows that f fibers through $G/\Gamma \to G/I$.
2. The equation $(*)$ implies that $\Gamma(X, d\mathcal{O}) \simeq \mathcal{L}ie(G/G')^*$. Hence the assertion.
3. From lemma 1.5.2 it follows that G/I has finite volume. Since G/I is an abelian group, this implies compactness.
4. This is a consequence of thm. 3.15.4.

$\square$

3.17. Induced actions on cohomology groups

Let G be a complex Lie group acting holomorphically on a complex space X. Every $g \in G$ induces a biholomorphic map $\mu(g) : X \to X$ which in turn induces a $\mathbb{C}$-linear automorphism on the cohomology groups $H^k(X,\mathbb{C})$ and $H^k(X,\mathcal{O})$. Thus we obtain induced actions of G on the cohomology groups $H^k(X,\mathbb{C})$ and $H^k(X,\mathcal{O})$. If

G is connected, then every $\mu(g)$ is homotopic to id_X. Hence the induced action on $H^\bullet(X,\mathbb{C})$ is trivial for every connected Lie group G.

Let us now consider *compact* complex spaces X. Then $H^k(X,\mathcal{O})$ is finite-dimensional. We want to show that for every complex Lie group acting holomorphically on X the induced action on $H^\bullet(X,\mathcal{O})$ is holomorphic. For this purpose we fix an element $g \in G$ and choose an open neighbourhood $V(g) \subset G$ which is biholomorphic to a ball. Then $V(g)$ is Stein and a Leray spectral sequence argument for the projection map $\pi : V \times X \to V$ yields that $H^k(V \times X,\mathcal{O})$ is naturally isomorphic to the space of holomorphic mappings from V to $H^k(X,\mathcal{O})$. The group action gives us a holomorphic map $\phi : V \times X \to X$ defined by $\phi(h,x) = h \cdot x$ inducing a $\mathbb{C}$-linear map

$$\phi^* : H^k(X,\mathcal{O}) \to H^k(V \times X,\mathcal{O}).$$

Now for any $\alpha \in H^k(X,\mathcal{O})$, $g \in G$ be the pulled-back element $f^*\alpha$ which corresponds to a holomorphic map $f : V \to H^k(X,\mathcal{O})$ such that $f(g) = g(\alpha)$. Therefore the induced action of G on $H^k(X,\mathcal{O})$ is holomorphic. Of course the same argumentation yields that G induces a holomorphic action on the cohomology groups of other canonically defined coherent sheaves like Ω^k, the sheaf of holomorphic k-forms.

Lescure gave a different proof for the holomorphicity of the induced actions on $H^q(X,\Omega^p)$ based on the Dolbeault-isomorphism $H^q(X,\Omega^p) \simeq H^{p,q}_{Dolb}(X)$ (see **[82]**).

It should be emphasized that, for a compact *Kähler* manifold, Hodge decomposition yields a *canonical* injection of the Dolbeault groups $H^q(X,\Omega^p)$ into $H^{p+q}(X,\mathbb{C})$. This implies that for any connected Lie group acting holomorphically on a compact Kähler manifold the induced action on the Dolbeault groups $H^q(X,\Omega^p)$ is trivial.

On the other hand, Lie group actions on non-Kähler manifolds need not be trivial. There are examples of Lescure (see **[81]**) of connected complex Lie groups acting on non-Kähler compact complexs such that the induced action on $H^1(X,\mathcal{O})$ is non-trivial. One such example is a solvmanifold, *i.e.*, a quotient of a solvable complex Lie group by a lattice.

Recently, D.N. Akhiezer proved that for a *reductive* linear algebraic group G with cocompact lattice Γ the G-action on $H^1(G/\Gamma,\mathcal{O})$ is always trivial **[3]**.

3.18. Existence of Zariski dense subgroups

As explained above, every lattice in a complex linear algebraic group is Zariski dense. Since there are obstructions to the existence of lattices (e.g. a Lie group containing a lattice must be unimodular), one might how these obstructions are related to the existence of discrete Zariski dense subgroups. However, it turns out that there are many linear algebraic groups which do admit Zariski dense discrete subgroups although they do not admit any lattice.

In particular, one can prove the following:

THEOREM 3.18.1. — *(see* **[156]***) Let G be a non-solvable connected complex linear-algebraic group.*

Then there exists a discrete Zariski dense subgroup Γ.

Since there are many such groups G which are not unimodular (e.g. any parabolic subgroup in a semisimple complex Lie group), it is clear that there are many complex linear algebraic groups admitting Zariski dense discrete subgroups, but no lattices.

The key idea for the proof of this result is to consider the projection $\pi : G \to G/R$ where R denotes the radical of G. The quotient G/R is a semisimple group. Using "proximal elements" one can construct free subgroups of G/R which are Zariski dense in S. (Alternatively one could also start with an arithmetic subgroup of $S = G/R$ and use the Tits-alternative (thm. 1.7.6) to deduce the existence of free subgroups.) Naturally, an embedding of a *free* group into G/R lifts to an embedding into G. In addition, it turns out that there is enough freedom in the choice of the subgroup and the lifting to construct a free Zariski dense discrete subgroup of G.

3.19. Group homomorphisms coinciding on subgroups of finite index

Later on, in our study of vector bundles, we will need a (rather technical) result concerning group homomorphisms which coincide on subgroups of finite index. Just for convenience we state and prove them in an elementary way although they are essentially contained in **[92]**.

LEMMA 3.19.1. — *Let Γ, H be groups, ρ, $\tilde{\rho} : \Gamma \to H$ group homomorphisms and assume that ρ and $\tilde{\rho}$ coincide on a subgroup of finite index Γ_1 in Γ.*

Then there exists a normal subgroup $\Gamma_0 \subset \Gamma$ of finite index and a map $\zeta_0 : \Gamma/\Gamma_0 \to H$ such that

$$\rho(\gamma) = \zeta(\gamma) \cdot \tilde{\rho}(\gamma) \quad \forall \gamma \in \Gamma,$$

where $\zeta : \Gamma \to H$ is the natural lift of ζ_0.

Furthermore $\zeta(\gamma)$ and $\rho(\lambda)$ commute for every $\gamma \in \Gamma$, $\lambda \in \Gamma_0$.

Proof. — Consider the Γ-action on the finite set Γ/Γ_1. Let Γ_0 denote the ineffectivity, *i.e.*, the kernel of the natural group homomorphism $\Gamma \to Perm(\Gamma/\Gamma_1)$, where for a given set S the group of all permutations of S is denoted by $Perm(S)$. Clearly Γ_0 is a normal subgroup of finite index in Γ.

Define $\zeta(\gamma) := \rho(\gamma)\tilde{\rho}(\gamma)^{-1}$. Now $\rho(\lambda) = \tilde{\rho}(\lambda)$ for $\lambda \in \Gamma_0$ implies $\zeta(\gamma\lambda) = \zeta(\gamma)$ for all $\gamma \in \Gamma$, $\lambda \in \Gamma_0$. Since Γ_0 is normal in Γ, we obtain furthermore $\zeta(\lambda\gamma) = \zeta(\gamma)$ for all $\gamma \in \Gamma$, $\lambda \in \Gamma_0$.

Now observe that

$$\rho(\lambda\gamma) = \zeta(\lambda\gamma)\tilde{\rho}(\lambda\gamma) = \zeta(\gamma)\tilde{\rho}(\lambda)\tilde{\rho}(\gamma)$$

and

$$\rho(\lambda\gamma) = \rho(\lambda)\rho(\gamma) = \zeta(\lambda)\widetilde{\rho}(\lambda)\zeta(\gamma)\widetilde{\rho}(\gamma) = \widetilde{\rho}(\lambda)\zeta(\gamma)\widetilde{\rho}(\gamma)$$

for all $\gamma \in \Gamma$, $\lambda \in \Gamma_0$. It follows that $\zeta(\gamma)$ and $\widetilde{\rho}(\lambda) = \rho(\lambda)$ commute for every $\gamma \in \Gamma$, $\lambda \in \Gamma_0$. $\square$

COROLLARY 3.19.2. — *Let G be a simply connected complex Lie group with $G = G'$, Γ a Zariski dense*[4] *discrete subgroup, $\Gamma_0 \subset \Gamma$ a subgroup of finite index, H a complex linear algebraic group and $\rho : \Gamma \to H$ a group homomorphism.*

Assume that $\rho|_{\Gamma_0}$ extends to a holomorphic group homomorphism $\rho_0 : G \to H$.

Then there exists a group homomorphism $\zeta : \Gamma \to H$ with finite image such that $\rho(\gamma) = \zeta(\gamma)\rho_0(\gamma)$ for all $\gamma \in \Gamma$. Furthermore $\rho_0(g)$ and $\zeta(\gamma)$ commute for every $\gamma \in \Gamma$ and $g \in G$.

Proof. — Due to lemma 3.19.1 it is clear that there exists a *map* $\zeta : \Gamma \to H$ such that $\rho(\gamma) = \zeta(\gamma)\rho_0(\gamma)$ for all $\gamma \in \Gamma$. Let $F = \zeta(\Gamma)$. Then F is centralized by $\rho_0(\Gamma_0)$. Since Γ_0 is of finite index in Γ, it is still Zariski dense in G. Hence F is centralized by $\rho(G_0)$. Now observe that

$$\zeta(gh)\rho_0(g)\rho_0(h) = \rho(gh) = \rho(g)\rho(h) = \zeta(g)\rho_0(g)\zeta(h)\rho_0(h)$$

for all $g, h \in \Gamma$. Since F is centralized by $\rho_0(G)$, it follows that $\zeta(gh) = \zeta(g)\zeta(h)$ for $g, h \in \Gamma$, *i.e.*, ζ is a group homomorphism. $\square$

Similarily one obtains the following variant.

COROLLARY 3.19.3. — *Let G be a connected complex Lie group, Γ a lattice, $\Gamma_0 \subset \Gamma$ a subgroup of finite index, H a complex linear algebraic group and $\rho : \Gamma \to H$ a group homomorphism.*

Assume that $\rho|_{\Gamma_0}$ extends to a holomorphic group homomorphism $\rho_0 : G \to H$.

Then there exists a group homomorphism $\zeta : \Gamma \to H$ with finite image such that $\rho(\gamma) = \zeta(\gamma)\rho_0(\gamma)$ for all $\gamma \in \Gamma$. Furthermore $\rho_0(g)$ and $\zeta(\gamma)$ commute for every $\gamma \in \Gamma$ and $g \in G$.

[4] A simply connected complex Lie group with $G = G'$ carries naturally the structure of an linear algebraic group, see prop. 1.9.1.

CHAPTER 4

SUBVARIETIES

4.1. Survey

In this chapter we are interested in the compact complex subspaces of complex parallelizable manifolds. A substantial part of the results in this chapter is joint work with A.T. Huckleberry, published in [**63**].

In the case of hypersurfaces, tori play a central role: the Albanese mapping $\alpha : X \to \mathrm{Alb}(X)$ is an equivariant surjective fibration $\alpha : X = G/\Gamma \to G/J = \mathrm{Alb}(X)$ and, for every complex hypersurface H in X, there exists a unique hypersurface $\widetilde{H}$ in $\mathrm{Alb}(X)$ with $H = \alpha^{-1}(\widetilde{H})$ ([GR]). Thus, in a certain sense, higher-codimensional subvarieties are of more interest.

Naturally, the easiest way of finding submanifolds is to find them as closed orbits of Lie subgroups. Recall that for every simply connected complex Lie group G and every lattice Γ both the radical and the nilradical of G have compact orbits (thm. 3.5.3). This is also true for the center of G (cor. 3.4.14). Furthermore, there are Cartan subgroups with a closed orbit (prop. 3.3.2) and often the orbits of the commutator group are closed (see §3.11). On the other hand, there are no continuous families. Given a group-theoretically parallelizable complex manifold $X = G/\Gamma$ and a point $x \in X$ there exists at most countably many compact orbits of connected Lie subgroups of G through x (see prop. 3.12.2).

If Z is a compact complex space with Kodaira-dimension $\kappa(Z)$, the canonical ring induces a surjective meromorphic map $\varphi_K : Z \rightharpoonup V$, where $\dim V = \kappa(Z)$ and the "generic" fibers F satisfy $\kappa(F) = 0$ (cf. [**146**]). One of our main goals here is, to group theoretically realize this "pluricanonical map" in the case of $Z \hookrightarrow X = G/\Gamma$. This is carried out in §4.8. For example, if either G is semisimple and X compact or G is solvable, then there is a closed complex subgroup $H < G$ which, up to covering spaces, can be identified with $\mathrm{Aut}_{\mathcal{O}}(Z)^0$, where $H \cdot \Gamma$ is closed, and the canonical map is given by the quotient $Z \to Z/H$. After a natural modification of Γ this fibration can be extended to a fibration of the ambient homogeneous space. It should be remarked that the results in §4.5 and §4.8 are known in the case of tori (cf. [**146**]).

Applying our results on the canonical map, in §4.9 we give upper bounds on the dimension of Z. For example, if G is simple, then $\operatorname{codim} Z \geq \sqrt{\dim G}$.

For nilmanifolds we prove that there are no transversal submanifolds in the following sense:

Assume that $X = G/\Gamma$ is a compact complex nilmanifold, $\Psi : X \to \mathrm{Alb}(X) = G/G'\Gamma$ the projection onto the Albanese. Let Z be an irreducible closed complex subspace of X. If $\Psi(Z) = \Psi(X)$, then $Z = X$.

This is used to show that for certain nilmanifolds every irreducible closed complex subspace must be an orbit. On the other there do exist nilmanifolds $X = G/\Gamma$ with a closed analytic subspace $Z \subset X$ of general type such that Z is not contained in any orbit of any proper connected Lie subgroup $H \subsetneq G$. We do not know whether this phenomen occurs for quotients of non-nilpotent Lie groups as well. In particular the following question is still open:

Does there exists a semisimple complex Lie group S with a discrete cocompact subgroup Γ and a compact subspace of general type $Z \subset G/\Gamma$ such that Z is not contained in a torus T with $Z \subset T \subset G/\Gamma$?

A special case of this question is the following: *Does there exists a compact Riemann surface C of genus $g \geq 2$ and a discrete cocompact subgroup Γ of $SL_2(\mathbb{C})$ such that C can be embedded into $SL_2(\mathbb{C})/\Gamma$?*

In §4.11 we study the Barlet cycle space, which is the complex-analytic analogue for the Chow scheme in algebraic geometry. It turns out that for G not nilpotent the cycle space has always non-compact irreducible components. (In contrast, for a compact Kähler manifold the cycle space has only compact irreducible components.)

Every compact complex parallelizable manifold contains a compact complex subtorus. On the other hand, it is a rather special property for a compact complex torus to be embeddable into a quotient of a complex semisimple Lie group: Only countably many compact complex tori admit such an embedding.

In §4.14 we show that the natural version of the Bloch conjecture is true in this setting: *Let $f : \mathbb{C} \to X$ be a holomorphic map. Then the complex-analytic Zariski closure $Z = \overline{f(\mathbb{C})}$ is the orbit of a subgroup H of G.* In the case of tori, this is a basic result of Green and Griffiths ([GG]) which we in fact use. We also provide a hyperbolicity criterion: *A closed complex subspace Z in a complex parallelizable manifold X is hyperbolic (in the sense of Kobayashi* [**75**]*) if and only if Z contains no compact complex tori.*

We demonstrate that all parallizable manifolds are "abelian connected" regardless of their fundamental group; this is in strong contrast to the Kähler situation (see §4.15).

4.2. Non-compact parallelizable manifolds

Our investigations focus mainly on compact complex parallelizable manifolds. However, it turns out that some results on compact complex subspaces Z of compact complex parallelizable manifolds X actually use only the compactness of Z and not the compactness of X. Therefore we include a short discussion of non-compact complex parallelizable manifolds.

Let X be a complex parallelizable manifold, *i.e.*, a (not necessarily compact) complex manifold whose tangent bundle is holomorphically trivial. If X admits only constant holomorphic functions this implies that the Lie algebra of holomorphic vector fields on X is finite–dimensional, in fact of the same dimension as X. We will see that in order to discuss compact complex subspaces of parallelizable manifolds it is enough to consider parallelizable manifolds X with $\mathcal{O}(X) = \mathbb{C}$. This is joint work with K. Oeljeklaus ([**113**]).

PROPOSITION 4.2.1. — *Let X be a complex manifold, Z a submanifold with $\mathcal{O}(Z) = \mathbb{C}$. Assume that X is locally homogeneous, i.e., the tangent bundle of X is spanned by global sections. Then X has a locally homogeneous complex submanifold Y with $Z \subset Y$ and $\mathcal{O}(Y) = \mathbb{C}$.*

COROLLARY 4.2.2. — *Let X be a complex parallelizable manifold, Z a compact complex subspace. Then X contains a complex parallelizable submanifold X_1 such that $\mathcal{O}(X_1) = \mathbb{C}$ and $Z \subset X_1$.*

Proof. — By the theorem there is a locally homogeneous submanifold X_1 with $Z \subset X_1$ and $\mathcal{O}(X_1) = \mathbb{C}$. Now locally homogeneous means that TX_1 is spanned by global sections. On the other hand the dual T^*X_1 is likewise spanned, because X_1 is a submanifold of a manifold X with trivial cotangent bundle T^*X. With $\mathcal{O}(X_1) = \mathbb{C}$ it follows that TX_1 is trivial, *i.e.*, X_1 is parallelizable. □

Proof of the proposition. — Let $X_0 = X$ and define recursively X_{k+1} as the connected component of

$$X'_{k+1} := \{x \in X_k : f(x) = f|_Z \quad \forall f \in \mathcal{O}(X_k)\}.$$

which contains Z. Then this sequence will terminate, and we set $Y = X_k$ for $k >> 0$. Clearly $\mathcal{O}(Y) = \mathbb{C}$. We have to show that Y is locally homogeneous. Assume that X_k is locally homogeneous. The Zariski tangent sheaf of X_{k+1} is defined by $df = 0$ for all $f \in \mathcal{O}(X_k)$. Let $\mathbf{X}$ be a holomorphic vector field on X_k which is tangent to X_{k+1} at some point $p \in X_{k+1}$. It follows that $df(\mathbf{X})(p) = 0$ and hence $df(\mathbf{X})|_{X_{k+1}} \equiv 0$ for all $f \in \mathcal{O}(X_k)$. This implies that $\mathbf{X}$ is tangential to X_{k+1} everywhere, *i.e.*, $\mathbf{X}$ restricts to a holomorphic vector field on X_{k+1}. Thus for every point $p \in X_{k+1}$ and every tangent vector $v \in T_pX_{k+1}$ there exists a holomorphic vector field $\mathbf{X}$ on X_{k+1} such that $\mathbf{X}_p = v$. It follows that X_{k+1} is locally homogeneous, too. Hence by induction on k it follows that Y is locally homogeneous. □

It should be emphasized that in fact the tangent bundle of Y is spanned by the restrictions of global vector fields on X. In particular, if X is homogeneous, then so is Y.

4.3. Closed orbits as subvarieties

Often submanifolds of a compact complex parallelizable manifold $X \simeq G/\Gamma$ arise as closed orbits of Lie subgroups. As we have already seen in the preceding chapter, many canonically defined subgroups have closed orbits. In particular the radical, the nilradical, the center and all the groups of the ascending central series have closed orbits. In most cases the orbits of the commutator group are closed as well. Besides characteristic subgroups, there is also a way to construct non-normal Lie subgroups with closed orbits using centralizers.

We will now deduce some existence results on closed submanifolds in general and embedded compact complex tori in particular.

PROPOSITION 4.3.1. — *Let G be a connected complex Lie group and Γ a lattice. Then the quotient manifold G/Γ contains a compact complex torus.*

Proof. — This follows from lemma 3.2.3 in combination with thm. 3.9.1. □

For $G = SL_2(\mathbb{C})$ the condition of Γ having finite covolume can be weakened.

PROPOSITION 4.3.2. — *Let Γ be a Zariski dense discrete subgroup of $SL_2(\mathbb{C})$.*

Then the quotient manifold $SL_2(\mathbb{C})/\Gamma$ contains a compact Riemann surface of genus one, i.e., a one-dimensional compact complex torus.

Proof. — Thanks to prop. 1.7.2 and lemma 1.7.12 we may assume that Γ is torsion-free. Since Γ is Zariski dense, it contains a semisimple element γ. Let $C = C(\gamma)$ denote the centralizer of γ in $SL_2(\mathbb{C})$. Then $C \simeq \mathbb{C}^*$ and $\{\gamma^k : k \in \mathbb{Z}\} \simeq \mathbb{Z}$, because γ is a semisimple element of infinite order. It follows that $C/(C \cap \Gamma)$ is a compact torus. □

PROPOSITION 4.3.3. — *Let X be a complex group-theoretically parallelizable manifold, i.e., a quotient of a connected complex Lie group G by a discrete subgroup Γ.*

Then the following conditions are equivalent.

1. *X has no non-trivial connected closed analytic subspaces.*
2. *X is a simple compact complex torus without non-constant meromorphic functions.*

Proof

1. Due to lemma 3.2.3 the manifold X contains a non-trivial closed analytic subspace if G is non-commutative. Thus we may assume $G \simeq (\mathbb{C}^n, +)$. If $X = G/\Gamma$ is non-compact, then either there exists an element $\alpha \in G$ with $\langle \alpha \rangle_{\mathbb{C}} \cap \langle \Gamma \rangle_{\mathbb{R}} = \{0\}$

or there exists an element $\alpha \in \Gamma$ such that $\langle\alpha\rangle_{\mathbb{C}} \cap \langle\Gamma\rangle_{\mathbb{R}}$ is real one-dimensional. In both cases $\langle\alpha\rangle_{\mathbb{C}} \Gamma/\Gamma$ defines a non-trivial closed analytic subspace of X. Finally, if X is a compact complex torus the absence of non-trivial analytic subspaces immediately implies that there are neither subtori nor hypersurfaces. Thus X must be a simple torus without non-constant meromorphic functions.

2. This direction is well-known, but for the sake of completeness we indicate how this can be deduced from results obtained in this book. One may use thm. 4.8.2 in order to deduce that a non-trivial closed analytic subspace Z of a simple torus X is necessarily of general type and therefore Moishezon. Then one considers the map

$$A_k : \underbrace{Z \times \cdots \times Z}_{k \text{ times}} \longrightarrow X, \quad (z_1, \ldots, z_k) \mapsto z_1 + \cdots + z_k.$$

Let Z_k denote the image of Z^k under A_k. Then $(Z_k)_k$ is an increasing sequence connected compact analytic subspaces of X. Thus there is a number N such that $Z_N = Z_k$ for all $k \geq N$. Now $Z_{2N} = Z_N$ implies that Z_N is a subtorus, thus $Z_N = X$ by simplicity of the torus X. But now Z being Moishezon and $A_N : Z^N \to X$ being surjective imply that X must be Moishezon. Therefore a simple torus of algebraic dimension zero can not contain a non-trivial connected closed analytic subset.

□

4.4. On closed orbits in the reductive case

A complex linear algebraic group is *reductive* if and only if it is the complexification of a maximal compact subgroup. If G is reductive and Γ is a cocompact discrete subgroup, then every element of Γ is semi-simple [**123**]). This allows a more precise description of closed orbits in the case where Γ is cocompact and not merely of finite covolume.

LEMMA 4.4.1. — *Let G be a reductive complex Lie group and Γ a cocompact discrete subgroup. Let H be a connected subgroup with a closed orbit in $X = G/\Gamma$. Then the algebraic Zariski closure of H is reductive.*

Proof. — Let R denote the radical of H and $\overline{R}$ algebraic Zariski closure of R in G. It suffices to show that $\overline{R}$ is reductive. Recall that the R-orbits are compact (see thm. 3.5.3). Note that $\overline{R}'$ is unipotent. Since Γ contains only semisimple elements, it follows that the intersection of $R \cap \Gamma$ with $\overline{R}'$ is trivial. Hence $R \cap \Gamma$ is abelian. This implies that R is abelian (see cor. 3.4.16). Thus $\overline{R}$ is abelian and therefore a product $A \times U$ with $A \simeq (\mathbb{C}^*)^k$ and $U \times (\mathbb{C})^l$. Now $R \cap \Gamma \subset A$. Using the compactness of $R/R \cap \Gamma$ this implies $R \subset A$. Hence $\overline{R} \subset A$, *i.e.*, $\overline{R} = A$. Thus $\overline{R}$ is reductive and hence $\overline{H}$ is likewise reductive. □

Note that the group H is perhaps not reductive.

EXAMPLE 4.4.2. — Let $G = (\mathbb{C}^*)^2$ and H be a closed connected subgroup of G which is isomorphic to $\mathbb{C}$, e.g. the image of the map $z \mapsto (e^z, e^{iz})$. Let Γ be a discrete subgroup of H which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Then Γ is cocompact in H and in G.

The following result is needed for our discussion of the extension of the "pluricanonical map" to the ambient homogeneous space.

LEMMA 4.4.3. — *Let G be a reductive complex Lie group and H a connected Lie subgroup whose algebraic Zariski closure is reductive. Then the normalizer group $N_G(H)$ is reductive.*

Proof. — Let $\overline{H}$ denote the Zariski closure of H and let Z be the center of $\overline{H}$. Observe that $\overline{H} = Z \cdot S$ with S semisimple and $H = Z_0 \cdot S$ for some Zariski dense subgroup Z_0 of Z. Let $N = N_G(\overline{H})$ denote the normalizer. The reductivity of $\overline{H}$ implies that $N^0 = \overline{H} \cdot C_G(\overline{H})$, where $C_G(\overline{H})$ denotes the centralizer. It follows that the center Z of H is contained in the center of N^0. This implies that H is normal in N^0. On the other hand $N_G(H) \subset N$, because H is Zariski dense in $\overline{H}$. Thus $N^0 \subset N_G(H) \subset N$. Therefore the reductivity of N implies that $N_G(H)$ is reductive. □

4.5. Subvarieties Z with $\kappa(Z) = 0$

Let Z be an n-dimensional connected complex manifold, $K = K_Z$ its canonical bundle, *i.e.*, $K_Z = \Lambda^n T_Z^*$, and $R_Z = \bigoplus_{k=0}^{\infty} \Gamma(Z, K^k)$ its *pluricanonical ring.* Unless $R_Z = \mathbb{C}$, in which case $\Gamma(Z, K^k) = \{0\}$ for all $k > 0$, the *Kodaira-dimension* $\kappa(Z)$ is defined to be the transcendence degree over $\mathbb{C}$ of the quotient field of R_Z. If $R_Z = \mathbb{C}$, then $\kappa(Z) := -\infty$. It follows that $\kappa(Z) \leq \dim Z$. If $\kappa(Z) = \dim Z$, then Z is said to be of *general type.* The Kodaira-dimension is a bimeromorphic invariant ([**146**]). Therefore it can be defined for an irreducible compact complex space Z by $\kappa(Z) := \kappa(\widehat{Z})$ where $\widehat{Z} \to Z$ denotes any desingularization. For the existence of a desingularization, see [**13, 55, 56**].

Using the pluricanonical ring, one constructs a pluri-canonical map (also called Iitaka-reduction). Its properties are summarized as follows (see [**146**] for details).

THEOREM 4.5.1 (Ueno). — *Let Z be an irreducible reduced compact complex space with $\kappa(Z) \geq 0$. Then there exists a surjective meromorphic map $\psi : Z \rightharpoonup V$ onto a compact complex manifold and a subset U of Z such that*

1. *U is the complement of at most countably many nowhere dense analytic subsets of Z.*
2. *For all $u \in U$ the fiber $F_u = \{\psi^{-1}(\psi\{u\})\}$ is irreducible and smooth. Furthermore $\kappa(F_u) = 0$ and $\dim F_u = \dim Z - \dim V$.*
3. *$\dim V = \kappa(Z)$.*

In general V need not be of general type.[1] Furthermore, often $U \neq Z$. It is an unsolved problem whether or not U is open in Z. As we will see for compact subspaces of (not necessarily compact) parallelizable manifolds the situation is much better. For instance, $U = Z$ in this case.

Recall that, given a compact complex subspace Z in a parallelizable manifold X, there exists a parallelizable submanifold X_1 with $Z \subset X_1 \subset X$ such that $\mathcal{O}(X_1) = \mathbb{C}$ (see §2). Hence wlog $\mathcal{O}(X) = \mathbb{C}$. Then $\mathbf{g} = \Gamma(X, TX)$ is a finite-dimensional Lie algebra and there is a canonical trivialization $TX \simeq X \times \mathbf{g}$. Using this trivialization for each $z \in Z$ the Zariski tangent space T_zZ may be regarded as a subvectorspace of $\mathbf{g}$. This yields a meromorphic (holomorphic, if Z is smooth) map γ from Z to the Grassmannian manifold $G(k, \mathbf{g})$, where $k = \dim_{\mathbb{C}}(Z)$, which we call the *Gauss map* for Z.

LEMMA 4.5.2. — *The Gauss map γ defined as above is a meromorphic map.*

Proof. — The graph Λ of the Gauss map may be defined as follows: $(z,p) \in \Lambda \subset Z \times G(k, \mathbf{g})$ iff $\mathbf{X} f_z = 0$ for all $f \in \mathcal{I}_{Z,z}$ and $\mathbf{X} \in V_p$, where $\mathcal{I}_Z$ is the ideal sheaf of Z and V_p the k-dimensional subvectorspace of $T_zX \simeq \mathbf{g}$ corresponding to $p \in G(k, \mathbf{g})$. Thus it is clear that the graph is an analytic subset of $Z \times G(k, \mathbf{g})$. □

The following observation is trivial but useful.

LEMMA 4.5.3. — *Let V, W be two different k-dimensional subvectorspaces of a vector space $\mathbf{g}$. Then there exists an alternating k-form ω such that $\omega|_V = 0$ but $\omega|_W \neq 0$.*

As a first step we give a description of a "generic pluricanonical fibre" in Z.

THEOREM 4.5.4. — *Let X be a parallelizable complex manifold and Z an irreducible compact complex subspace. Then $\kappa(Z) \geq 0$ and $\kappa(Z) = 0$ iff Z is a parallelizable submanifold.*

Proof. — Let $\widehat{Z}$ be a desingularization of the n-dimensional space Z. Pull-back of n-forms along $\widehat{Z} \to Z \to X$ yields a map $r : \Lambda^n \mathbf{g}^* \to \Gamma(\widehat{Z}, K)$. Therefore $\Gamma(\widehat{Z}, K) \neq \{0\}$, hence $\kappa(Z) \geq 0$.

We may assume $\mathcal{O}(X) = \mathbb{C}$. If there are points $p, q \in Z_{\mathrm{reg}}$ with $T_pZ \neq T_qZ$ (regarded as subvectorspaces of $\mathbf{g}$) then there are n-forms ω, μ induced from $\Lambda^n \mathbf{g}^*$ with $\omega(p) = 0 \neq \omega(q)$ and and $\mu(p) \neq 0 = \mu(q)$. Hence, $\kappa(Z) = 0$ implies that $T_pZ = T_qZ$ for all $p, q \in Z_{\mathrm{reg}}$. Since Z_{reg} is a not necessarily closed submanifold of X, it is clear that we obtain an integrability condition, *i.e.*, $\mathbf{h} = T_pZ$ for $p \in Z_{reg}$ is a Lie subalgebra of $\mathbf{g}$. Locally the vector fields of this Lie subalgebra are integrable and stabilize Z_{reg}, hence Z. Moreover these local orbits are open. This implies that Z is locally homogeneous, therefore $Z = Z_{reg}$. Hence $\kappa(Z) = 0$ implies that

[1] Examples with $V \simeq \mathbb{P}_1(\mathbb{C})$ are provided by certain elliptic surfaces.

Z is a parallelizable submanifold. Conversely parallelizability of Z obviously implies $\kappa(Z) = 0$. □

If X is a quotient of a complex Lie group by a discrete subgroup Γ, then every parallelizable submanifold is an orbit of a Lie subgroup. Thus in this case $Z \subset X$ is an orbit of a Lie subgroup $H \subset G$ if and only $\kappa(Z) = 0$. As a consequence, we deduce that the *analytic Zariski closure of an orbit is an orbit.*

COROLLARY 4.5.5. — *Let X be a group-theoretically parallelizable complex manifold, i.e., $X = G/\Gamma$ for a simply connected complex Lie group G with discrete subgroup Γ, and H a subgroup of G. Let Z denote the analytic Zariski closure of an orbit $H(x)$ of H in X, i.e., Z is the smallest complex subspace of X containing $H(x)$. Suppose that Z is compact. Then Z is the orbit of a complex Lie subgroup I of G.*

Proof. — Define $I = \mathrm{Stab}_G(Z)$. We have to show that I acts transitively on Z. Note that H acts transitively on the set of irreducible components of Z and that $H \subset I$. Thus we may replace Z by an irreducible component and H by the subgroup of H which stabilizes this component, *i.e.*, we may assume that Z is irreducible. For every $n \in \mathbb{N}$ the sections in K_Z^n induce a meromorphic map $\psi_n : Z \rightharpoonup \mathbb{P}_N$. Since these maps are canonically defined, they are equivariant for all automorphisms of Z. Thus ψ_n is H-equivariant. Note that for $n \gg 0$ the image is a variety of general type ([**146**]). Varieties of general type have finite automorphism groups. Thus the H-orbits in the image must be finite. But there is a Zariski dense H-orbit in Z, hence a Zariski dense H-orbit in the image. It follows that the image must be a point. Therefore ψ_n must be constant, *i.e.*, $\kappa(Z) \leq 0$ which implies that Z is an orbit. □

If Z is a subspace of Kodaira-dimension zero (*i.e.*, an orbit), then every compact subspace close to Z is likewise an orbit.

PROPOSITION 4.5.6. — *Let $X = G/\Gamma$ be a group-theoretically parallelizable complex manifold and $Z \subset X$ a compact orbit of a complex Lie subgroup $H \subset G$.*

Then there exists an open neighbourhood U of Z in X and an open neighbourhood W of e in G such that for every compact complex subspace $Y \subset U$ there exists an element $g \in W$ with $Y \subset gZ$.

Proof. — We start with proving the following assertion.

CLAIM 4.5.7. — There exists a open neighbourhood W of e in G such that for every $g \in W$ either $gZ \cap Z = \varnothing$ or $g \in H$.

Since Z is compact, there is a compact "fundamental domain" F, *i.e.*, a compact subset $F \subset H$ with $F\Gamma = H\Gamma$. Now let $W = \{g : g(F) \subset G \setminus (H\Gamma \setminus H)\}$. Then

$$\begin{aligned} gZ \cap Z \neq \varnothing &\iff gF\Gamma \cap H\Gamma \neq \varnothing \\ &\iff gF \cap H\Gamma \neq \varnothing \\ &\iff gF \cap H \neq \varnothing \quad (\text{for } g \in W) \\ &\iff g \in H \end{aligned}$$

Next let W_1 be an open neighbourhood of e in G such that $v \cdot w \in W$ for all $v, w \in W_1$. Then $W_1 \cdot Z$ is a disjoint union of translates of Z. A suitably choosen "slice" $T \subset W_1$ yields a biholomorphic map $T \times Z \xrightarrow{\sim} V$ to some open neighbourhood of Z in X. Hence Z admits an open neighbourhood which is biholomorphically equivalent to a direct product $Z \times T$ with T holomorphically separable. It follows that every compact complex subspace in $U = T \cdot Z$ is contained in a translate of Z. □

4.6. Embeddability of tori

Recall that for a complex Lie group G and a discrete subgroup Γ there exist only countably many connected complex Lie subgroups H with $H/(H \cap \Gamma)$ compact (prop. 3.12.2). This may be reformulated in the following way.

COROLLARY 4.6.1. — *Let X be a group-theoretically parallelizable complex manifold.*
Then up to translation by holomorphic automorphisms $\phi \in \mathrm{Aut}_{\mathcal{O}}(X)$ there exist only countably many compact complex parallelizable submanifolds.

In the semisimple case we can say more.

COROLLARY 4.6.2. — *Up to isomorphism there exist only countably many tori which admit an embedding into a compact quotient S/Γ of a semisimple complex Lie group by a discrete cocompact subgroup.*

Proof. — This is a consequence of the proposition, because there exist only countably many non-equivalent quotients of complex semisimple Lie groups by discrete cocompact subgroups (prop. 3.13.2). □

Thus it is a rather special property for a torus to be embeddable into a parallelizable manifold with semisimple automorphism group.

PROPOSITION 4.6.3. — *Let $T = \mathbb{C}^g/\Lambda$ be a torus, which is embedded (as submanifold) in a compact quotient of a semisimple complex Lie group S by a discrete subgroup Γ.*

Then there exists a number $k \in \mathbb{N}$ and a complex Lie group homomorphism ϕ : $\mathbb{C}^g \to (\mathbb{C}^*)^k$ with discrete kernel such that $\phi(\Lambda) \subset \left(\overline{\mathbb{Q}}^*\right)^k$.

Proof. — Any such torus is an orbit of a Lie subgroup $A \subset S$ which is commutative and whose Zariski closure in S is reductive (lemma 4.4.1). Therefore A is a Lie subgroup of a Cartan subgroup $H \simeq (\mathbb{C}^*)^k$. Now we have an embedding $\phi : A \to (\mathbb{C}^*)^k$. Moreover due to root theory for a given $a \in A$ all the $\phi_k(a)$ are algebraic numbers as soon as all the eigenvalues of $\mathrm{Ad}(a)$ are algebraic numbers. Here Ad denotes the adjoint representation. Now every semisimple complex Lie group may be defined over $\overline{\mathbb{Q}}$. Furthermore, there exists a $g \in G$ such that $g\Gamma g^{-1} \subset S(\overline{\mathbb{Q}})$ ([**123**], Th. 7.67). Hence for all $\gamma \in \Gamma$ all the eigenvalues of $\mathrm{Ad}(\gamma)$ are algebraic. This completes the proof. □

COROLLARY 4.6.4. — *Let $T = \mathbb{C}/\langle 1, \tau\rangle$ be an elliptic curve which may be embedded in a compact quotient $X = S/\Gamma$ of a semisimple complex Lie group by a discrete subgroup.*

Then τ is not an algebraic number.

Proof. — By the above result there exists $\alpha \in \mathbb{C}^*$ such that both e^α and $e^{\alpha\tau}$ are algebraic. On the other hand α and $\alpha\tau$ are clearly linearly independent over $\mathbb{Q}$. Bakers theorem now implies that α and $\alpha\tau$ are linearly independent over $\overline{\mathbb{Q}}$. Thus $\tau \notin \overline{\mathbb{Q}}$. □

More generally the proposition has the following consequence.

COROLLARY 4.6.5. — *Let $T = \mathbb{C}^g/\Lambda$ be a torus, which is embedded (as submanifold) in a compact quotient of a semisimple complex Lie group S by a discrete subgroup Γ.*

Then T does not admit complex multiplication (as defined in def. 9.5.2).

Proof. — This follows from proposition 4.6.3 in combination with proposition 9.13.3. □

4.7. Moishezon subspaces, Kähler subspaces and subspaces of general type

Subspaces which are "at the opposite extreme" to $\kappa(Z) = 0$ are those of *general type*, *i.e.*, subspaces with $\kappa(Z) = \dim Z$. Of course such spaces are Moishezon, *i.e.*, $\operatorname{tr}\deg_{\mathbb{C}} \mathcal{M}(Z) = \dim Z$. Now Moishezon spaces are in Fujiki's class $\mathcal{C}$. This is by definition the class of compact complex spaces which are holomorphic images of Kähler manifolds. Equivalently, a compact complex space is in class $\mathcal{C}$ if it is bimeromorphically equivalent to a Kähler manifold ([**147**]). Sometimes manifolds in class $\mathcal{C}$ are called *weakly Kähler.*

If Z is in class $\mathcal{C}$, then it is possible to prove strong statements regarding its Albanese mapping. Of course Z may be singular, so the Albanese $\alpha : Z \rightharpoonup \mathrm{Alb}(Z)$ is defined by the Albanese of a desingularization $\widehat{Z}$. Recall that a meromorphic mapping of a complex *manifold* into a space with a Stein universal cover, e.g. $\mathbb{C}^n$, is holomorphic. Thus the Albanese is defined independent of the desingularization.

PROPOSITION 4.7.1. — *Let Z be a compact subspace of a complex parallelizable manifold X. Let $\psi : Z \to \mathrm{Alb}(Z)$ be the Albanese. If Z is in class $\mathcal{C}$, then* $\mathrm{rank}(d\psi) = \dim Z$ *at all nonsingular points of Z, and in particular $\psi : Z \to \psi(Z)$ is an unramified covering if Z is smooth.*

Proof. — Let $\widetilde{Z}$ be a desingularization. Since Z is in class $\mathcal{C}$, it follows that every holomorphic 1-form on $\widetilde{Z}$ is closed and a pull-back from the Albanese torus. Therefore $\mathrm{Ker}(d\psi)$ is the space of those tangent vectors which are annihilated by all holomorphic 1-forms on $\widetilde{Z}$. Thus the first assertion is proven by considering holomorphic one-forms on X restricted to Z and lifted to $\widetilde{Z}$. For the second assertion note that any proper locally biholomorphic map is an unramified covering. □

COROLLARY 4.7.2. — *Let Z be a compact subspace of a parallelizable complex manifold X. Assume that Z is in class $\mathcal{C}$. Then* $\dim \mathrm{Alb}(Z) \geq \dim Z$*. Equality holds if and only if Z is a torus. In particular, if Z is of general type, then* $\dim \mathrm{Alb}(Z) > \dim Z$.

Proof. — If Z is of general type, then it is of course in class $\mathcal{C}$ – it is bimeromorphic to a projective algebraic manifold – and therefore, by the above, $\dim \mathrm{Alb}(Z) \geq \dim Z$. If $\dim \mathrm{Alb}(Z) = \dim Z$, then $\dim \Gamma(Z, \Omega^1) = \dim Z$. Hence $\kappa(Z) = 0$. Now theorem 4.5.4 implies that Z is smooth. Thus proposition 4.7.1 implies that Z is an unramified cover of the torus $\mathrm{Alb}(Z)$ and therefore itself a torus. □

For smooth subspaces we have the following refinement of the above result.

PROPOSITION 4.7.3. — *Let Z be a compact submanifold of a complex parallelizable manifold X. Then the following statements are equivalent:*

1. *Every holomorphic 1-form on Z is closed;*
2. *Z is in class $\mathcal{C}$;*
3. *Z is a Kähler manifold;*
4. *The Albanese $\psi : Z \to \psi(Z)$ is an unramified covering.*

Proof. — (4) $\Rightarrow$ (3) is proved by pulling back a Kähler form from $\mathrm{Alb}(Z)$. (3) $\Rightarrow$ (2) follows immediately from the definitions, and (3) $\Rightarrow$ (1) is well-known. (2) $\Rightarrow$ (4) is contained in prop. 4.7.1 above.

Finally, to prove (1) $\Rightarrow$ (3), let $\omega_1, \ldots, \omega_n$ be holomorphic 1-forms on X which form a basis of the space of sections of its cotangent bundle. Since their restrictions are closed, $\omega := \sum \omega_i \wedge \bar{\omega}_i$ defines a Kähler form on Z. □

The statement that the 1-forms are closed can be translated into a rather strong statement on the tangent space T_pZ, $p \in Z_{\mathrm{reg}}$. For a complex parallelizable manifold X with $\mathcal{O}(X) = \mathbb{C}$, there is a unique trivialization of the tangent bundle $TX \simeq X \times \Gamma(X, TX)$. In this way T_pZ may be regarded as subvectorspace of the Lie algebra of holomorphic vector fields $\Gamma(X, TX)$. We will now demonstrate that T_pZ forms an *abelian subalgebra* of $\Gamma(X, TX)$ for compact subspaces Z of class $\mathcal{C}$.

PROPOSITION 4.7.4. — *Let Z be a compact subspace of class $\mathcal{C}$ of X, where X is a complex parallelizable manifold with $\mathcal{O}(X) = \mathbb{C}$. Then, for all $p \in Z_{\mathrm{reg}}$, the tangent space T_pZ corresponds to an abelian subalgebra of the Lie algebra of holomorphic vector fields $\Gamma(X, TX)$.*

Proof. — Let $v, w \in T_pZ$ and let $\xi, \eta \in \Gamma(X, TX)$ be the corresponding vector fields (*i.e.*, $\xi_o = v$ and $\eta_p = w$). Let $\imath : Z \to X$ be the natural injection. If ω is a holomorphic 1-form on X, then

$$0 = d\imath^*(\omega)_p(v, w) = d\omega_p(\imath_*(v), \imath_*(w)) = d\omega(\xi, \eta) = \xi\omega(\eta) - \eta\omega(\xi) - \frac{1}{2}\omega([\xi, \eta]).$$

Note that $\omega(\eta)$ and $\omega(xi)$ are global holomorphic functions on X, hence constant. Thus the above equations yield $\omega([\xi, \eta]) = 0$. Since ω was choosen arbitrarily and T^*X is trivial (hence spanned by global sections), it follows that $[\xi, \eta] = 0$. □

4.8. The pluricanonical map

In this section we prove a structure theorem for the pluricanonical map of compact subvarieties of parallelizable manifolds.

THEOREM 4.8.1. — *Let Z be an irreducible compact complex subvariety of $X = G/\Gamma$, where G is a simply connected complex Lie group and Γ a discrete (not necessarily cocompact) subgroup. Then there exists a subgroup $\Gamma_0 \subset \Gamma$ and a complex Lie subgroup $\Gamma_0 \subset H \subset G$ such that:*

1. *The embedding $i : Z \hookrightarrow X$ can be lifted to an embedding $i_0 : Z \hookrightarrow G/\Gamma_0$;*
2. *The projection $\pi : G/\Gamma_0 \to G/H$ restricted to $i_0(Z)$ yields a pluricanonical map $\psi : Z \to Y$. All fibers of ψ are of Kodaira-dimension zero and moreover are isomorphic to H/Γ_0.*
3. *With $\Gamma \simeq \pi_1(X)$ the subgroup $\Gamma_0 \subset \Gamma$ is the image of the group homomorphism $\pi_1(Z) \to \pi_1(X)$ induced by the embedding map.*

In particular *all* fibers of the pluricanonical map have Kodaira-dimension zero.

Proof. — We may assume $e\Gamma \in Z$. Consider the canonical projection $\pi : G \to X$ and let Z_0 be the connected component of $\pi^{-1}(Z)$ which contains the identity e. Regard G to be acting on itself on the right. Let $\Gamma_0 := \mathrm{Stab}_\Gamma(Z_0)$. Then $\Gamma_0 \subset \Gamma \simeq \pi_1(X)$ is simply the image of the group homomorphism $\pi_1(Z) \to \pi_1(X)$ induced by the embedding $i : Z \hookrightarrow X$. Thus the embedding $i : Z \hookrightarrow X = G/\Gamma$ can be lifted to an embedding $i_0 : Z \hookrightarrow G/\Gamma_0$. In this way we may regard Z as a subvariety of G/Γ_0.

Let $\psi : Z \rightharpoonup V$ be a pluricanonical map and $U \subset Z$ the set of "generic" fibers as discussed in thm. 4.5.1. Let $\pi_0 : G \to G/\Gamma_0$ be the natural map and, for $g \in Z_0$, let $F_{\pi_0(g)}$ be the ψ-fiber through $\pi_0(g)$ whenever it is defined. For $g \in \pi_0^{-1}(U)$ it follows that $\kappa(F_{\pi_0(g)}) = 0$ and thus $g^{-1}F_{\pi_0(g)}$ is an orbit of a connected Lie subgroup $I(g)$ (theorem 4.5.4). Recall that there exists only countably many compact orbits of complex connected Lie subgroups through $e\Gamma_0$ (proposition 3.12.2). Recall further that U is the complement in Z of a countable union of nowhere dense analytic subsets and that $\pi_0^{-1}(U)$ is dense in the *connected* analytic subset $Z_0 \subset G$. Together these facts imply that there exists a connected complex Lie subgroup $I \subset G$ such that $gI\Gamma_0$ is contained in a ψ-fiber for all $g \in \pi_0^{-1}(Z) = Z_0$. This implies in particular that $gI\Gamma_0 = I\Gamma_0$ for $g \in \Gamma_0$, *i.e.*, Γ_0 normalizes I. Moreover it follows that the pluricanonical map (up to some bimeromorphic modification of the base space) coincides with the fibration of $G/\Gamma_0 \to G/I\Gamma_0$ restricted to Z. □

For subvarieties of certain parallelizable manifolds we can say more.

THEOREM 4.8.2. — *Assume in addition to the assumptions of the above theorem that either G is solvable or that G is reductive and X is compact.*
Then

1. *The base space Y can be embedded into a (not necessarily compact) complex parallelizable manifold;*
2. *The base space Y is of general type, i.e., $\kappa(Y) = \dim(Y)$;*
3. *H^0 is isomorphic to a covering group of $\mathrm{Aut}_{\mathcal{O}}(Z)^0$ and ψ realizes the geometric quotient for the $\mathrm{Aut}_{\mathcal{O}}(Z)^0$-action on Z, i.e., the fibers of ψ are exactly the orbits of $\mathrm{Aut}_{\mathcal{O}}(Z)^0$;*
4. *For every non-singular point $p \in Z \setminus Sing(Z)$ the tangent space T_pZ corresponds to a Lie subalgebra of $\Gamma(X, TX)$.*

Proof. — We preserve the notation introduced with the preceding theorem. Then the pluricanonical map $\psi : Z \to Y$ can be realized as restriction of a fibration $G/\Gamma_0 \to G/H$. Let N denote the normalizer of H^0 in G. Then $\Gamma_0 \subset N$, hence we obtain a map $Z \to G/N$. We claim that this map is constant. Since N is the normalizer of a connected Lie subgroup, it follows that G/N can be embedded equivariantly in some Grassmann manifold. If G is solvable, this implies that G/N is holomorphically separable (see e.g. **[59]**). Now let us consider the second case, *i.e.*, G is reductive and X is compact. Then it follows that N is reductive (lemma 4.4.3). This implies that G/N is Stein **[95]**. Thus in both cases G/N does not contain any positive-dimensional compact complex subvarieties, *i.e.*, the map $Z \to G/N$ must be constant. We may therefore replace G by N and hence assume that H^0 is normal in G.

Under this assumption G/H is a quotient manifold of G/H^0 by its discrete subgroup H/H^0. In other words: G/H is parallelizable.

Now assume that $Y \subset G/H$ is not of general type. Then we obtain a non-trivial pluricanonical map $\psi' : Y \to W$ which is the restriction of some fibration $G/H \to G/I$. This yields a surjective holomorphic map $f : Z \to W$ where all fibers are isomorphic to I/Γ_0. In particular, the fibers have Kodaira-dimension zero. However, for every connected holomorphic map $f : Z \to W$ of compact complex spaces the inequality $\kappa(Z) \leq \kappa(F) + \dim(W)$ holds for a generic fiber F (see **[146]**, Th.6.12). Thus $\dim(Y) = \kappa(Z) \leq \dim(W)$. Hence Y must be of general type.

Since H^0 is normal in G, it follows that $hg\Gamma_0 \in gH$ for all $g\Gamma_0 \in Z$. Since gH is the fiber of the pluricanonical map through $g\Gamma_0 \in Z$, this implies that the (left) H-action on G/Γ_0 stabilizes Z. Hence $H^0 \hookrightarrow \mathrm{Aut}_{\mathcal{O}}(Z)^0$ and furthermore it follows that the fibers of the pluricanonical map are just the orbits of H^0.

We have to show that $\mathrm{Aut}_{\mathcal{O}}(Z)^0$ is not larger than H^0. Recall that a proper connected holomorphic map is equivariant for any action of a connected complex Lie group (**[126]**, see also thm. 5.2.1). Moreover there exists a group homomorphism $\rho : \mathrm{Aut}_{\mathcal{O}}(Z)^0 \to G$ such that every automorphism $\phi \in \mathrm{Aut}_{\mathcal{O}}(Z)^0$ is given by left multiplication with $\rho(\phi)$ (see proposition B.1). Therefore $H^0 \neq \mathrm{Aut}_{\mathcal{O}}(Z)^0$ implies that there exists a positive-dimensional connected complex Lie group acting non-trivially on Y. This is impossible, because Y is of general type **[146]**, Cor.14.3.

Finally we prove statement (4). Observe that $\mathcal{L}ie(H) \subset T_pZ \subset \mathcal{L}ie(G)$. Now $T_p(Z)/\mathcal{L}ie(H)$ corresponds to the tangent T_qY for $q = \psi(p)$. Since Y is of general type, it follows from proposition 4.7.4 that T_qY is an (abelian) Lie subalgebra of $\mathcal{L}ie(G)/\mathcal{L}ie(H)$. This implies that $T_p(Z)$ is a Lie subalgebra of $\mathcal{L}ie(G)$. □

4.9. Dimension bounds for subvarieties

Here we restrict to the case where $X = G/\Gamma$ is compact, Γ is discrete and G is semi-simple.

Our goal here is to prove that the subvarieties of X have a high codimension, *i.e.*, there are no subvarieties of low codimension. This generalizes results of **[58]** on the non-existence of hypersurfaces. In order to handle the case of orbits, we begin with the following elementary rough estimate.

LEMMA 4.9.1. — *Let G be a simple linear algebraic group and H a proper connected Lie subgroup which is not parabolic. Then there exists a proper parabolic Lie subgroup P of G with* $\dim H < \dim P$ *and* $\mathrm{codim}_G H \geq \sqrt{\dim G}$.

Proof. — For any connected Lie subgroup H the commutator group H' is algebraic and equals the commutator group $\bar{H}'$ of the Zariski closure $\bar{H}$ **[20]**. Since $G = G'$, it follows that a proper connected Lie subgroup H can not be Zariski dense in G. Thus we may replace H by its Zariski closure and thereby assume that H is algebraic. Furthermore we may assume that H is maximal in the sense that it is not contained in any higher-dimensional non-parabolic subgroup of G except G itself.

Since H is algebraic, there exists a equivariant embedding of G/H into some projective space $\mathbb{P}_N$. Hence it is clear that there exists a number $n \in \mathbb{N}$ and an equivariant morphism $\phi : G/H \to \mathbb{P}_n$ with finite fibers such that for every $k < n$ every equivariant morphism from G/H to $\mathbb{P}_k$ has positive-dimensional fibers. We fix such a ϕ. Since H is not parabolic, G/H is not compact. Hence there exists a closed G-orbit G/Q in the boundary of $\phi(G/H)$. If $G \neq Q$, the proof is complete. Assume $G = Q$, *i.e.*, G has a fixed point p in $\mathbb{P}_n$. Complete reducibility of representations of G implies that there is a complementary stable hyperplane $H \simeq \mathbb{P}_{n-1}$ in $\mathbb{P}_n$ and an equivariant projection $\mathbb{P}_n \setminus \{p\} \to \mathbb{P}_{n-1}$. The minimality of $G/H \to \mathbb{P}_n$ implies that the induced morphism $G/H \to \mathbb{P}_{n-1}$ has one-dimensional fibers. Thus we obtain a fibration $G/H \to G/I$ with $H \subset I \subset G$ and $\dim I = \dim H + 1$. Now H being non-parabolic implies $\dim H \leq \dim G - 2$, hence $I \neq G$. Thus by our maximality assumption I is parabolic. The estimate $\mathrm{codim}_G H \geq \sqrt{\dim G}$ now follows from a case by case check of the dimensions of the maximal parabolic subgroups (use e.g. **[144]**). □

REMARK 4.9.2. — Maximal connected Lie subgroups of reductive subgroups have been classified by Dynkin **[38]**, **[39]**. Using this classification one could deduce sharp bounds on the dimension of Lie subgroups for every given semisimple Lie group. However, for our purposes here, this would not yield a result of qualitative difference.

Combining the above lemma with statement (4) of theorem 4.8.2 yields the following result.

THEOREM 4.9.3. — *Let G be a simple complex Lie group, Γ a discrete cocompact subgroup, and Z a proper complex subspace of $X = G/\Gamma$. Then* $\mathrm{codim}(Z) \geq \sqrt{\dim G}$.

If Z is in class $\mathcal{C}$, then for every $p \in Z_{\mathrm{reg}}$ the tangent space T_pZ is an abelian subalgebra of $\mathbf{g}$. As a consequence $\dim Z$ is bounded from above by the maximum dimension of an abelian subgroup.

THEOREM 4.9.4. — *Let G be a semi-simple complex Lie group, Γ a discrete subgroup, $X := G/\Gamma$, and Z a compact subvariety of X which is in class $\mathcal{C}$. Then* $\dim Z \leq \dim X/3$.

Proof. — The upper bound for the dimension of abelian Lie subgroups of type A_n ($SL_{n+1}(C)$) is a classical result of Schur **[132]**. Malcev **[88]** obtained similar results for all semi-simple Lie groups. They are summarized in the table below, where $a(G)$ denotes the maximal possible dimension for an abelian Lie subgroup.

Here [n] denotes the greatest integer smaller or equal to n.

From this it follows that $a(G) \leq (\dim G + \mathrm{rank}(G))/4$ and $a(G) \leq \dim G/3$ for any complex semi-simple Lie group G. □

4.10. Transversal Submanifolds in Nilmanifolds

We will demonstrate that for nilmanifolds there are no submanifolds transversal to the Albanese fibration. This may be regarded as a geometric analog for the following

Group	A_n	B_3	B_4	$B_n\,(n>4)$	C_n	D_4
$\dim G$	$(n+1)^2-1$	21	36	$2n^2+n$	$2n^2+n$	28
$a(G)$	$[(n+1)^2/4]$	5	7	$1+n(n-1)/2$	$n(n+1)/2$	6

Group	$D_n\,(n>4)$	E_6	E_7	E_8	F_4	G_2
$\dim G$	$2n^2-n$	78	133	248	52	14
$a(G)$	$n(n-1)/2$	16	27	36	9	3

well-known fact in group theory: *Let G be a nilpotent group, H a subgroup such that $H \cdot G' = G$. Then $H = G$.*

THEOREM 4.10.1. — *Let $X = G/\Gamma$ a compact nilmanifold, i.e., a quotient of a connected complex nilpotent Lie group G by a discrete cocompact subgroup Γ. Let $Y = X/G' = G/G'\Gamma = \mathrm{Alb}(X)$ and $\pi : X \to Y$ the natural projection. Assume that Z is a complex subspace of X with $\pi(Z) = Y$.*

Then $Z = X$.

We start with some preparation for the proof.

DEFINITION 4.10.2. — Let $E \to B$ be a torus principal bundle with structure group T. Then E is called *almost trivial* if there exists a finite subgroup Λ of the torus T such that $E/\Lambda \to B$ is trivial.

LEMMA 4.10.3. — *Let B be a complex manifold, $E \to B$ a torus principal bundle and $Z \subset E$ an irreducible complex subspace with* $\dim(Z) = \dim(B)$ *such that the projection of Z to B is surjective.*

Then E is almost trivial.

Proof. — Generically $\pi : Z \to B$ must be locally injective. Let S denote the union of all positive–dimensional components of the fibers of the projection $\pi : Z \to B$. Then S is a nowhere dense analytic subset and $M = \pi(S)$ is of codimension at least two in B. We claim: *If $E \setminus \pi^{-1}(M) \to B \setminus M$ is almost trivial, then $E \to B$ is almost trivial, too.* This holds, because triviality of $E/\Lambda \to B$ is equivalent to the existence of a section. For small enough $U \subset B$ the restriction of E to U is trivial, hence a section is simply a map $\sigma : U \to T$. However the universal covering of T is $\mathbb{C}^n$. Hence any holomorphic map $\sigma_0 : U \setminus M \to T$ extends through M. Thus we may neglect M and assume that all fibers of $E \to B$ are finite. Let N denote the number of points in a generic fiber. Let Λ be the (finite) subgroup of T containing all elements g with $Ng = 0$. Now for any N–tuple $a_1, \ldots, a_n$ in T the mean value $\frac{1}{N}\sum_i a_i$ is well-defined up to translation with elements in Λ. Note that taking the mean value is an operation invariant under an affine change of coordinates. Hence the mean value of all points in a given fiber of $Z \to B$ is a well-defined point in E/Λ. Thus we obtain a section in the bundle $E/\Lambda \to B$, *i.e.*, E is almost trivial. □

In the following $h_1(W)$ always denotes $\dim_{\mathbb{Q}}(H_1(W, \mathbb{Q}))$.

LEMMA 4.10.4. — *Let $E \to B$ be an almost trivial torus principal bundle. Then $h_1(E) > h_1(B)$.*

Proof. — The covering $E \to E/\Lambda$ induces a short exact sequence

$$1 \longrightarrow \pi_1(E) \longrightarrow \pi_1(E/\Lambda) \longrightarrow \Lambda \longrightarrow 1.$$

Since Λ is finite, it follows $h_1(E) \geq h_1(E/\Lambda)$. Furthermore $h_1(E/\Lambda) = h_1(T/\Lambda) + h_1(B)$ with $h_1(T/\Lambda) = h_1(T) > 0$. □

Now we are in a position the prove the theorem.

Proof of the theorem. — It is well-known ([**89**], see also cor. 2.2.3) that all the groups G^k of the (descending) central series have closed orbits. Hence we obtain a tower of torus principal bundles

$$X = X_0 \longrightarrow \cdots \longrightarrow X_n = G/G'\Gamma$$

given by $X_k = G/G^{n+1-k}$, where n is the smallest number for which $G^{n+1} = \{e\}$. We may refine this tower such that all the tori occuring as fibers are simple, *i.e.*, admit no subtori. Now $(\Gamma \cap G')/\Gamma'$ is finite, hence $h_1(X) = h_1(X_n)$. Observe that for every surjective proper holomorphic map between complex manifolds $f : V \to W$ the quotient $\pi_1(W)/f_*(\pi_1(V))$ is finite. Therefore $h_1(X_{k+1}) \leq h_1(X_k)$ for all k. Since $h_1(X_0) = h_1(X_n)$, it follows that all the numbers $h_1(X_k)$ coincide. By the above lemma this implies in particular that none of the torus principal bundles $X_k \to X_{k+1}$ can be almost trivial.

We will inductively show that Z is mapped surjectively on the X_i. By assumption it is mapped surjectively on X_n. Now let $E \to B$ be a torus–principal bundle (not almost trivial) with a simple torus as fiber and $Z \subset E$ with $\pi(Z) = B$. We want to show that $Z = E$. By lemma 4.10.3 we know already that the fibers can not be finite. Now assume that the fibers of $Z \to B$ are generically d–dimensional. For $x \in Z$ let Z_x denote the fiber of $Z \to B$ through x. Thanks to the principal action of the torus we may regard $T_x(Z_x)$ as a subvectorspace of the Lie algebra of the torus. Thus we obtain a meromorphic Gauss map form Z to some Grassmann manifold. Assume that generically Z_x is neither finite nor the whole torus T. Since T admits no subtori, this implies that the Gauss map is not constant along the fiber Z_x. Using the pull-back of a meromorphic function on the Grassmann manifold, it follows that Z admits a meromorphic function which generically is not constant on the fibers of $Z \to B$. But this implies that any level set Z^1 of this function is mapped surjectively on B. Inductively it follows that there exists a Z^k such that $Z^k \to B$ is surjective and generically finite. However this contradicts E not being almost trivial. Hence $Z = E$. □

For some nilmanifolds every subspace is an orbit.

PROPOSITION 4.10.5. — *Let G be a simply connected complex nilpotent Lie group, Γ a discrete cocompact subgroup, $X = G/\Gamma$, $A = \mathrm{Alb}(X) = G/G'\Gamma$.*

Assume that there exists a sequence of subtori

$$\{0\} = A_0 \subset A_1 \subset \cdots \subset A_n = A$$

such that A_{k+1}/A_k is a simple torus of algebraic dimension zero for all k. Then every closed irreducible complex subspace $Z \subset X$ is a parallelizable submanifold, i.e., an orbit.

Proof. — Let Z be a closed irreducible complex subspace of X. Let $Y \subset X$ be a minimal connected nilmanifold containing Z. The results of Chapter 9 imply that $\mathrm{Alb}(Y)$ likewise fulfills the condition stated above for $\mathrm{Alb}(X)$. This implies that every closed irreducible complex subspace of $\mathrm{Alb}(Y)$ is a subtorus. Now minimality of Y implies that $\pi : Z \to \mathrm{Alb}(Y)$ is surjective. Theorem 4.10.1 now yields $Z = Y$, *i.e.*, Z is parallelizable. □

4.11. The Cycle Space

For a complex space X the *cycle space* $\mathcal{C}$ in the sense of Barlet **[9]** parametrizes all pure-dimensional compact analytic cycles, *i.e.*, formal linear combinations $\sum_i n_i Z_i$ with $n_i \in \mathbb{N}$ and each Z_i being a compact irreducible reduced complex subspace of the same dimension in X. It corresponds to the Chow scheme in algebraic geometry. We will now study cycle spaces for parallelizable manifolds.

PROPOSITION 4.11.1. — *Let $X = G/\Gamma$ be a group–theoretically parallelizable manifold. Let Z be a compact parallelizable submanifold and $[Z]$ the corresponding point in the cycle space. Let S be the irreducible component of the cycle space $\mathcal{C}_X$ containing $[Z]$. Then G acts transitively on S, i.e., $[C] \in S$ if and only if $C = gZ$ for some $g \in G$.*

Proof. — Recall that $Z = H/(H \cap \Gamma)$ for some connected Lie subgroup H. Proposition 4.5.6 implies immediately that $[Z]$ has an open neighbourhood S^* in S which contains only translates of Z. Now let $V \subset X \times S$ be the associated universal cycle. On a Zariski -open subset U of V we may define a Gauss map in the following way: To $(x, [C])$ associate the tangent space $T_x C$ regarded as a subvectorspace of the Lie algebra $\mathbf{g}$. U may be choosen in such a way that it intersects all fibers of $V \to S$. We obtain a holomorphic map from U to a Grassmann manifold which depends only on $[C]$ but not on x provided $[C] \in S^*$. From the identity principle it follows that γ never depends on x. It follows that any cycle in S has the form $\sum_i n_i Z_i$ with Z_i

parallelizable and of the same dimension as Z. Now observe that the G–orbit through $[Z]$ in S is open and has dimension $codim(Z)$. The G–orbit through a cycle $\sum_i n_i Z_i$ has dimension greater or equal to the codimension of the Z_i in X. Since an irreducible space S can admit at most one open orbit of a complex Lie group G, it follows that the irreducible component S equals the G–orbit through $[Z]$. Moreover it follows that S is actually a connected component of the cycle space, because for every irreducible component S' of $\mathcal{C}_X$ the intersection $S \cap S'$ must be G-stable. □

We will now start to investigate compactness properties of the cycle space.

PROPOSITION 4.11.2. — *With the above notation we obtain $S = G/I$ with $I = H \cdot (\Gamma \cap N_G(H))$. Furthermore S is compact iff the following two conditions are fulfilled: H is normal in G and G/Γ is compact.*

Proof. — Since $H/(H \cap \Gamma)$ is compact, it follows that S is compact iff $G/(\Gamma \cap N_G(H))$ is compact. Thus S is compact iff $\Gamma_0 = \Gamma \cap N_G(H)$ is of finite index in Γ and G/Γ compact. It remains to show that Γ_0 being of finite index in Γ already implies that H is normal in G. Recall that $N_G(H)$ may be realized as the kernel of a linear representation ρ of G. Now $\rho(\Gamma)$ is Zariski -dense in $\rho(G)$ (thm. 3.4.1). This completes the proof. □

Thus we obtain many new examples of compact complex manifolds for which the cycle space has non-compact irreducible components.

In contrast, for large classes of compact complex manifolds it is known that all the irreducible components of the cycle space are compact, e.g. this is true for surfaces, Kähler manifolds and more generally all manifolds in class $\mathcal{C}$.

There are also other examples of compact complex manifolds whose cycle space has non-compact irreducible components. See **[26]** for examples based on twistor spaces.

COROLLARY 4.11.3. — *Let G be a simply connected complex Lie group, Γ a discrete cocompact subgroup. Assume that G is not nilpotent.*

Then the cycle space $\mathcal{C}_X$ of $X = G/\Gamma$ has a non-compact irreducible component.

Proof. — There is a Lie subgroup H with closed orbit $Z = H/(H \cap \Gamma)$ such that $\mathcal{L}ie(H)$ is a Cartan subalgebra of $\mathcal{L}ie(G)$ (prop. 3.3.2). By definition of a Cartan subalgebra $\mathcal{L}ie(H)$ is never an ideal in $\mathcal{L}ie(G)$ unless $\mathcal{L}ie(G)$ is nilpotent. □

For tori every irreducible component of the cycle space is compact, because tori are Kähler. For some nilmanifolds X there are non-compact irreducible components of $\mathcal{C}_X$, e.g. this is true for the standard Iwasawa manifold.

4.12. Fundamental groups of subspaces

A complex group-theoretically parallelizable manifold has a universal covering which is biholomorphic to a simply connected complex Lie group and thereby biholomorphic to a Stein manifold. It follows that a complex group-theoretically parallelizable manifold can not contain simply connected compact complex subspaces. Moreover it follows that for any compact complex subspace the embedding map is topologically non-trivial.

LEMMA 4.12.1. — *Let X be a complex group-theoretically parallelizable manifold and Z a compact complex subspace. Then the embedding map $i : Z \to X$ induces a group homomorphism $i_* : \pi_1(Z) \to \pi_1(X)$ with infinite image.*

Proof. — Write $X = G/\Gamma$ with G simply connected. Then $\Gamma \simeq \pi_1(X)$. By standard covering theory the embedding $i : Z \to X$ can be lifted to an embedding $j : Z \to G/(i_*\pi_1(Z))$. This yields the assertion, because G/Λ is Stein for every finite subgroup Λ. □

If $i_*\pi_1(Z)$ is commutative, then Z is contained in a torus.

PROPOSITION 4.12.2. — *Let G be a simply connected complex linear algebraic group, Γ a discrete cocompact subgroup, $Z \subset X = G/\Gamma$ a compact complex subspace.*

Assume that $i_*\pi_1(Z)$ is a commutative subgroup of $\pi_1(X) \simeq \Gamma$, where $i : Z \to X$ denotes the embedding.

Then there exists a submanifold T which is biholomorphic to a compact complex torus such that $Z \subset T \subset X$.

Proof. — We may assume $e\Gamma \in Z$. Let C denote the centralizer of $i_*\pi_1(Z)$ in G and A the center of C. Clearly $i_*\pi_1(Z) \subset A$. Since the orbit of C through $e\Gamma$ is closed, it follows that the A-orbit through $e\Gamma$ is closed as well. Now $i : Z \to X = G/\Gamma$ can be lifted to an embedding $j : Z \to G/(i_*\pi_1(Z))$ which yields a holomorphic map $Z \to G/A$. A result of Rosenlicht ([**128**], Th.3) states that the quotient of a linear algebraic group by a nilpotent algebraic group is always quasi-affine. Hence G/A is quasi-affine and therefore every holomorphic map from the compact complex space Z to G/A is constant. Thus Z is contained in the A-orbit $A/A \cap \Gamma \subset X$. But A is abelian, hence $A/A \cap \Gamma$ is a compact complex torus. □

For $G = SL_2(\mathbb{C})$ this can be improved.

COROLLARY 4.12.3. — *Let $G = SL_2(\mathbb{C})$, Γ a discrete cocompact subgroup and $Z \subset X = G/\Gamma$ a compact complex subspace.*

Then either $i_*\pi_1(Z)$ is Zariski dense in G or Z is a one-dimensional compact complex torus.

Proof. — We may assume that the Zariski closure H of $i_*\pi_1(Z)$ is not all of G. It is easy to check that a Borel subgroup in $SL_2(\mathbb{C})$ does not contain any discrete Zariski dense subgroup. Hence H is one-dimensional and therefore contains a commutative subgroup of finite index. Similar arguments to those above show that Z is contained in an H-orbit. Hence it coincides with a connected component of an H-orbit. □

4.13. More on subspaces of general type

The easiest way to construct subspaces of general type in a compact complex parallelizable manifolds is to take an orbit which is a compact complex torus and look for non-trivial subspaces of this torus. Under an (admittedly rather strong) additional topological assumption we have shown in the preceding section that this is the only way to construct subspaces of general type. Thus one might pose the question:

> *Does there exists a compact complex-parallelizable manifold X with a closed complex subspace of general type Z such that Z is not contained in any torus $T \subset X$?*

Campana and Flenner were the first to suggest constructing a counterexample using curves in nilmanifolds; their argumentation is based on the theory of Parshin ([**119**]) of generalized Jacobians.

Here we will use a different method to show that there are counterexamples.

PROPOSITION 4.13.1. — *There exists a complex nilmanifold X (i.e., a compact quotient of a simply connected complex nilpotent Lie group G and a discrete subgroup Γ) and a Riemann surface $C \subset X$ of genus $g \geq 2$ such that C is not contained in any proper parallalizable complex submanifold of X.*

Proof. — We start with the Iwasawa manifold $X_0 = U(\mathbb{C})/U(\mathbb{Z} \oplus i\mathbb{Z})$ where

$$U(A) = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in A \right\}$$

for $A = \mathbb{C}, \mathbb{Z} \oplus i\mathbb{Z}$. There is a fibration $\pi : X_0 \to Y$ given by

$$U(\mathbb{C})/U(\mathbb{Z} \oplus i\mathbb{Z}) \longrightarrow U(\mathbb{C})/[U(\mathbb{C}), U(\mathbb{C})]U(\mathbb{Z} \oplus i\mathbb{Z}).$$

This fibration realizes X_0 as a holomorphic E-principal bundle over the abelian surface $Y = E \times E$ where $E = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. Let $C \subset Y$ be a curve of genus $g \geq 2$ and let A denote the Jacobian of C. Then there exists a surjection $\tau : A \to Y$ and we define X_1 as the total space of the pulled-back E-principal bundle π, *i.e.*, $X_1 = A \times_Y X_0 \overset{\pi_1 = \tau^*\pi}{\to} A$. The E-principal bundle $\pi : X_0 \to Y$ admits a holomorphic connection (see §6.3). Hence $\pi : X_1 \to A$ admits a holomorphic connection, too. For dimension reasons this connection induces a *flat* connection on the restricted bundle $\pi' : \pi_1^{-1}(C) \to C \subset A$.

Thus π' is given by a representation of the fundamental group $\rho : \pi_1(C) \to E$. Now $\pi_1(C)$ is torsion-free for a Riemann surface C. It follows that π' is a topologically trivial E-principal bundle over C. Next let us consider the exact sequence of complex Lie groups

$$0 \longrightarrow \mathbb{Z} \oplus i\mathbb{Z} \to \mathbb{C} \longrightarrow E \longrightarrow 0.$$

For any complex space C, this yields an exact sequence of the corresponding sheaves of functions with values in the respective Lie

$$0 \longrightarrow \mathbb{Z} \oplus i\mathbb{Z} \to \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

The long exact cohomology sequences contains the following part:

$$\cdots \longrightarrow H^1(C, \mathcal{O}) \xrightarrow{\alpha} H^1(C, \mathcal{E}) \longrightarrow H^2(C, \mathbb{Z} \oplus i\mathbb{Z}) \longrightarrow \cdots$$

The cohomology group $H^1(C, \mathcal{E})$ parametrizes E-principal bundles over C. Thus π', being topologically trivial, must be induced by $\alpha(\phi)$ for some element $\phi \in H^1(C, \mathcal{O})$. Recall that $H^1(C, \mathcal{O}) \simeq H^1(A, \mathcal{O})$, since A is the Jacobian of C. Thus ϕ can be regarded as an element in $H^1(A, \mathcal{O})$. Using the group structure of E there is a structure of an abelian group on the cohomology group $H^1(A, \mathcal{E})$ parametrizing the E-principal bundles over A. Let $\xi \in H^1(A, \mathcal{E})$ denote the element defining $\pi_2 : X_2 \to A$ and for $t \in [0, 1]$ consider the E-principal bundle $p_t : Z_t \to A$ defined by $\xi - t\phi \in H^1(A, \mathcal{E})$. Recall that the obstruction to the existence of a holomorphic connection is given by an element in $H^1(A, \Omega^1)$ (see §6.3), that the bundle corresponding to ξ admits a holomorphic connection and that the natural map $H^1(A, \mathcal{O}) \to H^1(A, \Omega^1)$ vanishes, because A is a complex torus and therefore Kähler. It follows that all the E-principal bundles p_t admit holomorphic connections. This implies that the total spaces Z_t are again nilmanifolds. In particular the total space Z_1 is a nilmanifold realized as a E-principal bundle $p_1 : Z_1 \to A$. Now $C \subset A$ and by construction $(p_1)|_C$ is holomorphically trivial. Thus the embedding $C \hookrightarrow A$ can be lifted to an embedding $j : C \hookrightarrow Z_1$.

Finally, we claim that $j(C)$ is not contained in any proper parallelizable complex submanifold W of Z_1. Indeed, since A is the Jacobian of C, it is clear that $p_1(W) = A$ for every parallelizable complex submanifold $W \subset Z_1$ with $j(C) \subset W$. By thm. 4.10.1 this implies $W = Z_1$. □

REMARK 4.13.2. — In general, the above construction yields curves in nilmanifolds G/Γ where G is "degenerate" in the sense that G is a direct product with an abelian group. To avoid this, the construction has to be modified in such a way that one starts with a curve C whose Jacobian is isogeneous to $G/G'\Gamma$ for some nilmanifold $G/G'\Gamma$ where the center Z_G is contained in the commutator group G'. This is possible. For instance, the Jacobian of the Fermat curve

$$C = \{[x : y : z] \in \mathbb{P}_2(\mathbb{C}) : x^6 + y^6 + z^6\}$$

is isogeneous to E^{10} with $E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ with $\tau = \sqrt{3/4}i - 1/2$ (see **[76]** for this and other results on the Jacobian of Fermat curves). Now $\mathbb{Z} \oplus \tau\mathbb{Z}$ is the ring of algebraic integers of the cyclotomic field $K = \mathbb{Q}[\tau]$. Therefore $E^{10} \simeq G/G'\Gamma$, if G is the 11-dimensional Heisenberg group and $\Gamma = G(\mathbb{Z} \oplus \tau\mathbb{Z})$. It follows that C can be embedded into a deformation X of $X_1 = G/\Gamma$ in such a way that X is a complex nilmanifold and C is not contained in any proper parallelizable submanifold of X.

However, for $G = SL_2(\mathbb{C})$ the question is still open. If there exists a discrete cocompact subgroup Γ such that $X = SL_2(\mathbb{C})/\Gamma$ contains an irreducible compact complex subspace which is not a torus, then this subspace would of course be a (possibly singular) compact Riemann surface of genus $g \geq 2$. We have already seen that the embedding of a compact Riemann surface C with $g(C) \geq 2$ (if it exists) must be non-trivial on the homotopy level. This is not a useful obstruction, because there are many group homomorphisms from $\pi_1(C)$ to Γ. ($\pi_1(C)$ is almost free; to be more precise: it is generated by $2g$ generators $a_1, \dots, a_g, b_1, \dots, b_g$ subject only to the relation $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = e$.) Furthermore every group homomorphism from $\pi_1(C)$ to Γ is induced by a continuous mapping.

PROPOSITION 4.13.3. — *Let C be a compact Riemann surface of genus $g(C) \geq 1$, Γ a torsion-free discrete subgroup of $SL_2(\mathbb{C})$. Then the set of homotopy classes of continuous maps from C to $SL_2(\mathbb{C})/\Gamma$ is in one-to-one correspondence to* $\operatorname{Hom}(\pi_1(C), \Gamma)$.

For this we need two auxiliary results on Riemann surfaces.

LEMMA 4.13.4. — *Let C be a Riemann surface, K a connected, simply connected real Lie group. Then every K-principal bundle over C is (topologically) trivial.*

Proof. — Let $E \xrightarrow{K} C$ be a K-principal bundle. Assume $C \not\simeq \mathbb{P}_1(\mathbb{C})$. Let $\pi : \widetilde{C} \to C$ denote the universal covering. Then $\widetilde{C}$ is contractible, hence π^*E is trivial. Thus π^*E is given by a factor of automorphy $\alpha : \Gamma \times \widetilde{C} \to G$. This is a map α fulfilling

$$\alpha(\gamma\widetilde{\gamma}, x) = \alpha(\gamma, \widetilde{\gamma}x)\alpha(\widetilde{\gamma}, x).$$

The associated K-principal bundle over C is defined by $\widetilde{C} \times K/\sim$ with $(x, k) \sim (\gamma x, \alpha(\gamma, x)k)$ for all $x \in \widetilde{C}$, $\gamma \in \Gamma$, $k \in K$. This bundle is trivial if and only if there exists a map $\phi : \widetilde{C} \to K$ such that

$$(*) \qquad \phi(x) = \phi(\gamma x) = \alpha(\gamma, x)$$

for all $x \in \widetilde{C}$, $\gamma \in \Gamma$. (The trivialization is induced by $(x, k) \mapsto (\Gamma x, \phi(x)^{-1}k)$.) Thus we have to construct such a map ϕ. Recall that C may be equipped with the structure of a real two-dimensional CW-complex. This is a disjoint decomposition of C into cells fulfilling certain additional properties, cf. e.g. **[85]**. Since cells are simply connected, for each cell $\sigma \subset C$ the pre-image $\pi^{-1}(\sigma) \subset \widetilde{C}$ is a disjoint union of cells which are freely permuted by the Γ-action. Thus we have a natural induced structure of a CW-complex on $\widetilde{C}$ such that for each cell σ belonging to this structure and every $\gamma \in \Gamma \setminus \{e\}$ the transform $\gamma(\sigma)$ is again a cell belonging to this CW-complex structure. Moreover $\sigma \cap \gamma(\sigma) = \varnothing$. Let $\widetilde{C}_k$ denote the k-skeleton, *i.e.*, the union of all cells of dimension $\leq k$. The above considerations imply that $\widetilde{C}_k$ contains a closed CW-subcomplex $\widetilde{C}_k^*$ with $\widetilde{C}_{k-1} \subset \widetilde{C}_k^*$ such that for each k-cell σ there exists a unique $\gamma \in \Gamma$ for which $\gamma(\sigma) \subset \widetilde{C}_k^*$. In other words: The natural map $\Gamma \times (\widetilde{C}_k^* \setminus \widetilde{C}_{k-1}) \to \widetilde{C}_k \setminus \widetilde{C}_{k-1}$ is bijective. Now we claim that we can construct the desired map $\phi : \widetilde{C} \to K$ by starting with an arbitrary map from $\widetilde{C}_0^* \to K$ and then extending this map along the inclusions

$$\widetilde{C}_0^* \subset \widetilde{C}_0 \subset \widetilde{C}_1^* \subset \widetilde{C}_1 \subset \widetilde{C}_2^* \subset \widetilde{C}_2 = \widetilde{C}.$$

To extend from $\widetilde{C}_k^*$ to $\widetilde{C}_k$ we simply *use* the desired relation $(*)$. For the inclusion $\widetilde{C}_{k-1} \subset \widetilde{C}_k^*$ it suffices to take any continuous extension. However, we have to show that there exists such a continuous extension. Recall that K is required to be connected and simply connected. This implies that for any relative CW-complex (X, A) with $\dim_{\mathbb{R}}(X) \leq 2$ any continuous map $f : A \to K$ extends to a continuous map $\widetilde{f} : X \to K$.

The proof for the case $C \simeq \mathbb{P}_1(\mathbb{C})$ follows easily from the fact that $\mathbb{P}_1(\mathbb{C})$ is obtained from S^1 by the suspension functor. We omit the details, since we do not need this case anyway. (Since $\mathbb{P}_1(\mathbb{C})$ is simply connected, it is clearly impossible to embed $\mathbb{P}_1(\mathbb{C})$ into a quotient of a complex Lie group by a discrete subgroup.) □

LEMMA 4.13.5. — *Let C be a Riemann surface. Then every continuous map from C to $SU_2(\mathbb{C})$ is homotopic to a constant map.*

Proof. — Every continuous map between differentiable manifolds is homotopic to a differentiable map. Hence every continuous map from C to $SU_2(\mathbb{C})$ is homotopic to a non-surjective map. Now $SU_2(\mathbb{C})$ is homeomorphic to S^3, hence every non-surjective map is homotopic to a constant map. □

Proof of the proposition. — Let $K \simeq SU_2(\mathbb{C})$ denote a maximal compact subgroup of $SL_2(\mathbb{C})$. Since Γ is torsion-free and discrete, $\Gamma \cap K = \{e\}$. Thus we have a K-principal bundle $\pi : X = SL_2(\mathbb{C})/\Gamma \xrightarrow{K} M = K \backslash SL_2(\mathbb{C})/\Gamma$ and M is an Eilenberg-MacLane space $K(\Gamma, 1)$. Hence $[C, M] = \mathrm{Hom}(\pi_1(C), \Gamma)$. Now for every continuous map $f : C \to M$ we have the pull-backed K-principal bundle over C which is trivial. Therefore f lifts to a map from C to M. Finally we have to show that two maps $f, g : C \to X$ are homotopic if and only if $\pi \circ f$ and $\pi \circ g$ are homotopic. By homotopy lifting, $\pi \circ f \sim \pi \circ g$ implies $f \sim h$ for some map $h : C \to X$ with $\pi \circ h = \pi \circ g$. Thus $h(x) = R_{\rho(x)}(g(x))$ $(x \in C)$ for some map $\rho : C \to K$, where R denotes the principal K-right action on X. Now ρ is homotopic to a constant map. Hence $f \sim h \sim g$. □

4.14. Holomorphic maps $f : \mathbb{C} \to X$

The following result ("Bloch's Conjecture") was proved by Green and Griffiths [**47**]: *Let $f : \mathbb{C} \to X$ be a holomorphic map of the complex numbers into a compact complex torus. Then the complex-analytic Zariski closure $W = \overline{f(\mathbb{C})}$ is a subtorus.* (The complex-analytic Zariski closure of a subset S in a complex space X is the smallest closed complex-analytic subset of X containing S.) We will generalize this to holomorphic mappings from $\mathbb{C}$ to a compact parallelizable manifold.

THEOREM 4.14.1. — *Let X be a compact parallelizable complex manifold and $f : \mathbb{C} \to X$ a holomorphic map. Then the complex-analytic Zariski closure $W = \overline{f(\mathbb{C})}$ is an orbit of a complex Lie subgroup of $G := \mathrm{Aut}_{\mathcal{O}}(X)^0$.*

Proof. — By the theorem of Green and Griffiths $\alpha : W \to \mathrm{Alb}(W)$ is surjective. Hence it suffices to prove the following statement:

> *Let $X = G/\Gamma$ be a compact complex parallelizable manifold, Z an irreducible compact complex subspace. Then either Z is parallelizable or $Z \to \mathrm{Alb}(Z)$ is not surjective.*

Thus we will now assume that Z is a compact complex subspace of $X = G/\Gamma$ with $Z \to \mathrm{Alb}(Z)$ surjective and deduce that this implies parallelizability of Z. First we note that cor. 4.7.2 implies that Z is not of general type, *i.e.*, $\kappa(Z) < \dim Z$. Consider the map $\psi : Z \to Y$ as given in theorem 4.8.1. Then the image of $\tilde{f} = \psi \circ f : \mathbb{C} \to Y$ is dense in Y with respect to the analytic Zariski topology. If we are in a position to apply theorem 4.8.2, then it follows that Y is of general type and $Y \to \mathrm{Alb}(Y)$ is surjective. Hence Y is a point, *i.e.*, Z is parallelizable. This completes the proof for the case where G is reductive or solvable.

It remains to handle the case where G is neither solvable nor reductive. Here we assume that G is minimal in the following sense: *If $\tilde{G}$ is a complex Lie subgroup of G with Z contained in a closed $\tilde{G}$-orbit, then $\tilde{G} = G$.*

We begin by considering the radical fibration $\tau : G/\Gamma \to G/R\Gamma = S/\Lambda$, where R is the radical and S a maximal connected semi-simple Lie subgroup. Since S is reductive it follows that the image of Z under τ is an orbit. By minimality of G it follows that Z is mapped surjectively on S/Λ. Going back to the proof of theorem 4.8.2 we note that in order to deduce the statements of theorem 4.8.2 it is enough to show that G/N is holomorphically separable (with $N = N_G(H^0)$ defined as in theorem 1.10.1). Now, the manifold G/N is naturally embedded in a Graßmann manifold. Let $\overline{G}$ denote the algebraic Zariski closure of G in the automorphism group of this Graßmann manifold and $\overline{N}$ the isotropy group of $\overline{G}$ at the neutral point eN. Furthermore let $\overline{R}$ denote the radical of $\overline{G}$. Then the $\overline{R}$-orbits in $\overline{G}/\overline{N}$ are closed and $\overline{R}\,\overline{N}/\overline{R}$ is a closed algebraic subgroup of $\overline{G}/\overline{R} = S$. Recall that Z may be embedded in G/Γ_0 with $\Gamma_0 \subset N$. It follows that $\Lambda_0 = R\Gamma_0/R$ is a discrete cocompact subgroup of S with $\Lambda_0 \subset RN/R$. By thm. 3.4.1 this implies that Λ_0 is Zariski dense. Hence RN/R is also Zariski dense in

S. Thus $\overline{R}\,\overline{N}/\overline{R} = S$, *i.e.*, $\overline{G} = \overline{R}\,\overline{N}$. Therefore $\overline{R}$ acts transitively on $\overline{G}/\overline{N}$. It follows that G/N is a subset of an orbit of a solvable group in a Graßmann manifold. Since such orbits are holomorphically separable, it follows that G/N is holomorphically separable. This completes the proof. □

The theorem can be generalized to deal with holomorphic maps from arbitrary complex Lie groups instead of $\mathbb{C}$.

PROPOSITION 4.14.2. — *Let X be a compact parallelizable complex manifold, H a connected complex Lie group and $f : H \to X$ a holomorphic map. Then the complex-analytic Zariski closure $W = \overline{f(H)}$ is an orbit of a complex Lie subgroup of $G := \mathrm{Aut}_{\mathcal{O}}(X)^0$.*

Proof

For every element v in the Lie algebra $\mathcal{L}ie(H)$ let S_v denote the set $\{f(\exp(vt)) : t \in \mathbb{C}\}$. By the theorem the complex-analytic Zariski closures $\overline{S_v}$ are orbits of complex Lie subgroups of G containing $f(e)$. Recall that there exist only countably many connected Lie subgroups of G with compact orbits through a given point (prop. 3.12.2). Therefore there exists an element $v_0 \in \mathcal{L}ie(H)$ and a subset $\Omega \subset \mathcal{L}ie(H)$ such that Ω is dense in the analytic Zariski topology and $\overline{S_{v_0}} = \overline{S_v}$ for all $v \in \Omega$. Density of Ω implies that $S_v \subset Z$ for all $v \in \mathcal{L}ie(H)$, hence density of $\exp(\mathcal{L}ie\,H)$ in H implies that $f(H) \subset \overline{S_{v_0}}$. Hence $\overline{f(H)} = \overline{S_{v_0}}$ and the proof is complete. □

As a consequence of the theorem we obtain a characterization of hyperbolic submanifolds.

COROLLARY 4.14.3. — *Let $X = G/\Gamma$ be a compact complex parallelizable manifold and Z a complex subspace. Then Z is hyperbolic in the sense of Kobayashi if and only if it contains no compact complex torus.*

Proof. — By a result of Brody ([**25**]), Z is hyperbolic if and only if every holomorphic map from $\mathbb{C}$ to Z is constant. Hence Z is hyperbolic if and only if it contains no parallelizable complex submanifold. Finally recall that every compact complex parallelizable manifold contains a compact complex torus (follows from theorem 3.10.1). □

For submanifolds of parallelizable manifolds we thus obtain an affirmative answer to the following general question.

QUESTION 4.14.4. — Given a compact complex manifold Z, are the following conditions equivalent?

1. Z is hyperbolic in the sense of Kobayashi;
2. There is no non-constant holomorphic map from compact complex torus to Z.

Serge Lang [**80**] conjectured that the answer is positive is true at least for projective manifolds. In general it is obvious that (1) implies (2). Furthermore (2) implies that every curve $C \subset Z$ is of genus ≥ 2 and therefore hyperbolic (note that condition (2) excludes rational curves, because there is a surjective map from a torus to $\mathbb{P}_1(\mathbb{C})$).

We would like to remark that this characterization of hyperbolic submanifolds can be proved in a more elementary way, in particular without using the result of Green and Griffiths.

By the reparametrization results of Brody **[25]** it suffices to prove the following:

> *Let h be a hermitian metric on Z and $f : \mathbb{C} \to Z$ be a non-constant holomorphic map with bounded derivative (with respect to h). Then Z contains a parallelizable submanifold.*

But this is an immediate consequence of the lemma below, because the complex-analytic Zariski -closure of an orbit is parallelizable provided it is compact (cor. 4.5.5).

LEMMA 4.14.5. — *Let G, H be complex Lie groups, $f : G \to H$ be a holomorphic map with $f(e_G) = e_H$. Assume that the derivative Tf is bounded with respect to some G– resp. H–invariant hermitian metrics on G and H.*

Then f is already a Lie group homomorphism.

Proof. — Consider $Tf : TG \to TH$. It induces a map $F : G \times \mathbf{g} \to \mathbf{h}$ which is linear in the second variable and such that $F(\cdot, v) : G \to \mathbf{h}$ is bounded. By Liouville's theorem $F(\cdot, v)$ is constant for all v. Thus we obtain a linear map $\phi : \mathbf{g} \to \mathbf{h}$ and a dual map $\phi^* : \mathbf{h}^* \to \mathbf{g}^*$. The latter is induced by pulling back holomorphic 1-forms. Hence

$$\begin{aligned}\alpha([\phi X, \phi Y]) = d\alpha(\phi X, \phi Y) = \phi^* d\alpha(X, Y)& \\ = d\phi^*\alpha(X, Y) = \phi^*([X, Y]) &= \alpha(\phi[X, Y])\end{aligned}$$

for all $\alpha \in \mathbf{h}^*$. Thus $[\phi X, \phi Y] = \phi[X, Y]$, *i.e.*, ϕ is a Lie algebra homomorphism. □

4.15. Abelian connectedness

Let $\mathcal{S}$ be a class of groups, e.g. the class of all abelian groups. Then one can pose the following question:

> *Let X be a compact complex manifold and assume that for every pair of points $x, y \in X$ there exists a finite family of compact complex spaces Z_i with $\pi_1(Z_i) \in \mathcal{S}$ and holomorphic maps $f_i : Z_i \to X$ such that $\cup_i f_i(Z_i)$ is connected and contains x and y.*
>
> *Does this imply $\pi_1(X) \in \mathcal{S}$?*

For Kähler manifolds there are some positive results. In particular the answer is positive for the class of finite groups as well as for the class of almost abelian groups, see **[27, 28]**.

For parallelizable manifolds the situation is quite different: Given a quotient of a complex Lie group G by a lattice Γ it is always posssible to connnect two points in G/Γ by a chain of tori, regardless whether $\pi_1(G/\Gamma)$ is abelian or not.

PROPOSITION 4.15.1. — *Let G be a complex Lie group, Γ a lattice, and $x, y \in X = G/\Gamma$.*

Then there exists a finite family of compact complex tori Z_i embedded in X such that the union $Z = \cup Z_i$ is connected and contains both x and y.

Proof. — There is a natural equivalence relation on X given as follows: $x \sim y$ iff x and y can be connected by a chain of compact complex tori embedded in X. Clearly

$$(*) \qquad (x \sim y) \iff (\phi(x) \sim \phi(y))$$

for all $x, y \in X$, $\phi \in \mathrm{Aut}_{\mathcal{O}}(X)$. Let $I = \{g \in G : g\Gamma \sim e\Gamma\}$. Due to $(*)$ the assumption $g, h \in I$ implies

$$e\Gamma \sim g\Gamma = ge\Gamma \sim gh\Gamma$$

and

$$e\Gamma = g^{-1}g\Gamma \sim g^{-1}e\Gamma.$$

Thus $g, h \in I$ implies $g^{-1}, gh \in I$, *i.e.*, I is a subgroup. Now the statement follows from thm. 3.10.1. $\square$

CHAPTER 5

HOLOMORPHIC MAPPINGS

5.1. Survey

In this chapter we study holomorphic mappings from and to parallelizable manifolds. Special emphasis is given to holomorphic self-maps.

In certain contexts holomorphic mappings are automatically equivariant. In particular, if $X = G/\Gamma$ is a compact complex parallelizable manifold, Y a normal complex space and $f : X \to Y$ is a surjective connected holomorphic map, then Y is also parallelizable. In fact $Y \simeq G/H\Gamma$ for some normal connected complex Lie subgroup $H \subset G$ such that $F : G/\Gamma \to G/H\Gamma$ is simply the natural projection.

Furthermore we prove that for a lattice Γ in a semisimple complex Lie group G every surjective holomorphic self-map f of the complex manifold $X = G/\Gamma$ with $f(e\Gamma) = e\Gamma$ is already an automorphism and of the form

$$f : g\Gamma \mapsto \phi(g)\Gamma,$$

where ϕ denotes an automorphism of G (as complex Lie group) with $\phi(\Gamma) = \Gamma$. As a consequence we obtain that for semisimple G the automorphism group of such a complex manifold G/Γ has only finitely many connected components and its connected component is isomorphic to $G/(Z \cap \Gamma)$ where Z denotes the (finite) center of G.

If G is not semisimple, there may exist surjective holomorphic self-maps of G/Γ which are not bijective. However, they can be non-bijective only in the direction of the nilradical. To give a precise statement, let us assume that G is a simply connected complex Lie group, N its nilradical, Γ a lattice in G, and $f : G/\Gamma \to G/\Gamma$ a surjective holomorphic map with $f(e\Gamma) = e\Gamma$.

Then there exists an automorphism ϕ of G with $\phi(\Gamma) \subset \Gamma$ such that $f(g\Gamma) = \phi(g)\Gamma$ and a commutative diagram of holomorphic mappings

$$\begin{array}{ccc} G/\Gamma & \xrightarrow{f} & G/\Gamma \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G/N\Gamma & \xrightarrow[F]{\sim} & G/N\Gamma \end{array}$$

with $F(gN\Gamma) = \phi(g)N\Gamma$ and $\pi(g\Gamma) = gN\Gamma$ such that F is biholomorphic.

Finally, we show that meromorphic maps to complex parallelizable manifolds are automatically holomorphic.

Most of the results contained in this chapter have been published in [**158**].

5.2. Maps to parallelizable manifolds are equivariant

We have already seen (lemma 3.4.3) that a compact complex parallelizable manifold $X = G/\Gamma$ never admits a non-constant G-equivariant map to a projective space $\mathbb{P}_n(\mathbb{C})$. This is particularly useful in studying holomorphic maps, because proper connected surjective holomorphic maps are automatically equivariant. This is a result which basically goes back to Blanchard [**14**]. In the formulation cited below it is due to Remmert and van de Ven [**126**].

THEOREM 5.2.1. — *Let $f : X \to Y$ be a surjective proper holomorphic map with connected fibers between normal complex spaces. Assume that there exists a holomorphic action of a connected complex Lie group G on X.*

Then there exists a holomorphic G-action on Y for which f is equivariant.

COROLLARY 5.2.2. — *Let G by a connected complex Lie group, $\Gamma \subset G$ a lattice, Y a normal complex space and $f : X = G/\Gamma \to Y$ a surjective connected proper holomorphic map.*

Then there exists a connected normal complex Lie subgroup $H \subset G$ such that $H\Gamma$ is closed in G and there is a biholomorphic map $G/H\Gamma \simeq Y$ such that f is simply the canonical projection map from G/Γ to $G/H\Gamma$. Furthermore $G/H\Gamma = (G/H)/(\Gamma/(H\cap\Gamma))$ is parallelizable and $\Gamma/(H\cap\Gamma)$ is a lattice in G/H.

Proof. — By the theorem above the G-action on X induces a G-action on Y. This action is transitive, since G acts transitively on X and $f : X \to Y$ is surjective. Thus $Y \simeq G/I$ for a closed complex Lie subgroup $I \subset G$ with $\Gamma \subset I$. The connected component $H = I^0$ of I is normalized by I, hence by Γ. Thus H must be normal in G (cor. 3.4.7). Therefore G/H is a complex Lie group and $Y \simeq G/I$ is isomorphic to the quotient of this complex Lie group G/H by its discrete subgroup $\Gamma/(H\cap\Gamma)$. Finally $\Gamma/(H\cap\Gamma)$ is a lattice in G/H by lemma 1.5.2. □

CONJECTURE 5.2.3. — *In the formulation of the above corollary the condition of f being proper is not necessary.*

(If G/Γ is compact, then f is automatically proper. Hence this conjecture concerns only quotients by non-cocompact lattices.)

Next we present a basic lemma on holomorphic mappings between parallelizable manifolds.

LEMMA 5.2.4. — *Let X be a complex manifold on which a simply connected complex Lie group G acts. Let $Y = H/\Lambda$ be a group-theoretically parallelizable complex manifold, $f : X \to Y$ a holomorphic map and assume that every holomorphic function on X is constant.*

Then f is equivariant with respect to a homomorphism of complex Lie groups $\phi : G \to H$.

Proof. — We have to show that the G-fundamental vector fields on X can be pushed forward to H-fundamental vector fields on Y. There are H-fundamental vector fields $v_1, \ldots, v_n$ on Y (with $n = \dim_{\mathbb{C}}(Y)$) which give a global trivialization of TY. Now let w be a fundamental G-vector field on X. Using the trivialization of TY we obtain holomorphic functions g_i on X such that $f_*(w_x) = \sum_i g_i(x) v_i$ for every $x \in X$. Since every holomorphic function on X is constant, it follows that the pushed-forward tangent vectors $f_*(w_x)$ constitute a vector field $\sum_i g_i v_i$ with $g_i \in \mathbb{C}$. Thus we obtain a homomorphism of Lie algebras $f_* : \mathcal{L}ie\, G \to \mathcal{L}ie\, H$. Since G was assumed to be simply connected, there exists an associated homomorphism of Lie groups $\phi : G \to H$. □

We are now able to deduce a structure theorem for holomorphic mappings between complex parallelizable manifolds.

THEOREM 5.2.5. — *Let $f : X \to Y$ be a holomorphic map between group-theoretically parallelizable complex manifolds $X \simeq G/\Gamma$, $Y \simeq H/\Lambda$. Assume that every holomorphic function on X is constant.*

Then there exists a homomorphism of complex Lie groups $F : G \to H$ with $F(\Gamma) \subset \Lambda$ and an element $x \in H$ such that $f(g\Gamma) = x \cdot F(g)\Lambda$.

Two such pairs (F, x), (F', x') correspond to the same map if and only if there exists an element $\lambda \in \Lambda$ such that $x' = x\lambda$ and $F'(g) = \lambda^{-1} F(g) \lambda$.

Proof. — Choose $x \in H$ such that $f(e\Gamma) = x\Lambda$. By the above lemma there exists a homomorphism of complex Lie groups $\phi : G \to H$ such that

$$f(g\Gamma) = f(g \cdot e\Gamma) = \phi(g) \cdot x\Lambda.$$

Let F be defined by $F(g) = x^{-1}\phi(g)x$. For $\gamma \in \Gamma$ we obtain

$$x\Lambda = f(e\Gamma) = f(\gamma\Gamma) = xF(\gamma)\Lambda.$$

Hence $F(\Gamma) \subset \Lambda$. Finally consider two maps f, f' given by $f(g\Gamma) = xF(g)\Lambda$ (resp. $f'(g\Gamma) = x'F'(g)\Lambda$). If $f = f'$, then $f(e\Gamma) = f'(e\Gamma)$ implies $x' = x\lambda$ for some $\lambda \in \Lambda$. Now

$$\zeta : g \mapsto F(g)^{-1} x^{-1} x' F'(g)$$

is a continuous map defined on the connected group G. The equality $f = f'$ implies that $\zeta(g) \in \Lambda$ for all $g \in G$. Hence ζ is constant. Since $\zeta(e) = \lambda$, it follows that $F' = \lambda^{-1} F \lambda$. □

COROLLARY 5.2.6. — *Let G be a simply connected complex Lie group, Γ a discrete subgroup such that $X = G/\Gamma$ has no non-constant holomorphic functions. Let $f : X \to X$ be a holomorphic self-map such that $D(f)$ has maximal rank at one point.*

Then f is surjective, $D(f)$ has maximal rank everywhere and there exists an automorphism $\phi \in \mathrm{Aut}(G)$ with $\phi(\Gamma) \subset \Gamma$ and an element $x \in G$ such that $f(g\Gamma) = x\phi(g)\Gamma$.

Proof. — Due to the theorem f is equivariant. This implies that f is surjective and $D(f)$ has maximal rank everywhere as soon as $D(f)$ has maximal rank in one point. The theorem furthermore implies that we can find a homomorphism of complex Lie groups $\phi : G \to G$ with discrete kernel which has the desired properties. However, for a simply connected Lie group such a ϕ is necessarily an automorphism. This follows from the usual correspondence between simply connected Lie groups and Lie algebras because such a ϕ evidently induces an automorphism of the Lie algebra of G. □

DEFINITION 5.2.7. — Given a complex Lie group G and a discrete subgroup Γ, the set of all holomorphic Lie group automorphisms ϕ of G with $\phi(\Gamma) = \Gamma$ is denoted by $\mathrm{Aut}(G, \Gamma)$.

Note that $\mathrm{Aut}(G, \Gamma)$ embedds naturally into $\mathrm{Aut}(G)$ which in turn can be embbeded into $GL(\mathcal{L}ie\, G)$. In this way $\mathrm{Aut}(G, \Gamma)$ carries a natural structure of a Lie group. The connected component $\mathrm{Aut}(G, \Gamma)^0$ consists of all $\phi \in \mathrm{Aut}(G)$ with $\phi|_\Gamma = \mathrm{id}_\Gamma$. If Γ is a lattice in a linear complex Lie group G, then $\phi|_\Gamma = \mathrm{id}|_\Gamma$ implies $\phi = \mathrm{id}_G$ (prop. 3.7.5). Thus $\mathrm{Aut}(G, \Gamma)$ is a countable discrete group for a lattice Γ in a linear complex Lie group G.

COROLLARY 5.2.8. — *Let G be a simply connected complex Lie group and Γ a discrete subgroup such that G/Γ has no non-constant holomorphic functions. Let $A_0 = \mathrm{Aut}(G, \Gamma) \ltimes G$ be the natural semidirect product given by the natural action of $\mathrm{Aut}(G, \Gamma)$ on G.*

Then $\mathrm{Aut}(G/\Gamma) = A_0/\Gamma$ where Γ is embedded in A_0 by

$$\gamma \mapsto (\mathrm{int}_\gamma, \gamma)$$

(with int_γ denoting conjugation by γ).

COROLLARY 5.2.9. — *Let G be a connected complex Lie group and Γ a discrete subgroup such that G/Γ has no non-constant holomorphic functions. Denote the center of G by Z_G.*

Then $\mathrm{Aut}_{\mathcal{O}}(X)^0 = G/(Z_G \cap \Gamma)$, where $\mathrm{Aut}_{\mathcal{O}}(X)^0$ denotes the connected component of the group of holomorphic automorphisms of the complex manifold $X = G/\Gamma$.

5.3. The group of connected components of $\mathrm{Aut}(X)$

We will now study the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$, where $\mathrm{Aut}(X)^0$ denotes the connected component of e (with respect to the compact-open topology). We will describe this quotient group and show that it is always countable and that, if G is semisimple, it is finite.(1) Our previous results imply that $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is in fact isomorphic to $\mathrm{Aut}(G,\Gamma)/\mathrm{Int}(\Gamma)$. Hence we have to consider the size of $\mathrm{Aut}(G,\Gamma)$.

LEMMA 5.3.1. — *Let G be a simply connected complex Lie group and Γ a discrete subgroup. Assume that every holomorphic function on G/Γ is constant.*
Then $\mathrm{Aut}(G,\Gamma)$ naturally embeds into $\mathrm{Aut}(\Gamma)$ and is countable.

Proof. — Let $\phi \in \mathrm{Aut}(G,\Gamma)$ and assume that ϕ acts trivially on Γ. Then

$$\zeta(g\Gamma) = g \cdot \phi(g^{-1})$$

defines a holomorphic map from G/Γ to G with $\zeta(e\Gamma) = e$. But G (being a simply connected complex Lie group) is holomorphically separable (cor. 1.11.3). It follows that $\zeta \equiv e$ which implies $\phi = \mathrm{id}$. Hence $\mathrm{Aut}(G,\Gamma)$ embeds into $\mathrm{Aut}(\Gamma)$. On the other hand $\mathrm{Aut}(G,\Gamma)$ is a closed subgroup of the Lie group $\mathrm{Aut}(G)$. These facts together imply that $\mathrm{Aut}(G,\Gamma)$ is countable. □

COROLLARY 5.3.2. — *Let G be a simply connected complex Lie group, Γ a discrete subgroup. Assume that every holomorphic function on $X = G/\Gamma$ is constant.*
Then $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is countable.

We will now restrict to the case where G is semisimple and prove that in this case $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is even finite. Here it is crucial that almost all automorphisms of semisimple Lie groups are inner. In fact we prove finiteness of $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ for all Lie groups G with $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$ finite. But first let us make a general remark and then proceed to an example.

The remark is the following. For a simply connected Lie group G the automorphism group $\mathrm{Aut}(G)$ is isomorphic to the automorphism group of its Lie algebra and thereby easily seen to be linear algebraic. In particular, $\mathrm{Aut}(G)$ has only finitely many connected components. Thus in order to check the finiteness of $\mathrm{Out}(G)$ it suffices to check that $\mathrm{Aut}(G)$ and $\mathrm{Int}(G) = G/Z$ have the same dimension.

Now let us produce an example of a non-semisimple Lie group for which $\mathrm{Out}(G)$ is finite and which does admit a lattice.

EXAMPLE 5.3.3. — Let $A \in SL(2,\mathbb{Z})$ with $|trace(A)| \neq 2$. Then A generates an infinite cyclic discrete subgroup Λ in $SL(2,\mathbb{C})$. Let T denote the Zariski closure of Λ in $SL(2,\mathbb{C})$. Now T is a maximal torus of $SL(2,\mathbb{C})$ and the semidirect product

(1)There are complex manifolds X for which $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is not countable. For example, let $S = \{0\} \times \mathbb{Z} \subset \mathbb{C}^2$. Then $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is uncountable for $X = \mathbb{C}^2 \setminus S$, because the whole permutation group of S embeds into $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ ([**127**], Prop.3.1).

$G_0 = T \ltimes (\mathbb{C}^2, +)$ induced by the natural action of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ admits a lattice, viz. $\Lambda \ltimes (\mathbb{Z}^2, +)$. One can easily check that $\operatorname{Out}(G) \simeq \mathbb{Z}/2\mathbb{Z}$, where G denotes the universal covering of G_0.

LEMMA 5.3.4. — *Let G be a complex Lie group, Γ a lattice and assume that* $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Int}(G)$ *is finite.*

Then $\operatorname{Aut}(G,\Gamma)/\operatorname{Int}(\Gamma)$ *is finite.*

Proof. — Let H denote the subgroup of all $\phi \in \operatorname{Aut}(G,\Gamma)$ which are given by inner automorphisms of G. Then $\operatorname{Aut}(G,\Gamma)/H \subset \operatorname{Out}(G)$ and $H \simeq N/Z$, where Z denotes the center of G and N the normalizer of $\Gamma \subset G$. Observe that N^0 centralizes Γ. Since $\operatorname{Ad}(\Gamma)$ is Zariski dense in $\operatorname{Ad}(G)$ (thm. 3.4.1), it follows that $N^0 = Z^0$. Thus N/Z^0 is a discrete subgroup in G/Z^0. Therefore $Z^0\Gamma/Z^0$ is also discrete in G/Z^0. Since the covolume of $Z^0\Gamma/Z^0$ in G/Z^0 is obviously finite, $Z^0\Gamma/Z^0$ has finite index in N/Z^0. Consequently $N/\Gamma Z^0$ is finite. This completes the proof of the finiteness of $\operatorname{Aut}(G,\Gamma)/\operatorname{Int}(\Gamma)$. □

Using the isomorphism $\operatorname{Aut}(X)/\operatorname{Aut}(X)^0 \simeq \operatorname{Aut}(G,\Gamma)/\operatorname{Int}(\Gamma)$ for lattices in simply connected complex Lie groups this lemma implies the following result.

THEOREM 5.3.5. — *Let G be a simply connected complex Lie group for which* $\operatorname{Out}(G)$ *is finite (e.g. G semisimple), Γ a lattice and $X = G/\Gamma$.*

Then $\operatorname{Aut}(X)/\operatorname{Aut}(X)^0$ *is finite.*

5.4. Self-maps

Here we will demonstrate that except in the directions of the nilradical, a surjective self-map of a parallelizable manifold is necessarily biholomorphic.

We start with the following observation.

LEMMA 5.4.1. — *Let G be a real Lie group, Γ a lattice and ϕ an automorphism of G with $\phi(\Gamma) \subset \Gamma$.*

Then $\#\,\Gamma/\phi(\Gamma) = |\det \operatorname{Ad}(\phi)|$ *where* $\operatorname{Ad}(\phi)$ *denotes the associated automorphism of the Lie algebra $\mathcal{L}ie(G)$.*

Proof. — The Haar measure on G can be realized by integration over a G-invariant volume form ω which can be considered as an element in $\wedge^d \mathcal{L}ie(G)^*$ (with $d = \dim G$). Thus

$$\int_{G/\phi(\Gamma)} \omega = \int_{G/\Gamma} \phi^*\omega$$

and $\omega = |\det(\operatorname{Ad}(\phi))|\phi^*\omega$. □

COROLLARY 5.4.2. — *Let G be a semisimple Lie group, Γ a lattice and ϕ an automorphism of G with $\phi(\Gamma) \subset \Gamma$.*

Then $\phi(\Gamma) = \Gamma$.

COROLLARY 5.4.3. — *Let S be a semisimple complex Lie group, Γ a lattice and f a surjective holomorphic self-map of the complex manifold G/Γ.*

Then f is bijective, i.e., an automorphism.

However, for non-semisimple Lie groups it is possible that $\phi(\Gamma) \subsetneq \Gamma$.

EXAMPLE 5.4.4. — For a commutative ring A define

$$G_A = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in A \right\},$$

Then $\Gamma = G_{\mathbb{Z}+i\mathbb{Z}}$ is a lattice in the complex Lie group $G = G_{\mathbb{C}}$ and

$$\phi : \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & nx & n^2 z \\ & 1 & ny \\ & & 1 \end{pmatrix}$$

is an automorphism of G with $\phi(\Gamma) \subsetneq \Gamma$, if $n \in \mathbb{N}$ with $n \geq 2$.

More generally, we have the following construction.

LEMMA 5.4.5. — *Let $\mathbf{g}$ be a nilpotent Lie algebra, defined over $\mathbb{Q}$. Assume that $\mathbf{g}$ is graded, i.e., there are vector subspaces $\mathbf{g}_k$ of $\mathbf{g}$ (with $k \in \mathbb{N}$) such that $\mathbf{g} = \bigoplus_k \mathbf{g}_k$ as a vector space and $[\mathbf{g}_m, \mathbf{g}_n] \subset \mathbf{g}_{m+n}$ for all m, n (again everything being defined over $\mathbb{Q}$).*

Then the simply connected nilpotent complex Lie group G corresponding to $\mathbf{g} \otimes_{\mathbb{Q}} \mathbb{C}$ admits a lattice Γ and an automorphism ϕ such that $\phi(\Gamma) \subsetneq \Gamma$ (provided $\mathbf{g}_0 \subsetneq \mathbf{g}$).

Proof. — For any positive natural number N we can define an automorphism Φ_N of $\mathbf{g}$ if we set $\Phi_N(v) = N^k v$ for $v \in \mathbf{g}_k$ and extend linearly to the whole of $\mathbf{g}$. The grading being defined over $\mathbb{Q}$ allows us to find a $\mathbb{Z}$-module Λ inside $\mathbf{g}$ such that $\Lambda \otimes_Z \mathbb{Q} \simeq \mathbf{g}$ and $\Lambda = \bigoplus_k (\mathbf{g}_k \cap \Lambda)$. Thus $\Phi_N(\Lambda) \subset \Lambda$. By a result of Malcev (see thm. 2.2.1) $\exp(\Lambda + i\Lambda)$ generates a lattice Γ in G. Evidently $\phi(\Gamma) \subset \Gamma$ if ϕ is an automorphism of G corresponding to Φ_N for some N. On the other hand $|\det \Phi_N| > 1$ whenever $N > 1$ and $\mathbf{g} \neq \mathbf{g}_0$. Therefore $\phi(\Gamma) \neq \Gamma$. □

REMARK 5.4.6. — There are nilpotent $\mathbb{Q}$-Lie algebras which do not admit any non-trivial grading [**36**].

We prove that for arbitrary (not necessarily nilpotent) complex Lie groups the inclusion $\phi(\Gamma) \subset \Gamma$ can be non-bijective only in the directions of the nilradical.

We begin with a lemma on lattice-preserving automorphisms of nilpotent Lie groups.

LEMMA 5.4.7. — *Let N be a simply connected real nilpotent Lie group, Γ a lattice, and ϕ an automorphism of N with $\phi(\Gamma) \subseteq \Gamma$. Let $\chi_\phi(t)$ denote the characteristic polynomial of the induced automorphism of $\mathcal{L}ie(N)$.*

Then $\chi_\phi \in \mathbb{Z}[t]$.

Proof. — First let us consider the case where N is abelian. In this case $N \simeq \mathbb{R}^n$ and $\Gamma \simeq \mathbb{Z}^n$ in suitable coordinates. Hence $\phi \in M(n, \mathbb{Z})$ and evidently $\chi_\phi \in \mathbb{Z}[t]$.

Now the general proof can be deduced by induction. Induction is justified by Malcev's result (see cor. 2.2.3) that for every lattice Γ in N and every term N^k of the central series the intersection $N^k \cap \Gamma$ is a lattice in N^k. □

PROPOSITION 5.4.8. — *Let G be a simply connected solvable real Lie group, and $(\Gamma_k)_{k\in\mathbb{N}}$ a sequence of lattices in G.*

Let I^0 denote the connected component of the closure of the union $\cup_k \Gamma_k$. Then I^0 is contained in the nilradical N of G.

Proof. — For $g \in G$ let $\mathrm{Ad}(g)$ denote the adjoint automorphism of $\mathcal{L}ie(G)$ and denote the characteristic polynomial of $\mathrm{Ad}(g)|_{\mathcal{L}ie\, N}$ by χ_g. Now $N \cap \Gamma_k$ is a lattice in N (thm. 3.5.3). Thus for every k and every $\gamma \in \Gamma_k$ the characteristic polynomial χ_γ of $\mathrm{Ad}(\gamma)|_{\mathcal{L}ie\, N}$ is contained in $\mathbb{Z}[x]$. By continuity this implies that for every $g \in I^0$ the characteristic polynomial of $\mathrm{Ad}(g)|_{\mathcal{L}ie\, N}$ is the same as for the identity map, *i.e.*, $\mathrm{Ad}(g)|_{\mathcal{L}ie\, N}$ is unipotent. Since N contains the commutator group of G, $\mathrm{Ad}(g)|_{\mathcal{L}ie\, N}$ is unipotent if and only if $g \in N$. It follows that I^0 is contained in the nilradical. □

LEMMA 5.4.9. — *Let G denote a Lie group, Γ a lattice, ϕ an automorphism of G with $\phi(\Gamma) \subset \Gamma$.*

Then either $\phi(\Gamma) = \Gamma$ or $\widehat{\Gamma} = \cup_{k\geq 0}\phi^{-k}(\Gamma)$ is a non-discrete subgroup of G.

Proof. — First note that $\cup_k \phi^{-k}(\Gamma)$ is a subgroup of G, because it is the union of an ascending sequence of subgroups. Let I denote the closure of $\cup_k \phi^{-k}(\Gamma)$ in G and denote its connected component by I^0. Evidently I^0 is normalized by Γ. Now $\mathrm{Ad}(G)$ and $\mathrm{Ad}(\Gamma)$ have the same Zariski closure in $\mathrm{Aut}(\mathcal{L}ie\, G)$. Thus $\mathrm{Ad}(G)$ must stabilize $\mathcal{L}ie\, I$ and consequently I^0 is a normal Lie subgroup of G. We will now consider the quotient Lie group G/I^0. Since $\Gamma \subset I$, the quotient G/I admits an invariant finite measure (see lemma 1.5.2). Hence I/I^0 is a lattice in G/I^0. On the other hand $I^0\Gamma$ also is of finite covolume and therefore $\Gamma_0 = \Gamma/(\Gamma \cap I^0)$ is likewise a lattice. It follows that Γ_0 is a lattice in G/I^0 with $\cup_k \phi_0^{-k}(\Gamma_0)$ being discrete (where ϕ_0 is the automorphism of G/I^0 induced by ϕ). Thanks to the preceding lemma we may now conclude $\phi_0(\Gamma_0) = \Gamma_0$, *i.e.*, $\phi(\Gamma)I^0 = \Gamma I^0$. This completes the proof since $I^0 \subset N$. □

We are now in a position to prove the following theorem.

THEOREM 5.4.10. — *Let $X = G/\Gamma$ be a complex-parallelizable manifold with finite volume. Let f be a surjective holomorphic self-map. Let N denote the nilradical of G.*

Then there is a commutative diagram

$$\begin{array}{ccc} G/\Gamma & \xrightarrow{f} & G/\Gamma \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ G/N\Gamma & \underset{F}{\xrightarrow{\sim}} & G/N\Gamma \end{array}$$

where F is an automorphism of the complex manifold $G/N\Gamma$.

Proof. — There is an automorphism ϕ of the complex Lie group G with $\phi(\Gamma) \subset \Gamma$ and an element $a \in G$ such that f is given by

$$f : g\Gamma \mapsto a\phi(g)\Gamma.$$

We have to show that $\phi(N\Gamma) = N\Gamma$. Recall that by Mostow's result (thm. 3.5.3) both the radical R and the nilradical N of G have compact orbits in G/Γ. Thus $R\Gamma/R$ is a lattice in the semisimple group G/R, implying $\phi(R\Gamma) = R\Gamma$. On the other hand $R \cap \Gamma$ is a lattice in R. Hence $\phi((R \cap \Gamma)N) = (R \cap \Gamma)N$. Together these facts yield $\phi(N\Gamma) = N\Gamma$ as desired. □

5.5. Meromorphic maps

The results obtained so far on holomorphic maps to parallelizable manifolds are likewise valid for meromorphic maps for the simple reason that every meromorphic map to a complex parallelizable manifold *is* already holomorphic.

PROPOSITION 5.5.1. — *Let X be a complex manifold, G a complex Lie group, $\Gamma \subset G$ a discrete subgroup and $f : X \to Y = G/\Gamma$ a meromorphic map. Then $f : X \to Y$ is already holomorphic.*

Proof. — Indeed, f is holomorphic outside an analytic set $E \subset X$ of codimension two. Now, for any simply connected open subset $U \subset X$, $U \setminus E$ is still simply connected. It follows that $f|_{U\setminus E}$ lifts to a holomorphic map to the universal covering G of Y. Since G is Stein (cor. 1.11.3), this lift can be realized by holomorphic functions. By Hartog's theorem it then follows that f was holomorphic on U in the first place. □

CHAPTER 6

VECTOR BUNDLES

6.1. Survey

First we collect some basic properties of connections and homogeneous bundles. A bundle E over a G-space X is called homogeneous if the action of G on X can be lifted to an action of an extension of G on E. In order to determine whether a homogeneous vector bundle admits a flat connection we need to control this extension. Therefore we study non-trivial extensions of complex Lie groups, introducing the notion of a "essential extension". These considerations allow us to prove that certain homogeneous vector bundles are automatically flat.

In particular we prove the following:

THEOREM 6.1.1. — *Let G be a connected complex Lie group, $-\Gamma \subset G$ a lattice, $X = G/\Gamma$ the quotient manifold and E be a homogeneous holomorphic vector bundle over X*

Then E admits a flat holomorphic connection if one of the following conditions is fulfilled:

1. *G is semisimple;*
2. *E is a vector bundle of rank two and the radical R_G of G is nilpotent;*
3. *E is a line bundle.*

For the case where G/Γ is compact this result is contained in [**153**].

6.2. Sections of Homogeneous Vector Bundles

Here we discuss section in homogeneous vector bundles. We start by recalling the definition of a homogeneous vector bundle.

DEFINITION 6.2.1. — Let $E \to X$ be a vector bundle, G a group acting on X. The bundle E is called *homogeneous* with respect to the G-action, if for every $g \in G$

the induced automorphism $\lambda_g : X \to X$ can be lifted to a bundle automorphism $\widetilde{g} : E \to E$.

We do not require any uniqueness of $\widetilde{g}$ and in fact the lack thereof may imply that the G-action can not be lifted to E.

We will prove that a homogeneous vector bundle over a quotient $X = G/\Gamma$ of a connected complex Lie group G by a lattice Γ admits global sections only inasmuch as it is trivial.

Since we do not want to assume G/Γ to be compact, we first have to show that the space of sections is finite-dimensional.

PROPOSITION 6.2.2. — *Let G be a connected complex Lie group, Γ a lattice and $E \to X = G/\Gamma$ a homogeneous vector bundle.*

Then $\Gamma(X, E)$ is finite-dimensional.

Proof. — We first need a result on the structure of G/Γ.

CLAIM 6.2.3. — Let G be a connected complex Lie group and Γ a lattice. Then there exists a G-equivariant holomorphic surjective map π from X onto a compact complex parallelizable manifold Y such that the algebraic dimension of the fibers equals zero.

Proof. — The algebraic reduction of X maps X onto a compact complex torus T. If the fibers have algebraic dimension larger than zero, we continue by considering the algebraic reduction of the fiber. This yields us a holomorphic surjection of X onto a compact complex parallelizable manifold (which is a torus bundle over a torus). We may continue in this way and for dimension reasons we will arrive at a holomorphic surjective map from X onto a compact complex parallelizable manifold for which the fibers have algebraic dimension zero. □

We will now discuss E restricted to a fiber of π.

CLAIM 6.2.4. — Let F be a complex manifold of algebraic dimension zero (*i.e.*, every meromorphic function on F is constant) and $E \to F$ a vector bundle of rank r. Then $\dim \Gamma(F, E) \leq r$.

Proof. — Assume the contrary. Then there exists a number d with $1 \leq d \leq r$ and sections $\sigma_0, \dots, \sigma_d$ such that

1. The sections $(\sigma_i)_{0 \leq i \leq d}$ are linearly independent as elements in $\Gamma(F, E)$.
2. There exists a point $x \in F$ such that the vector subspace of E_x spanned by $(\sigma_i(x))_{1 \leq i \leq d}$ has dimension d.
3. For every point $x \in F$ the vector subspace of E_x spanned by $(\sigma_i(x))_{0 \leq i \leq d}$ has dimension at most d.

But in this case one of the meromorphic functions

$$f_i = (\sigma_0 \wedge \sigma_1 \wedge \cdots \wedge \widehat{\sigma_i} \wedge \cdots \wedge \sigma_d)/(\sigma_1 \wedge \cdots \wedge \sigma_d)$$

must be non-constant, contradicting the assumption of F having algebraic dimension zero. $\square$

Now let $X = G/\Gamma$, $E \to X$ and $\pi : X \to Y$ as above. Since $\dim(F, E) \leq \operatorname{rank}(E)$ for every fiber F of π, we may conclude that the direct image sheaf $\pi_* E$ is finitely generated as a $\mathcal{O}_Y$-module sheaf. For homogeneity reasons it is locally free and therefore coherent. Since Y is compact, it follows that $\Gamma(Y, \pi_* E) \simeq \Gamma(X, E)$ is finite-dimensional. $\square$

Now we are in a position to prove the following structure theorem on sections of homogeneous vector bundles.

PROPOSITION 6.2.5. — *Let G be a connected complex Lie group, Γ a lattice and $E \to X = G/\Gamma$ a homogeneous vector bundle.*

Then E contains a G-invariant vector subbundle E_0 which is trivial as a holomorphic vector bundle such that $\Gamma(X, E) = \Gamma(X, E_0)$.

Proof. — The sections of E generate a coherent subsheaf E_0 of E. By homogeneity this subsheaf is locally free. Evidently it is invariant under the G-action. Thus we obtain a G-equivariant map from X to the projective space $\mathbb{P}(\Gamma(X, E_0)^*)$ (which is finite-dimensional due to the preceding proposition). This map is constant (lemma 3.4.3) and hence E_0 is trivial as a holomorphic vector bundle. $\square$

6.3. Generalities on connections and homogeneous vector bundles

Here we collect some well-known basic facts on connections.

A connection D on a vector bundle is a rule for differentiating sections. A precise definition may be given in the following way:

DEFINITION 6.3.1. — Let E be a holomorphic vector bundle, $\mathcal{E}$ the sheaf of holomorphic sections in E and Ω^1 the sheaf of holomorphic one-forms. A holomorphic *connection* D on a holomorphic vector bundle $E \to X$ is a morphism of sheaves $D : \mathcal{E} \to \Omega^1 \otimes \mathcal{E}$ such that $D(f\sigma) = (df) \otimes \sigma + f(D\sigma)$ for every open subset U and $f \in \mathcal{O}(U)$, $\sigma \in \mathcal{E}(U)$.

For every holomorphic connection D on the trivial bundle there exists a $r \times r$-matrix of one-forms $A_{i,j}$ such that for a section $s = (s_1, \ldots, s_r)$, $D(s) = ds + A(s)$ (with $A(s) = \left(\sum_j A_{1j} \otimes s_j, \ldots, \sum_j A_{rj} \otimes s_j\right)$). Given a holomorphic vector bundle with local trivialization on an open covering U_i this implies that there is a correspondence between holomorphic connections and a collection of $r \times r$-matrices of one-forms A^i on U_i fulfilling certain transition relations. This yields that there is an obstruction to the existence of a holomorphic connection given by an element in the cohomology group $H^1(X, \Omega^1 \otimes \mathrm{Hom}(E, E))$. For a line bundle L given by a cocycle $\phi_{ij} \in \mathcal{O}^*(U_{ij})$ this obstruction is given by the element in $H^1(X, \Omega^1 \otimes \mathrm{Hom}(L, L)) \simeq H^1(X, \Omega^1)$ given by $(1/\phi_{ij})d\phi_{ij} = d\log\phi_{ij}$.

6.3.1. Connections and Group Actions. — A connection D on a vector bundle E over a manifold X induces a direct sum decomposition of the tangent bundle $TE = VE \oplus HE$, where VE consists of the vertical tangent vectors (those tangent to the fibers of $E \to X$) and HE consists of the *horizontal* tangent vectors. Here a tangent vector $v \in T_pE$ is called *horizontal* if there exists an open neighbourhood U of $x = \pi(p)$ in X and a section $\sigma \in \mathcal{E}(U)$ such that $D_{\pi_* v}\sigma = (D\sigma)(\pi_* v) = 0$ and v is tangent to $\{\sigma(x) : x \in U\}$.

This decomposition yields a natural way of lifting vector fields from the base manifold X to horizontal vector fields on E. Moreover the construction implies that horizontal vector fields induce local 1-parameter groups of vector bundle automorphisms of E.

Next we note

LEMMA 6.3.2. — *Let M, N be manifolds, $\mathbf{X}$, $\mathbf{Y}$ vector fields on M resp. N. Let f be a function on M. Assume that $\mathbf{X}$ and $\mathbf{Y}$ are globally integrable.*

Then the vector field $\mathbf{X} + f\mathbf{Y}$ on $M \times N$ is globally integrable.

(For notational simplicity the function f and the vector fields $\mathbf{X}$ and $\mathbf{Y}$ are identified with there respective lifts to $M \times N$.)

Proof. — Let $\mu : I \times M \to M$, $\nu : I \times N \to N$ (with $I = \mathbb{R}$ or $I = \mathbb{C}$) be the induced one-parameter groups, *i.e.*, $\mu_* \frac{\partial}{\partial t} = \mathbf{X}$ and $\nu_* \frac{\partial}{\partial t} = \mathbf{Y}$. Then $\eta : I \times M \times N \to M \times N$ defined by $\eta = (\mu(t, m), \nu(f(n)t, n))$ yields the desired one-parameter group. □

Now consider a vector field $\mathbf{X}$ lifted to a horizontal vector field $\widetilde{\mathbf{X}}$ on a vector bundle E with respect to a connection. Locally the vector bundle is trivial and with respect to any trivialization $\widetilde{\mathbf{X}} = \mathbf{X} + f\mathbf{Y}$ where f is a function on the base manifold and $\mathbf{Y}$ a fundamental vector field for the action of $GL_n(\mathbb{C})$ on the fiber $\mathbb{C}^n$ of the vector bundle. Thus using the preceding lemma one can deduce the following assertion.

PROPOSITION 6.3.3. — *Let M be a manifold, $E \to M$ a vector bundle, D a connection and $\widetilde{\mathbf{X}}$ the lift of a vector field $\mathbf{X}$ on M with respect to D.*

Then $\widetilde{\mathbf{X}}$ is globally integrable on E if and only if $\mathbf{X}$ is globally integrable on M. Furthermore if $\widetilde{\mathbf{X}}$ is globally integrable, then the induced one-parameter subgroup acts on E by vector bundle automorphisms.

We will frequently need the fact that, for principal bundles over certain complex manifolds, the group of bundle automorphisms is a Lie group. For compact base manifolds there is the following result of Morimoto.

PROPOSITION 6.3.4 (Morimoto, [**101**]). — *Let X be a compact complex manifold, H a complex Lie group, $E \to X$ a (holomorphic) H-principal bundle. Let G denote the group of bundle automorphism, i.e., the group of all biholomorphic self-maps of E which commute with the H-principal right action.*

Then G is a complex Lie group with respect to compact-open topology.

However, we are also interested in the case where the base manifold is just a quotient of a complex Lie group by a (not necessarily cocompact) lattice.

PROPOSITION 6.3.5. — *Let G be a connected complex Lie group, Γ a lattice, H a complex Lie group, $E \to X = G/\Gamma$ a G-homogeneous H-principal bundle and let* $\mathrm{Aut}(E)^*$ *denote the subgroup of* $\mathrm{Aut}(E)$ *generated by 1-parameter groups of principal bundle automorphisms of E.*

Then $\mathrm{Aut}(E)^*$ *is a complex Lie group.*

Proof. — We claim that $\mathrm{Aut}(E)^*$ is associated to a finite-dimensional Lie algebra. Indeed, $\mathrm{Aut}(X)$ is finite-dimensional, since $\mathrm{Aut}(X)^0 \simeq G/(Z_G \cap \Gamma)$ (cor. 5.2.9), thus it suffices to consider only those automorphisms which induce the identity on the base manifold. 1-parameter groups of such automorphisms correspond to sections in $\mathrm{Ad}(E)$ where $\mathrm{Ad}(E)$ denotes the vector bundle over X associated to the H-principal bundle E via the group homomorphism $\mathrm{Ad} : H \to GL(\mathcal{L}ie\, H)$. This is evidently a homogeneous vector bundle. Therefore the space of global sections is finite-dimensional by prop. 6.2.2. □

There are several properties being equivalent to "homogeneous" for vector bundles.

LEMMA 6.3.6. — *Let X be a complex manifold, $E \to X$ a holomorphic vector bundle and $G \subset \mathrm{Aut}_{\mathcal{O}}(X)$.*

Then the following conditions are equivalent

1. *E is homogeneous (with respect to the G-action);*
2. *For every $g \in G$ the bundle $L_g^* E \to M$ obtained by pull-back via the translation $L_g : X \to X$ is isomorphic to E;*
3. *Let $\widetilde{G}$ denote the group of all vector bundle automorphisms of E which induce on X an automorphism contained in G. Then $\widetilde{G} \to G$ is surjective.*

Let L denote the group of all vector bundle automorphisms inducing the identity map on X. Then condition 3 states that we have a short exact sequence of groups

$$1 \longrightarrow L \longrightarrow \widetilde{G} \xrightarrow{\alpha} G \longrightarrow 1$$

Note that $\widetilde{G}$ can be choosen as a finite-dimensional Lie group, if X is biholomorphic to a quotient G/Γ of a connected complex Lie group G by a lattice Γ (thanks to prop. 6.3.5). In order to lift the G-action on X to an action on E we need a splitting of this sequence. In general this sequence need not be split. For instance, let $E = \mathcal{O}(1)$ be the ample generator of the Picard group on $\mathbb{P}_1$. With $G = \mathrm{Aut}_{\mathcal{O}}(\mathbb{P}_1) = PSL_2(\mathbb{C})$

we obtain $L \simeq \mathbb{Z}_2 \ltimes \mathbb{C}^2$ and $\widetilde{G} \simeq SL_2(\mathbb{C}) \ltimes \mathbb{C}^2$ and the sequence $1 \to L \to \widetilde{G} \to G \to 1$ is not split.

The following two propositions are well-known. Nevertheless we sketch the proofs for completeness.

PROPOSITION 6.3.7. — *Let X be a complex manifold on which a connected complex Lie group G acts holomorphically. Let E be a holomorphic vector bundle which admits a holomorphic connection.*

Then E is homogeneous.

Proof. — Using the connection one can lift vector fields and thereby one-parameter groups of automorphisms. Since G is connected, it is generated by its one-parameter subgroups. □

PROPOSITION 6.3.8. — *Let G be a connected complex Lie group, Γ a lattice $X = G/\Gamma$, $G = \mathrm{Aut}_{\mathcal{O}}(X)^0$ and E a homogeneous vector bundle. Then E admits a holomorphic connection.*

Proof. — Let F denote the connected component of the automorphism group of E. By definition of homogeneity the natural group homomorphism $F \to G$ is surjective. Let $H \subset \mathcal{L}ie(F)$ denote a vector subspace of $\mathcal{L}ie(F)$ such that the induced projection map $H \to \mathcal{L}ie(G)$ is a vector space isomorphism. Then the fundamental vector fields on E associated to elements $\mathbf{X} \in \mathbf{H}$ yield a horizontal distribution $HE \subset TE$ which determines a connection. □

6.4. Factors of automorphy

Factors of automorphy are used to study fiber bundles $E \to X$ which become trivial after pulling back to the universal covering. For our purposes this theory is particularly useful, because many holomorphic bundles over complex parallelizable manifolds become trivial after pulling back to the universal covering.

PROPOSITION 6.4.1. — *Let G be a simply connected complex Lie group. Let E be a holomorphic vector bundle over G. Assume that at least one of the following conditions is fulfilled.*

1. *E is topologically trivial;*
2. *G is solvable;*
3. *$G/R \simeq SL_2(\mathbb{C})$, where R denotes the radical of G;*
4. *E is a line bundle.*
5. *E admits a holomorphic connection;*
6. *E is homogeneous.*

Then E is holomorphically trivial.

Proof. — Every simply connected complex Lie group is a Stein manifold (cor. 1.11.3). Hence Grauert's Oka-principle ([**44**]) implies that every topologically trivial vector bundle is already holomorphically trivial.

The topologically classification of vector bundles is given by homotopy classes of continuous mappings to *classifying spaces*. Since every simply connected solvable Lie group is contractible as a topological space, this implies (2).

In case (3), G is homotopy-equivalent to the 3-sphere S^3 on which every vector bundle is topologically trivial (see §7.13).

A topological classification of complex line bundles over a manifold X is given by $H^2(X, \mathbb{Z})$. Now for a simply connected Lie group G we have $\pi_1(G) = \pi_2(G) = \{e\}$ [**30**]. Thus $H_1(G, \mathbb{Z}) = \{0\} = H_2(G, \mathbb{Z})$ by the Hurewicz isomorphism. In particular $\mathrm{Ext}(H_1(G, \mathbb{Z}), \mathbb{Z}) = \{0\} = \mathrm{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z})$. Therefore $H^2(G, \mathbb{Z}) = \{0\}$ by a universal coefficient theorem (see e.g. [**139**]).

Finally for the case (5) $\iff$ (6) note that in this case the G-action on G may be lifted to an action of an extension Lie group of G on E. Let S be a maximal connected semisimple Lie subgroup of G. Simply-connectedness of G forces S to be simply connected. It follows that every Lie group extension of S is split. Therefore the S-action on G lifts to an action of S on E and similarily to an action on the associated $GL_r(\mathbb{C})$-principal bundle P. Now the S-orbits in P yield sections in $P|_{S(x)}$. Hence for $x \in G$ the restricted principal bundle $P|_{S(x)}$ is trivial. Since S is a deformation-retract of G, this implies that P (hence E) is topologically and therefore holomorphically trivial. □

We now summarize basic facts on factors of automorphy. This theory is valid both in the differentiable and the holomorphic category.

Let X be a (differentiable or complex) manifold, $\pi : \widetilde{X} \to X$ the universal covering and $\Gamma = \pi_1(X)$, acting on $\widetilde{X}$ from the right on $\widetilde{X}$ by deck transformations. Furthermore let H be a Lie group, acting from the left on a manifold F. Let $E \to X$ denote a fiber bundle with F as fiber and H as structure group. Assume that $\pi^*E \to \widetilde{X}$ is trivial.

Then there exists a map $f : \Gamma \times \widetilde{X} \to G$, called *factor of automorphy* such that $E \simeq (\widetilde{X} \times F)/\sim$ with $(x, z) \sim (x', z')$ if and only if $(x', z') = (x\gamma^{-1}, f(\gamma, x)z)$ for some $\gamma \in \Gamma$. Conversely, given an arbitrary map $f : \Gamma \times \widetilde{X} \to G$ one can define a bundle in this way, provided $\sim$ is a equivalence relation. This is equivalent to the condition

$$(Z) \qquad f(\tilde{\gamma}\gamma, x) = f(\tilde{\gamma}, x\gamma^{-1}) \cdot f(\gamma, x)$$

for all $\tilde{\gamma}, \gamma \in \Gamma$, $x \in \widetilde{X}$.

Two factors of automorphy f, f' define isomorphic bundles if and only if there exists a map $g : \widetilde{X} \to G$ such that

$$(B) \qquad g(x\gamma^{-1}) \cdot f'(\gamma, x) = f(\gamma, x) \cdot g(x)$$

for all $x \in \widetilde{X}$, $\gamma \in \Gamma$.

Thus the set of all bundles with fiber F and structure group G over X becoming trivial after pulling back to the universal covering may be identified with equivalence classes of maps $f : \Gamma \times \widetilde{X} \to G$ fulfilling (Z) with $f \sim f'$ if and only if there exists a map $g : \widetilde{X} \to G$ such that (B) is fulfilled.

Using group cohomology this may be reformulated as follows: *There is a one-to-one correspondence of bundles over X becoming trivial after pulling back to the universal covering and the Galois cohomology set $H^1(\Gamma, M)$ where M denotes the Γ-module of mappings from $\widetilde{X} \to G$ with the Γ-action induced by the usual action by deck transformations on $\widetilde{X}$.*

6.4.1. Flat bundles. — The simplest case of factors of automorphy $f : \Gamma \times \widetilde{X} \to G$ are those which are constant in the second variable, *i.e.*, $f(\gamma, x) = f_0(\gamma)$. In this case condition (Z) reduces to $f_0(\widetilde{\gamma}\gamma) = f_0(\widetilde{\gamma}) \cdot f_0(\gamma)$.

Thus these special factors of automorphy simply correspond to group homomorphisms $f_0 \in \mathrm{Hom}(\pi_1(X), G)$. Bundles arising in this way are called *bundles given by a representation of the fundamental group* or *flat bundles.*

A vector bundle is given by a representation of the fundamental group if and only if it admits a flat connection, *i.e.*, a connection with vanishing curvature ([**6**]).

It is a natural idea to construct non-trivial bundles via representations of the fundamental group. In this case the equivalence for factors of automorphy (B) takes a special form.

LEMMA 6.4.2. — *Let $\rho, \widetilde{\rho} : \pi_1(X) \to G$ be a group homomorphisms. Then the induced bundles E, $\widetilde{E}$ are equivalent if and only if there exists a mapping $g : \widetilde{X} \to G$ with*

$$g(x\gamma^{-1}) = \rho(\gamma^{-1}) \cdot g(x) \cdot \widetilde{\rho}(\gamma)$$

for all $\gamma \in \pi_1(X)$, $x \in \widetilde{X}$.

In other words: *Two G-principal bundles defined by group homomorphisms $\rho, \widetilde{\rho} : \pi_1(X) \to G$ are equivalent as G-principal bundles if and only if there exists an $\pi_1(X)$-equivariant map from the universal covering to G where the $\pi_1(X)$-action on G is given by*

$$\gamma : g \mapsto \rho(\gamma^{-1}) \cdot g \cdot \widetilde{\rho}(\gamma).$$

WARNING 6.4.3. — Every G-principal bundle defined by a group homomorphism $\rho : \pi_1(X) \to G$ admits a flat connection in a canonically way. Here we discussed only equivalence as *G-principal bundles.* Instead, one can discuss equivalence as *flat G-principal bundles*, *i.e.*, ask for bundle isomorphisms which are compatible with the flat connection. It turns out that two G-principal bundles defined by group homomorphisms ρ, $\widetilde{\rho}$ are isomorphic *as flat G-principal bundles* if and only if $\rho = \widetilde{\rho}$.

6.5. Group extensions

We have already emphasized that, given a homogeneous bundle E over a homogeneous manifold $X = G/H$, one can in general not lift the G-action. Instead of lifting the action of G one has to pass to an extension group $\widetilde{G}$. Naturally one would like to keep this extension as small as possible. In particular if $\widetilde{G}$ contains a subgroup such that the natural projection to G is still surjective, $\widetilde{G}$ should be replaced be this subgroup. We now formalize this.

DEFINITION 6.5.1. — Let

$$1 \longrightarrow L \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

be a short exact sequence of connected Lie groups.

Such an extension is called *essential* if for every Lie subgroup $H \subset \widetilde{G}$ either $H = \widetilde{G}$ or $H \cdot L \neq \widetilde{G}$.

This implies some restriction on L.

LEMMA 6.5.2. — *Let*

$$1 \longrightarrow L \to \widetilde{G} \longrightarrow G \longrightarrow 1$$

be an essential extension of connected Lie groups.

If G is semisimple, then $L = \{e\}$.

Proof. — This is a consequence of Levi-Malcev-decomposition applied to $\widetilde{G}$. □

PROPOSITION 6.5.3. — *Let*

$$1 \longrightarrow L \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

be an essential extension of connected Lie groups.

Then $L \subset R'$ where R' denotes the commutator group of the radical R of $\widetilde{G}$.

In particular L is nilpotent.

Proof. — Let $S \cdot R$ be a Levi-Malcev-decomposition for $\widetilde{G}$, *i.e.*, S is a maximal connected semisimple Lie subgroup and R the radical. Let S_L denote a maximal connected semisimple Lie subgroup of L. Using conjugation we may assume that $S_L \subset S$. Now S_L is a normal Lie subgroup of the semisimple Lie group S. Due to semisimplicity there exists a normal Lie subgroup $S_0 \subset S$ such that $S_0 \cdot S_L = S$ and $S_0 \cap S_L$ is discrete. Now $G_0 = S_0 \cdot R$ constitutes a Lie subgroup of $\widetilde{G}$ with $G_0 \cdot L = \widetilde{G}$. Hence $G_0 = \widetilde{G}$ by the assumption of essentiality. Therefore $S_L = \{e\}$, *i.e.*, L is solvable.

We consider the adjoint action on $\mathcal{L}ie(\widetilde{G})$. As a linear representation of a semisimple group $\mathrm{Ad}(S)$ is completely reducible. Clearly $\mathcal{L}ie(L)$, $\mathcal{L}ie(R)$ and $\mathcal{L}ie(R')$ are all stable under $\mathrm{Ad}(S)$. It follows that there is a direct sum decomposition as a vector space $\mathcal{L}ie(R) = V \oplus W$, where V and W are $\mathrm{Ad}(S)$-stable vectorspaces such that $\mathcal{L}ie(L) = V \oplus \mathcal{L}ie(R' \cap L)$ and $\mathcal{L}ie(R') \subset W$. Now $\mathcal{L}ie(R') \subset W$ implies that W is actually a Lie subalgebra. Since it is $\mathrm{Ad}(S)$-stable, it follows that $\mathcal{L}ie(S) \oplus W$ is a Lie subalgebra of $\mathcal{L}ie(\widetilde{G})$. Now the assumption of $(*)$ being an essential extension implies that $\mathcal{L}ie(S) \oplus W = \mathcal{L}ie(\widetilde{G})$. It follows that $V = \{0\}$, hence $\mathcal{L}ie(L) \subset \mathcal{L}ie(R')$. □

PROPOSITION 6.5.4. — *Let*

$$1 \longrightarrow L \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

be an essential extension of connected Lie groups.

Assume that the radical $R(G)$ of G is nilpotent. Then the radical $R = R(\widetilde{G})$ of $\widetilde{G}$ is also nilpotent.

Proof. — By arguments similar to those in the preceding proof, there is a direct sum decomposition $\mathcal{L}ie(R) = \mathcal{L}ie(R \cap G') \oplus V$ of $\mathrm{Ad}(S)$-stable vectorspaces, where S is a maximal connected semisimple Lie subgroup of $\widetilde{G}$. In order to show that R is nilpotent, it suffices to prove that $\mathrm{ad}(v)|_{\mathcal{L}ie\, R}$ is nilpotent for every $v \in \mathcal{L}ie(R)$ (Engel's theorem). Clearly $\mathrm{ad}(v)|_{\mathcal{L}ie\, R}$ is a nilpotent for every $v \in \mathcal{L}ie(G' \cap R)$. Thus it suffices to prove that $\mathrm{ad}(v)$ is nilpotent for $v \in V$. For a given $v \in V$ set

$$\Phi_v = \{w \in \mathcal{L}ie(\widetilde{G}) : \exists n : \mathrm{ad}(v)^n(w) = 0\}$$

Thus Φ_v is the largest vector subspace of $\mathcal{L}ie(\widetilde{G})$ on which $\mathrm{ad}(v)$ is nilpotent. We claim that Φ_v is a Lie subalgebra. Indeed, if $\mathrm{ad}(v)^n(x) = 0$ and $\mathrm{ad}(v)^m(y) = 0$ for some $n, m \in \mathbb{N}$, $x, y \in \mathcal{L}ie(\widetilde{G})$, repeated application of the Leibnitz rule yields $\mathrm{ad}(v)^{n+m-1}[x,y] = 0$. Recall that V is $\mathrm{Ad}(S)$-stable and $V \cap \mathcal{L}ie(G') = \{0\}$. Hence $[v,s] = 0$ for $s \in \mathcal{L}ie(S)$. It follows that $\mathcal{L}ie(S) \subset \Phi_v$. Furthermore nilpotency of R/L implies that $\mathcal{L}ie(R) \subset \mathcal{L}ie(L) + \Phi_v$. Therefore $\Phi_v + \mathcal{L}ie(L) = \mathcal{L}ie(\widetilde{G})$. Since the extension was assumed to be essential, it follows that $\Phi_v = \mathcal{L}ie(\widetilde{G})$, *i.e.*, $\mathrm{ad}(v)$ is nilpotent. □

These two propositions imply that a number of properties of Lie groups are preserved under essential extensions.

COROLLARY 6.5.5. — *The following properties of a connected Lie group are preserved by essential extensions:*

1. *G is solvable resp. nilpotent;*
2. *G is perfect, i.e., $G = G'$;*
3. *G/R contains no simple factor isomorphic to $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$, where R denotes the radical.*
4. *The radical of G is nilpotent.*
5. *G is semisimple.*

So far we deduced some general results on essential extension of connected Lie groups. We will now provide a more specialized result needed later in order to prove that every homogeneous line bundle is flat.

PROPOSITION 6.5.6. — *Let*

$$(*) \qquad 1 \longrightarrow L \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

be an extension of connected complex Lie groups.

Assume that L is a non-compact central Lie subgroup of $\widetilde{G}$ and that $\widetilde{G}$ admits a discrete subgroup Λ with $L \cap \Lambda = \{e\}$ finite, $L \cdot \Lambda$ closed and that $\widetilde{G}/L\Lambda$ admits a $\widetilde{G}$-left invariant probability measure.

Then $\widetilde{R}' \cap L$ is discrete, where $\widetilde{R}$ denotes the radical of $\widetilde{G}$. In particular, $(*)$ is not essential.

Proof. — Observe that $\widetilde{R}\Lambda$ is closed in $\widetilde{G}$, because $\widetilde{R}\Lambda/L$ is closed in $\widetilde{G}/L \simeq G$ by the theorem of Mostow (see thm. 3.5.3).

Hence without losing generality we may assume that $\widetilde{G}$ is solvable. Then $\widetilde{G}/L\Lambda$ is compact (cor. 3.6.3). Now L is a connected commutative Lie group. Hence there exists a discrete subgroup $\Pi \subset L$ with L/Π compact. Let $\Gamma = \Pi \cdot \Lambda$. This is a group, because Π is central. It is easily verified that Γ is discrete and $\widetilde{G}/\Gamma$ compact. Solvability of $\widetilde{G}$ implies that $\widetilde{G}' \cap \Gamma$ is a subgroup of finite index in Γ' (prop. 3.11.2). But $\Gamma' = \Lambda'$ due to centrality of Π. Hence $\Gamma' \cap L = \{e\}$ implying finiteness of $(\widetilde{G}' \cap \Gamma \cap L)$. Observe that the $\widetilde{G}'$-orbits in $\widetilde{G}/\Gamma$ are closed and that $L/(\Gamma \cap L)$ is compact. Therefore it follows that $\widetilde{G}' \cap L$ is discrete. Using prop. 6.5.3 it follows that $(*)$ is not essential. □

6.6. Flatness theorems

We will now translate the results on essential Lie group extensions to theorems stating that certain homogeneous bundles are automatically flat. First we develop some auxiliary results which establish the relationship between essential Lie group extensions and non-flatness of homogeneous bundles.

LEMMA 6.6.1. — *Let G be a connected complex Lie group, Γ a discrete subgroup, $X = G/\Gamma$, S a complex Lie group and $E \to X$ a homogeneous S-principal bundle.*

Assume that the G-action on X can be lifted to a G-action on E. Then $E \to X$ admits a flat holomorphic connection.

Proof. — A connection on a principal bundle can be described in terms of a subbundle $H \subset TE$ of "horizontal" vector fields. This subbundle has to satisfy the following conditions:

1. $TE = H \oplus V$ where V denotes the subbundle of *vertical* tangent vectors, *i.e.*, the subbundle of all tangent vectors which are tangent to the fibers of the projection $E \to X$.

2. H must be invariant under the S-principal right action on E.

If the G-action on X lifts to a G-action on E, then such a subbundle H is given by $H_x = T_x(G \cdot x)$. Furthermore in this case H is clearly involutive, because it is the tangent bundle of a foliation. This implies that the curvature of the associated connection is zero. □

COROLLARY 6.6.2. — *Let G be a connected complex Lie group, $\Gamma \subset G$ a lattice, S a complex Lie group and $E \to X = G/\Gamma$ a homogeneous S-principal bundle.*

Then either E admits a flat holomorphic connection or there exists an essential extension of connected Lie groups

$$1 \longrightarrow L \longrightarrow \tilde{G} \xrightarrow{\tau} G \longrightarrow 1$$

such that $\dim_{\mathbb{C}}(L) > 0$ *and the $\tilde{G}$-action on X induced by τ lifts to an action on E.*

Proof. — Note that by prop. 6.3.5 there exists a (finite-dimensional) Lie group $\tilde{G}$ acting on E in such a way that for every single element $g \in G$ there is a lift $\tilde{g} \in G$ acting compatibly. Hence the assertion is a consequence of the lemma. □

COROLLARY 6.6.3. — *Let G be a simply connected semisimple complex Lie group, Γ a lattice and $E \to X = G/\Gamma$ a homogeneous S-principal bundle*

Then E admits a flat connection and consequently is given by a representation of the fundamental group $\rho : \pi_1(X) \to S$.

This follows, because a semisimple Lie group does not admit any essential extension.

For a homogeneous bundle E without flat connection we can say more about the action of the extension group $\tilde{G}$ on E.

LEMMA 6.6.4. — *Let G be a connected complex Lie group, Γ a lattice, H a complex Lie group and $E \to X = G/\Gamma$ a homogeneous H-principal bundle.*

Let $1 \to L \to \tilde{G} \to G \to 1$ be an extension of connected Lie groups such that the $\tilde{G}$-action on X lifts to E. Assume that $\tilde{G}$ acts almost effectively on E.

Then L acts almost freely on E, i.e., all the isotropy groups are discrete.

Proof. — For every $x \in X$ let V_x denote the set of all $g \in L$ which act trivially on the fiber E_x. Then V_x is a complex Lie subgroup of L and we obtain a holomorphic map $x \mapsto \mathcal{L}ie(V_x)$ which is a G-equivariant holomorphic map from X to a Grassmann manifold, hence constant. Thus $V_x^0 = V_y^0$ for all $x, y \in X$. Since $\tilde{G}$ acts almost effectively on E, it follows that $V_x^0 = 1$, *i.e.*, L acts almost effectively on every fiber.

Now observe that L consists of bundle automorphisms, *i.e.*, commutes with the principal H-right action on E. For any group H the only self-maps $f : H \to H$ commuting with all right translations are the left translations. Hence for every fiber E_x there is a group homomorphism $\rho : L \to H$ and an isomorphism $E_x \simeq H$ such that L acts by left translations via ρ. Since L acts almost effectively on every fiber, it follows that L acts almost freely on every fiber. □

COROLLARY 6.6.5. — *Under the assumptions of the lemma it is true that* $\dim_{\mathbb{C}}(\widetilde{G}) \leq \dim_{\mathbb{C}}(G) + \dim_{\mathbb{C}}(H) = \dim_{\mathbb{C}}(E)$.

THEOREM 6.6.6. — *Let $H = \mathbb{C}^*$ or $H = \mathbb{C}$, G a connected complex Lie group, Γ a lattice and $X = G/\Gamma$.*

Then every homogeneous H-principal bundle over X is flat.

Proof. — Let E be a homogeneous H-principal bundle. Due to the homogeneity of E there is an extension of connected complex Lie groups

$$1 \longrightarrow L \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

such that the $\widetilde{G}$-action on X lifts to an action on E (use prop. 6.3.5). We may require that $\widetilde{G}$ acts effectively on E.[1] Now corollary 6.6.5 implies that $\dim_{\mathbb{C}}(\widetilde{G}) \leq \dim_{\mathbb{C}}(G) + 1$. If $\dim_{\mathbb{C}}(\widetilde{G}) = \dim_{\mathbb{C}}(G)$, the bundle is flat. Hence we may assume that $\dim_{\mathbb{C}}(\widetilde{G}) = \dim_{\mathbb{C}}(G) + 1$. Due to lemma 6.6.4 this implies that $\widetilde{G}$ acts transitively on E. We have to study the H-action in detail. Since H is commutative, right and left translation on H coincide. Therefore every element $g \in L$ determines a map $\phi_g : X \to H$ such that g is acting on the fiber E_x as right translation by $\phi_g(x)$. Now $H \subseteq \mathbb{C}$, hence ϕ_g is given by a holomorphic function. But $X = G/\Gamma$ does not admit any non-constant holomorphic function (thm. 3.7.1). This implies in particular that L acts freely on every fiber. Thus we are now able to invoke prop. 6.5.6 with an isotropy group of the $\widetilde{G}$-action on E as Λ. It follows that the extension $1 \to L \to \widetilde{G} \to G \to 1$ can not be essential. Hence E is flat. □

COROLLARY 6.6.7. — *Let G be a connected complex Lie group, Γ a lattice, $X = G/\Gamma$ and L a homogeneous line bundle.*

Then L is flat, i.e., admits a flat holomorphic connection.

6.7. Homogeneous vector bundles of rank two

In many cases a homogeneous vector bundle of rank two is automatically flat. The key reason for this is that $GL_2(\mathbb{C})$ contains no interesting nilpotent subgroup and essential extensions always involve nilpotent groups.

THEOREM 6.7.1. — *Let G be a connected complex Lie group, Γ a lattice, and E a homogeneous vector bundle of rank two over $X = G/\Gamma$. Assume that the radical of G is nilpotent.*

Then E is flat.

[1] To obtain an effective action of $\widetilde{G}$ we have to drop our usual assumption that G is simply connected.

As preparation for the proof we present several technical results on Lie groups.

LEMMA 6.7.2. — *Let N be a nilpotent connected Lie group, H a closed subgroup with N/H compact.*

Then N'/H' is compact, too.

Proof. — There is no loss in generality, if we replace N by the universal covering $\widetilde{N}$ and H by its pre-image under the projection $\pi : \widetilde{N} \to N$.

By a criterion of Malcev, compactness of N/H is equivalent to the existence of a continuous faithful unipotent representation $\rho : N \to GL_n(\mathbb{R})$ such that $\rho(N)$ and $\rho(H)$ have the same Zariski closure in $GL_n(\mathbb{C})$ ([**89**], [**123**]). Let I denote this Zariski closure. Then both $\rho(N)'$ and $\rho(H)'$ are Zariski dense in I' (use lemma 1.9.3). Thus $\rho(N')$ and $\rho(H')$ have the same Zariski closure, and Malcev's criterion implies that N'/H' is compact. □

PROPOSITION 6.7.3. — *Let N be a connected nilpotent Lie group and H a closed Lie subgroup such that N/H is compact and $H^0 \subset N'$.*

Then $H^0/(H^0 \cap H')$ is compact, too.

Proof. — By the preceding lemma, compactness of N/H implies that N'/H' is compact. Now H/H^0 is discrete, hence $H^0 \subset H'H^0 \subset H$ implies that $H'H^0$ is closed. Thus $H'H^0/H'$ is a closed subset of N'/H' and therefore compact. Finally note that $H'H^0/H' \simeq H^0/(H^0 \cap H')$. □

COROLLARY 6.7.4. — *Let N be a connected nilpotent complex Lie group and H a closed complex Lie subgroup with N/H compact and $H^0 \subset N'$. Let $\rho : H \to GL_2(\mathbb{C})$ be a holomorphic group homomorphism.*

Then $H^0 \subset \ker \rho$.

Proof. — Let I denote the Zariski closure of $\rho(H)$ in $GL_2(\mathbb{C})$. Then I is a nilpotent algebraic subgroup of $GL_2(\mathbb{C})$. For any such I the connected component I^0 is commutative. Hence H admits a subgroup H_1 of finite index such that $\rho(H_1)$ is commutative. It follows that $H_1' \subset \ker \rho$. Now H_1' is of finite index in H', hence $H^0/(H^0 \cap H_1')$ is compact. Thus $\rho(H^0)$ is compact.

Since ρ was assumed to be holomorphic, $\rho(H^0)$ must be a complex Lie subgroup of $GL_2(\mathbb{C})$. Therefore compactness of $\rho(H^0)$ implies $H^0 \subset \ker \rho$. □

Proof of the theorem. — For any such vector bundle E there exists an extension of connected Lie groups

$$1 \longrightarrow L \longrightarrow \widetilde{G} \xrightarrow{\tau} G \longrightarrow 1$$

and a holomorphic group homomorphism $\rho : H = \tau^{-1}(\Gamma) \to GL_2(\mathbb{C})$ such that E arises as a fibered product $\widetilde{G} \times_H \mathbb{C}^2$. Furthermore we may assume that $L \subset \widetilde{R}'$ where $\widetilde{R}$ denotes the radical of $\widetilde{G}$. Moreover $\widetilde{R}$ is nilpotent due to prop. 6.5.4. Recall that the R-orbits on G/Γ are closed (thm. 3.5.3). Hence $\widetilde{R}/(\widetilde{R} \cap H)$ is compact and we may apply the preceding lemma to $\widetilde{R} \cap H \subset \widetilde{R}$ and conclude that $H^0 = L \subset \ker \rho$. This implies that E is given by a representation of the fundamental group

$$\pi_1(G/\Gamma) = \pi_1(\widetilde{G}/H) \to H/H^0 \xrightarrow{\rho} GL_2(\mathbb{C})$$

□

CHAPTER 7

FLAT BUNDLES

7.1. Survey

Our goal here is to classify flat vector bundles or more generally flat principal bundles. As we previously pointed out, every flat bundle is induced by a representation of the fundamental group $\rho : \pi_1(X) \to GL_n(\mathbb{C})$ (resp. $\rho : \pi_1(X) \to G$ for flat G-principal bundles). Two representations $\rho_i : \pi_1(X) \to G$ yield the same holomorphic G-principal bundle if and only if there exists a holomorphic map f from the universal covering $\widetilde{X}$ to the structure group G which is $\pi_1(X)$-equivariant in the following sense:

$$f(x\gamma) = \rho_1(\gamma^{-1}) \cdot f(x) \cdot \rho_2(\gamma).$$

Naturally one desires to obtain a more accessible criterion. One easily shows that a representation $\rho : \Gamma \to H$ induces a trivial holomorphic H-principal bundle on G/Γ if and only if ρ can be extended to a holomorphic group homomorphism $\widetilde{\rho} : G \to H$. It is another matter to obtain a criterion which determines whether or not two given non-trivial flat bundles are isomorphic.

For three special cases we are able to provide such a criterion.

First, a special result covers commutative structure groups. In particular we prove the following: *Let $X = G/\Gamma$, assume G/R contains no $SL_2(\mathbb{C})$-factor. Let H be a commutative complex Lie group and $H_{\mathbb{R}}$ a real form which contains the maximal compact subgroup of H. Then there is a one-to-one correspondence between flat H-principal bundles over X and* $\mathrm{Hom}(\Gamma, H_{\mathbb{R}})$. In particular this implies that homogeneous line bundles are parametrized by $\mathrm{Hom}(\Gamma, S^1)$.

Second, we consider representations with bounded images. *If the images $\rho_i(\pi_1(X))$ are both relatively compact in G, then the induced holomorphic principal bundles are equivalent if and only if ρ_1 and ρ_2 are conjugate by an element in G.* (This is true for every compact complex manifold X, parallelizable or not.)

Thirdly, we deduce the main result of this section which gives a complete classification of homogeneous principal bundles over $X = S/\Gamma$ with S semisimple without

$SL_2(\mathbb{C})$-factors. Superrigidity applies here. Hence for every complex linear algebraic group H and every group homomorphism $\rho : \Gamma \to H$ there exists a unique continuous group homomorphism $\widetilde{\rho} : S \to H$ such that $\widetilde{\rho}|_\Gamma$ and ρ coincide on a subgroup of finite index. We call ρ *essentially antiholomorphic* if $\widetilde{\rho}$ is antiholomorphic. We prove that there is a one-to-one correspondence between homogeneous H-principal bundles and conjugacy classes of essentially antiholomorphic representations of $\Gamma \to H$.

Subsequently we use these results in order to prove the following statement: *Every positive-dimensional compact complex manifold admits a non-trivial holomorphic vector bundle.* This answers a question of Bănică.

We study in substantial detail the question how many representations with finite image exist and obtain two results in different directions: First, for every compact complex parallelizable manifold $X = G/\Gamma$ and every element $\gamma \in \Gamma \simeq \pi_1(X)$ there exists a number $n \in \mathbb{N}$ and a group homomorphism $\rho : \Gamma \to GL_n(\mathbb{C})$ with finite image such that $\rho(\gamma) \neq I$. Second, for G semisimple without $SL_2(\mathbb{C})$-factors there is a result in the converse direction: For fixed $X = G/\Gamma$ and a fixed connected Lie group H there exist only finitely many group homomorphisms from $\pi_1(X)$ to H with finite image (up to conjugacy).

Finally we study the topological nature of flat bundles. In a certain sense flat bundles are automatically almost topologically trivial. For instance, it is easy to see that for flat vector bundles all the Chern classes must vanish. We prove the following statement: *Let $X = G/\Gamma$ and assume that G/R has no $SL_2(\mathbb{C})$-factors. Then for every homogeneous H-principal bundle E over X there exists a finite covering $\tau : X_1 \to X$ such that $\tau^*E \to X_1$ is topologically trivial.* (This is stronger than the statement on Chern classes, because $\tau^* : H^\bullet(X, \mathbb{C}) \to H^\bullet(X_1, \mathbb{C})$ is injective for every finite covering $\tau : X_1 \to X$.)

Most of the results of this chapter are contained in **[159]**, for the special case where G/Γ is compact in **[153]**. The existence result for non-trivial holomorphic vector bundles over arbitrary compact complex manifolds is published in **[152]**.

7.2. A triviality criterion

PROPOSITION 7.2.1. — *Let G be a connected complex Lie group, H a complex Lie group and $\Gamma \subset G$ a discrete subgroup. Let $\rho : \Gamma \to H$ be a group homomorphism. Assume that every holomorphic function on G/Γ is constant.*

Then the H-principal bundle $E \to G/\Gamma$ induced by ρ is holomorphically trivial if and only if ρ extends to a holomorphic group homomorphism $\widetilde{\rho} : G \to H$.

Proof. — Assume that there is a trivializing map $\phi : G \to H$, *i.e.*, a holomorphic map $\phi : G \to H$ such that $\phi(g\gamma) = \phi(g)\rho(\gamma)$ for all $g \in G$ and $\gamma \in \Gamma$. Upon replacing ϕ by $\widetilde{\phi}(g) \stackrel{def}{=} \phi(e)^{-1}\phi(g)$, we may assume that $\phi(e) = e$. Define $\alpha : G \times G \to H$ by

$$\alpha(g_1, g_2) = \phi(g_1)\phi(g_2)\phi(g_1 g_2)^{-1}.$$

Then $\alpha(g_1,e)=\alpha(e,g_2)=e$ and $\alpha(g_1,g_2\gamma)=\alpha(g_1,g_2)$ for all $g_1,g_2\in G$ and $\gamma\in G$. We consider now the induced maps $\alpha_g : G/\Gamma\to H$ given by $\alpha_g : x\Gamma\mapsto\alpha(g,x)$. Since $\alpha_e\equiv e$, it is clear that all the maps α_g are homotopic to a constant map. Hence these maps may be lifted to the universal covering $\widetilde{H}$ of H, *i.e.*, there exist holomorphic maps $\widetilde{\alpha}_g : G/\Gamma\to\widetilde{H}$ such that $\alpha_g=\pi\circ\widetilde{\alpha}_g$ where $\pi:\widetilde{H}\to H$ is the universal covering map. But $\widetilde{H}$ is a simply connected complex Lie group and therefore Stein. Hence the maps $\widetilde{\alpha}_g$ are constant. It follows that the maps α_g are likewise constant. Thus $\alpha(g,x)=\alpha(g,e)=e$ for all $g,x\in G$. □

7.3. Abelian representations of the fundamental group

In this section, we investigate flat principal bundles with commutative structure group over parallelizable manifolds. The key advantage in this case is that for a *commutative* group H (and any group G) the set of group homomorphisms $\mathrm{Hom}(G,H)$ is again a (commutative) group in a natural way, viz. by $(\rho_1\cdot\rho_2)(g)=\rho_1(g)\cdot\rho_2(g)$. Let X be a complex manifold, H a commutative complex Lie group and $\rho_i:\pi_1(X)\to H$ $(i=1,2)$ group homomorphisms defining H-principal bundles $E_i\to X$. Then $E_1\simeq E_2$ if and only if the H-principal bundle defined by $\rho_1\cdot\rho_2^{-1}$ is trivial. Therefore the above triviality criterion yields the following classification of flat H-principal bundles for commutative structure group H.

PROPOSITION 7.3.1. — *Let G be a simply connected complex Lie group and Γ a discrete subgroup such that every holomorphic function on G/Γ is constant. Furthermore let H a connected commutative complex Lie group,* $\mathrm{Hom}(G,H)$ *(resp.* $\mathrm{Hom}(\Gamma,H)$*) the group of all homomorphisms of complex Lie groups. Let $r:\mathrm{Hom}(G,H)\to\mathrm{Hom}(\Gamma,H)$ denote the natural restriction homomorphism.*

Then there is a one-to-one correspondence between flat H-principal bundles over $X=G/\Gamma$ and the quotient group $\mathrm{Hom}(\Gamma,H)/r\mathrm{Hom}(G,H)$.

(Murakami [**109**] proved this for the case where G/Γ is a torus.)

If H has a real form which contains its maximal compact subgroup and $SL_2(\mathbb{C})$-factors are not present, we may improve this result.

PROPOSITION 7.3.2. — *Let G be a simply connected complex Lie group, Γ a lattice and H a connected commutative complex Lie group. Assume that $H_{\mathbb{R}}$ is a real form of H such that $H_{\mathbb{R}}$ contains the maximal compact subgroup K of H. Assume moreover that no factor of G/R_G is isomorphic to $SL_2(\mathbb{C})$, where R_G denotes the radical of G.*

Then there is a one-to-one correspondence between H-principal bundles on $X=G/\Gamma$ admitting a flat connection and $\mathrm{Hom}(\Gamma,H_{\mathbb{R}})$.

Proof. — The assumptions imply in particular that $\Gamma/(G'\cap\Gamma)$ is discrete in G/G' and that $(\Gamma\cap G')/\Gamma'$ is finite (cor. 3.11.5).

Note that $\Gamma/\Gamma' \simeq H_1(X,\mathbb{Z})$ is a finitely generated abelian group, hence isomorphic to $A \times \mathbb{Z}^{2n}$ for some finite abelian group A. Now

$$\mathrm{Hom}(\Gamma, H) = \mathrm{Hom}(A, H) \oplus \mathrm{Hom}(\mathbb{Z}^{2n}, H).$$

Note that $\mathrm{Hom}(A,H) = \mathrm{Hom}(A,H_{\mathbb{R}})$, because $H_{\mathbb{R}}$ is required to contain the maximal compact subgroup of H. Moreover there is a one-to-one correspondence between $\mathrm{Hom}(\mathbb{Z}^{2n}, H)$ and $\mathrm{Hom}_{\mathbb{R}}(G/G', H)$. Now $\mathrm{Hom}_{\mathbb{C}}(G/G', H) \cap \mathrm{Hom}_{\mathbb{R}}(G/G', H_{\mathbb{R}}) = \{0\}$, because a holomorphic mapping with totally real image is constant. Using this, dimension reasons imply

$$\mathrm{Hom}_{\mathbb{R}}(G/G', H) = \mathrm{Hom}_{\mathbb{C}}(G/G', H) \oplus \mathrm{Hom}_{\mathbb{R}}(G/G', H_{\mathbb{R}}).$$

It follows that $\mathrm{Hom}(\Gamma/\Gamma') \simeq \mathrm{Hom}(\Gamma/\Gamma', H_{\mathbb{R}}) \oplus \mathrm{Hom}_{\mathbb{C}}(G/G', H)$. □

COROLLARY 7.3.3. — *Let G be a simply connected complex Lie group and Γ a lattice. Assume that no factor of G/R_G is isomorphic to $SL_2(\mathbb{C})$, where R_G denotes the radical of G.*

Then there is a one-to-one correspondence between homogeneous complex line bundles over G/Γ and $\mathrm{Hom}(\Gamma, S^1)$.

Proof. — Every homogeneous line bundle over G/Γ is flat and for $H = \mathbb{C}^*$ the subgroup $S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$ is evidently a real form $H_{\mathbb{R}}$ containing the maximal compact subgroup. □

For $G \simeq SL_2(\mathbb{C})$ it is possible that $\mathrm{rank}_{\mathbb{Z}}(\Gamma/\Gamma') > 0$ for a discrete cocompact subgroup Γ. In this case $\mathrm{Hom}(\Gamma, S^1) \neq \mathrm{Hom}(\Gamma, \mathbb{C}^*)$. Since evidently $\mathrm{Hom}(SL_2(\mathbb{C}), \mathbb{C}^*) = 1$, proposition 7.3.1 now implies that in this case there exists flat line bundles on $X = SL_2(\mathbb{C})/\Gamma$ which can not be defined by a *unitary* representation $\rho : \Gamma \to S^1$.

7.4. Bounded representations

Here we study bounded representations, *i.e.*, representations with relatively compact image. First we want to mention some ways in which bounded representations arise.

1. If Γ is a lattice in a simply connected complex Lie group G, then Γ is finitely generated and G is linear. In this case a theorem of Malcev (prop. 1.7.5) implies that Γ is *residually finite* , *i.e.*, for every element $\gamma \in \Gamma$ there exists a finite group F and a group homomorphism $\rho : \Gamma \to F$ such that $\rho(\gamma) \neq e$. Since every finite group embeds in some linear group, this yields many representations of Γ with finite image.
2. If $H_1(G/\Gamma) \simeq \Gamma/\Gamma'$ is non-trivial, Γ admits many group homomorphisms to $S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$.

3. Let K be a number field, $\mathcal{O}_K$ its ring of algebraic integers, S a semisimple K-group and $\mathcal{S}$ resp. $\mathcal{T}$ the set of all archimedean valuations v such that G is K_v-isotropic resp. K_v-anisotropic. Assume that $K_v \simeq \mathbb{C}$ for all $v \in \mathcal{S}$ and that $\mathcal{T}$ is not empty. Then $G = \Pi_{v\in\mathcal{S}} S(K_v)$ is a complex Lie group, $U = \Pi_{v\in\mathcal{T}} S(K_v)$ is a compact real Lie group and $\Gamma = S(\mathcal{O}_K)$ is a lattice in G which admits an injective group homomorphism to U. Now every representation of U induces a representation of Γ with relatively-compact image.

THEOREM 7.4.1. — *Let G be a connected complex Lie group, R its radical, Γ a discrete subgroup of G, H a Stein Lie group and $\rho_i : \Gamma \to H$ group homomorphisms with relatively compact image.*

Assume that $\mathrm{Ad}(\Gamma)$ *and* $\mathrm{Ad}(G)$ *have the same Zariski closure in $GL(\mathcal{L}ie\, G)$ and that $\Gamma \cap R$ is cocompact in R.*

Let E_1 and E_2 denote the H-principal bundles over G/Γ induced by ρ_1 resp. ρ_2. Then $E_1 \sim E_2$ if and only if ρ_1 and ρ_2 are conjugate by an element $h \in H$.

Proof. — Recall that $E_1 \sim E_2$ holds iff there exists a holomorphic map $\phi : G \to H$ such that

$$\phi(g\gamma) = \rho_1(\gamma)^{-1}\phi(g)\rho_2(\gamma) \tag{1}$$

for all $g \in G$, $\gamma \in \Gamma$.

Let K_i denote the closure of $\rho_i(\Gamma)$ in H. The sets K_i are compact subgroups. Since H is Stein, it admits a strictly plurisubharmonic exhaustion function τ. We may assume that τ is invariant under the $K_1 \times K_2$-action on H given by $(k_1, k_2) : h \mapsto k_1^{-1}hk_2$ (because we can replace τ by the function obtained by averaging τ over the $K_1 \times K_2$-orbits). Then (1) implies $\tau(\phi(g\gamma)) = \tau(\phi(g))$. Thus we obtain a plurisubharmonic function on G/Γ. However, under the assumptions of the theorem every plurisubharmonic function on G/Γ is constant (This follows from thm. 3.7.1 and prop. 3.7.2). Thus $g \mapsto \tau(\phi(g))$ is constant. Since τ is strictly plurisubharmonic, this implies that ϕ is constant. □

COROLLARY 7.4.2. — *Let G, Γ and H be as in the above theorem and let $\rho : \Gamma \to H$ be a group homomorphism with relatively compact image.*

Then the induced H-principal over G/Γ is holomorphically trivial if and only if $\rho \equiv e$.

For *compact* complex manifolds we do not really need the parallelizability assumption at this point.

THEOREM 7.4.3. — *Let X be a compact complex manifold. Let H be a Stein complex Lie group. Let $\rho_1, \rho_2 : \Gamma = \pi_1(X) \to H$ be group homomorphisms such that the images $\rho_i(\Gamma)$ are relatively compact in H.*

Then the induced bundles E_1, E_2 are holomorphically equivalent if and only if there exists an element $g \in H$ such that

$$\rho_1(\gamma) = g \cdot \rho_2(\gamma) \cdot g^{-1}$$

for all $\gamma \in \Gamma$, i.e., if and only if ρ_1 and ρ_2 are conjugate.

Recall that every complex Lie subgroup of $GL_n(\mathbb{C})$ is Stein ([**96**]).

Proof. — Assume that the bundles E_1 and E_2 defined by ρ_i are holomorphically equivalent. Then there exists a holomorphic map $f : \widetilde{X} \to H$ such that

$$(*) \qquad f(x\gamma) = \rho_1(\gamma^{-1}) \cdot f(x) \cdot \rho_2(\gamma).$$

The images $\rho_i(\Gamma)$ are subgroups of H which constitute relatively-compact subsets. Recall that for every subgroup in a topological group the closure is again a subgroup. Hence the closures of the $\rho_i(\Gamma)$ form compact subgroups K_i of H. Now $K = K_1 \times K_2$ is a compact group with an action on the Stein manifold H defined by

$$\mu(k_1, k_2) : g \mapsto k_1^{-1} \cdot g \cdot k_2.$$

In this form $(*)$ reads as

$$f(x\gamma) = \mu(\rho(\gamma))(f(x))$$

with $\rho = (\rho_1, \rho_2)$. Thanks to the proposition of the preceding section it follows that such a map f must be constant. If f is constant with value $g \in H$, then $(*)$ translates to

$$\rho_1(\gamma) = g \cdot \rho_2(\gamma) \cdot g^{-1}.$$

□

For the special case $H = \mathbb{C}^*$ the theorem implies the well-known fact that two line bundles on a compact complex manifold X given by unitary representations $\rho, \rho' : \pi_1(X) \to S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$ are holomorphically equivalent if and only if $\rho = \rho'$.

7.5. Antiholomorphic maps and actions of unipotent groups

The purpose of this section is to collect several auxiliary results on algebraic group actions. These results will be needed in the subsequent section on essentially antiholomorphic representations.

PROPOSITION 7.5.1. — *Let Z be a quasi-affine variety, $\bar{H}$ a connected commutative linear-algebraic group (both defined over $\mathbb{C}$) and H a connected Zariski dense complex Lie subgroup of $\bar{H}$. Assume that there is a regular action $\mu : \bar{H} \times Z \to Z$, a map*

$\alpha : H \to Z$ with relatively compact image and an antiholomorphic map $\phi : H \to Z$ such that

$$\alpha(h) = \mu(h)(\phi(h)) \tag{2}$$

for all $h \in H$. Let U denote the unipotent radical of $\bar{H}$.

Then both $\alpha(H)$ and $\phi(H)$ are contained in a single $\mu(\bar{H})$-orbit W, U acts trivially on this orbit and the maps α, ϕ are homomorphisms of real Lie groups from H to W with respect to the natural group structure on W with $\phi(e)$ as neutral element.

Proof. — We start with a discussion of *semi-invariant* functions on Z.

DEFINITION 7.5.2. — Let Z be a variety, G an algebraic group acting on Z. A regular function f on Z is called *semi-invariant* if there exists a character χ of G such that $f(gz) = \chi(g)f(z)$ for all $z \in Z$ and $g \in G$.

Now let f be a semi-invariant for the $\bar{H}$-action on Z. Then

$$f(\alpha(h)) = \chi(h)f(\phi(h)) \tag{B}$$

for a character χ of $\bar{H}$ and all $h \in H$. This implies

$$|f(\alpha(h))|^2 = |\chi(h)f(\phi(h))|^2 = |\chi(h)\overline{f(\phi(h))}|^2$$

The left side is bounded while the right side is the absolute value of a holomorphic function. Hence there is a constant c such that $|f \circ \alpha| \equiv c$ and $\chi \cdot \overline{f \circ \phi} \equiv c$. It follows that $f \circ \alpha$ and $f \circ \phi$ vanish either everywhere or nowhere.

LEMMA 7.5.3. — *Let W be a quasi-affine algebraic variety and G a connected solvable linear algebraic group acting regularly on W. Assume that for every semi-invariant f the zero-set $V(f)$ is either empty or the whole of W.*

Then G acts transitively on W.

Proof. — Assume the contrary. Then W must contain a proper invariant algebraic subvariety Y. Now the ideal I_Y is a non-trivial invariant subvectorspace of the space of regular functions $\mathbb{C}[W]$. Recall that every $f \in \mathbb{C}[W]$ is contained in a finite-dimensional subvectorspace. Using this fact the theorem of Lie implies that I_Y contains a one-dimensional invariant subvectorspace S. Now any $f \in S \setminus \{0\}$ is a semi-invariant on W vanishing on Y but not vanishing everywhere. Contradiction! □

Applying this lemma to our situation we may conclude that both $\alpha(H)$ and $\phi(H)$ are contained in a single $\bar{H}$-orbit W.

Since $\bar{H}$ is commutative, the homogeneous $\bar{H}$-space W has a canonical structure as a commutative group which is unique up to the choice of the neutral element.

CLAIM 7.5.4. — The U-action on W is trivial.

Proof. — If not, there is a non-trivial regular U-equivariant map $\tau : W \to \mathbb{C}$. Using the assumption that H is Zariski dense in $\bar{H}$, we may find a one-parameter subgroup $\gamma(t)$ in H such that $\tau(\mu(\gamma(t))(x)) = t + \tau(x)$ for $x \in W$. But then $t \mapsto \tau(\alpha(\gamma(t)))$ is a bounded function on $\mathbb{C}$ which may be represented as

$$\tau(\alpha(\gamma(t))) = \tau(\mu(\gamma(t))(\phi(t))) = t + \underbrace{\tau(\phi(\gamma(t)))}_{\text{antiholo.}}. \tag{3}$$

This is a contradiction, because the left side is bounded while the right side is a non-constant harmonic function on $\mathbb{C}$. Thus the U-action must have been trivial. □

Now W is a homogeneous space of the reductive commutative group $\bar{H}/U$. By choosing a point in W as neutral element, W inherits a structure as reductive commutative algebraic group. Let us choose $\phi(e) = \alpha(e)$ as neutral element. Then every character of W is an semi-invariant for the $\bar{H}$-action and consequently our previous considerations imply that $\chi \circ \phi$ is a real Lie group homomorphism from H to $\mathbb{C}^*$ for every character χ of W. It follows that ϕ (and hence also α) are real Lie group homomorphisms. □

7.6. Essentially antiholomorphic representations

We continue our preparations for the desired classification theorem of flat vector bundles over $X = G/\Gamma$ with G semisimple without $SL_2(\mathbb{C})$-factors. We need some basic information on *essentially antiholomorphic representations.*

DEFINITION 7.6.1. — Let G and H be complex Lie groups, and $\Gamma \subset G$ a discrete subgroup. A group homomorphism $\rho : \Gamma \to H$ is called *essentially antiholomorphic* if there exists an antiholomorphic Lie group homomorphism $\zeta : G \to H$ and a map $\xi : \Gamma \to H$ with relatively compact image $\xi(\Gamma) \subset H$ such that

$$\rho(\gamma) = \zeta(\gamma) \cdot \xi(\gamma) \tag{**}$$

for all $\gamma \in \Gamma$.

REMARK 7.6.2. — Let G, Γ, H as above, $\Gamma_0 \subset \Gamma$ a subgroup of finite index, $\rho : \Gamma \to H$ a group homomorphism. Assume that there exists an antiholomorphic Lie group homomorphism $\rho_0 : G \to H$ such that $\rho|_{\Gamma_0} = \rho_0|_{\Gamma_0}$. Then ρ is essentially antiholomorphic (see lemma 3.19.1).

PROPOSITION 7.6.3. — *Let G be a connected complex Lie group, Γ a lattice, H a Stein Lie group and $\rho : \Gamma \to H$ an essentially antiholomorphic representation with maps ζ, ξ given as above.*

Then

1. *$\xi : \Gamma \to H$ is a group homomorphism.*
2. *Both ξ and ζ are uniquely determined by ρ.*
3. *$\xi(\gamma)$ and $\zeta(g)$ commute for all $g \in G$, $\gamma \in \Gamma$.*

Proof. — The equation (**) combined with $\rho(\gamma\delta) = \rho(\gamma)\rho(\delta)$ implies

$$(A) \qquad \zeta(\delta)^{-1}\xi(\gamma)\zeta(\delta) = \xi(\gamma\delta)\xi(\delta)^{-1}$$

for all $\gamma, \delta \in \Gamma$.

Let A be a connected complex Lie subgroup of G for which there exists a compact set $F \subset G$ such that $A \subset F\Gamma$. For $\gamma \in \Gamma$ we define an antiholomorphic map $\phi_\gamma : A \to H$ by

$$\phi_\gamma(a) = \zeta(a)\xi(\gamma)\zeta(a)^{-1}.$$

If $\overline{\xi(\Gamma)} = K$, then $\phi_\gamma(A) \subset \zeta(F) \cdot K \cdot K^{-1} \cdot \zeta(F)$. Since the latter set is compact, it follows that ϕ_γ is constant. This implies that $\zeta(A)$ commutes with $\xi(\Gamma)$. Since G is generated as a group by such subgroups A (thm. 3.10.1), this yields the third assertion. The first assertion is an immediate corollary. To check unicity, let ζ, $\widetilde{\zeta}$ be antiholomorphic representations and ξ, $\widetilde{\xi}$ be bounded maps such that $\zeta\xi = \widetilde{\zeta}\widetilde{\xi}$. This implies $\widetilde{\zeta}^{-1}\zeta = \widetilde{\xi}\xi^{-1}$. This is now a bounded antiholomorphic map, hence constant, and in fact constant with value e. Therefore $\widetilde{\zeta} = \zeta$ and $\widetilde{\xi} = \xi$. □

COROLLARY 7.6.4. — *Let G be a complex Lie group, Γ a lattice and $\rho : \Gamma \to GL(V)$ an irreducible representation which is essentially antiholomorphic.*

Then there exist vector spaces V_i $(i = 1, 2)$, an antiholomorphic representation $\rho_0 : G \to GL(V_1)$ and a relatively compact representation $\zeta : \Gamma \to GL(V_2)$ such that $\rho = \rho_0|_\Gamma \otimes \zeta$.

Thus for a given essentially antiholomorphic representation ρ we have a unique antiholomorphic representation ρ_0 of G and a unique relatively compact representation ζ of Γ. These are called the *associated antiholomorphic resp. bounded* representations of ρ.

THEOREM 7.6.5. — *Let G be a connected complex Lie group, H a linear complex Lie group, Γ a lattice and $\rho, \widetilde{\rho} : \Gamma \to H$ essentially antiholomorphic representations. Then the induced flat bundles are isomorphic as holomorphic H-bundles if and only if there exists a constant $c \in H$ such that $\rho|_{\Gamma\cap G'} = c\widetilde{\rho}|_{\Gamma\cap G'}c^{-1}$.*

Proof. — There is no loss in generality in assuming that G is simply-connected. The assumption of the induced flat bundles being holomorphically equivalent is equivalent to the statement that there exists a holomorphic map $\phi : G \to H$ such that

$$(4) \qquad \phi(g\gamma) = \xi(\gamma)^{-1}\zeta(\gamma)^{-1}\phi(g)\widetilde{\zeta}(\gamma)\widetilde{\xi}(\gamma)$$

for all $g \in G$ and $\gamma \in \Gamma$ where $\rho = \zeta \cdot \xi$ resp. $\widetilde{\rho} = \widetilde{\zeta} \cdot \widetilde{\xi}$ are the respective decompositions as products of an antiholomorphic and a bounded factor. We define a map $\alpha : G \to H$ by

$$(5) \qquad \alpha(g) = \zeta(g)\phi(g)\widetilde{\zeta}(g)^{-1}$$

Then

$$\alpha(g\gamma) = \zeta(g)\zeta(\gamma)\xi(\gamma)^{-1}\zeta(\gamma)^{-1}\phi(g)\widetilde{\zeta}(\gamma)\widetilde{\xi}(\gamma)\widetilde{\zeta}(\gamma)^{-1}\widetilde{\zeta}(g)^{-1} \tag{6}$$
$$= \xi(\gamma)^{-1}\zeta(g)\phi(g)\widetilde{\zeta}(g)^{-1}\widetilde{\xi}(\gamma) = \xi(\gamma)^{-1}\alpha(g)\widetilde{\xi}(\gamma)$$

(Recall that $\zeta(G)$ and $\xi(\Gamma)$ commute).

Claim 7.6.6. — Let A be a connected commutative complex Lie subgroup of G such that there exists a compact subset $F_A \subset G$ such that $A \subset F_A\Gamma$. Let $c = \phi(e) = \alpha(e) \in H$ and $W = \mu(G)(c)$ where μ is the action given by

$$\mu(g) : x \mapsto \zeta(g)^{-1}x\widetilde{\zeta}(g) \tag{7}$$

Let η denote the Γ-action on H given by

$$\eta(\gamma) : x \mapsto \xi(\gamma)^{-1}x\widetilde{\xi}(\gamma) \tag{8}$$

Then $\alpha(A) \subset W$, $\phi(A) \subset W$ and $\eta(A\cap\Gamma)$ stabilizes W. If in addition A is contained in G', then $\mu(A)$ acts trivially on W.

Proof. — Since H is assumed to be linear, we may embedd it into some $GL(n,\mathbb{C})$. Then an application of prop. 7.5.1 (with $Z = GL(n,\mathbb{C})$, $H = A$ and the Zariski closure of $\mu(H)$ in $\mathrm{Aut}(GL(n,\mathbb{C}))$ as $\bar{H}$) yields $\alpha(A) \subset W$ and $\phi(A) \subset W$.

By definition of α the inclusion $(A\cap\Gamma) \subset W$ implies that $\eta(\gamma)(c) \in W$ for $\gamma \in A\cap\Gamma$. Since $\eta(\Gamma)$ commutes with $\mu(G)$, it follows that $\eta(A \cap \Gamma)$ stabilizes W. Finally, since G was assumed to be simply connected, this implies that G' is unipotent. Hence the last assertion also follows from prop. 7.5.1. □

Claim 7.6.7. — The group $\mu(G')$ acts trivially on W and in particular stabilizes the point $\phi(e)$.

This is immediate, because G' is generated by unipotent subgroups with a bounded orbit (prop. 3.10.2).

Next we discuss "translates" of ϕ. For $x \in G$ let ϕ_x denote the map defined by $\phi_x(g) = \phi(xg)$. Then $\phi_x : G \to H$ is a holomorphic map fulfilling (4). Thus we may apply the above considerations and conclude that $\mu(G')(\phi_x(e)) = \phi_x(e)$. Since $\phi_x(e) = \phi(x)$, it follows that $\mu(G')$ stabilizes $\phi(G)$ pointwise. Hence

$$\phi(g\gamma) = \eta(\gamma)(\phi(g)) \tag{9}$$

for $g \in G$, $\gamma \in G' \cap \Gamma$. By assumption $\xi(\Gamma)$ and $\widetilde{\xi}(\Gamma)$ are contained in compact subgroups K resp. $\widetilde{K}$ of H. Since H is Stein, it admits a strictly plurisubharmonic exhaustion function τ. By averaging with respect to Haar measure we may assume that τ is invariant with respect to the $K \times \widetilde{K}$-action on H given by ?. Then we obtain

$$\tau(\phi(g\gamma)) = \tau(\phi(g)) \tag{10}$$

for $g \in G$, $\gamma \in G' \cap \Gamma$. Now every plurisubharmonic function on $G'/(G' \cap \Gamma)$ is constant (see prop. 3.11.10). Hence ϕ is constant along the G'-orbits on G. In particular $\phi|_{G'}$ is constant and this yields the assertion with $c = \phi(e)$. □

PROPOSITION 7.6.8. — *Let G be a complex Lie group, $\Gamma \subset G$ a lattice, H a complex linear algebraic group and $\rho : G \to H$ an antiholomorphic Lie group homomorphism.*

Then the flat bundle over G/Γ defined by $\rho|_\Gamma$ is trivial if and only if there exists a number n and holomorphic Lie group homomorphisms $\rho_1 : G \to (\mathbb{C}^*)^n$ and $\rho_2 : (\mathbb{C}^*)^n \to H$ such that $\rho = \rho_2 \circ \zeta \circ \rho_1$ where ζ denotes complex conjugation on $T = (\mathbb{C}^*)^n$ and $\rho_1(\Gamma) \subset (\mathbb{R}^*)^n$.

Proof. — By prop. 7.2.1 triviality of the induced bundle is equivalent to the existence of a holomorphic Lie group homomorphism $\phi : G \to H$ such that $\phi|_\Gamma = \rho|_\Gamma$. Hence one direction of the statement is clear: If there is a torus T and holomorphic Lie group homomorphisms ρ_1 and ρ_2 as specified in the proposition, then $\phi \overset{def}{=} \rho_2 \circ \rho_1$ has the desired properties.

Let us assume that the bundle is trivial and that $\phi : G \to H$ is the trivializing holomorphic Lie group homomorphism. Then $\phi|_{G'}$ is constant by the preceding theorem. This implies that $\rho(\gamma) = e$ for all $\gamma \in G' \cap \Gamma$. Hence $\rho(\Gamma)$ is commutative. Let A denote the Zariski closure of $\rho(\Gamma)$ in H. Then H/A is a quasi-affine variety (because A is commutative). Thus $\phi : G \to H$ induces a holomorphic map from G/Γ to H/A which must be constant. We may assume that $\phi(e) = e$. Then $\phi(G) \subset A$. Furthermore $\rho(G) \subset A$ by the density theorem. Hence we may assume that $H = A$, *i.e.*, we may assume that H is a commutative linear algebraic group. Thus $H \simeq (\mathbb{C})^d \times (\mathbb{C}^*)^g$ and we may discuss these factors separately.

Let us first discuss the case $H = \mathbb{C}$. Then $\rho - \phi$ defines a Γ-invariant harmonic function on G and hence must be constant, which is impossible unless both ρ and ϕ are constant.

Now let us discuss the case $H = \mathbb{C}^*$. Let $\zeta : H \to H$ denote complex conjugation. Then $f(g\Gamma) = |\phi(g)^{-1}\zeta \circ \rho(g)|^2$ defines a plurisubharmonic function on G/Γ which must be constant. It follows that the holomorphic map $F : G \to H$ given by $g \mapsto \phi(g)^{-1}\zeta \circ \rho(g)$ is likewise constant. Thus $\phi = \zeta \circ \rho$. Since $\phi|_\Gamma = \rho|_\Gamma$, it follows that $\rho(\Gamma) \subset \mathbb{R}^*$.

The general case (*i.e.*, $H \simeq (\mathbb{C})^d \times (\mathbb{C}^*)^g$) is now an easy consequence. □

7.7. Vector Bundles

To transfer our results on principal bundles to vector bundles, we need the following generalized Schur lemma.

LEMMA 7.7.1. — *Let Γ be a group, V a vector space and $\zeta, \xi : \Gamma \to GL(V)$ group homomorphisms such that $\zeta(\gamma)$, $\xi(\delta)$ commute for all $\gamma, \delta \in \Gamma$. Assume that V is an irreducible Γ-module for $\rho = \zeta \cdot \xi$.*

Then there exists vector spaces V_1, V_2 and irreducible representations $\zeta_0 : \Gamma \to GL(V_1)$, $\xi_0 : \Gamma \to GL(V_2)$ such that $(V, \rho) \simeq (V_1, \zeta_0) \otimes (V_2, \xi_0)$.

LEMMA 7.7.2. — *Let $A \subset GL(n, \mathbb{C})$ be a connected commutative complex Lie subgroup, $\Lambda \subset A$ a lattice, Q a projective manifold with $b_1(Q) = 0$ on which A acts holomorphically and $x \in Q$. Assume that for every $\lambda \in \Lambda$ there exists a sequence of natural numbers n_k such that $\lim n_k = \infty$ and $\lim(\lambda^{n_k}(x)) = x$.*

Then x is fixed point for the A-action on Q.

Proof. — The assumptions on Q imply that $\mathrm{Aut}(Q)^0$ is a linear algebraic group. Let H denote the Zariski closure of A in $\mathrm{Aut}(Q)^0$. Assume that x is not a fixed point for the A-action. The H-orbit through x is a locally closed subset of Q and isomorphic to the quotient H/I where $I = \{h \in h : h(x) = x\}$. This quotient H/I is again a linear algebraic group (because H is commutative). It follows that the image of A in I/H under the natural morphism $\tau : A \to I/H$ can not be relatively compact. Hence there is an element $\lambda \in \Lambda$ such that the sequence $\tau(\lambda^n)$ in H/I contains no convergent subsequence. Since the H-orbits in Q are locally closed, this implies that no subsequence of $\lambda^n(x)$ can converge to x. This contradicts the assumptions of the lemma. Hence x must be an A-fixed point. □

PROPOSITION 7.7.3. — *Let G be a connected complex Lie group, Γ a lattice, V a complex vector space and $\rho : \Gamma \to GL(V)$ be an essentially antiholomorphic representation, with antiholomorphic part ζ and bounded part ξ. Assume that V contains a $\rho(\Gamma)$-invariant subvectorspace W.*

Then W is already invariant under $\zeta(G)$ and $\xi(\Gamma)$.

COROLLARY 7.7.4. — *Let G be a connected complex Lie group, Γ a lattice, V a complex vector space, $\rho : \Gamma \to GL(V)$ an essentially antiholomorphic representation and W a $\rho(\Gamma)$-stable subvectorspace of V.*

Then the restricted representation $\rho' : \Gamma \to GL(W)$ is likewise essentially antiholomorphic.

Proof. — Let $k = \dim W$. We consider the induced actions on the Grassmann manifold Q of k-dimensional subvectorspaces of V. For simplicity, they are also denoted by ρ, ζ and ξ. Let $x = [W] \in Q$. Since $\xi(\Gamma)$ is bounded, it is clear that for every element $\gamma \in \Gamma$ there exists a sequence of natural numbers n_k such that $\lim n_k = \infty$ and $\lim \xi(\gamma^{n_k}) = e$. Then $\rho = \zeta \cdot \xi$ implies that $\lim \zeta(\gamma^{-n_k})(x) = x$. By lemma 7.7.2 it follows that x is a fixed point for every connected commutative complex Lie subgroup $A \subset G$ with $A/(A \cap \Gamma)$ compact. By thm. 3.10.1 this implies that x is a fixed point for $\zeta(G)$ (and hence for $\xi(\Gamma)$, too). □

LEMMA 7.7.5. — *Let Γ be a group, $\rho_i : \Gamma \to GL(V_i)$ representations on complex vector spaces for $i = 1, 1$ and assume that*

1. *Both V_i are irreducible Γ-modules with respect to the representations ρ_i.*
2. *The Zariski closure H of $\rho_1(\Gamma)$ in $GL(V_1)$ is connected.*
3. *The image $\rho_2(\Gamma)$ is finite.*

Then $V_1 \otimes V_2$ is an irreducible Γ-module with respect to $\rho_1 \otimes \rho_2$.

Proof. — Let $\Gamma_0 = \ker \rho_2$. Since Γ/Γ_0 is finite and H is connected, it is clear that $\rho_1(\Gamma_0)$ is Zariski dense in $GL(V_1)$. Hence V_1 is an irreducible Γ_0-module while Γ_0 acts trivially on V_2. It follows that every Γ_0-invariant subvectorspace of $V_1 \otimes V_2$ has the form $V_1 \otimes W$ for some subvectorspace $W \subset V_2$. Clearly, such a $V_1 \otimes W$ is Γ-invariant only if $W = \{0\}$ or $W = V_2$. Thus $V_1 \otimes V_2$ is an irreducible Γ-module. □

COROLLARY 7.7.6. — *Let G be a connected complex Lie group, $\Gamma \subset G$ a lattice, $\rho_1 : G \to GL(V_1)$ an antiholomorphic representation, $\rho_2 : \Gamma \to GL(V_2)$ a representation with relatively compact and Zariski connected image and $\rho_3 : \Gamma \to GL(V_3)$ be a representation with finite image.*

Assume that all the representations ρ_i are irreducible.

Then $(\rho_1|_\Gamma) \otimes \rho_2 \otimes \rho_3 : \Gamma \to GL(V_1 \otimes V_2 \otimes V_3)$ is likewise irreducible.

7.8. The classification of homogeneous vector bundles

We make use of Margulis' superrigidity theorem in the following form:

THEOREM 7.8.1. — *Let S be a simply connected semisimple complex Lie group and $\Gamma \subset S$ a lattice. Assume that there does not exist a normal Lie subgroup $S_0 \subset S$ such that $S_0 \simeq SL(2, \mathbb{C})$ and $S_0/(S_0 \cap \Gamma)$ is of finite volume.*

Then there exists a compact real semisimple Lie group K and a group homomorphism $j : \Gamma \to K$ such that for every complex-algebraic group H and every group homomorphism $\alpha : \Gamma \to H$ there exist continuous group homomorphisms $\zeta : S \to H$, $\xi : K \to H$ and $\nu : \Gamma \to H$ such that

1. *$\alpha(\gamma) = \zeta(\gamma) \cdot \xi(j(\gamma)) \cdot \nu(\gamma)$ for all $\gamma \in \Gamma$.*
2. *The image $\nu(\Gamma)$ is finite.*
3. *$\zeta(s)$, $\xi(k)$ and $\nu(\gamma)$ commute for every $s \in S$, $k \in K$ and $\gamma \in \Gamma$.*

Note that for every continuous group homomorphism ζ from a complex semisimple Lie group S to a complex algebraic group H there exists a holomorphic group homomorphism ζ_0 and an antiholomorphic group homomorphism ζ_1 such that $\zeta = \zeta_0 \cdot \zeta_1$.

Using Margulis' theorem and our previous results we obtain the following classification.

THEOREM 7.8.2. — *Let S be a simply connected semisimple complex Lie group and $\Gamma \subset S$ a lattice. Assume that there does not exist a normal Lie subgroup $S_0 \subset S$ such that $S_0 \simeq SL(2,\mathbb{C})$ and $S_0/(S_0 \cap \Gamma)$ is of finite volume.*

Then there exists a compact real semisimple Lie group K and a group homomorphism $j : \Gamma \to K$ such that there is a one-to-one correspondance between irreducible holomorphic vector bundles on S/Γ admitting a flat connection and triples (ρ, ζ, ν) where

1. *All ρ, ζ and ν are irreducible representations of Γ.*
2. *ρ extends to an antiholomorphic representation of G.*
3. *ζ fibers through a representation of K.*
4. *ν is a representation with finite image.*

7.9. Sections in Vector Bundles given by antiholomorphic representations

Recall that homogeneous vector bundles admit global section only inasmuch as they are trivial (prop. 6.2.5). Thus the triviality criterion (prop. 7.6.8) enables us to obtain a complete description of the global sections in a flat bundle given by an antiholomorphic representation.

PROPOSITION 7.9.1. — *Let G be a connected complex Lie group, Γ a lattice and ρ be a holomorphic representation of G on a complex vector space V. Let $V_1 = V^{G'}$ denote the vector subspace of all $v \in V$ which are fixed by $\rho(G')$. Let Σ denote the subset of all $v \in V_1$ such that v is an eigenvector with real eigenvalue for every $\rho(\gamma)$ ($\gamma \in \Gamma$) and let V_0 denote the subvectorspace of V spanned by Σ.*

Let E and E_0 denote the flat vector bundles on $X = G/\Gamma$ which is induced by the representation $\bar{\rho}|_\Gamma$ on V resp. V_0.

Then E_0 is a holomorphically trivial vector bundle and $\Gamma(X, E_0) = \Gamma(X, E)$.

Proof. — Combine prop. 7.6.8 and prop. 6.2.5. □

For antiholomorphic representations we are now able to give a precise description of the global sections of the associated vector bundle.

PROPOSITION 7.9.2. — *Let G be a connected complex Lie group, Γ a lattice, V a complex vector space and $\rho : G \to GL(V)$ be an antiholomorphic representation. Let E denote the flat vector bundle over G/Γ which is induced by $\rho|_\Gamma$.*

Let Σ denote the set of all vectors $v \in V$ which are invariant under $\rho(G')$ and a $\rho(\gamma)$-eigenvector with real eigenvalue for every $\gamma \in \Gamma$.

Then $H^0(G/\Gamma, E) \simeq \langle\Sigma\rangle_{\mathbb{C}}$ where $\langle\Sigma\rangle_{\mathbb{C}}$ denotes the complex vector space spanned by Σ.

Proof. — By construction $\langle\Sigma\rangle_{\mathbb{C}}$ is the largest $\rho(G)$-invariant vector subspace of V inducing a holomorphically trivial vector subbundle of E. □

7.10. Subbundles of flat bundles

THEOREM 7.10.1. — *Let G be a connected complex Lie group, Γ a lattice and $E \to G/\Gamma$ be a flat vector bundle given by an essentially antiholomorphic representation $\rho : G \to GL(n, \mathbb{C})$.*

Assume that every meromorphic function on $X = G/\Gamma$ is constant.

Let $L \subset E$ be a vector subbundle. Then L also admits a flat connection.

If $G = G'$, then L is parallel with respect to the given flat connection on E.

Proof. — Let $k = \mathrm{rank}(L)$. By passing to $\Lambda^k L \subset \Lambda^k E$ we may assume that L is a line bundle. The flat connection on E yields a canonical way to lift the G-action on X to a G-action on E. The subbundle $E_0 = \oplus_{g \in G} g^* L$ is a G-invariant subbundle of E and therefore parallel with respect to the flat connection on E. Moreover it is given by an essentially antiholomorphic representation (see cor. 7.7.4). We may thus assume $E_0 = E$. Let $d = \mathrm{rank}(E_0)$ and choose $g_0, \ldots, g_d \in G$ such that the line subbundles $L_i = g_i^* L$ are in general position, *i.e.*, such that for every number m with $0 \le m \le d$ and every choice of $0 \le i_1 < \cdots < i_m \le d$ the subsheaf of E spanned by $L_{i_1} + \cdots + L_{i_m}$ has rank m.

CLAIM 7.10.2. — The line bundles $L_0, \ldots, L_d$ are in general position at every point and therefore yield a trivialization of $\mathbb{P}(E_0)$.

Proof. — Let U be an open neighbourhood of a point such that all the L_i admit nowhere vanishing sections σ_i. For every $k \in \{0, \ldots, d\}$ we obtain a section α_k in $\wedge^d E_0$ by $\alpha_k = \wedge_{i \neq k} \sigma_i$. Let $\alpha = \otimes_k \alpha_k$. Then α is a section in the line bundle $\otimes^{d+1}(\wedge^d E_0)$ and is vanishing exactly where the $L_0, \ldots, L_d$ fail to be in general position. Thus the set of all points in X where the $L_0, \ldots, L_d$ are not in general position constitute an analytic hypersurface of X. But the assumptions on X imply that X contains no hypersurfaces. Hence $L_0, \ldots, L_d$ are in general position everywhere. □

Thus $\mathbb{P}(E_0)$ is a holomorphically trivial $\mathbb{P}_{d-1}$-bundle. Observe that $\mathbb{P}(E_0)$ is defined by an essentially antiholomorphic group homomorphism $\widetilde{\rho} : \Gamma \to PGL(d, \mathbb{C})$. Using thm. 7.6.5 it follows that $\widetilde{\rho}|_{G' \cap \Gamma}$ is trivial. Hence $\widetilde{\rho}(\Gamma)$ constitutes a commutative subgroup of $PGL(d, \mathbb{C})$. This implies that there is a fixed point for the $\widetilde{\rho}(\Gamma)$-action on $\mathbb{P}(V)$. Corresponding to this fixed point there exists a sub-line bundle $L' \subset E$ which is G-invariant and therefore parallel to the given flat connection on E.

CLAIM 7.10.3. — The line bundles L' and L are isomorphic (as holomorphic line bundles).

Proof. — Recall that $\mathbb{P}(E)$ is holomorphically trivial and that every meromorphic function on X is constant. It follows that there is a unique trivialization of $\mathbb{P}(E)$ and that every section of $\mathbb{P}(E) \to X$ is constant with respect to this trivialization. This implies that any two line subbundles of E are holomorphically isomorphic. □

Thus L is isomorphic to a flat line bundle. However, it is parallel with respect to the given flat connection on E if and only if it is G-invariant. This in turn is equivalent to the assertion that L corresponds to a fixed point of the Γ-action on $\mathbb{P}(V)$. This action is trivial, if $G = G'$. □

7.11. Finite representations of the fundamental group

Here we will prove that compact complex parallelizable manifolds have many non-trivial holomorphic vector bundles which are induced by representations of the fundamental group with finite image.

We have seen already that given a connected complex Lie group H and a representation of the fundamental group $\rho : \pi_1(X) \to H$ with finite, but non-trivial image the induced H-principal bundle over X is not holomorphically trivial.

Now we encounter the following question: Given a manifold X which is not simply connected, does there always exist a non-trivial finite representation of the fundamental group? This is indeed the case for homogeneous complex manifolds, but not for arbitrary complex manifolds.

First we would like to recall that for every quotient $X = G/\Gamma$ of a connected complex Lie group G by a lattice Γ the fundamental group $\pi_1(X)$ is *residually finite* (see cor. 1.12.4), *i.e.*, for every element $\gamma \in \pi_1(X)$ there exists a group homomorphism $\rho : \pi_1(X) \to S$ to a finite group S such that $\rho(\gamma) \neq e_S$.

In fact there exists a non-trivial finite representation for arbitrary homogeneous complex manifolds (not necessarily parallelizable).

PROPOSITION 7.11.1. — *Let G be a complex Lie group, H a closed complex Lie subgroup. Assume that $X = G/H$ is not simply connected.*

Then there exists a non-trivial group homomorphism from $\pi_1(X)$ to a finite group.

Proof. — There is no loss in generality in assuming G to be simply connected. Then $\pi_1(G/H) \simeq H/H^0$. (As usual H^0 denotes the connected component of e in H.) Furthermore G may be embedded into a general linear group $GL_n(\mathbb{C})$ (see prop. 1.11.1). Let N denote the normalizer $N_{GL_n(\mathbb{C})}\left(H^0\right)$ of H^0 in $GL_n(\mathbb{C})$. The normalizer of a connected Lie subgroup of a linear algebraic group may be realized as isotropy group of a certain action on a Grassmann manifold induced by the adjoint representation. This implies that the normalizer of a connected Lie subgroup of linear algebraic group is always a linear algebraic subgroup. Now let $\bar{H}^0$ denote the closure of H^0 in $GL_n(\mathbb{C})$ with respect to the algebraic Zariski topology. Then $N/\bar{H}^0$ is a linear algebraic group, hence admits a faithful linear representation. It follows that either H/H^0 admits a non-trivial finite representation fibering through $H/(H \cap \bar{H}^0)$ or $H \subset \bar{H}^0$. But $\bar{H}^0/H^0$ is commutative. Hence in the second case H/H^0 is a finitely generated abelian group. Every finitely generated abelian group is isomorphic to some direct product $\mathbb{Z}^k \times A$ with A finite and therefore clearly admits a non-trivial finite representation. □

For non-homogeneous manifolds there is no such result.

EXAMPLE 7.11.2. — There exists a three-dimensional compact complex manifold with infinite fundamental group such that every finite representation of the fundamental group is trivial.

Proof. — By a result of Higman **[54]** there exists a finitely presentable group without any non-trivial subgroup of finite index. Furthermore there is a recent result of Taubes **[140]** (based on twistor space constructions) that for every finitely presentable group there exists a three-dimensional compact complex manifold with this group as fundamental group. □

7.12. On the number of finite representations

In the preceding section we proved that for a lattice Γ in a simply connected complex Lie group G there always exist some group homomorphisms to finite groups.

For the case where G is semisimple without $SL_2(\mathbb{C})$-factors we will now demonstrate that there are only finitely many non-equivalent group homomorphisms with finite image from Γ to a fixed Lie group. This implies that for a given number $r \in \mathbb{N}$ there are only finitely many non-equivalent homogeneous vector bundles of rank r over $X = G/\Gamma$.

This implies that under some assumptions for each $r \in \mathbb{N}$ there are only finitely many non-equivalent flat vector bundles of rank r over a compact parallelizable manifold.

THEOREM 7.12.1. — *Let S be a semisimple, simply connected complex Lie group without $SL_2(\mathbb{C})$-factor and Γ a lattice.*

Then for every Lie group H with finitely many connected components there exists only finitely many non-conjugate group homomorphisms $\rho : \Gamma \to H$. (Two group homomorphisms ρ, τ are conjugate if there exists an element $h \in H$ such that $\rho(\gamma) = h\tau(\gamma)h^{-1}$ for all $\gamma \in \Gamma$.)

Moreover, for every such H there exists a normal subgroup of finite index $\Gamma_H \subset \Gamma$ such that $\Gamma_H \subset \ker \rho$ for every $\rho \in \mathrm{Hom}(\Gamma, H)$.

COROLLARY 7.12.2. — *Let S be a semisimple, simply connected complex Lie group without $SL_2(\mathbb{C})$-factor and Γ a lattice. Let H be a complex Lie group with finitely many connected components.*

Then there exists only finitely man non-isomorphic H-principal bundles over S/Γ admitting a flat connection.

For the proof of the theorem we need some preparation. First we recall that by Margulis superrigidity it suffices to consider representations with finite image.

LEMMA 7.12.3. — *Let S be a connected linear Lie group, Γ a finitely generated infinite subgroup. Let $\Lambda_k = \bigcap_\rho \ker\rho$ where ρ runs over all group homomorphisms from Γ to finite groups of order $\leq k$.*

Then Λ_k is an infinite normal subgroup of Γ.

Proof. — Normality is obvious. By Selberg's theorem (see prop. 1.7.2) we know that Γ contains a torsion-free subgroup of finite index. It follows that Γ contains a subgroup $H \simeq (\mathbb{Z}, +)$. Let S denote the set of all elements $\gamma \in \Gamma$ which are $k^!$-divisible, *i.e.*, for which there exists an element $\lambda \in \Gamma$ with $\lambda^{(k^!)} = \gamma$. With $H \simeq \mathbb{Z}$ we have $k^!\mathbb{Z} \subset H \cap S$, hence S is an infinite set. Now $S \subset \Lambda_k$, hence Λ_k is infinite. □

Furthermore we need the result that for a given finite group and a given Lie group with only finitely many connected components there exist only finitely many group homomorphisms up to conjugacy.

PROPOSITION 7.12.4. — *Let G be a finite group, H a (real) Lie group with finitely many connected components.*

Then up to conjugacy there exist only finitely many group homomorphisms from G to H.

Proof. — Clearly the image of such a group homomorphism $f : G \to H$ is finite, hence compact. Since all the maximal compact Lie subgroups of H are conjugate, there is no loss of generality in assuming that H is compact.

Let us assume that there exists an infinite sequence of pairwise non-conjugate group homomorphisms $f_n : G \to H$. Since H is compact, we may choose a subsequence such that $f_n(x)$ converges for every $x \in G$. Evidently the limit map $f : G \to H$ is again a group homomorphism. Theorem 5.3 of **[100]** now implies that for all n sufficiently large the image group $f_n(G)$ is conjugate to a subgroup of $f(G)$. This contradicts the assumption that all the f_n are pairwise non-conjugate, because $f(G)$ has only finitely many subgroups and every such subgroup has only finitely many automorphisms. □

PROPOSITION 7.12.5. — *Let S be a simply connected semisimple complex Lie group without $SL_2(\mathbb{C})$-factors, Γ a lattice, H a (real) Lie group with finitely many connected components.*

Then there exists a normal subgroup Γ_H of finite index in Γ such that $\Gamma_H \subset \ker\rho$ for every group homomorphism $\rho : \Gamma \to H$ with finite image.

Proof. — Let $k = k(H)$ as in cor. 1.7.9. Then $\rho(\Lambda_k)$ must be abelian for every group homomorphism $\rho : \Gamma \to H$ with finite image. By Margulis' theory every infinite normal subgroup of Γ is of finite index. Hence Λ_k is of finite index in Γ. Moreover the commutator group Λ_k' is of finite index in Λ_k, hence of finite index in Γ. Thus $\Gamma_n := \Lambda_k'$ has the desired property. □

Now the proof of the theorem is an immediate consequence of the fact that a finite group (such as Γ/Γ_n) has only finitely many non-equivalent irreducible representations.

By use of superrigidity we obtain in particular.

COROLLARY 7.12.6. — *Let S be a simply connected semisimple complex Lie group without $SL_2(\mathbb{C})$-factors, Γ a lattice and H a Lie group with finitely many connected components.*

Then there exist only finitely many non-conjugate group homomorphisms from Γ to H.

COROLLARY 7.12.7. — *Let S be a simply connected semisimple complex Lie group without $SL_2(\mathbb{C})$-factors and Γ a discrete cocompact subgroup in S.*

Then for each $r \in \mathbb{N}$ there exist only finitely many non-equivalent homogeneous vector bundles of rank r over $X = S/\Gamma$.

Superrigidity results for lattices $\Gamma \subset S$ are often stated in the form that for certain groups H every group homomorphism from a lattice to H either has finite (or relatively compact) image or extends to a group homomorphism from S to H.

Our theorem implies such a result for torsion groups.

COROLLARY 7.12.8. — *Let S be a simply connected semisimple complex Lie group, without $SL_2(\mathbb{C})$-factors, Γ a lattice, T a n-torsion group for some $n \in \mathbb{N}$, i.e., $g^n = e$ for all $g \in T$.*

Then for every group homomorphism $\rho : \Gamma \to T$ the image $\rho(\Gamma)$ is finite.

7.13. Existence of vector bundles

Bănică posed the question, whether every compact complex manifold admits a non-trivial holomorphic vector bundle. We are now in a position to give an affirmative answer.

THEOREM 7.13.1. — *Let X be a positive-dimensional compact complex manifold.*

Then X admits a non-trivial holomorphic vector bundle E with $\operatorname{rank}(E) \leq \dim X$.

Proof. — If the holomorphic tangent bundle of X is non-trivial, there is nothing to prove. Thus we may assume that X is a parallelizable, *i.e.*, $X \simeq G/\Gamma$ where G is a simply connected complex Lie group and Γ a discrete cocompact subgroup. If $G \neq G'$, then the Albanese torus $\mathrm{Alb}(X)$ is non-trivial, and there are non-trivial line bundles on X obtained by pull-back from $\mathrm{Alb}(X)$. Finally, if $G = G'$, let ρ denote the representation of the fundamental group $\pi_1(X) \simeq \Gamma$ which is the restriction of the complex conjugate of the adjoint representation of G. thm. 7.6.5 implies that in this case the vector bundle E induced by ρ is non-trivial. Furthermore, the construction of ρ implies $\operatorname{rank}(E) = \dim(X)$. □

Given the fact that every compact complex manifold admits a non-trivial holomorphic vector bundle, there are two natural questions. First: Is the statement on the rank optimal? Second: Does similar results hold in other categories?

Concerning the first question we note that already for surfaces it is possible that a compact complex manifold does not admit any non-trivial line bundle, namely this happens for K3-surfaces of algebraic dimension zero.

Concerning the second question we remark that clearly every projective manifold admits a line bundle L with $c_1(L) \neq 0$.

However, in the differentiably category there is no general existence theorem for non-trivial vector bundles. In fact, the three-dimensional sphere S^3 is an example for a compact differentiable manifold X such that every principal bundle (with finite-dimensional structure group) over X is trivial. To see this, note that S^3 is the suspension of S^2. This implies that (topologically) H-principal bundles over S^3 are classified by homotopy classes of continuous maps from S^2 to H [**6**]. But $\pi_2(H) = \{1\}$ for every Lie group H ([**30**]). Thus S^3 does not admits any non-trivial vector bundle.

One may also asked whether it is possible to drop or weaken the compactness assumption. Some assumption is certainly necessary, e.g. on a non-compact Riemann surface every holomorphic vector bundle is trivial. At least our methods yield that quotients of complex Lie groups by lattices always admit non-trivial holomorphic vector bundles, even if the quotient is non-compact.

7.14. Topological structure of flat bundles

Our goal here is to prove that for a compact complex parallelizable manifold every homogeneous vector bundle is almost topologically trivial. In particular we will see that for a homogeneous vector bundle over a compact complex parallelizable manifold all the Chern classes vanish.

This result is specific to parallelizable manifolds, e.g. for homogeneous rational manifolds like $\mathbb{P}_n(\mathbb{C})$ every line bundle is homogeneous. In contrast, for parallelizable manifolds, homogeneous bundles are close to flat bundles. In fact for many special cases we proved that a homogeneous bundle is automatically flat.

Flat vector bundles are always close to be topologically trivial. This is true, because the presence of a flat connection implies that all Chern classes are zero. Now every (continuous) complex vector bundle of rank k over a compact differentiable manifold X is obtained as a pull-back of the universal vector bundle over the classifying space $BU(k)$ via a continuous map $f : X \to BU(k)$. Topological triviality of the vector bundle is equivalent to the assumption that f is homotopic to a constant map. Vanishing of all Chern classes is equivalent to the property that the induced homomorphism of cohomology rings

$$f^* : H^\bullet(BU(k), \mathbb{C}) \longrightarrow H^\bullet(X, \mathbb{C})$$

is trivial. In this sense flat vector bundles are always almost topologically trivial.

First we prove a general statement on flat bundles with solvable structure group (which is possibly well-known). Namely, if X is a compact differentiable manifold and E a flat bundle with solvable structure group, then there exists a finite covering $\tau : X_1 \to X$ such that $\tau^* E$ is topologically trivial.

We then proceed to prove the main theorems of this section. We will prove the following: *Let H be a complex linear Lie group, E a homogeneous H-principal bundle over a compact complex parallelizable manifold X. Then there exists a finite covering $\tau : X_1 \to X$ and a flat H-principal bundle $E_1 \to X_1$ such that there is a topological isomorphism $\tau^* E \simeq E_1$.* This implies that for every homogeneous vector bundle over a compact complex parallelizable manifold the Chern classes vanish.

For compact complex tori every flat vector bundle is topologically trivial. However already for nilmanifolds it is quite possible that $H_1(X, \mathbb{Z})$ contains torsion. In this case it follows that there exist flat complex line bundles which are not topologically trivial.

We begin with some considerations about line bundles. As usual let $\mathrm{Pic}(X)$ denote the group of holomorphic line bundles over a complex manifold X.

LEMMA 7.14.1. — *Let X be a compact complex manifold, A the torsion part of $H^2(X, \mathbb{Z})$ and $\delta : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ the natural group homomorphism.*

Then for every flat line bundle L we have $\delta(L) \in A$. Conversely for every element $\alpha \in A$ there exists a flat line bundle L over X with $\delta(L) = \alpha$.

There is an isomorphism $\mathrm{Ext}(H_1(X, \mathbb{Z}), \mathbb{Z}) \simeq A$ and a non-functorial isomorphism between A and the torsion part of $H_1(X, \mathbb{Z}) \simeq \pi_1(X)^{ab}$.

Proof. — Since X is compact, the homology groups are finitely generated. It follows that $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^r \times T$ for some finite abelian group T. The short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 0$$

yields a long exact Ext-sequence

$$\cdots \longrightarrow \mathrm{Hom}(T, \mathbb{C}) \longrightarrow \mathrm{Hom}(T, \mathbb{C}^*) \longrightarrow \mathrm{Ext}(T, \mathbb{Z}) \longrightarrow \mathrm{Ext}(T, \mathbb{C}) \longrightarrow \cdots$$

Now $\mathrm{Hom}(T, \mathbb{C}) = 0$ and $\mathrm{Ext}(T, \mathbb{C}) = 0$, hence there is an isomorphism between $\mathrm{Hom}(T, \mathbb{C}^*)$ and $\mathrm{Ext}(T, \mathbb{Z})$. Furthermore by the universal coefficient theorem for cohomology

$$0 \longrightarrow \mathrm{Ext}(H_1(X, \mathbb{Z}), \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0$$

The group $\mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$ is free. Hence the finite group $\mathrm{Ext}(H_1(X, \mathbb{Z}), \mathbb{Z}) = \mathrm{Ext}(T, \mathbb{Z}) \simeq \mathrm{Hom}(T, \mathbb{C}^*)$ is isomorphic to the torsion part of $H^2(X, \mathbb{Z})$. □

This lemma implies in particular that every torsion element of $H^2(X, \mathbb{Z})$ is contained in the image of $\delta : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$. It follows that there exists a topologically non-trivial flat line bundle as soon as $H_1(X, \mathbb{Z})$ contains torsion. This happens already for nilmanifolds. For instance, there is a three-dimensional nilmanifold X with $\mathrm{Ext}(H_1(X), \mathbb{Z}) \simeq \mathbb{Z}_2$ given by $X = G/\Gamma$ with

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{C} \right\}$$

and

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y \in \mathbb{Z} \oplus i\mathbb{Z}; z \in \mathbb{Z} \oplus \frac{1}{2} i\mathbb{Z} \right\}.$$

PROPOSITION 7.14.2. — *Let M be a compact differentiable manifold and $E \to M$ an H-principal bundle for some connected Lie group H. Assume that E is given by a representation of the fundamental group $\rho : \pi_1(M) \to H$ and that H is solvable.*

Then there exists a finite covering $\pi : M_1 \to M$ such that $\pi^* E \to M$ is (differentiably) trivial.

If $H_1(M, \mathbb{Z})$ is torsion-free, then E itself is trivial.

Proof. — First we want to remark that a differentiable bundle is differentiably trivial if and only if it is topologically trivial.

Let K be a maximal compact Lie subgroup of H. Since H is solvable, K is commutative, hence $K \simeq (S^1)^m$. Now the structure group of E can be reduced to K, hence it suffices to prove the statement for $H = K$. Considering one factor after the other, it is enough to deal with the case $H \simeq S^1$.

Now we consider the exact sequences of sheaves $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$ and $1 \to \mathbb{Z} \to \mathcal{E} \to \mathcal{S} \to 1$ where as usual $\mathbb{Z}$, $\mathbb{R}$ and S^1 denote the sheaves of locally constant functions with values in that group. We obtain a commutative diagramm of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{S} & \longrightarrow & 0 \end{array}$$

which induce a commutative cohomology diagram

$$\begin{array}{ccccccc} H^1(M, \mathbb{R}) & \xrightarrow{\beta} & H^1(M, S^1) & \xrightarrow{\delta} & H^2(M, \mathbb{Z}) & \longrightarrow & H^2(M, \mathbb{R}) \\ \downarrow & & \downarrow p & & \| & & \downarrow \\ 0 = H^1(M, \mathcal{E}) & \longrightarrow & H^1(M, \mathcal{S}) & \xrightarrow{\sim} & H^2(M, \mathbb{Z}) & \longrightarrow & H^2(M, \mathcal{E}) = 0 \end{array}$$

Next recall that $H^1(M,A) \simeq \mathrm{Hom}(H_1(M),A)$ for A an abelian group (resp. the sheaf of locally constant functions with values in that group). Thus $H^1(M,S^1)$ classifies homomorphisms $\rho : \pi_1(M) \to S^1$ (observe that $H_1(M)$ is the abelianization of $\pi_1(M)$). On the other hand $H^1(M,\mathcal{S})$ classifies the S^1–principal bundles. But now $\mathrm{Ext}(H_1(M),\mathbb{Z}) = 0$ implies that $\beta : H^1(M,\mathbb{R}) \to H^1(M,S^1)$ is surjective, *i.e.*, $\delta : H^1(M,S^1) \to H^2(M,\mathbb{Z})$ is the zero map. Since $H^k(M,\mathcal{E}) = 0$ for $k = 1,2$ implies $H^1(M,\mathcal{S}) \simeq H^2(M,\mathbb{Z})$, it follows that the map $p : H^1(M,S^1) \to H^1(M,\mathcal{S})$ is the zero map. Thus any S^1–principal bundle is trivial if it is induced by a representation of the fundamental group. □

REMARK 7.14.3

1. In the statement of the proposition, the choice of the covering depended on E. Actually it is possible to choose a finite covering $\tau : M_1 \to M$ such that τ^*E is (differentiably) trivial for all flat H–principal bundles for all connected solvable Lie groups H. It suffices to choose M_1 in the following way: Let F denote a torsion-free subgroup of finite index in $H_1(M,\mathbb{Z})$ (which exists, because $H_1(M,\mathbb{Z})$ is finitely generated for M compact). Then let $\Gamma = \phi^*(F)$ with $\phi : \pi_1(M) \to H_1(M,\mathbb{Z})$ and let $M_1 \to M$ be the covering corresponding to $\Gamma \subset \pi_1(M)$.
2. The assumption of compactness of the manifold M is essential for the statement. For instance, let $\Gamma = \mathbb{Q}/\mathbb{Z}$, M a manifold with $\pi_1(M) \simeq \Gamma$ and $L \to M$ the line bundle defined by the natural embedding of the fundamental group $\pi_1(M) = \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$. Then π^*L is topologically non-trivial for any covering $\pi : M_1 \to M$ except the universal covering.

THEOREM 7.14.4. — *Let X be a compact complex parallelizable manifold, $E \to X$ a homogeneous vector bundle.*

Then all the Chern classes $c_k(E)$ vanish.

REMARK 7.14.5

1. For flat vector bundles it is obvious that all the Chern classes do vanish, because the Chern classes of a vector bundle may be calculated in terms of the curvature of a connection and by definition a flat vector bundle has a connection with vanishing curvature.
2. For an arbitrary holomorphic vector bundle E over a compact complex manifold X, the vanishing of all Chern classes does *not* imply that there exists a finite covering $\tau : X_1 \to X$ with $\tau^*E \to X_1$ topologically trivial. For example, there exists a vector bundle over $\mathbb{P}_2$ such that all Chern classes vanish, but the bundle is not topologically trivial (see [**7**]).
3. Parallelizability of X implies that homogeneous bundles are closely related to flat bundles. For arbitrary homogeneous vector bundles over arbitrary compact

homogeneous manifolds the corresponding assertion is clearly wrong, e.g. every line bundle over $\mathbb{P}_n$ is homogeneous.

4. It is absolutely essential to require that the structure group is linear. For instance, the Iwasawa-manifold is a torus-principal bundle over a torus. This is a homogeneous bundle with non-linear structure group, and it is certainly not topologically isomorphic to a flat bundle.

The proof of the theorem results from the lemmata below.

We recall the standard fact from fiber bundle theory that every principal bundle with contractible structure group is (topologically) trivial. This can be generalized to the following statement:

Let $G = S \ltimes U$ be a semi-direct product of Lie groups with U contractible. Let $\pi : G \to S \simeq G/U$ denote the natural projection.

Then every G-principal bundle E is topologically isomorphic to the S-principal bundle E' associated via $\pi : G \to S$.

This is a particularly useful formulation for our purposes, because if E is homogeneous with respect to some group action on the base space, then evidently E' is homogeneous as well.

LEMMA 7.14.6. — *Let $X = G/\Gamma$ be a compact complex parallelizable manifold, H a connected linear complex Lie group and S a maximal connected semisimple Lie subgroup of H.*

Let $E \to X$ be a homogeneous H-principal bundle. Then there exists a finite covering $\tau : X_1 \to X$ and a homogeneous S-principal bundle $E' \to X_1$ such that $\tau^* E$ and E' are topologically isomorphic (as H-principal bundles).

Proof. — Fix an embedding $H \hookrightarrow GL_N(\mathbb{C})$ and let $\bar{H}$ denote the Zariski closure of H in $GL_N(\mathbb{C})$. The commutator group H' of H is linear algebraic [**20**]. Therefore $\bar{H}/H'$ is a commutative linear algebraic group. Since $H/H' \subset \bar{H}/H'$, it follows that $H/H' \simeq \mathbb{C}^k \times (\mathbb{C}^*)^l$ for some $k, l \in \mathbb{N}$. Thus the H/H'-principal bundle over X associated to E is isomorphic to a direct product of $\mathbb{C}$- and $\mathbb{C}^*$-principal bundles. We have already seen that every homogeneous $\mathbb{C}^*$-principal bundle over X is flat and therefore becomes trivial after passing through a finite covering. Moreover any $\mathbb{C}$-principal bundle is topologically trivial. Hence there exists a finite covering $\tau : X_1 \to X$ such that the structure group of $\tau^* E$ can be reduced to H'. Using the Levi-decomposition of H' and the observation made before the lemma, we may further reduce the structure group to S. □

We are now in a position to prove that every homogeneous bundle over a parallelizable manifold is topologically equivalent to a flat one.

PROPOSITION 7.14.7. — *Let $X = G/\Gamma$ be a compact complex parallelizable manifold and H a connected linear complex Lie group.*

Let $E \to X$ be a homogeneous H-principal bundle. Then there exists a finite covering $\tau : X_1 \to X$ and a flat *H-principal bundle E_1 over X_1 such that τ^*E and E_1 are topologically isomorphic.*

Proof. — By the above lemma we may assume that H is semisimple and thereby in particular that H is linear algebraic. Since E is homogeneous, there exists an essential extension of connected complex Lie groups

$$1 \longrightarrow L \longrightarrow \widetilde{G} \xrightarrow{\sigma} G \longrightarrow 1$$

such that the G-action on $X = G/\Gamma$ lifts to a $\widetilde{G}$-action on E. Let $I = \sigma^{-1}(\Gamma)$. Then E is isomorphic to a fiber product $E = \widetilde{G} \times_I H \to X = \widetilde{G}/I$ given by some group homomorphism $\rho : I \to H$. Now we may replace H by the Zariski closure of $\rho(I)$ in H. By passing to a finite cover of X we may assume that this is connected. A new application of the above lemma yields that we may further reduce the structure group to a maximal connected semisimple Lie subgroup of $\overline{\rho(I)}$. Iterating this process, it is clear that we may simultanously assume that H is semisimple and connected and that $\rho(I)$ is Zariski dense in H.

Let $\widetilde{R}$ denote the radical of R. Since $\rho(I)$ is Zariski dense, it is clear that the Zariski closure of $\rho(I \cap \widetilde{R})$ in H is a normal solvable subgroup, hence finite. Once again replacing X by a finite cover, we may assume that $I \cap \widetilde{R} \subset \ker \rho$. But $L \subset \widetilde{R}$ (because the extension $(*)$ is essential, cf. proposition 5?), hence $\rho : I \to H$ actually fibers through $\Gamma \simeq I/L$. Thus we obtain a flat bundle. □

This proposition implies immediately that for a homogeneous vector bundle over a compact complex parallelizable manifold the Chern classes vanish. (Note that for a finite covering $\tau : X_1 \to X$ the induced homomorphism of cohomology rings $\tau^* : H^\bullet(X, \mathbb{C}) \to H^\bullet(X_1, \mathbb{C})$ is injective (see e.g. **[37]**, Prop. VIII.10.10). Hence $c_k(\tau^*E) = 0$ implies $c_k(E) = 0$.)

CHAPTER 8

DEFORMATIONS AND COHOMOLOGY

8.1. Survey

This chapter is concerned with cohomology groups and deformations of complex parallelizable manifolds. Throughout this chapter the comparison between the parallelizable and the Kähler situation serves as a guiding line.

For a compact Kähler manifold X there exists a compact complex torus $\mathrm{Alb}(X)$ and a holomorphic map $\pi : X \to \mathrm{Alb}(X)$ such that

1. For every holomorphic map f from X to a compact complex torus T there exists an element $a \in T$ and a holomorphic Lie group homomorphism $F : \mathrm{Alb}(X) \to T$ such that $f(x) = a \cdot F \circ \pi(x)$ for all $x \in X$.
2. The holomorphic map $\pi : X \to \mathrm{Alb}(X)$ induces isomorphisms on the cohomology groups $H^1(\cdot, \mathbb{R})$, $\Gamma(\cdot, d\mathcal{O})$ and $H^1(\cdot, \mathcal{O})$.

Here we prove a similar statement for parallelizable manifolds. Although parallelizable manifolds are never Kähler unless they are tori, it is possible to obtain results similar to the Kähler case as long as reductive factors of rank 1 are absent. In particular we prove the following.

THEOREM 8.1.1. — *Let G be a connected complex linear algebraic group and Γ a lattice. Assume that there exists no surjective algebraic group homomorphism from G to $PSL_2(\mathbb{C})$ or $\mathbb{C}^*$.*

Then there exists a compact complex torus A and a surjective holomorphic map $\pi : X \to A$ such that

1. *For every connected complex Lie group T and every holomorphic map $f : X \to T$ there exists an element $a \in T$ and a holomorphic Lie group homomorphism $F : A \to T$ such that $f(x) = a \cdot F(\pi(x))$ for all $x \in X$.*
2. *The map $\pi : X \to A$ induces isomorphisms on the cohomology groups $H^1(\cdot, \mathbb{C})$ and $\Gamma(\cdot, d\mathcal{O})$.*
3. *Every meromorphic function on X is a pull-back from A.*
4. *If X is compact, π induces an isomorphism of $H^1(\cdot, \mathcal{O})$.*

Furthermore, we determine $H^1(X, \mathcal{O})$ for cocompact lattices in arbitrary complex linear algebraic groups.

THEOREM 8.1.2. — *Let G be a connected complex linear algebraic group, $\Gamma \subset G$ a lattice and $X = G/\Gamma$. Let $G = S \cdot R$ be a Levi-Malcev-decomposition, N the nilradical, and $A = [S, R] \cdot N'$. In addition, let W denote the maximal subvectorspace of $\mathcal{L}ie(R'A/A)$ such that $\mathrm{Ad}(\gamma)|_W$ is a semisimple linear endomorphism with only real eigenvalues for every $\gamma \in \Gamma$.*

Then $\dim H^1(G/\Gamma, \mathcal{O}) = \dim(G/G') + b_1(\Gamma/(R \cap \Gamma)) + \dim(W)$.

Using this result we deduce the following vanishing theorem.

THEOREM 8.1.3. — *Let G be a connected complex Lie group, Γ a discrete cocompact subgroup and $X = G/\Gamma$.*

Then $H^1(X, \mathcal{O}) = \{0\}$ iff $b_1(X) = 0$.

Since $H^1(X, \mathcal{T}) \simeq H^1(X, \mathcal{O}^n)$ for an n-dimensional parallelizable manifold, this implies that a compact complex parallelizable manifold admits infinitesimal deformations if and only if $b_1(X) > 0$. In addition, we prove that for $b_1(X) > 0$ there actually exist small deformations. Thus we obtain the following result which characterizes rigidity.

THEOREM 8.1.4. — *Let G be a connected complex Lie group, Γ a discrete cocompact subgroup and $X = G/\Gamma$ the complex quotient manifold. Then the following properties are equivalent.*

1. *X admits no infinitesimal deformations (i.e., there are no non-trivial families over* $\mathrm{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$*).*
2. *X admits no small deformations (i.e., there are no non-trivial families over the unit disk).*
3. $b_1(X) = 0$.
4. $H^1(X, \mathcal{O}) = \{0\}$

This characterization of rigidity of parallelizable manifolds generalizes a number of earlier results. The existence of non-trivial deformations of positive-dimensional tori is classical. Deformations of solv-manifolds have been studied by Nakamura [**110**]. Raghunathan proved the rigidity of quotients of G/Γ for G semisimple without *rank* 1-factors [**122**]. Ghys proved the theorem for $G = SL_2(\mathbb{C})$ ([**42**]).

Using the results on $H^1(X, \mathcal{O})$, we investigate the Hodge-Frölicher spectral sequence and properties of line bundles related to this sequence. Again, we compare the parallelizable situation with the Kähler setup. Finally we use the Serre-construction to prove the existence of non-homogeneous vector bundles over $SL_2(\mathbb{C})/\Gamma$ for discrete cocompact subgroups Γ with $b_1(\Gamma) = 0$.

8.2. Basic results on $H^1(G/\Gamma, \mathcal{O})$

The following theorem of Akhiezer generalizes earlier results of Raghunathan [**122**].

THEOREM 8.2.1 (Akhiezer, [**3**]). — *Let G be a reductive complex linear algebraic Lie group and Γ a cocompact lattice. Let $X = G/\Gamma$.*

Then there are natural isomorphisms of G-modules

$$H^q(X, \Omega^p) = H^{p,q}_{Dolb.}(X) \simeq \bigwedge^p (\mathcal{L}ie\, G) \otimes H^q(\Gamma, \mathbb{C})$$

with G acting on $\mathcal{L}ie\, G$ by the adjoint representation and acting trivially on $H^q(\Gamma, \mathbb{C})$.

In particular $H^1(X, \mathbb{C}) \simeq H^1(X, \mathcal{O})$ and the G-action on $H^1(X, \mathcal{O})$ is trivial.

COROLLARY 8.2.2. — *Let Γ be a discrete cocompact subgroup of $SL_2(\mathbb{C})$. Then* $\dim H^1(SL_2(\mathbb{C})/\Gamma, \mathcal{O}) = \operatorname{rank}_{\mathbb{Z}} (\Gamma/\Gamma')$.

This is also proved independently by Ghys [**42**].

COROLLARY 8.2.3. — *Let Γ be a discrete cocompact subgroup in $SL_2(\mathbb{C})$. Assume that Γ/Γ' is finite.*

Then $H^1(SL_2(\mathbb{C})/\Gamma, \mathcal{O}) = 0$.

This consequence has also been obtained independently by Rajan [**124**] and the author [**153**].

Ghys deduced the above mentioned result on $H^1(SL_2(\mathbb{C})/\Gamma, \mathcal{O})$ in a study on deformations of the complex structure of such quotients (see §8.3).

We will also employ the follwing result of Kodaira.

THEOREM 8.2.4 (Kodaira, see [**110**]). — *Let G be a simply connected nilpotent complex Lie group and Γ a lattice.*

Then the projection map $\pi : X = G/\Gamma \to G/G'\Gamma$ induces an isomorphism of the cohomology groups $\pi^* : H^1(Y, \mathcal{O}) \tilde{\to} H^1(X, \mathcal{O})$.

Kodaira's proof of this fact is based on a study of harmonic forms on nilmanifolds.

8.3. Deformations of $SL_2(\mathbb{C})/\Gamma$

Deformations of arbitrary quotients of $SL_2(\mathbb{C})$ by discrete cocompact subgroups have been studied in detail by Ghys ([**42**]).

Fix a discrete cocompact subgroup $\Gamma \subset SL_2(\mathbb{C})$ and a set of generators $\alpha_1, \ldots, \alpha_r$ of Γ. Then the set of all group homomorphisms $\operatorname{Hom}(\Gamma, SL_2(\mathbb{C}))$ can be identified with an algebraic subvariety $\mathcal{R}_\Gamma$ of $(SL_2(\mathbb{C}))^r$ via the natural map

$$\rho \mapsto (\rho(\alpha_1), \ldots, \rho(\alpha_r)).$$

Ghys proved that there exists an open neighborhood W of e in $\mathcal{R}_\Gamma$ such that for every $u \in W$ the Γ-action on $SL_2(\mathbb{C})$ given by

$$\gamma : x \mapsto u(\gamma)^{-1} \cdot x \cdot \gamma$$

is free and properly discontinuously and the respective quotient manifold X_u is diffeomorphic to $SL_2(\mathbb{C})$. For $u, \tilde{u} \in W$ the quotient manifolds X_u and $X_{\tilde{u}}$ are biholomorphic if and only if $u = g\tilde{u}g^{-1}$ for some $g \in SL_2(\mathbb{C})$. In particular, X_u is not biholomorphic to $SL_2(\mathbb{C})$ if $u \not\equiv e$. Moreover, X_u is not parallelizable for $u \not\equiv e$.

On the other hand, every sufficiently small deformation of the complex manifold $SL_2(\mathbb{C})/\Gamma$ arises in this way.

If $b_1(\Gamma) > 0$, then every open neighbourhood U of e in W contains an element u such that $u(\Gamma)$ is a non-trivial abelian subgroup of $SL_2(\mathbb{C})$.

If $b_1(\Gamma) > 1$, then every open neighbourhood U of e in W contains an element u such that $u(\Gamma)$ is a non-trivial non-abelian subgroup of $SL_2(\mathbb{C})$.

On the other hand, if $b_1(\Gamma) = 0$, then e is an isolated point in W by Weil rigidity ([**150**]). If $b_1(\Gamma) = 1$, then there is an open neighbourhood V of e in W such that $u(\Gamma)$ is abelian for all $u \in V$.

8.4. Leray spectral sequence

Let $f : X \to Y$ be a holomorphic map between complex spaces. There is a Leray spectral sequence for the structure sheaf $\mathcal{O}_X$. The respective lower term sequence yields the following.

$$0 \longrightarrow H^1(Y, \mathcal{R}^0 f_* \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^0(Y, \mathcal{R}^1 f_* \mathcal{O}_X) \longrightarrow H^2(Y, \mathcal{R}^0 f_* \mathcal{O}_X)$$

Assume that f is connected and proper. Then $\mathcal{R}^0 f_* \mathcal{O}_X = \mathcal{O}_Y$. Furthermore, if $\dim H^k(f^{-1}\{p\}, \mathcal{O}) = r$ for a natural number k and all $p \in Y$, then, by Grauert's theorem, $\mathcal{R}^k f_* \mathcal{O}_X$ is a locally free coherent sheaf of rank r. This implies in particular the following observation.

LEMMA 8.4.1. — *Let $f : X \to Y$ be a proper connected holomorphic map and assume that for every fiber F_p the induced cohomology map $i^* : H^1(X, \mathcal{O}) \to H^1(F_p, \mathcal{O})$ vanishes. Assume furthermore that* $\dim H^1(F_p, \mathcal{O})$ *does not depend on $p \in Y$.*

Then $\alpha : H^1(X, \mathcal{O}) \to H^0(\mathcal{R}^1 f_* \mathcal{O})$ is the zero map.

8.4.1. Leray spectral sequence for flat bundles

PROPOSITION 8.4.2. — *Let $\pi : E \to X$ be a holomorphic fiber bundle with typical fiber F and structure group S. Assume that F is connected and compact, that S acts on F in such a way that there exists an invariant hermitian metric and that $E \to X$ admits a flat holomorphic connection.*

Then $\alpha : H^1(E, \mathcal{O}) \to H^0(X, \mathcal{R}^1 \pi_* \mathcal{O})$ is surjective.

Proof. — Let $\eta \in H^0(X, \mathcal{R}^1\pi_*\mathcal{O})$. Then there exists an open cover $\mathcal{U} = (U_i)_i$ by contractible open Stein subsets of X such that η is given by $\eta_i \in H^1(\pi^{-1}(U_i), \mathcal{O})$. Using the Dolbeault-isomorphism we may choose corresponding $\bar{\partial}$-closed $(0,1)$-forms ω_i on $W_i = \pi^{-1}(U_i)$. Of course these forms ω_i are not unique. Now, if there is a *canonical* way to choose ω_i, then the forms ω_i coincide on the intersection of the U_i and yield a globally defined $\bar{\partial}$-closed $(0,1)$-form on E, implying that η is contained in the image of $H^1(E, \mathcal{O})$.

The assumptions made in the proposition allow us to choose a hermitian metric on each fiber in such a way that for every contractible Stein open subset $U \subset X$ we obtain a trivialization $E|_U \simeq U \times F$ which is compatible with the flat connection and such that the choosen hermitian metric on each fiber is just the pull-back of one fixed S-invariant hermitian metric on F. Then there is a canonical way to choose the forms ω_i. Namely the forms ω_i are to be choosen such that they annihilate horizontal vector fields (with respect to the connection) and are harmonic if restricted to a fiber of π. □

LEMMA 8.4.3. — *Let G be a complex Lie group, Γ a discrete subgroup and A a connected commutative normal complex Lie subgroup such that $A/(A \cap \Gamma)$ is compact. Assume that the short exact sequence of Lie algebras*

$$0 \longrightarrow \mathcal{L}ie(A) \longrightarrow \mathcal{L}ie(G) \longrightarrow \mathcal{L}ie(G/A) \longrightarrow 0$$

is split.

Then $\pi : E = G/\Gamma \to G/A\Gamma = B$ is a torus bundle with flat holomorphic connection.

Proof. — The flat connection is induced by the splitting of the Lie algebra sequence. □

COROLLARY 8.4.4. — *Under the assumptions of the lemma the induced map*

$$H^1(E, \mathcal{O}) \longrightarrow H^0(B, \mathcal{R}^1\pi_*\mathcal{O})$$

is surjective.

We will also need a description of the structure of $\mathcal{R}^1 f_*\mathcal{O}$ for torus bundles.

Let $f : E \to Y$ be a locally trivial holomorphic fiber bundle with a compact complex torus T is typical fiber. Let $V = \Omega^1(T)$ denote the vector space of holomorphic 1-forms on T. Then there is a exact sequence

$$1 \longrightarrow \underbrace{\mathrm{Aut}^0(T)}_{=T} \longrightarrow \mathrm{Aut}(T) \xrightarrow{\zeta} GL(V)$$

Let $\mathcal{U} = \{U_i\}$ be a trivializing open cover of Y such that E is given by transition functions $\phi_{ij} : U_i \cap U_j \to \mathrm{Aut}(T)$.

CLAIM 8.4.5. — Under the above assumptions $\mathcal{R}^1 f_* \mathcal{O}_E$ is a locally free coherent sheaf on Y isomorphic to the sheaf of sections in the flat vector bundle given by the transition functions $\psi_{ij} : U_i \cap U_j \to GL(V)$ defined by $\psi_{ij} = \overline{\zeta \circ \phi_{ij}}$.

Proof. — This is a consequence of the Dolbeault-isomorphism. □

We apply this to parallelizable manifolds.

PROPOSITION 8.4.6. — *Let G be a complex Lie group, Γ a discrete subgroup and A a normal abelian complex Lie subgroup. Assume that $A/(A \cap \Gamma)$ is compact. Denote the natural projection map $E = G/\Gamma \to G/A\Gamma = B$ by π.*

In this case $\mathcal{R}^1\pi_*\mathcal{O}$ is a flat vector bundle of rank $\dim A$ over B induced by the representation $\rho : \Gamma \to GL(\mathcal{L}ie\, A^*)$ given by $\gamma \mapsto \overline{\mathrm{Ad}^*(\gamma)}$.

LEMMA 8.4.7. — *Let $\pi : X \to Y$ be a finite holomorphic covering.*

Then $\pi^* : H^1(Y, \mathcal{O}) \to H^1(X, \mathcal{O})$ is injective.

Proof. — If ω is a $\bar\partial$-closed $(0,1)$-form on Y and $\bar\partial f = \pi^*\omega$ for a function f on X, then $\bar\partial g = \omega$ for $g(y) = \frac{1}{d}\sum_{\pi(x)=y} f(x)$ (where d denotes the degree of π). □

8.5. Description of $H^1(X, \mathcal{O})$

THEOREM 8.5.1. — *Let G be a connected complex Lie group, $\Gamma \subset G$ a lattice and $X = G/\Gamma$. Let $G = S \cdot R$ be a Levi-Malcev-decomposition, N the nilradical, and $A = [S, R] \cdot N'$. Furthermore let W denote the maximal linear subspace of $\mathcal{L}ie(R'A/A)$ such that $\mathrm{Ad}(\gamma)|_W$ is a semisimple linear endomorphism with only real eigenvalues for every $\gamma \in \Gamma$.*

Then $\dim H^1(G/\Gamma, \mathcal{O}) \le \dim(G/G') + b_1(\Gamma/(R \cap \Gamma)) + \dim(W)$. *Equality holds, if G is linear algebraic.*

REMARK 8.5.2. — We do not know any example of a quotient manifold of a connected complex Lie group by a lattice which is *not* biholomorphic to a quotient of a *linear algebraic* $\mathbb{C}$-group by a lattice.

COROLLARY 8.5.3. — *Let G be a simply connected complex Lie group and $\Gamma \subset G$ be a cocompact lattice. Assume that the radical R is nilpotent and that the semisimple group G/R contains no factor S_0 such that $S_0/(S_0 \cap R\Gamma)$ is compact and $S_0 \simeq SL_2(\mathbb{C})$.*

Then $H^1(G/\Gamma, \mathcal{O}) \simeq H^1(G/G'\Gamma, \mathcal{O})$.

Proof. — If R is nilpotent, then $R = N$ and therefore $A = R'A$. Since $A \subset H \subset R'A$, it follows that $A = H$. Furthermore the assumption on S implies that $H^1(G/R\Gamma, \mathcal{O}) = 0$. Thus $\dim H^1(G/\Gamma, \mathcal{O}) = \dim G/G'$. The assumption on S also implies that $G'\Gamma$ is closed in G. Thus $H^1(G/G'\Gamma, \mathcal{O}) \simeq H^1(G/\Gamma, \mathcal{O})$. □

We now prove the theorem.

Proof. — We study the sequence of fibrations

$$1 \longrightarrow G/\Gamma \xrightarrow{\pi_1} G/N'\Gamma \xrightarrow{\pi_2} G/A\Gamma \xrightarrow{\pi_3} G/(G'\cap R)\Gamma \xrightarrow{\pi_4} G/R\Gamma \longrightarrow 1$$

First we must verify the existence of these fibrations, *i.e.*, we have to show that N', A, $G'\cap R$ and R all have closed orbits. Results of Mostow (thm. 3.5.3) imply that R and N have closed orbits. By classical results of Malcev (see **[89]** or §2.2) this implies closedness of the N'-orbits. From prop. 3.11.2 we obtain closedness of the $G'\cap R$-orbits. Finally note that $A=(G^k\cap R)N'$ for k sufficiently large. Hence closedness of A-orbits follows from thm. 3.11.4.

We note that all these projections π_i are surjective proper holomorphic maps and for each i we will study the lower term sequence of the Leray spectral sequence for the sheaf $\mathcal{O}$.

CLAIM 8.5.4. — The induced map $\pi_1^*: H^1(G/N'\Gamma,\mathcal{O}) \to H^1(G/\Gamma,\mathcal{O})$ is an isomorphism.

Proof. — By a theorem of Kodaira (see **[110]**) there is an isomorphism

$$H^1(N/N'(N\cap\Gamma),\mathcal{O}) \simeq H^1(N/(N\cap\Gamma),\mathcal{O}).$$

It follows that the embedding $i: N'/(N'\cap\Gamma) \hookrightarrow N/(N\cap\Gamma)$ induces the zero map between the respective cohomology groups $H^1(\cdot,\mathcal{O})$. Thus the group homomorphism $j^*: H^1(G/\Gamma,\mathcal{O}) \to H^1(N'/(N'\cap\Gamma))$ induced by the inclusion map $j: N'/(N'\cap\Gamma) \to G/\Gamma$ must vanish as well. By homogeneity it follows that for every fiber F of p the induced cohomology map $i^*: H^1(G/\Gamma,\mathcal{O}) \to H^1(F,\mathcal{O})$ is zero. Therefore

$$H^1(G/\Gamma,\mathcal{O}) \longrightarrow H^0(G/N'\Gamma,\mathcal{R}^1p_*\mathcal{O})$$

vanishes (lemma 8.4.1) and $H^1(G/\Gamma,\mathcal{O}) \simeq H^1(G/N'\Gamma,\mathcal{O})$. □

CLAIM 8.5.5. — The induced map $\pi_2^*: H^1(G/A\Gamma,\mathcal{O}) \to H^1(G/N'\Gamma,\mathcal{O})$ is an isomorphism.

Proof. — Recall that A is defined as $A=[S,R]N'$. Since $[S,R]\subset R\cap G'\subset N$, it is clear that A/N' is abelian. This implies that for $\pi: G/N'\Gamma \to G/A\Gamma$ the higher direct image sheaf $\mathcal{R}^1\pi_*\mathcal{O}$ is the coherent sheaf associated to a flat vector bundle given by a representation ρ which arises in the following way: ρ is the restriction of the complex conjugation of the representation from G on $GL(\mathcal{L}ie(A/N')^*)$ induced by the coadjoint representation. By construction no linear subspace of $\mathcal{L}ie(A/N')$ is invariant under $\mathrm{Ad}(S)$. It follows that $\mathcal{R}^1\pi_*\mathcal{O}$ does not admit global sections. Hence the claim. □

The lower term sequence of the Leray spectral sequence for π_3 yields

$$H^1(G/(G'\cap R)\Gamma,\mathcal{O}) \to H^1(G/A\Gamma,\mathcal{O}) \xrightarrow{\alpha} H^0(G/(G'\cap R)\Gamma,\mathcal{R}^1(\pi_3)_*\mathcal{O}) \overset{\phi}{\simeq} W$$

The isomorphism ϕ is a consequence of prop. 7.9.2.

CLAIM 8.5.6. — If G is linear algebraic, then α is surjective.

Proof. — Using cor. 8.4.4 it suffices to show that the short exact sequence of complex Lie groups

$$(*) \qquad 1 \longrightarrow (G' \cap R)/A \longrightarrow G/A \longrightarrow G/(G' \cap R) \longrightarrow 1$$

is split. Let V denote the unipotent radical of G. Then $G' \cap R \subset V \subset N$. Furthermore $A = [S,R]N'$ implies that V/A is commutative. Let L denote a Levi subgroup of G, *i.e.*, a maximal connected reductive subgroup of G. Then L is a reductive group, acting (by conjugation) linearly on the vector group V/A. Hence there is a L-invariant subvector space $W \subset V$ such that $V = W \oplus (G' \cap R)$. Thus $G = (L \ltimes W) \ltimes V$ and we obtain a splitting of $(*)$. □

Finally we have to discuss the lower term sequence for π_4.

$$1 \longrightarrow H^1(G/R\Gamma, \mathcal{O}) \longrightarrow H^1(G/(G' \cap R)\Gamma, \mathcal{O}) \xrightarrow{\beta} H^0(G/R\Gamma, \mathcal{R}^1(\pi_4)_*\mathcal{O})$$

First we note that the exact sequence of Lie algebras

$$0 \longrightarrow \mathcal{L}ie(R/(G' \cap R)) \longrightarrow \mathcal{L}ie(G/(G' \cap R)) \longrightarrow \mathcal{L}ie(G/R) \longrightarrow 0$$

is always split. Hence β is surjective. Furthermore the adjoint action of G on $\mathcal{L}ie(G)$ induces the trivial action on $Lie(R/(G' \cap R))$. Thus $\mathcal{R}^1(\pi_4)_*\mathcal{O}$ is a free $\mathcal{O}$-module sheaf with rank equal to $\dim_{\mathbb{C}} H^1(R/(G' \cap R)\Gamma, \mathcal{O})$. Note that $G/G' \simeq R/(G' \cap R)$ and that $R/(G' \cap R)$ is a compact complex torus. It follows that $\dim H^0(G/R\Gamma, \mathcal{R}^1(\pi_4)_*\mathcal{O}) = \dim(G/G')$.

Finally we recall that by Akhiezer's theorem (thm. 8.2.1) we have the equality

$$\dim H^1(G/R\Gamma, \mathcal{O}) = \dim H^1(G/R\Gamma, \mathbb{C}) = b_1(\Gamma/(R \cap \Gamma)).$$

□

8.5.1. The vanishing criterion

PROPOSITION 8.5.7. — *Let G be a complex Lie group and Γ a discrete cocompact subgroup.*

Then $H^1(G/\Gamma, \mathcal{O}) = 0$ if and only if $b_1(G/\Gamma) = 0$.

Proof. — The exponential sequence yields an embedding

$$H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathcal{O}).$$

Hence $H^1(X, \mathcal{O}) = 0$ implies $b_1(X) = 0$ for $X = G/\Gamma$.

Conversely let us assume $b_1(X) = 0$. Then $\mathrm{Hom}(\Gamma, \mathbb{Z}) = 0$. Since Γ is a lattice, this implies $\mathrm{Hom}(G, \mathbb{C}) = 0$. Hence $G = G'$ and consequently $R = [S,R]$. This implies in particular that $R'A = A$ for $A = [S,R]N'$. Hence $b_1(\Gamma/(R \cap \Gamma))$, $\dim(G/G')$ and $\dim R'A/A$ all equal zero and $H^1(X, \mathcal{O}) = \{0\}$ follows by the above theorem. □

COROLLARY 8.5.8. — *Let X be a compact complex parallelizable manifold. Then $H^1(X, \mathcal{T}) = 0$ iff $b_1(X) = 0$.*

8.6. Topological invariance of $\dim H^1(X, \mathcal{O})$

For compact Kähler manifolds $\dim H^1(X, \mathcal{O})$ equals $b_1(X)$ and therefore depends only on the fundamental group of X. Nakamura has shown (thus answering a question of Iitaka) that for arbitrary compact complex manifolds $\dim H^1(X, \mathcal{O})$ may jump within a smooth family (see **[110]**). He gave an example of a parallelizable complex manifold X_0 which admits small deformations X_t for which $\dim H^1(X_t, \mathcal{O}) \neq \dim H^1(X, \mathcal{O})$. In this example the X_t are no longer parallelizable for $t \neq 0$. This is not by coincidence. As an application of the description of $H^1(X, \mathcal{O})$ obtained in the preceding section we deduce the following result concerning the topological invariance of $H^1(X, \mathcal{O})$.

THEOREM 8.6.1. — *Let G, H be a complex linear algebraic groups, Γ and Λ discrete cocompact subgroups of G resp. H and $X = G/\Gamma$ and $Y = H/\Lambda$ the respective quotient manifolds.*

Assume that $\pi_1(X) \simeq \pi_1(Y)$.

Then $\dim H^1(X, \mathcal{O}) = \dim H^1(Y, \mathcal{O})$.

Proof. — Let G_0 denote a complex-linear algebraic group, G ist universal cover, Γ_0 a discrete cocompact subgroup of G_0 and Γ its preimage in G under the natural projection $G \to G_0$.

We have to show that $\dim H^1(G/\Gamma, \mathcal{O})$ is completely determined by Γ.

We observe that simply connected complex Lie groups are linear and discrete subgroups in linear groups admit maximal normal solvable and maximal normal nilpotent subgroups which we denote by $\mathrm{rad}(\cdot)$ resp. $n(\cdot)$. The usual density theorems for lattices in complex Lie groups imply that $\mathrm{rad}(\Gamma)$ and $n(\Gamma)$ contain $R \cap \Gamma$ resp. $N \cap \Gamma$ as subgroups of finite index (where R resp. N denotes the radical resp. nilradical of G).

Among all subgroups of finite index of $n(\Gamma)$ we choose one for which the commutator group, which we call $n_1(\Gamma)$ has minimal $\mathbb{Z}$-rank (see e.g. **[121]**, Def. 2.9. for the notion of $\mathbb{Z}$-rank for finitely generated nilpotent groups). Then $n_1(\Gamma)$ is commensurable to $N' \cap \Gamma$. Similarly we choose $r_1(\gamma)$. Next recall that for a lattice Λ in a semisimple Lie group S the image $\rho(\Lambda)$ is Zariski dense in $\rho(S)$. This holds for every *real* representation ρ. Besides $[S, N]N' = [G', N]N'$, since $G' \cap R \subset N$. It follows that $\Gamma_1 := [\Gamma', n(\Gamma)]n_1(\Gamma)$ must be a lattice in $[S, N]N'$.

Now $b_1(G/R\Gamma) = b_1(\Gamma/\mathrm{rad}(\Gamma))$,

$$\dim G/G' = \dim R/(R \cap G') = \dim R/[S, N]R' = \mathrm{rank}_{\mathbb{Z}}(\mathrm{rad}(\Gamma)/\Gamma_1 r_1(\Gamma))$$

and $\dim_{\mathbb{C}}(W)$ equals half the dimension over $\mathbb{Q}$ of the $\mathbb{Q}$-vector space W_0 where W_0 is the maximal $\mathbb{Q}$-linear subspace of $(r_1(\Gamma)\Gamma_1/\Gamma_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the natural linear transformation $\tau(\gamma)$ induced by conjugation is diagonalizable over $\mathbb{R}$ with only real eigenvalues for every $\gamma \in \Gamma$.

Thus $\dim H^1(G/\Gamma, \mathcal{O})$ depends only on Γ. □

8.7. Small deformations

So far we proved that there exist infinitesimal deformations iff $b_1(X) > 0$. Now we shall discuss small deformations. By actually constructing a deformation family we will show that every compact parallelizable complex manifold X with $b_1(X) > 0$ admits non-trivial small deformations.

THEOREM 8.7.1. — *Let G be a connected complex Lie group, Γ a discrete cocompact subgroup and $X = G/\Gamma$ the quotient manifold. Assume that $b_1(X) > 0$.*

Then there exist small deformations of X, i.e., there is a proper flat holomorphic family $Y \to \Delta_1$ with $Y_0 \simeq X$ and $Y_t \not\simeq X$ for $t \neq 0$.

Proof. — We will reduce the general theorem to two special cases, namely $G \simeq SL_2(\mathbb{C})$ and the case where G is commutative.

CLAIM 8.7.2. — Under the above assumptions there exists a connected complex Lie group H which is either commutative or isomorphic to $SL_2(\mathbb{C})$, a cocompact discrete subgroup $\Lambda \subset H$ and a G-equivariant holomorphic surjection $\pi : X \to Z = H/\Lambda$. Furthermore $b_1(Z) > 0$.

Proof. — Let R denote the radical of G. By Mostow's theorem there is a fibration $\rho : G/\Gamma \to G/R\Gamma$ (thm. 3.5.3). If $b_1(G/R\Gamma) > 0$, arithmeticity results for lattices in semisimple Lie groups (see [**92**]) imply that there exists a fibration $\pi_0 : G/R\Gamma \to SL_2(\mathbb{C})/\Lambda$ such that $\pi = \pi_0 \circ \rho$ is the desired surjection. On the other hand, if $b_1(G/R\Gamma) = 0$, then $G'\Gamma$ is closed in G and Γ' is a subgroup of finite index in $G' \cap \Gamma$ (see thm. 3.11.4). Thus in this case $G/\Gamma \to G/G'\Gamma$ yields a surjection onto a positive-dimensional torus. □

CLAIM 8.7.3. — Let G be either $SL_2(\mathbb{C})$ or $(\mathbb{C}^n, +)$, Γ a discrete cocompact subgroup, $X = G/\Gamma$ the quotient manifold and $\tau : \Gamma \to (\mathbb{Z}, +)$ a surjective group homomorphism.

Then there exists an open neighbourhood U of e in G such that for every $u \in U$ the Γ-action on G given by

$$\gamma : x \mapsto u^{-\tau(\gamma)} \cdot x \cdot \gamma$$

is free and properly discontinuous. Moreover U contains a subset A of measure zero such that the quotient manifold X_u is not biholomorphic to X for any $u \notin A$.

Proof. — For $G = (\mathbb{C}^n, +)$ this is easy to check and for $G = SL_2(\mathbb{C})$ it has been proved by Ghys [**42**]. □

Thus we deduced that we have a fibration $G/\Gamma \to H/\Lambda$ and non-trivial small deformations of H/Λ. We need to show that these deformations can be lifted to non-trivial deformations of G/Γ.

This is achieved by the following statement.

CLAIM 8.7.4. — Let G, H be connected Lie groups, Γ resp. Λ discrete cocompact subgroups in G resp. H, $\pi : G \to H$ a surjective Lie group homomorphism with $\pi(\Gamma) = \Lambda$, $\tau : \Lambda \to \mathbb{Z}$ a surjective group homomorphism and $u \in G$.

Assume that the Λ-action on H given by

$$\lambda : h \mapsto (\pi(u))^{-\tau(\lambda)} \cdot h \cdot \lambda$$

is free and properly discontinous.

Then the Γ-action on G given by

$$\gamma : u^{-\tau(\pi(\gamma))} \cdot g \cdot \gamma$$

is also free and properly discontinuous.

Proof. — In order to show that the action on G is properly discontinuous, we have to verify that for every compact subset $K \subset G$ the set

$$S = \{\gamma \in \Gamma : u^{-\tau(\pi(\gamma))} \cdot K \cdot \gamma \cap K \neq \varnothing\}$$

is finite. Now $\pi(K)$ is compact and hence

$$S_1 = \{\lambda \in \Lambda : (\pi(u))^{-\tau(\lambda)} \cdot \pi(K) \cdot \lambda \cap \pi(K) \neq \varnothing\}$$

must be finite. Thus $M = \{u^{-\tau(\lambda)} : \lambda \in S_1\}$ is likewise finite. Now

$$S \subset \{\gamma \in \Gamma : M \cdot K \cdot \gamma \cap K \neq \varnothing\}.$$

Both $M \cdot K$ and K are compact and consequently S is finite. The freeness can be checked in a similar way. □

8.8. Parallelizable deformations

The deformations of parallelizable manifolds obtained in the preceding section are, in general, no longer parallelizable. Indeed, if G is a semisimple complex Lie group, then (up to conjugcay) there are only countably many lattices (see prop. 3.13.2). In particular, if $G = SL_2(\mathbb{C})$ and Γ is a cocompact lattice with $b_1(G/\Gamma) > 0$, then $X = G/\Gamma$ can be deformed within the class of compact complex manifolds, but not within the class of compact complex parallelizable manifolds. Thus it is reasonable to ask for conditions under which there exist parallelizable deformations.

PROPOSITION 8.8.1. — *Let G be a simply connected complex Lie group, Γ a lattice, Z the center of G and $Z_{\mathbb{R}}$ a totally real Lie subgroup of Z.*

For $\rho \in \mathrm{Hom}(\Gamma, Z_{\mathbb{R}})$ let $\Gamma_\rho = \{\gamma \cdot \rho(\gamma) : \gamma \in \Gamma\}$ and $X_\rho = G/\Gamma_\rho$.

Then there is a non-empty open subset $\Omega \subset \mathrm{Hom}(\Gamma, Z_{\mathbb{R}})$ such that Γ_ρ is discrete and a lattice for $\rho \in \Omega$. For any two elements $\rho, \zeta \in \Omega$ the following properties are equivalent:

1. $\rho = \zeta$,
2. *There is a biholomorphic map $\phi : X_\rho \to X_\zeta$ inducing the identity map on $\pi_1(X_\rho) \simeq \Gamma_\rho \simeq \Gamma \simeq \Gamma_\zeta \simeq \pi_1(X_\zeta)$.*

Proof. — If there is such a biholomorphic map ϕ, it lifts to a complex Lie group automorphism $\phi' \in \mathrm{Aut}(G)$ (see thm 5.2.5). Since G is assumed to be simply connected, the center Z is isomorphic to some $(\mathbb{C}^d, +)$ and $G \simeq V \times Z$ as complex manifold, where V is a Stein manifold. We may assume $Z_{\mathbb{R}} \subset \mathbb{R}^d$. Now let τ_1 be a strictly plurisubharmonic exhaustion function on V and τ_2 be the function on $\mathbb{C}^d$ given by $\tau_2(z_1, \ldots, z_d) = \sum_i \Im(z_i)^2$. Then $\tau(v, z) = \tau_1(v) + \tau_2(z)$ defines a strictly plurisubharmonic function on G with $\tau^{-1}(\{0\}) = \mathbb{R}^d \subset Z$. Moreover τ is invariant under left or right multiplication by elements of $Z_{\mathbb{R}}$. Let $F : G \to \mathbb{R}$ be defined by $F(g) = \tau\left(\phi'(g)g^{-1}\right)$. Then

$$F(g\gamma) = \tau\left(\phi'(g)\phi'(\gamma)\gamma^{-1}g^{-1}\right) = \\ = \tau\left(\rho(\gamma)\zeta(\gamma)^{-1} \cdot \phi'(g)g^{-1}\right) = \tau\left(\cdot\phi'(g)g^{-1}\right) = F(g)$$

Thus F defines a stricly plurisubharmonic function on G/Γ. However, on G/Γ every plurisubharmonic function is constant (see thm. 3.7.1). Hence F is constant. This implies that $\phi' \equiv \mathrm{id}_G$ (because τ is *strictly* plurisubharmonic). □

COROLLARY 8.8.2. — *Let G be a complex Lie group with positive-dimensional center and $\Gamma \subset G$ a cocompact lattice. Then there exists a non-trivial deformation family of $X = G/\Gamma$ within the class of compact complex parallelizable manifolds.*

It should be mentioned that a positive-dimensional center is (together with $b_1(\Gamma) > 0$) a sufficient condition for the existence of parallelizable deformations, but not a necessary one.

EXAMPLE 8.8.3. — Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Then $A \in SL_2(\mathbb{Z})$ with an eigenvalue $\sqrt{5/4} + 3/2$. For any $\tau \in H^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ let $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ and $X_\tau = (\mathbb{C}^* \times E_\tau \times E_\tau)/(\mathbb{Z})$ with the $\mathbb{Z}$-action on $\mathbb{C}^* \times E_\tau \times E_\tau$ generated by

$$(x, v, w) \mapsto \left(\lambda x, A \cdot \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)\right).$$

This yields a smooth family $(X_\tau)_{\tau \in H^+}$ of compact parallelizable manifolds. However, on each of this compact complex parallelizable manifold the same three-dimensional solvable Lie group G acts and the center of this group is discrete.

8.9. Examples

1. Let $\Lambda \subset SL_2(\mathbb{C})$ be a cocompact lattice with a surjective group homomorphism $\rho : \Lambda \to \mathbb{Z}$ (see e.g. [**84**] for existence of such lattices). Let $G = SL_2(\mathbb{C}) \times \mathbb{C}$ and $\Gamma = \{(s, x) \in G : s \in \Gamma,\ x - \sqrt{2}\rho(s) \in \mathbb{Z}[i]\}$. Then Γ is a lattice in G such that $G'\Gamma$ is not closed in G. Furthermore $b_1(G/\Gamma) = \operatorname{rank}_{\mathbb{Z}}(\Lambda/\Lambda') + 2 > 0$ and $\dim H^1(G/\Gamma, \mathcal{O}) = \operatorname{rank}_{\mathbb{Z}}(\Lambda/\Lambda') + 1 > 0$ although every holomorphic map from G/Γ to a torus is constant.
2. Nakamura ([**110**]) gave an example of a three-dimensional solvmanifold $X = G/\Gamma$ with $\dim G/G' = 1$ and $\dim H^1(G/\Gamma, \mathcal{O}) = 3$. We will give another such example which in addition demonstrates that $\dim H^1(G/\Gamma, \mathcal{O})$ may jump within a commensurability class of Γ.

   Let p be a non-square positive natural number, $L = \mathbb{Q}[\sqrt{p}]$ and $K = L[i]$. By Dirichlet's theorem the group $\mathcal{O}_L^*$ of units of L contains an element of infinite order α. We may assume that $N_{L/Q}(\alpha) = 1$. Fix two embeddings $\sigma_1, \sigma_2 : K \to \mathbb{C}$ such that $\sigma_1 \neq \overline{\sigma_2}$, but $\sigma_1(i) = i$ and $\sigma_2(i) = -i$. Now $\sigma = (\sigma_1, \sigma_2) : K \to \mathbb{C}^2$ embedds $\mathcal{O}_K$ in $\mathbb{C}^2$ as a lattice. Let Λ_i $(i = 1, 2)$ be the subgroups of $\mathcal{O}_K^*$ generated by α resp. α and i. Now σ induces an embedding of the groups Λ_i into $GL(2, \mathbb{C})$. Note that $\det \sigma(\alpha) = N_{L/\mathbb{Q}}(\alpha) = 1$ and $\det \sigma(i) = 1$. Hence both $\sigma_i(\Lambda_i)$ are lattices in a Cartan subgroup T of $SL_2(\mathbb{C})$. Let $\Gamma_i = \Lambda \ltimes \mathcal{O}_K$. Then the groups Γ_i are lattices in a three-dimensional solvable complex Lie group $G = \mathbb{C}^* \ltimes (\mathbb{C}^2, +)$. By construction, the Γ_1-action on $\mathbb{C}^2$ is totally real, while $\sigma(i) \in \Gamma_2$ acts on $\mathbb{C}^2$ without real eigenvalues.

   Therefore we obtain a $2 : 1$ covering of three-dimensional solvmanifolds $G/\Gamma_1 \to G/\Gamma_2$ such that $\dim H^1(G/\Gamma_1, \mathcal{O}) = 3$ but $\dim H^1(G/\Gamma_2, \mathcal{O}) = 1$.

8.10. Powers of line bundles

Here we confirm that certain properties of line bundles are not affected by passing to a finite tensor power. In particular this is true for the existence of (flat) holomorphic connections as well as for the existence of a holomorphic structure on a topological complex line bundle.

LEMMA 8.10.1. — *Let X be a complex manifold, L a holomorphic line bundle, $k \in \mathbb{N}$ Then L admits a holomorphic connection resp. flat holomorphic connection if and only if L^k admits such a connection.*

Proof. — A connection D on L induces a connection $\widetilde{D}$ on L^k via

$$(*) \qquad \widetilde{D}(\sigma) = k\widetilde{\sigma} \otimes \sqrt[k]{\widetilde{\sigma}}^{-1} \otimes D\left(\sqrt[k]{\widetilde{\sigma}}\right)$$

The k-th root of a section in L^k is a section in L which is well-defined only up to multiplication with a k-th root of unity. Taking the same branch of the k-th root at both occurances of the k-th root in $(*)$ makes the formula well-defined.

In the converse direction a connection $\widetilde{D}$ on L^k yields a connection D on L via

$$(**) \qquad D(\sigma) = \frac{1}{k}\sigma^{1-k} \otimes \widetilde{D}(\sigma^k).$$

This is exactly the reverse procedure of $(*)$.

In a joint local trivialization D and $\widetilde{D}$ are given by $D(\sigma) = \omega \otimes \sigma + d\sigma$ resp. $\widetilde{D}(\sigma) = \widetilde{\omega} \otimes \sigma + d\sigma$ for some one-forms ω, $\widetilde{\omega}$. From $(*)$ resp. $(**)$ one can deduce $\widetilde{\omega} = k\omega$. Since $d\omega$ resp. $d\widetilde{\omega}$ yields the curvature form, it follows that D is flat if and only if $\widetilde{D}$ is flat. □

COROLLARY 8.10.2. — *Let X be a complex manifold. Assume that the Picard group* $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*)$ *finite.*

Then every holomorphic line bundle over X admits a flat holomorphic connection and is therefore induced by a representation $\rho : \pi_1(X) \to S^1$.

LEMMA 8.10.3. — *Let X be a complex manifold, $L \to X$ a topological complex line bundle (i.e., a complex line bundle with continuous transition functions), $k \in \mathbb{N}$.*

Then L admits a structure of a holomorphic line bundle if and only if L^k does.

Proof. — Topological line bundles are parametrized by $H^2(X, \mathbb{Z})$. An element $\alpha \in H^2(X, \mathbb{Z})$ corresponds to a line bundle admitting a holomorphic structure if and only if it is mapped to zero by the natural group homomorphism $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})$. This implies the assertion, because $H^2(X, \mathcal{O})$ is a complex vector space and therefore a torsion-free abelian group. □

8.11. The Hodge-Frölicher spectral sequence

The Hodge-Frölicher spectral sequence ([**41**], see also [**48**]) relates Dolbeault- and DeRham-Cohomology on complex manifolds as follows:

$$H^{p+q}(X, \mathbb{C}) = E_\infty^{p+q} \quad \Leftarrow \quad E_\bullet^{p,q}$$

with

$$E_1^{p,q} = H^{p,q}_{Dolb}(X, \mathbb{C}) \simeq H^q(X, \Omega^p).$$

It is a fundamental fact in Kähler geometry that this spectral sequence collapses in $E_1^{p,q}$ for every compact Kähler manifold. More generally it collapses in $E_1^{p,q}$ for all compact manifolds for which the Hodge-decomposition holds. This includes all Moishezon manifolds and moreover all manifold in Fujiki's class $\mathcal{C}$. It also collapses in $E_1^{p,q}$ for all compact complex surfaces. Now $E_1^{1,0}$ is isomorphic to the vector space of all holomorphic one-forms $H^0(X, \Omega^1)$, whereas $E_2^{1,0}$ is isomorphic to the vector space $H^0(X, d\mathcal{O})$ of all *closed* holomorphic one-forms. Hence $E_1^{1,0} \neq E_2^{1,0}$ for all compact complex parallelizable manifolds other than tori. Thus for compact complex parallelizable manifolds other than tori the Hodge-Frölicher spectral sequence cannot collapse before $E_2^{p,q}$. A result basically due to Sakane ([**33**], Thm.9) states that for nilmanifolds it does collapse in $E_2^{p,q}$.

For quotients of $SL_2(\mathbb{C})$ by discrete cocompact subgroups the Hodge-Frölicher spectral sequence likewise collapses in $E_2^{p,q}$; this can be deduced from the results of Akhiezer [**3**] on the Dolbeault cohomology of such quotients.

8.12. Line bundles: Comparison with Kähler manifolds

Now we want to compare the above results with the situation for compact Kähler manifolds.

For compact complex manifolds X we consider the following properties for line bundles:

(α) : There is a line bundle $\widetilde{L} \in \mathrm{Pic}^0(\mathrm{Alb}(X))$ such that $\mathrm{Alb}^*\widetilde{L} \simeq L^k$ for some $k \in \mathbb{N}$;
(β') : L admits a flat holomorphic connection;
(β) : L is given by a representation of the fundamental group;
(γ) : L is admits a holomorphic connection and L^k is topologically trivial for some $k > 0$,
(δ) : A tensor power L^k is topologically trivial for some $k \in \mathbb{N}$;
(δ') : A tensor power $L' = L^k$ for some $k \in \mathbb{N}$ is *divisible* as an element in the group $\mathrm{Pic}(X)$, *i.e.*, for every $m \in \mathbb{N}$ there exists a line bundle $H \in \mathrm{Pic}(X)$ such that $H^m \simeq L'$.

For arbitrary compact complex manifold there are the following implications:

$$\alpha \iff \beta' \implies \beta \implies \gamma \implies \delta \iff \delta'$$

Proof. — By the theorem of Appell-Humbert (see e.g. [**108**]) every topologically trivial line bundle on a torus is induced by a representation of the fundamental group. Hence $\alpha \Rightarrow \beta'$. The equivalence $\beta \Leftrightarrow \beta'$ is a classical result of Atiyah [**5**]. $\beta \Rightarrow \gamma \Rightarrow \delta$ is obvious. Finally note that the group of topologically trivial line bundles $\mathrm{Pic}^0(X)$ admits a surjection $H^1(X, \mathcal{O}) \to \mathrm{Pic}^0(X)$. Hence $\mathrm{Pic}^0(X)$ is a divisible group. On the other hand topologically line bundles are parametrized by $H^2(X, \mathbb{Z})$ which is a finitely generated abelian group. This suffices to prove $\delta \Leftrightarrow \delta'$. □

The converse implications hold only under special assumptions.

PROPOSITION 8.12.1. — *$\alpha \Leftarrow \beta$ holds if and only if the projection on the Albanese $X \to A = \mathrm{Alb}(X)$ induces an isomorphism $H^1(X, \mathbb{C}) \simeq H^1(A, \mathbb{C})$.*

$\beta \Leftarrow \gamma$ holds if and only if the natural group homomorphism $H^1(X,\mathcal{O}) \to H^1(X,d\mathcal{O})$ induced by $f \mapsto df$ is the zero homomorphism. This happens if and only if $E_2^{0,1} \simeq E_3^{0,1}$ for the Hodge-Frölicher spectral sequence.

$\gamma \Leftarrow \delta$ holds if and only if the natural group homomorphism $H^1(X,\mathcal{O}) \to H^1(X,\Omega^1)$ (induced by the $\mathbb{C}$-module sheaf homomorphism given by $f \mapsto df$) is zero. This is equivalent to $E_1^{0,1} \simeq E_2^{0,1}$ for the Hodge-Frölicher spectral sequence.

$\alpha \Leftarrow \delta$ holds if and only if the projection map to the Albanese induces an isomorphism $H^1(X,\mathcal{O}) \simeq H^1(\mathrm{Alb}(X),\mathcal{O})$.

Proof. — Clearly $\alpha \Leftrightarrow \beta'$ holds if and only if the groups $\mathrm{Hom}(\pi_1(\mathrm{Alb}(X)),\mathbb{C}^*)$ and $\mathrm{Hom}(\pi_1(X),\mathbb{C}^*)$ coincide up to torsion. This is equivalent to $H^1(X,\mathbb{C}) \simeq H^1(\mathrm{Alb}(X),\mathbb{C})$, because $H^1(M,\mathbb{C}) \simeq \mathrm{Hom}(\pi_1(M),\mathbb{C})$ for any topological space M.

If a line bundle L is topologically trivial, then there is a cocycle $f_{ij} \in \mathcal{O}(U_{ij})$ (for some Leray cover $(U_i)_i$) such that L is defined by $g_{ij} = \exp(f_{ij})$. Explicit calculations show that there exists a holomorphic connection if and only if the cohomology class of $d\log g_{ij} = dg_{ij}/g_{ij} = df_{ij}$ in $H^1(X,\Omega^1)$ vanishes. (Here Ω^1 denotes the sheaf of holomorphic one-forms.) A flat holomorphic connection exists if and only if $d\log g_{ij}$ defines the zero cohomology class in $H^1(X,d\mathcal{O})$, where $d\mathcal{O}$ denotes the sheaf of d-closed holomorphic one-forms. For the Hodge-Frölicher spectral sequence it is well-known that $E_1^{p,q} \simeq H^q(X,\Omega^p)$ with $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ induced by ordinary differentiation $d : \Omega^p \to \Omega^{p+1}$. Hence $E_1^{0,1} \simeq E_2^{0,1}$ if and only if the natural group homomorphism $H^1(X,\mathcal{O}) \to H^1(X,\Omega^1)$ is zero. Now we consider the long exact cohomology sequence induced by the short exact sequence of sheaves $0 \to \mathbb{C} \to \mathcal{O} \to d\mathcal{O} \to 0$. We obtain

$$0 \longrightarrow H^0(X,d\mathcal{O}) \longrightarrow H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O}) \longrightarrow H^1(X,d\mathcal{O}) \longrightarrow \cdots$$

Let V denote the subvectorspace of $H^1(X,\mathcal{O})$ for which there exists a flat holomorphic connection on the associated line bundle. Then (non-functorially) $H^1(X,\mathbb{C}) \simeq H^0(X,d\mathcal{O}) \oplus V$. On the other hand the Hodge-Frölicher spectral sequence yields $E_\infty^1 \simeq E_3^{1,0} \oplus E_3^{0,1}$ with $E_\infty^1 \simeq H^1(X,\mathbb{C})$ and $E_3^{1,0} \simeq E_2^{1,0} \simeq H^0(X,d\mathcal{O})$. In this way $V \simeq E_3^{0,1}$. Hence $\beta \Leftarrow \gamma$ if and only if $E_2^{0,1} = E_3^{0,1}$.

Finally $\alpha \Leftarrow \delta$ is clearly equivalent to $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(\mathrm{Alb}(X))$. Using the exponential sequence it is easy to verify that this holds if and only if $H^1(X,\mathcal{O}) \simeq H^1(\mathrm{Alb}(X),\mathcal{O})$. □

Thus all the terms $E_r^{0,1}$ of the Hodge-Frölicher spectral sequence are significant for line bundles and the existence of (flat) holomorphic connections. ($E_r^{0,1} \simeq E_3^{0,1}$ for $r \geq 3$).

COROLLARY 8.12.2. — *For a compact Kähler manifold X all the properties α, β, γ, δ are equivalent.*

For compact complex parallelizable manifolds the picture is more complicated.

PROPOSITION 8.12.3. — *Let $X = G/\Gamma$ be a compact complex parallelizable manifold. Let $S = G/R$ where R denotes the radical of G. Then one has the following implications*

$\alpha \Leftarrow \beta$ holds if and only if $G'\Gamma$ is closed and $(\Gamma \cap G')/\Gamma'$ is finite. In particular this is true, if S contains no $SL_2(\mathbb{C})$-factor.

$\beta \Leftarrow \gamma$ holds for every compact complex parallelizable manifold.

Finally, if R is nilpotent and no simple factor of G/R is isomorphic to $SL_2(\mathbb{C})$, then all the properties α, β, γ and δ are equivalent.

Proof. — The statement on the implication "$\alpha \Leftarrow \beta$" follows from cor. 3.11.5. For implication "$\beta \Leftarrow \gamma$" for compact parallelizable manifolds see cor. 6.6.7. The final statement of the proposition is implied by cor. 8.5.3. □

There are parallelizable examples with $\alpha \not\Leftarrow \beta$ and $\gamma \not\Leftarrow \delta$. Namely, if $X = G/\Gamma$ with $G = SL_2(\mathbb{C})$ and Γ/Γ' is infinite, then $\dim H^1(X, \mathbb{C})$ is larger than $H^1(\mathrm{Alb}(X), \mathbb{C})$ ($\mathrm{Alb}(X)$ is trivial, because $G = G'$). Hence $\alpha \not\Leftarrow \beta$ in this case.

Nakamura has calculated $H^1(X, \mathcal{O})$ explicitly for certain solvmanifolds (see **[110]**). This yields an example with $\gamma \not\Leftarrow \delta$.

As mentioned above $\beta \Leftarrow \gamma$ for compact Kähler manifolds as well as for all compact complex parallelizable manifolds. However, this does not hold for arbitrary compact complex manifolds. Cordero, Fernandez and Gray produced an example for which $E_2^{0,1} \neq E_3^{0,1}$ (**[32, 33]**). It is a four-dimensional compact complex manifold, diffeomorphic to a real nilmanifold, but not complex parallelizable. The above considerations show that on this manifold there exists a topologically trivial line bundle which does admit a holomorphic connection, but no flat holomorphic connection.

Property (γ) stated that the line bundle has a holomorphic connection *and* a finite tensor power of the bundle is topologically trivial. One may ask whether the presence of a holomorphic connection *implies* that a finite power is topologically trivial. For compact Kähler manifolds as well as for compact complex parallelizable manifolds this is indeed the case.

LEMMA 8.12.4. — *Let X be a compact manifold, which is Kähler or complex parallelizable. Let L be a line bundle which admits a holomorphic connection. Then there exists a $k \in \mathbb{N}$ such that L^k is topologically trivial.*

Proof. — If X is complex-parallelizable, then the existence of a holomorphic connection implies that there exists a flat holomorphic connection by thm. 6.6.6.

Now assume that X is Kähler. The presence of a holomorphic connection implies that there exists a d-closed $(2,0)$-form representing $c_1(L)$. On the other hand every line bundle admits a hermitian connection, hence there is also a d-closed $(1,1)$-form representing $c_1(L)$. Thus Hodge-decomposition forces $c_1(L) = 0$.

Thus $c_1(L) = 0$ in both cases. For a compact manifold this implies that a finite power of L is topologically trivial. □

However, we do not know whether there is such an implication for an arbitrary compact complex manifold. (There is no such implication for a non-compact manifold. For instance, if X is a Stein manifold, then every line bundle admits a holomorphic connection, because $H^1(X, \Omega^1) = 0$.)

8.13. The first Chern class

Using the results on homogeneous vector bundles, we will prove the following:

THEOREM 8.13.1. — *Let S be a semisimple complex Lie group, Γ a discrete cocompact subgroup, E a holomorphic vector bundle over S/Γ.*
Then $c_1(E) = 0$.

The proof will be obtained through an analysis of the S-action on various cohomology spaces.

8.13.1. Group actions on $H^1(X, \mathcal{O}^*)$

LEMMA 8.13.2. — *Let G be a complex Lie group acting on a complex space X by biholomorphic transformations. Consider*

$$(*) \qquad \cdots \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow \cdots$$

There is an induced G-action on these cohomology groups (by group automorphisms) such that $(*)$ is equivariant. Moreover, the G-action on the complex vector space $H^1(X, \mathcal{O})$ is holomorphic.

Proof. — Functoriality properties imply the existence of the induced actions. Holomorphicity of the G-action on $H^1(X, \mathcal{O})$ has been discussed in §3.17. □

Before continuing our argumentation we recall some language. Given a complex space X the group $H^1(X, \mathcal{O}^*)$ is denoted by $\mathrm{Pic}(X)$, the image of the natural group homomorphism $H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ is denoted by $NS(X)$, and the kernel by $\mathrm{Pic}^0(X)$. Thus there is a short exact sequence

$$(*) \qquad 0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow NS(X) \longrightarrow 0$$

LEMMA 8.13.3. — *For every complex space X the short exact sequence $(*)$ is split.*

Proof. — The obstruction to such a splitting is given by an element in the group $\mathrm{Ext}(NS(X), \mathrm{Pic}^0(X))$. As a complex vector space the additive group $H^1(X, \mathcal{O})$ is *divisible*, *i.e.*, for every element g and every natural number n there exists an element h such that $nh = g$. There is a surjective group homomorphism $H^1(X, \mathcal{O}) \to \mathrm{Pic}^0(X)$. Hence $\mathrm{Pic}^0(X)$ is divisible, too. This implies $\mathrm{Ext}(NS(X), \mathrm{Pic}^0(X)) = 0$. □

A similar arguments yields the following.

LEMMA 8.13.4. — *Let A be a commutative Lie group and A^0 the connected component of e. Then A is isomorphic (as a Lie group) to $A^0 \times A/A^0$.*

Proof. — Since A/A^0 is discrete, it suffices to prove that $0 \to A^0 \to A \to A/A^0 \to 0$ splits as a sequence of abstract groups. But connected commutative Lie groups are divisible, hence $\mathrm{Ext}(A/A^0, A^0) = 0$. □

LEMMA 8.13.5. — *Let A be a commutative complex Lie group and A^0 the connected component of $e \in A$. Let S be a connected semisimple complex Lie group acting on A by holomorphic group automorphisms.*

Then the natural projection $\pi : A^S \to A/A^0$ is surjective (where A^S denotes the fixed-point set).

Proof. — The projection $A \to A/A^0$ is equivariant. Let G denote the group of all holomorphic group automorphisms of A inducing the identity map on A/A^0. Evidently $S \subset G$, since A/A^0 is discrete and S connected. There is an exact sequence

$$1 \longrightarrow C \longrightarrow G \longrightarrow \mathrm{Aut}(A^0) \longrightarrow 1.$$

The group C consists of all holomorphic group automorphisms of $A = A^0 \times \Lambda$ which may be written in the form $(a, \lambda) \mapsto (a + f(\lambda), \lambda)$ for some $f \in \mathrm{Hom}(\Lambda, A^0)$. Clearly C is a commutative normal Lie subgroup of G. Furthermore G is a semidirect product $\mathrm{Aut}(A^0) \ltimes C$. The Levi-Malcev-decomposition for the connected component G^0 of G implies that all maximal connected semisimple Lie subgroups of G are conjugate. Since C is normal and commutative, it follows that S is conjugate to a subgroup of $\mathrm{Aut}(A^0)$ by an element $\phi \in G \subset \mathrm{Aut}(A)$. Thus $\phi(\Lambda) \subset A^S$ which implies surjectivity of $A^S \to A/A^0$. □

PROPOSITION 8.13.6. — *Let X be a complex space, S a connected semisimple complex Lie group acting on X. Let $NS(X)$ denote the image of $H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$. Assume that $H^1(X, \mathcal{O})$ is finite-dimensional.*

Then the induced group homomorphism $H^1(X, \mathcal{O}^*)^S \to NS(X)$ is surjective.

Proof. — Since S is connected, every $s \in S$ induces an automorphism $L_s : X \to X$ which is homotopic to the identity map. Therefore the S-action on $H^\bullet(X, \mathbb{Z})$ is trivial. Let V denote the complex subvectorspace of $H^1(X, \mathcal{O})$ spanned by $H^1(X, \mathbb{Z})$. Then S fixes V point-wise. Let W denote the image of V in $H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$. Then $\mathrm{Pic}^0(X)/W \simeq H^1(X, \mathcal{O})/V$ is a finite-dimensional complex vector space on which S acts holomorphically. Now we recall that $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0$ splits. Endowing $NS(X)$ with the discrete topology, we obtain a commutative complex Lie group $A \simeq \mathrm{Pic}(X)/W$ with $A^0 \simeq \mathrm{Pic}^0(X)/W$ on which S acts holomorphically. Now the preceding lemma implies surjectivity of $\mathrm{Pic}(X)^S \to NS(X)$. □

COROLLARY 8.13.7. — *Let S be a connected complex semisimple Lie group acting holomorphically on a complex space X with $\dim_{\mathbb{C}} H^1(X, \mathcal{O}) < \infty$.*

Then every holomorphic line bundle is topologically equivalent to a homogeneous line bundle.

COROLLARY 8.13.8. — *Let S be a simply connected complex semisimple Lie group, Γ a discrete cocompact subgroup, $X = S/\Gamma$.*

Then $NS(X)$ is finite. Furthermore there is a subgroup $\Gamma_0 \subset \Gamma$ of finite index such that $\pi^* L$ is topologically trivial for every line bundle $L \in \mathrm{Pic}(X)$, where π denotes the finite covering $S/\Gamma_0 \to X$.

REMARK 8.13.9. — Despite the finiteness of $NS(X)$ the group $H^2(X, \mathbb{Z})$ is not necessarily finite. For instance there is a discrete cocompact subgroup $\Gamma \subset SL_2(\mathbb{C})$ such that $b_1(SL_2(\mathbb{C})/\Gamma) = 1$ (see A.§3). By the Künneth formula this implies that $H^2(X, \mathbb{Z})$ is infinite for $X = (SL_2(\mathbb{C})/\Gamma) \times (SL_2(\mathbb{C})/\Gamma)$.

Proof of the corollary. — Since S is semisimple, every homogeneous vector bundle is flat. For a flat line bundle, the corresponding element in $H^2(X, \mathbb{Z})$ is a torsion element. Thus the above corollary implies that $NS(X)$ is a torsion group. Compactness of S/Γ implies that $H^2(X, \mathbb{Z})$ and therefore $NS(X)$ is finitely generated. A finitely generated abelian torsion group is finite. Thus $NS(X)$ is finite. By the universal coefficient theorem there is a (non-functorial) isomorphism between the torsion part of $H^2(X, \mathbb{Z})$ and the torsion part of $H_1(X, \mathbb{Z})$ for compact X. Recall that a flat line bundle is induced by a representation $\rho : H_1(X, \mathbb{Z}) \to \mathbb{C}^*$ and is topologically trivial if and only if ρ vanishes on the torsion part of $H_1(X, \mathbb{Z})$. Let $\tau : \Gamma = \pi_1(X) \to H_1(X, \mathbb{Z})$ denote the natural projection, Λ a torsion-free subgroup of finite index in $H_1(X, \mathbb{Z})$ (which exists, because $H_1(X, \mathbb{Z})$ is finitely generated) and $\Gamma_0 = \tau^{-1}(\Lambda)$. Then Γ_0 does the job. □

The theorem now follows using the fact that $c_1(E) = c_1(\det E)$ for any vector bundle E.

For later application we note some further consequences.

PROPOSITION 8.13.10. — *Let S be a complex semisimple Lie group, Γ discrete cocompact, $X = S/\Gamma$, Z a compact Kähler manifold, $f : Z \to X$ a holomorphic map and $L \in \mathrm{Pic}(X)$.*

Then either $f^* L$ is holomorphically trivial or $H^0(Z, f^* L) = \{0\}$.

Proof. — Since Z is Kähler, every non-trivial line bundle with non-trivial section has non-vanishing first Chern class contrary to $f^* c_1(L) = 0$. □

One may pose the following question: *Given a topological complex line bundle L on a complex manifold X does there exists a holomorphic structure on L?* This is equivalent to the question whether the group homomorphism

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z})$$

is surjective. We can answer this for a special case.

LEMMA 8.13.11. — *Let $S = SL_2(\mathbb{C})$, Γ a discrete cocompact subgroup with Γ/Γ' finite, $X = S/\Gamma$.*

Then $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ *is surjective.*

Proof. — This group homomorphism is part of a long exact sequence

$$\cdots \longrightarrow \mathrm{Pic}(X) \xrightarrow{c} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}) \longrightarrow \cdots$$

Now $H^2(X, \mathcal{O})$ is a complex vectorspace, hence torsion-free. Thus surjectivity of c is implied by finiteness of $H^2(X, \mathbb{Z})$. Since X is compact, finiteness of $H^2(X, \mathbb{Z})$ is equivalent to $H^2(X, \mathbb{C}) = \{0\}$ which follows from proposition B.7. □

8.14. The Serre-construction

Here we want to describe a method of construction which yields the existence of non-homogeneous vector bundles of rank two over $X = SL_2(\mathbb{C})/\Gamma$ for Γ discrete, cocompact with Γ/Γ' finite.

Our goal is to prove the following

THEOREM 8.14.1. — *Let $S = SL_2(\mathbb{C})$, Γ a torsion-free discrete cocompact subgroup with Γ/Γ' finite and C an elliptic curve in $X = S/\Gamma$.*

Then there exists a unique non-homogeneous holomorphic vector bundle E of rank two with as section $\sigma \in H^0(X, E)$ such that $C = \{x : \sigma(x) = 0\}$.

Furthermore $E|_C$ is trivial.

Note. — Every quotient $X = SL_2(\mathbb{C})/\Gamma$ of $SL_2(\mathbb{C})$ by a discrete cocompact subgroup Γ contains an elliptic curve (prop. 4.3.2, see also prop. 4.15.1).

A key element in the proof of the theorem is the vanishing theorem on $H^1(X, \mathcal{O})$ derived earlier.

Finally we investigate deformations of the vector bundles obtained in this way. It turns out that there are only the obvious deformations. Every small deformation is obtained by translation. Thus it is not possible to obtain further examples of non-homogeneous vector bundles by deformation.

PROPOSITION 8.14.2. — *Let S,Γ, X, E be as above, P a complex space $U \to P \times X$ a holomorphic vector bundle, $p \in P$ such that $U_p \simeq E$ where $U_p \to X$ is the vector bundle obtained from U by embedding X into $P \times X$ as $\{p\} \times X$. Furthermore let $\widetilde{E} \to S \times X$ denote the vector-bundle $\widetilde{E} = \mu^* E$ obtained by pull-back via the multiplication map $\mu : S \times X \to X$.*

Then there exists an open neighbourhood W of p in P and a holomorphic map $f : W \to S$ such that $F^*\widetilde{E} = U$ with $F = (f, \mathrm{id}_X)$.

A key step for this result on deformations is the calculation of certain cohomology groups.

LEMMA 8.14.3. — *Let E be as above. Then $\dim_{\mathbb{C}} H^k(X, E^* \otimes E) = 2$ for $0 \leq k \leq 3$ and*

$$\dim_{\mathbb{C}} H^k(X, E) = \begin{cases} 1 & \text{for } k = 0, 3 \\ 0 & \text{for } k = 1, 2 \end{cases}$$

We also prove a result in the converse direction.

THEOREM 8.14.4. — *Let $S = SL_2(\mathbb{C})$, Γ be a torsion-free discrete cocompact subgroup with Γ/Γ' finite and E a non-homogeneous vector bundle on $X = S/\Gamma$ of rank 2. Assume that there exists a non-trivial section $\sigma \in H^0(X, E)$. Assume in addition that the zero-set $\{\sigma = 0\}$ is irreducible and of multiplicity one. Then $\{x : \sigma(x) = 0\}$ is an elliptic curve and the bundle E may be reconstructed by the Serre-construction.*

Proof. — First recall that there are no hypersurfaces in X, hence $C = \{\sigma = 0\}$ is a curve. Observe that, given a vector field $\mathbf{X} \in \Gamma(X, TX)$ and a section $\sigma \in \Gamma(X, E)$ we may calculate $\mathbf{X}(\sigma)$ in a local trivialization. Of course the result depends on the trivialization. But, (this is easy to check explicitly) the restriction of $\mathbf{X}(\sigma)$ to the zero-set $V(\sigma) = \{\sigma = 0\}$ does not! Hence for any vector field $\mathbf{X}$ on X we obtain a section $\mathbf{X}\sigma \in \Gamma(C, E|_C)$. Moreover, if two vector fields $\mathbf{X}$ and $\mathbf{Y}$ span the normal bundle $N_{C,X}$ at some point $p \in C$, then $\mathbf{X}\sigma$ and $\mathbf{Y}\sigma$ span $E|_C$ at p. Thus $\mathbf{X}\sigma \wedge \mathbf{Y}\sigma$ defines a non-trivial section ζ in $\det E|_C$. This section ζ must vanish at the points where $\mathbf{X}$ and $\mathbf{Y}$ fail to span the normal bundle $N_{C,X}$. Using this, one can show that either $N_{C,X}$ is trivial or $\det E|_C$ admits a section which vanishes at some point $q \in C$, but not everywhere. However, the second case is impossible. To see this, let $\pi : \widehat{C} \to C$ denote the normalization of C. Then $\widehat{C}$ is a Kähler manifold and we may apply prop. 8.13.10. Thus the normal bundle has to be trivial. This implies that C is parallelizable, *i.e.*, an elliptic curve. □

We will apply the so-called Serre-construction in order to obtain non-homogeneous vector bundles over certain quotients $SL_2(\mathbb{C})/\Gamma$ (Γ discrete).

In the context of complex-projective spaces the Serre-construction is well-known, see e.g. **[52]**, **[114]**, **[137]**. Nevertheless we will briefly recall the main arguments in order to underline that they are still valid in a non-projective setup.

We formulate now the general result which we will use.

THEOREM 8.14.5 (Serre-Construction). — *Let X be a complex manifold, Y a complex submanifold*[1] *of codimension two with $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}$ and L a line bundle over X such that $\mathcal{L} \simeq \det \mathcal{I}_Y/\mathcal{I}_Y^2$ where $\mathcal{L}$ denote the sheaf of holomorphic sections in L and $\mathcal{I}_Y$ the ideal sheaf of Y. Assume that $H^2(X, L) = \{0\}$.*

Then there exists a vector bundle E of rank two over X with a section $\sigma \in \Gamma(X, E)$ such that $Y = \{x : \sigma(x) = 0\}$. (If moreover $H^1(X, L) = \{0\}$, then E is unique.)

Assume in addition that there exists no section in L^* vanishing on Y but not vanishing everywhere. Then $\Gamma(X, E)$ is generated by σ as $\Gamma(X, \mathcal{O}_X)$-module.

For parallelizable manifolds this has the following consequence.

COROLLARY 8.14.6. — *Let $X = SL_2(\mathbb{C})/\Gamma$ with Γ discrete and H a one-dimensional subgroup with compact orbit $Y = H/(H \cap \Gamma)$. Assume $H^1(X, \mathcal{O}_X) = \{0\}$.*

Then there exists a holomorphic vector bundle E of rank two on X with a section $\sigma \in \Gamma(X, E)$ such that $Y = \{\sigma = 0\}$. Moreover $\Gamma(X, E) = \mathbb{C}\sigma$, if Γ is Zariski dense in $SL_2(\mathbb{C})$.

Proof. — Any subvectorspace V transversal to $\mathcal{L}ie(H)$ in $\mathcal{L}ie(G)$ yields a trivialization of the normal bundle $N_{Y,X}$. Hence we may choose L as the trivial line bundle. Since X is parallelizable, K_X is trivial. Hence Serre-duality implies that $H^2(X, L) = H^2(X, \mathcal{O}_X)$ is dual to $H^1(X, \mathcal{O}_X)$, thus zero by assumption. Therefore we can apply the theorem. Finally recall that the density assumption implies that there are no hypersurfaces on X **[58]**. Thus any section in the line bundle L^* vanishing on Y must vanish everywhere. □

This implies the first theorem, because we proved already that $H^1(X, \mathcal{O}) = 0$ under the assumptions of the theorem.

The statement on the sections $\Gamma(X, E) = \mathbb{C}\sigma$ has interesting consequences. It implies that the curve C used to construct E may be recovered from E simply as the zero-set of any non-trivial section. This clearly implies that E is not a homogeneous vector bundle. Moreover it implies that there are infinitely many non-isomorphic non-homogeneous vector bundles of rank two on X, because there are infinitely many elliptic curves on X.

Proof of the theorem. — Instead of E itself we construct $\mathcal{E}^*$, the coherent sheaf of sections in the dual bundle E^*. The sheaf $\mathcal{E}^*$ is constructed as an extension of coherent sheafs

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

Such extensions are classified by $\mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{L})$. In our situation ($\mathcal{L}$ locally free, $\mathcal{I}_Y$ is a coherent sheaf which locally admits a free resolution of length one) a lemma of Serre

[1] Actually the condition of smoothness of Y may be relaxed to the requirement that Y should be a local complete intersection.

implies that the coherent sheaf associated to an element $e \in \mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{L})$ is locally free if and only if e_y generates $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{L})_y$ as a $\mathcal{O}_{X,y}$-module sheaf at every point $y \in Y$. By Nakayama's lemma this is equivalent to the assumption that e_y generates $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{L})(y)$ as complex vectorspace. (Here, as usual, $\mathcal{S}(x) = \mathcal{S}_x/m_x\mathcal{S}_x$ for any coherent sheaf $\mathcal{S}$, where m_x is the maximal ideal of the local ring $\mathcal{O}_{X,x}$.)

In order to find such an $e \in \mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{L})$, we will prove $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{L}) \simeq \mathcal{O}_Y$ and that the natural map

$$\mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{L}) \longrightarrow \Gamma(\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{L})) \simeq \Gamma(\mathcal{O}_Y) \simeq \mathbb{C}$$

is surjective.

There is a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{I}_Y, \mathcal{L})) \Rightarrow E_\infty^{p+q} = \mathrm{Ext}^{p+q}(\mathcal{I}_Y, \mathcal{L})$$

yielding the following exact sequence ("lower term sequence")

$$\begin{aligned}(11)\quad 0 \longrightarrow H^1(X, \mathcal{H}om(\mathcal{I}_Y, \mathcal{L})) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{L}) \longrightarrow \\ \longrightarrow H^0(X, \mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{L})) \longrightarrow H^2(X, \mathcal{H}om(\mathcal{I}_Y, \mathcal{L})) \longrightarrow \cdots\end{aligned}$$

Since Y is a smooth submanifold of codimension two (actually needed: locally complete intersection of codimension two), we know that

$$\mathcal{E}xt^k(\mathcal{O}_Y, \mathcal{S}) \simeq \begin{cases} 0 & \text{for } k = 0, 1 \\ \mathcal{H}om(\det \mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{S}) & \text{for } k = 2 \end{cases}$$

for every locally free sheaf $\mathcal{S}$, thus in particular for $\mathcal{S} = \mathcal{L}$. Now we deduce the long exact $\mathcal{E}xt^k(\cdot, \mathcal{L})$ sequence associated to

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

obtaining

$$0 \longrightarrow \mathcal{H}om(\mathcal{O}_Y, \mathcal{L}) \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{L}) \longrightarrow \mathcal{H}om(\mathcal{I}_Y, \mathcal{L}) \longrightarrow \mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{L}) \cdots$$

Thus $\mathcal{E}xt^k(\mathcal{O}_X, \mathcal{L}) = 0$ for $k > 0$ implies $\mathcal{E}xt^k(\mathcal{I}_Y, \mathcal{L}) \simeq \mathcal{E}xt^{k+1}(\mathcal{O}_Y, \mathcal{L})$ for all $k > 0$. In particular $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{L}) \simeq \mathcal{H}om(\det \mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{L}) \simeq \mathcal{O}_Y$ where the second equivalence follows from our assumption $L|_Y \simeq \det I_Y/I_Y^2$. Furthermore $\mathcal{H}om(\mathcal{O}_Y, \mathcal{L})$ is the zero sheaf, since $\mathcal{L}$ is locally free and $\mathcal{O}_Y$ is a torsion $\mathcal{O}_X$-module sheaf. Hence

$$\mathcal{H}om(\mathcal{I}_Y, \mathcal{L}) \simeq \mathcal{H}om(\mathcal{O}_X, \mathcal{L}) \simeq \mathcal{L}.$$

This means that locally every $\mathcal{O}_X$-module sheaf homomorphism from $\mathcal{I}_Y$ to $\mathcal{L}$ is obtained via embedding $\mathcal{I}_Y \hookrightarrow \mathcal{O}_X$ and multiplying $f_x \in (\mathcal{I}_Y)_x \subset \mathcal{O}_{X,x}$ by a section $s_x \in L_x$. Summarizing we obtain

$$0 \longrightarrow H^1(X, \mathcal{L}) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{L}) \longrightarrow H^0(\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{L})) \longrightarrow H^2(X, L)$$

with $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{L}) \simeq \mathcal{O}_Y$.

Finally we have to study sections in E. For this purpose we consider the long exact $\mathcal{E}xt(\cdot, \mathcal{O}_X)$-sequence associated to $0 \to \mathcal{L} \to \mathcal{E}^* \to \mathcal{I}_Y$. Since $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{O}_X) \simeq \mathcal{O}_Y$ we obtain

$$0 \longrightarrow \mathcal{H}om(\mathcal{I}_Y, \mathcal{O}_X) \simeq \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^* \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

Now for any surjective sheaf homomorphism $\mathcal{L}^* \to \mathcal{O}_Y$ the kernel sheaf must be $\mathcal{L}^* \otimes \mathcal{I}_Y$. Therefore

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, E) \longrightarrow \Gamma(X, \mathcal{L}^* \otimes \mathcal{I}_Y) \longrightarrow \cdots$$

Since by assumption there are no sections in $\mathcal{L}^*$ vanishing in Y but not vanishing everywhere, it follows that $\Gamma(X, \mathcal{O}_X) \xrightarrow{\alpha} \Gamma(X, \mathcal{E})$ is an isomorphism. Let $\sigma = \alpha(1)$. By construction σ is a section in $\mathcal{E}$ inducing a $\mathcal{O}_X$-sheaf homomorphism from $\mathcal{E}^*$ to $\mathcal{O}_X$ which fibers $\mathcal{E}^* \to \mathcal{I}_Y \to \mathcal{O}_X$ where the first map is surjective and the second map is ordinary inclusion. Hence $\{x : \sigma(x) = 0\} = Y$. □

8.15. Deformations of vector bundles

For any holomorphic vector bundle E over a fixed compact complex space X there exists a versal deformation space and the Zariski tangent space is exactly the space $H^1(X, \mathrm{Hom}(E, E))$ [**40**]. Thus we are led to calculate the dimension of this cohomology group.

LEMMA 8.15.1. — *Let $S = SL_2(\mathbb{C})$, Γ a discrete cocompact subgroup such that Γ/Γ' is finite, $X = S/\Gamma$, $Y \subset X$ an elliptic curve and $E \to X$ a holomorphic vector bundle of rank two with a section $\sigma \in \Gamma(X, E)$ such that $Y = \{x : \sigma(x) = 0\}$.*

Then $\dim_{\mathbb{C}} H^1(X, \mathrm{Hom}(E, E)) = 2$.

Proof. — Exploitation of the long exact cohomology sequence associated to

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

yields

$$\dim_{\mathbb{C}} H^k(X, \mathcal{I}_Y) = \begin{cases} 0 & \text{for } k = 0, 1 \\ 1 & \text{for } k = 2, 3. \end{cases}$$

Using duality between $H^k(X, E)$ and $H^{3-k}(X, E^*)$, an evaluation of

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y \longrightarrow 0$$

yields

$$\dim_{\mathbb{C}} H^k(X, E) = \begin{cases} 1 & \text{for } k = 0, 3 \\ 0 & \text{for } k = 1, 2 \end{cases}$$

Now we apply $\mathcal{H}om(\mathcal{E}^*, \cdot)$ to

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y \longrightarrow 0$$

and obtain

$$(*) \qquad 0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{H}om(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{E} \otimes \mathcal{I}_Y \longrightarrow 0$$

(The higher $\mathcal{E}xt$-sheaves $\mathcal{E}xt^k(\mathcal{E}, \mathcal{S})$ with $k \geq 1$ vanish for every coherent sheaf $\mathcal{S}$ because $\mathcal{E}$ is locally free.)

$$(**) \qquad 0 \longrightarrow \mathcal{E} \otimes \mathcal{I}_Y \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{O}_Y \longrightarrow 0$$

Using the triviality $E|_Y$ the last sequence yields

$$\dim_{\mathbb{C}} H^k(X, \mathcal{E} \otimes \mathcal{I}_Y) = \begin{cases} 1 & \text{for } k = 0, 3 \\ 2 & \text{for } k = 1, 2 \end{cases}$$

Finally exploitation of $(*)$ yields

$$\dim_{\mathbb{C}} H^k(X, \mathrm{Hom}(E, E)) = 2 \text{for all } k = 0, 1, 2, 3.$$

□

A family of vector bundles over a complex space X is given by a vector bundle $U \to S \times X$. For every point $s \in S$ this gives a vector bundle $U_s \to X$ as pull-back via the natural embedding $X \hookrightarrow G \times X$ given by $x \mapsto (s, x)$. Now any vector bundle E over a G-space X yields a family $\widetilde{E} \to G \times X$ by $\widetilde{E} = \mu^* E$ where $\mu : G \times X \to X$ is the map defining the group action. Given a section $\sigma \in H^0(X, E)$ with $C = \{x : \sigma(x) = 0\}$ one obtains sections $\sigma_g \in H^0(X, E_g)$ with $g(C) = \{x : \sigma(x) = 0\}$.

In our situation it follows that E_g is not isomorphic to $E_{\widetilde{g}}$ unless $g(C) = \widetilde{g}(C)$. Hence there is already a two-dimensional effective deformation induced by translations. Since the Zariski tangent space of the versal deformation space is two-dimensional, too, it follows that every small deformation is obtained by a translation.

CHAPTER 9

ON THE STRUCTURE OF COMPLEX NILMANIFOLDS

9.1. Survey

A *nilmanifold* is a compact complex parallelizable manifold $X = G/\Gamma$ which is a quotient of a *nilpotent* complex Lie group G by a discrete subgroup Γ. Such a nilmanifold is a tower of torus principal bundles. We are interested in the structure of those tori occuring in this tower and prove that unless the nilmanifold is somewhat degenerate, they are all isogenous to products of simple tori with complex multiplication. Conversely, every torus with complex multiplication occurs in the tower of a nilmanifold in a non-trivial way. Here a torus T is defined as a *torus with complex multiplication* if $\mathrm{End}_{\mathbb{Q}}(T)$ contains a number field K such that $[K:\mathbb{Q}] = 2\dim_{\mathbb{C}}(T)$. This implies that K is totally complex.

Tori with complex multiplication are rather special tori. A torus has complex multiplication if and only if it "arithmetic" in a certain sense (see theorem. 9.5.10). Up to isomorphism there exist only countably many tori with complex multiplication. A one-dimensional torus $\mathbb{C}/\langle 1,\tau\rangle$ has complex multiplication if and only if $\tau^2 \in \mathbb{Q}+\tau\mathbb{Q}$. However, a torus with complex multiplication is not necessarily an abelian variety. In fact, for every totally complex number field K which is not a *CM*-field there exists a torus T with complex multiplication by K such that T admits no non-constant meromorphic functions at all. (A totally complex number field K is called a *CM*-field, if it contains a totally real subfield K^+ with $[K:K^+] = 2$.) We use this to give an example of a non-trivial nilmanifold without non-constant meromorphic functions.

Now we present the main result.

THEOREM 9.1.1. — *Let G be a connected, simply connected complex nilpotent Lie group and Γ a discrete subgroup such that $X = G/\Gamma$ is compact. Let G^k denote the descending central series (i.e., $G^0 = G$, $G^{k+1} = [G, G^k]$) and C_k the ascending central series (i.e., $C_0 = \{e\}$, $C_{k+1} = \{c : cgc^{-1}g^{-1} \in C_k \ \forall g \in G\}$).*

Let

$$X = X^n \xrightarrow{T_n} X^{n-1} \xrightarrow{T_{n-1}} \cdots \xrightarrow{T_2} X^1 = T_1 = \mathrm{Alb}(X)$$

denote the tower of torus principal bundles associated to the descending central series, (i.e., $X^n = G/G^n\Gamma$, $T_n = G^{n-1}/G^n\Gamma \cap G^{n-1}$) and

$$X = X_0 \xrightarrow{S_1} X_1 \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} X_{n-1} = S_n$$

the tower of torus principal bundles associated to the ascending central series. (It is well-known (see cor. 2.2.3 and **[10][123]***) that all the $G^k\Gamma$ and $C_k\Gamma$ are closed in G.)*

Then for $k > 1$ both S_k and T_k are isogenous to a product of simple tori with complex multiplication.

If we impose a non-degeneracy condition on the nilmanifold, we can deduce certain properties of the Albanese torus $\mathrm{Alb}(X)$. For the statement of this result we need to explain the notion of a *decomposition series* for a torus T. This is a series of subtori $\{e\} = T_0 \subset T_1 \subset \cdots \subset T_m = T$ such that all the quotients T_{k+1}/T_k are positive-dimensional and simple. These quotients T_{k+1}/T_k are called *simple factors* for this composition series. Up to ordering and isogenies, they are independent of the particular choice of a composition series, *i.e.*, depend only on T. If T is algebraic, then it is isogenous to a direct product of these simple factors.

PROPOSITION 9.1.2. — *Assume the notations of the above theorem.*

Assume that $Z = C_1 \subset G' = G^1$. Then the composition series both for $\mathrm{Alb}(X)$ *and for any torus embedded in X as complex submanifold has the property that every simple factor admits complex multiplication.*

If in addition $C \subset G'$ with $C = \{c \in G : cgc^{-1}g^{-1} \in G^2\}$, then $\mathrm{Alb}(X)$ *is isogenous to a product of simple tori with complex multiplication.*

The condition $C \subset G'$ holds in particular in the following two cases:

1. G is a generalized Heisenberg group, *i.e.*, $Z = G'$.
2. G is a maximal unipotent subgroup in a complex semisimple Lie group.

(In the second case $C \subset G'$ can be verified easily using the theory of root systems for semisimple Lie groups.)

EXAMPLE 9.1.3. — Let G be a complex Heisenberg group, *i.e.*, a simply connected nilpotent complex Lie group G with one-dimensional center Z. (This implies $G' = Z$.) Let Γ be a discrete cocompact subgroup. Then $X = G/\Gamma \xrightarrow{C} G/G'\Gamma = \mathrm{Alb}(X)$ is the Albanese, and our theorem implies that $C = G'/(G' \cap \Gamma)$ is an elliptic curve with complex multiplication and $\mathrm{Alb}(X)$ is isogenous to a direct product of copies of C.[(1)]

(1) The article *On the Picard Group of a compact complex nilmanifold* in the *Rocky Mt. J. Math.* 17, 65–76 (1987) contains some alleged examples of nilmanifolds which would provide counterexamples to this conclusion. These examples are based on the assumption that certain subgroups (given in terms of their generators) are discrete. But actually they fail to be discrete.

In the converse direction, we confirm that every torus with complex multiplication occurs in such a way.

THEOREM 9.1.4. — *Let* $\mathbf{g}$ *be a nilpotent* $\mathbb{Q}$*-Lie algebra,* K *a totally complex number field,* $[K : \mathbb{Q}] = 2g$.

Then there exists a simply connected nilpotent complex Lie group G, *and a discrete cocompact subgroup* Γ *such that* $\mathcal{L}ie(G)$ *is isomorphic to a direct product of* g *copies of* $\mathbf{g} \otimes_{\mathbb{Q}} \mathbb{C}$ *and every torus* $G^k/(G^{k+1}\Gamma)$ *is a direct product of tori with complex multiplication by* K.

If K *is not a CM-field, then* G *and* Γ *may be choosen in such a way that none of the tori* $G^k/(G^{k+1}\Gamma)$ *admits a non-constant meromorphic function.*

Considering number fields which are neither totally real nor totally complex, one can generalize this method of construction in order to obtain interesting non-compact nilmanifolds X such that every holomorphic function on X is constant, but every compact analytic subset is finite.

We have another type of converse to thm. 9.1.1 which characterizes nilmanifolds among solvmanifolds.

THEOREM 9.1.5. — *Let* G *be a simply connected solvable complex Lie group,* Γ *a discrete cocompact subgroup. Assume that the center* Z *is contained in the commutator group* G'.

Then G *is nilpotent if and only if every simple factor in a decomposition series of* $\mathrm{Alb}(X) \simeq G/G'\Gamma$ *has complex multiplication.*

(By a result of Barth and Otte (see cor. 3.11.3) $G'\Gamma$ is closed in G, hence $\mathrm{Alb}(X) \simeq G/G'\Gamma$.) For investigating the structure of tori occuring in the tower of a nilmanifold, it is essential to use the structure theory for tori. For abelian varieties, there is a very satisfactory structure theory. However, we need a structure theory for arbitrary tori, not only the algebraic ones. Many results for abelian varities do not hold for arbitrary tori. Most important, an arbitrary torus is not necessarily isogenous to a product of simple tori. Thus we first have to develop a structure theory in the non-algebraic setting.

One tool is the above mentioned composition series.

PROPOSITION 9.1.6. — *Every torus* T *admits a* composition series, *i.e., an ascending sequence of subtori* $\{e\} = T_0 \subset \cdots \subset T_m = T$ *such that all the quotient* $S_k = T_{k+1}/T_k$ *are positive-dimensional and simple. The sequence* S_k *is called the sequence of simple factors for this composition series.*

If there exists another composition series, the sequence of simple factors differs only by ordering and isogenies.

If there is a surjective morphism of tori $T \to T'$ or an injective morphism of tori $T' \to T$, then the sequence of simple factors for T' is contained in the sequence of simple factors for T (up to ordering and isogenies).

An algebraic torus always equals the sum of all its simple subtori. In contrast, an arbitrary torus is generated by all its irreducible subtori. Here a torus T is called irreducible, if it is not generated by its proper subtori. We prove that irreducible non-simple tori can not occur in our context. The reason is the following:

PROPOSITION 9.1.7. — *Let $T = \mathbb{C}^g/\Lambda$ be an irreducible torus, E a commutative subalgebra of $\mathrm{End}_{\mathbb{Q}}(T)$. Assume that $\Lambda \otimes \mathbb{Q}$ is a principal E-module, i.e., generated as E-module by one element.*

Then T is simple and E is a number field.

Commutativity of E is essential, cf. the example in section §9.6.

For a nilpotent group, the commutator map $\zeta(g,h) = ghg^{-1}h^{-1}$ is especially useful. In particular it yields a $\mathbb{Z}$-bilinear map $G/G' \times G/G' \to G'/G^2$. This enables us to conclude that the tori we encounter have large endomorphism algebras. Furthermore these algebras must contain large commutative subalgebras. This follows from an auxiliary, purely algebraic result on bilinear forms on modules. Then the above stated result implies that all tori occuring in the tower of a nilmanifold must be semisimple, *i.e.*, isogenous to a direct product if simple tori. Moreover these simple tori must have complex multiplication.

For solvable, non-nilpotent groups it is quite possible that $G' = G^2$. Hence the $\mathbb{Z}$-bilinear map from $G/G' \times G/G' \to G'/G^2$ is not useful in this context. Thus the proof of thm. 9.1.1 does not hold for solvable non-nilpotent groups.

In order to prove the converse statement, thm. 9.1.5, we note that for a non-nilpotent Lie group the adjoint representation is never nilpotent. From this we deduce the following structure theorem on solvmanifolds.

PROPOSITION 9.1.8. — *Let G be a simply connected solvable complex Lie group, N its nilradical, Γ a discrete cocompact subgroup in G, $T = G/N\Gamma$ ($N\Gamma$ is closed in G by a result of Mostow (see thm. 3.5.3)). Let $T = \mathbb{C}^g/\Lambda$.*

Then there exists a homomorphism of complex Lie groups $\phi : \mathbb{C}^g \to (\mathbb{C}^*)^k$ with discrete kernel such that $\phi(\Lambda)$ is contained in the $\overline{\mathbb{Q}}$-rational points, i.e., $\phi(\Lambda) \subset \left(\overline{\mathbb{Q}}^*\right)^k$.

It should be noted that such a Lie group homomorphism ϕ is very much transcendental and thus does not carry algebraic numbers to algebraic numbers.

Using Baker's theorem from transcental number theory we prove the following

PROPOSITION 9.1.9. — *Let Λ be a lattice in $\mathbb{C}^g$ such that $\Lambda \subset \overline{\mathbb{Q}}^g$. Let $\phi : \mathbb{C}^g \to (\mathbb{C}^*)^k$ be a complex Lie group homomorphism such that $\phi(\Lambda) \subset (\overline{\mathbb{Q}}^*)^k$.*

Then ϕ is constant.

Now any torus with complex multiplication admits a lattice, such that all elements in the lattice are $\overline{\mathbb{Q}}$-rational. Therefore this result implies that $G/N\Gamma$ is never a torus with complex multiplication. From this we deduce thm. 9.1.5.

9.2. A remark on the commutator group

Under special circumstances the commutator map $\zeta : (g, h) \mapsto ghg^{-1}h^{-1}$ is a group homomorphism in both variables.

LEMMA 9.2.1. — *Let G be a group, H a subgroup and assume that $[G, H]$ is central. Then $\zeta : G \times H \to G$ defined by*

$$\zeta : (g, h) \mapsto ghg^{-1}h^{-1}$$

is a group homomorphism in both variables.

Note that the assumption implies in particular that $\zeta(g, h) = e$ if $g \in G'$ or $h \in H'$, because $[G, H]$ is abelian.

Proof. — This follows by explicit calculation.

$$\zeta(g_1g_2, h) = g_1g_2h(g_1g_2)^{-1}h^{-1} = g_1 \left(g_2hg_2^{-1}h^{-1}\right) hg_1^{-1}h^{-1} = \tag{12}$$

$$= g_1hg_1^{-1}h^{-1}g_2hg_2^{-1}h^{-1} = \zeta(g_1, h)\zeta(g_2, h). \tag{13}$$

A similar calculation yields $\zeta(g, h_1h_2) = \zeta(g, h_1)\zeta(g, h_2)$. □

As usual, let G^k resp. C_k denote the descending resp. ascending central series of a given group G, *i.e.*, $G^0 = G$, $C_0 = \{e\}$, $G^{k+1} = [G, G^k]$ and C_{k+1}/C_k is the center of G/C_k.

By the lemma, the commutator map yields $\mathbb{Z}$-bilinear maps

$$G/G' \times G^{k-1}/G^k \longrightarrow G^k/G^{k+1}$$

and

$$G/G' \times C_{k-1}/C_k \longrightarrow C_k/C_{k+1}.$$

In the first case, the image of the map generates G^k/G^{k+1} as a $\mathbb{Z}$-module (*i.e.*, as an abelian group). In the second case the map is non-degenerate in the second variable in the following sense: For each $c \in C_{k-1} \setminus C_k$ there exists an element $g \in G$ such that $gcg^{-1}c^{-1} \in C_k \setminus C_{k+1}$. (This follows immediately from the definition of the C_k.)

9.3. Nilmanifolds

We recall the fundamental fact that for nilmanifolds many natural subgroups have closed orbits. In particular this true for all the subgroups of the descending and ascending central series (see cor. 2.2.3, [**89**]).

From this one may deduce the following fact.

THEOREM 9.3.1. — *Let $X = G/\Gamma$ be a complex nilmanifold, i.e., a quotient of a connected complex nilpotent Lie group G by a discrete cocompact subgroup Γ.*

Then there exists a tower of torus principal bundles

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0,$$

where X_0 is a torus, given by $X_i = G/G^{i+1}\Gamma$.

For the ascending central series the assumption of cocompactness of Γ can be weakened.

THEOREM 9.3.2 (Barth-Otte [**10**]). — *Let G be a connected complex nilpotent Lie group, Γ a closed Lie subgroup such that every holomorphic function on G/Γ is constant.*

Then all the C_k have closed orbits in G/Γ.

There is an example due to K. Oeljeklaus of a quotient G/Γ of a complex nilpotent Lie group G by a discrete subgroup Γ such that every holomorphic function on G/Γ is constant, but the G'-orbits are not closed ([**112**], p.64). Hence this theorem applies only to the ascending central series and not to the descending central series. The central goal of this chapter is to answer the question: Which tori can actually occur in this tower, if $X = G/\Gamma$ is non-trivial, e.g. not a direct product with a torus?

9.4. Compact complex tori

In order to study complex nilmanifolds, we need some basic facts on compact complex tori. For abelian varieties, all the results in this section are trivial or well-known. However, complex nilmanifolds are never projective manifolds unless they are already abelian. Therefore it would be quite artificial to restrict our attention to nilmanifolds with projective Albanese.

A major difficulty in dealing with non-algebraic tori arises from the fact that the Poincaré Reducibility Theorem does not hold for arbitrary non-algebraic torus.

In what follows a *torus* is always a compact complex torus, *i.e.*, a quotient of a complex vector space $\mathbb{C}^n$ by a lattice Λ of rank $2n$. We always fix a base-point e in a torus. Then there is a unique structure as a commutative complex Lie group with e as neutral element. A morphism between tori is holomorphic map, which we require to take the neutral element to the neutral element. It is well-known that every such morphism is a Lie group homomorphism and linear, if pull-backed to the universal covering.

A very important notion for tori is that of an *isogeny.* An isogeny between two tori S, T is a holomorphic correspondence, given by a graph $\Phi \subset S \times T$ such that both projections $\Phi \to S$ and $\Phi \to T$ are finite unramified coverings. Isogenies define an equivalence relation among tori and it is often useful to think of two tori being essentially equal if there is an isogeny between them. Isogenies between two given tori S and T are not necessarily maps, hence not contained in $\mathrm{Hom}(S,T)$. However, there is a natural 1-1 correspondence between isogenies and invertible elements in $\mathrm{Hom}(S,T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We call a torus T *algebraic* if it is a projective manifold. This is equivalent to assume that T is Moishezon. It is also equivalent to assume that T is the complex space associated to an abelian variety defined over $\mathbb{C}$. Furthermore $T = \mathbb{C}^g/\Lambda$ is algebraic if and only if there exists a *Riemann form* H. A Riemann form H is a positive definite Hermitian form with $\Im H(\Lambda \times \Lambda) \subset \mathbb{Z}$.

DEFINITION 9.4.1. — A torus T is *simple* if it does not contain any subtorus S with $0 < \dim(S) < \dim(T)$.

A torus T is *semisimple* if it is isogenous to a direct product of simple tori.

By a theorem of Poincaré every algebraic torus is semisimple. However, there are non-algebraic tori which are not semisimple.

Many results on abelian varieties generalize easily to semisimple tori. For instance, it is a direct consequence of the definition that every subtorus of a semisimple torus is again semisimple.

LEMMA 9.4.2. — *Let*

$$(*) \qquad 0 \longrightarrow T_1 \longrightarrow T_2 \xrightarrow{\alpha} T_3 \longrightarrow 0$$

be an exact sequence of tori. Assume that T_2 is semisimple. Then all of the T_i are semisimple. Moreover there exists an element $s \in \mathrm{Hom}(T_3, T_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\alpha \circ s = \mathrm{id}_{T_3}$, i.e., $(*)$ splits on the isogeny level.

Thus quotients and subtori of semisimple tori are semisimple.

This enables us to make the following definition.

DEFINITION/PROPOSITION 9.4.3. — Let T be a torus. Let T_0 be the smallest subtorus for which T/T_0 is semisimple. Then the projection $\pi_{ss} : T \to T_{ss} = T/T_0$ is called *semisimple reduction* of T.

Proof. — Let T_1, T_2 be subtori of T with T/T_i $(i = 1,2)$ semisimple. Then $T/(T_1 \cap T_2)$ may be embedded in $T/T_1 \times T/T_2$. Therefore $T/(T_1 \cap T_2)$ is again semisimple. This implies that T_0 is uniquely determined. □

By the preceding lemma it is clear that $\dim T_{ss} > 0$ for every positive-dimensional torus.

LEMMA 9.4.4. — *Every torus T admits a non-constant morphism to a simple torus.*

Proof. — By induction on $\dim(T)$. If $\dim(T) = 1$, then T is simple. If T is not simple (hence $\dim(T) > 1$), there is a subtorus S and by induction hypothesis T/S admits a non-constant morphism to a simple torus. □

The semisimple reduction fulfills a universality property.

LEMMA 9.4.5. — *Let T, S be tori, $f : T \to S$ a morphism, S semisimple. Then there exists a morphism $f_0 : T_{ss} \to S$ such that $f = f_0 \circ \pi_{ss}$.*

Proof. — Let T' denote the kernel of f. Then T/T' is contained in the semisimple torus S, hence semisimple itself. It follows that $\ker \pi_{ss} \subset T'$ by the construction of the semisimple reduction. □

Similarily, it follows that $T \to T_{ss}$ defines a covariant functor from the category of complex tori to the category of semisimple complex tori.

Another variation on this theme yields the following.

LEMMA 9.4.6. — *Let T be a torus, S a simple torus. Then there exists a surjective morphism π form T to some torus T' such that*

1. *T' is isogenous to a direct product of copies of S,*
2. *Every morphism from T to S fibers through $\pi : T \to T'$.*

We mentioned already that every torus admits a non-constant morphism onto a simple torus. By induction this fact allows the definition of a *composition series.*

DEFINITION 9.4.7. — A *composition series* for a torus T is a sequence of subtori

$$\{e\} = T_0 \subset T_1 \subset \cdots \subset T_d = T$$

such that T_i/T_{i-1} is simple.

The sequence of simple tori $S_i = T_i/T_{i-1}$ is called the sequence of simple factors of the composition series T_i.

Owing to the preceding lemma, it is clear that every torus has a composition series. However, we need some kind of uniqueness.

LEMMA 9.4.8. — *Let T be torus, T_i and T'_i two composition series. Then the two sequences of simple factors coincide up to ordering and isogenies.*

Proof. — We will prove this by induction on $\dim(T)$. Note that the lemma is trivially true for $\dim(T) = 1$.

Let $Z = T_1$. This is a simple subtorus of T. Consider the morphisms $\tau_i : Z \to T'_i/T'_{i-1} = S'_i$. Since Z and S'_i are simple, every τ_i is either constant or an isogeny.

It is clear that exactly τ_i is an isogeny for exactly one number $i = i_0$. Now we have two composition series Z_i and Z'_i for T/S as follows: $Z_i = T_{i+1}/S$ for all i, $Z'_i = T'_i/(S \cap T'_i)$ for $i < i_0$ and $Z'_i = T'_{i+1}/(S \cap T'_{i+1})$ for $i \geq i_0$. The statement now follows from the induction hypothesis. $\square$

Thus the sequence of simple factors does not depend on the particular choice of a composition series, if we ignore ordering and isogenies. This legitimizes the following definition.

DEFINITION 9.4.9. — A simple torus S is called a *simple factor* for a torus T, if it is isogenous to a simple factor of a composition series of T.

In a similar way to the proof of the above lemma one can prove the following:

LEMMA 9.4.10. — *Let S, T be tori. Assume that there is a surjective morphism $\tau : S \to T$ or an injective morphism $i : T \to S$.*

Then (up to ordering and isogenies) the sequence of simple factors for T is a subsequence of the sequence of simple factors for S.

DEFINITION 9.4.11. — A torus T is *irreducible* if T is not the sum of all its proper subtori.

A semisimple torus is irreducible if and only if it is simple.

LEMMA 9.4.12. — *A torus T is irreducible if and only if there is a subtorus T_0 with $\{e\} \subseteq T_0 \subsetneq T$ such that T/T_0 is simple and T_0 contains every non-trivial subtorus of T.*

Proof. — Obviously T is irreducible if there exists such a T_0. Conversely, let T be an irreducible torus. Then T_0 defined as the sum of all proper subtori does the job. $\square$

COROLLARY 9.4.13. — *Let T be a torus which is not irreducible.*

Then there exists a surjective morphism from T to a semisimple torus S which is not simple.

LEMMA 9.4.14. — *Every torus T equals the sum of all its irreducible subtori.*

Proof. — By induction on $\dim(T)$ as follows. Let T be a torus. If there exist subtori $T_i \subsetneq T$ with $T = \oplus_i T_i$, the statement follows by induction. If there exists no such family of subtori, then T is already irreducible. $\square$

Caveat: These irreducible subtori may have large intersection.

DEFINITION 9.4.15. — For an irreducible torus T with T_0 as above the quotient T/T_0 is called the *dominant simple factor* for T.

LEMMA 9.4.16. — *Let $f : S \to T$ be a surjective morphism of tori, S irreducible. Then T is irreducible, too. Furthermore the dominant simple factors for S and T are isogenous.*

The relevance of dominant simple factors relies on the following fact.

LEMMA 9.4.17. — *Let T be a torus, S a simple factor.*

Then T contains an irreducible subtorus U with a dominant simple factor isogenous to S.

Proof. — Consider all subtori Z_i of T for which S is a simple factor. Choose U as one of the Z_i with the minimal possible dimension. Then S does not occur as simple factor for any subtorus of U. This implies that S is not a simple factor for the sum of all proper subtori of U. Since S is a simple factor for U, it follows that U is irreducible. □

LEMMA 9.4.18. — *Let T be a torus, T_i a family of irreducible subtori with $T = \sum_i T_i$. Let $T \to T/S = T_{ss}$ be the semisimple reduction. Then every simple factor of T_{ss} is a dominant simple factor for some T_i.*

Proof. — Let A be a simple factor of T_{ss}. Then there exists a surjective morphism τ from T to a torus S' isogenous to S. If T_i is irreducible with a dominant simple factor not isogenous to S, then the restriction of τ to T_i must be constant. Now $T = \sum_i T_i$ implies that there is a T_i such that $\tau|_{T_i}$ is not constant. □

Recall that, given a compact complex parallelizable manifold X, there exists a holomorphic surjective map f from X onto an abelian variety A such that f induces an isomorphism between the respective function fields (thm. 3.15.4).

This has an immediate consequence for simple tori.

COROLLARY 9.4.19. — *Let T be a simple torus. Then either T is algebraic or every meromorphic function on T is constant.*

Some tori admit closed complex subspace which are not subtori. This has implications for the decomposition series.

PROPOSITION 9.4.20. — *Let T be a torus, S_i the sequence of simple factors for a decomposition series $\{0\} = T_0 \subset \cdots \subset T_n = T$.*

Assume that T contains a closed complex subspace Z which is not a subtorus. Then at least one of the S_i is algebraic, i.e., isomorphic to a complex abelian variety.

Proof. — There exists a subtorus $A \subset T$ such that Z is A-invariant and Z/A is a complex space of general type contained in the quotient torus T/A (**[146]**). Let $\langle Z/A\rangle$ denote the subtorus of T/A generated by Z/A. For some $n \in \mathbb{N}$ there is a surjective morphism $(Z/A)^n \to \langle Z/A\rangle$. Hence $\langle Z/A\rangle$ is Moishezon and therefore an abelian variety. Now every simple factor for T/A is also simple factor for T. Hence there must be a simple factor for T which is isomorphic to an abelian variety. □

9.5. Endomorphism algebras

In this section we summarize basic facts on the endomorphism algebra of a torus. Most of this is well-known, see e.g. [**79**], [**135**].

For each torus T the set $\mathrm{End}(T) = \mathrm{Hom}(T,T)$ is a ring with composition as multiplication and pointwise addition in T as addition in $\mathrm{End}(T)$. Thus $\mathrm{End}(T)$ is a $\mathbb{Z}$-algebra. However, often it is easier to work with the $\mathbb{Q}$-algebra $\mathrm{End}_{\mathbb{Q}}(T) = \mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. For instance, the $\mathbb{Z}$-algebra $\mathrm{End}(T)$ is not invariant under isogenies.

Furthermore, *an endomorphism ϕ of a torus T yields an invertible element in $\mathrm{End}_{\mathbb{Q}}(T)$ if and only if $\ker \phi$ is finite.*

Consider a torus T as quotient of complex vectorspace $V = \mathbb{C}^g$ by a lattice Λ. Given an endomorphism $\phi : T \to T$, this canonically induces endomorphisms on the tangent space at e of T, which we may identify with V, as well as on the fundamental group of T, which we may identify with Λ. Thus we obtain natural representations of $\mathbb{Q}$-algebras

$$\rho_r : \mathrm{End}_{\mathbb{Q}}(T) \to \mathrm{End}_{\mathbb{Q}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$$

(often called the rational representation) and

$$\rho_a : \mathrm{End}_{\mathbb{Q}}(T) \to \mathrm{End}_{\mathbb{C}}(V)$$

(sometimes called the analytic representation). These representations are related, because $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is canonically isomorphic to the $2g$-dimensional real vectorspace obtained from $V \simeq \mathbb{C}^g$ by restriction of scalars. It follows that $\rho_r \simeq \rho_a \oplus \bar{\rho}_a$ considered as representations in $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2g})$.

THEOREM 9.5.1. — *Let $T = \mathbb{C}^n/\Lambda$ be a simple torus. Then $E = \mathrm{End}_{\mathbb{Q}}(T)$ is a skewfield with $\dim_{\mathbb{Q}}(E) \leq \dim_{\mathbb{R}}(T)$.*

Sketch of the proof. — Simplicity of T implies that E is an integral domain. Furthermore, E is a finite-dimensional over $\mathbb{Q}$, because $\rho_r : E \to \mathrm{End}(\Lambda \otimes \mathbb{Q})$ is injective. Now a finite-dimensional algebra over a field is necessarily a skewfield, provided it is an integral domain. Finally, the dimension bound follows because $\Lambda \otimes \mathbb{Q}$ is a E-left vector space. □

For our applications the most important case involves tori with complex multiplication.

DEFINITION 9.5.2. — A torus T (not necessarily simple) has *complex multiplication* (by a number field K) if $\mathrm{End}_{\mathbb{Q}}(T)$ contains a number field K with $[K : \mathbb{Q}] = 2\dim_{\mathbb{C}}(T)$ as subring.

Caveat: This notion is often reserved to algebraic tori, a restriction which would be unnatural for us.

EXAMPLE 9.5.3. — A one-dimensional torus $T = \mathbb{C}/\langle 1, \tau\rangle_{\mathbb{Z}}$ is a torus with complex multiplication if and only if $\tau^2 = p\tau + q$ for some $p, q \in \mathbb{Q}$.

Complex multiplication is a rather rare phenomenon among tori.

LEMMA 9.5.4. — *Let T be a torus with complex multiplication. Then T is isogenous to some $S \times \cdots \times S$, S simple.*

Proof. — For dimension reasons, every torus T contains a simple subtorus S. Now let A denote the subtorus of T which is the sum of all subtori isogenous to S. Recall that a morphism from a simple torus to any torus is either constant or an isogeny to the image. Therefore A is stabilized by all endomorphisms of T. However, the assumption of complex multiplication implies that $K \subset \mathrm{End}_{\mathbb{Q}}(T)$ does not stabilize any proper subtorus. Hence $A = T$. □

COROLLARY 9.5.5. — *Let T be a torus with complex multiplication.*

Then either T is algebraic or $a(T) = 0$, i.e., every meromorphic function is constant.

Let $T = V/\Lambda$ be a torus with complex multiplication by a complex number field K. It is important to study the natural representations of $K \subset \mathrm{End}_{\mathbb{Q}}(T)$.

LEMMA 9.5.6. — *Let T be a torus with complex multiplication by a number field K. Let $2g = [K : \mathbb{Q}]$ and let $\tau_1, \ldots, \tau_{2g} : K \to \mathbb{C}$ denote the $2g$ distinct embeddings of K into $\mathbb{C}$.*

Then $\rho_r : K \to \mathrm{End}_{\mathbb{Q}}(\Lambda \otimes \mathbb{Q})$ is isomorphic to the representation $\oplus_{i=1}^{2g} \tau_i$.

Proof. — This is immediate, since $\Lambda \otimes \mathbb{Q}$ is a one-dimensional K-vectorspace. □

COROLLARY 9.5.7. — *Let T be a torus with complex multiplication by a number field K, $2g = [K : \mathbb{Q}]$. Then there exist g distinct, pairwise non-conjugate embeddings $\sigma_1, \ldots \sigma_g \to \mathbb{C}$ such that $\rho_a \simeq \oplus_{i=1}^{g} \sigma_i$. In particular K is totally complex, i.e., admits no real embedding.*

Proof. — Recall $\rho_r \simeq \rho_a \oplus \bar{\rho}_a$. It follows that ρ_a is isomorphic to the direct sum of a choice of g embeddings $\sigma_i : K \to \mathbb{C}$ such that every embedding of K in $\mathbb{C}$ equals one of the σ_i or $\bar{\sigma}_i$. The existence of such a choice implies that K is totally complex. □

Thus for each torus with complex multiplication by a (totally complex) number field K one obtains a *type* in the sense of the definition below.

DEFINITION 9.5.8. — Let K be a totally complex number field, $[K : \mathbb{Q}] = 2g$. A *type* σ for K is a choice of g complex embeddings which are mutually distinct and non-conjugate.

Since there are g pairs of conjugate embeddings, there are exactly 2^g types for a given totally complex number field K of degree $2g$.

DEFINITION 9.5.9. — Let K be a totally complex number field and σ a type. A torus T has complex multiplication of type (K, σ) if the complex linear representation ρ_a : $K \to \mathrm{End}(\mathbb{C}^g)$ induced by the representation on the tangent space at e is equivalent to the sum of the complex embeddings of K contained in σ.

THEOREM 9.5.10. — *Let K be a totally complex number field, $\mathcal{O}_K$ the ring of algebraic integers and σ a type.*

Then $\Lambda = \sigma(\mathcal{O}_K) = \{(\sigma_1(p), \dots, \sigma_g(p)) : p \in \mathcal{O}_K\}$ is a lattice in $\mathbb{C}^g$ such that V/Λ is a torus with complex multiplication type (K, σ).

Conversely, every torus T with complex multiplication type (K, σ) is isogenous to V/Λ.

The following consequence will be needed later on.

COROLLARY 9.5.11. — *Let T by a g-dimensional torus with complex multiplication. Then there exists a lattice $\Gamma \subset \mathbb{C}^g$ such that $T \simeq \mathbb{C}^g/\Gamma$ and $\Gamma \subset \overline{\mathbb{Q}}^g$, where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.*

Given a torus with complex multiplication, there are criteria how to determine whether the torus is simple and whether it is algebraic.

DEFINITION 9.5.12. — Let k, K be a totally complex number fields, $k \subset K$. A type σ for K is *lifted from* k, if there is type σ_k for k such that σ consists of all complex embeddings of K whose restriction to k is contained in σ_k.

PROPOSITION 9.5.13. — *Let T be a torus with complex multiplication type (K, σ). Then T is simple if and only if σ is not lifted from a proper totally complex subfield $k \subsetneq K$.*

There are totally complex number fields such that every type is lifted from a proper subfield. For instance, let p be a prime number, $K = \mathbb{Q}(\sqrt{p}, i)$. Then every type of K is lifted either from $\mathbb{Q}(i)$ or from $\mathbb{Q}(\sqrt{p}i)$. Thus, given a totally complex number field K, it is not always possible to find a *simple* torus with complex multiplication by K.

Now we recall the characterization of the *algebraic* tori with complex multiplication. For this we need the notion of a *CM*-field. (*CM* stands for "complex multiplication").

DEFINITION 9.5.14. — A totally complex number field K is called a *CM-field* if and only if it is the quadratic extension of a totally real number field.

It is easy to see that K is a *CM*-field, if it admits a totally real subfield K^+ and a totally imaginary element α such that $K = K^+(\alpha)$.

THEOREM 9.5.15 ([**78**], Th.4.4,p.19). — *Let T be a torus with complex multiplication type (K, σ). Then T is algebraic if and only if σ is lifted from a CM-field.*

Thus for the construction of non-algebraic tori with complex multiplication one needs a totally complex number field K with a type not lifted from a *CM*-subfield. We will show that such a type always exists unless K itself is a *CM*-subfield.

LEMMA 9.5.16. — *Let K be a number field which contains a CM-subfield K_0.*

Then K contains a CM-subfield K' such that every CM-subfield of K is contained in K'.

Proof. — Let K^+ denote the subfield of K which consists of all totally real elements of K. Let α be a totally imaginary element in K_0 such that $K_0 = (K^+ \cap K_0)(\alpha)$. Define $K' = K^+(\alpha)$. Let K_1 be an arbitrary *CM*-subfield of K. Then $K_1 = (K^+ \cap K_1)(\beta)$ for a totally imaginary element β. Now $\alpha\beta$ is totally real, hence contained in K^+. It follows that $K_1 \subset K^+(\beta) = K^+(\alpha) = K'$. □

COROLLARY 9.5.17. — *Let K be a totally complex number field which is not a CM-field. Then K has a type which is not lifted from a CM-subfield.*

Proof of the corollary. — If K has no *CM*-subfields, there is nothing to prove. Otherwise let K' be the maximal *CM*-subfield provided by the lemma. Then every type which lifts from a subfield of K' may be lifted through K'. Since $2h = [K' : \mathbb{Q}] < 2g = [K : \mathbb{Q}]$, it follows that there are $2^g - 2^h > 0$ types not lifted from a *CM*-subfield. □

COROLLARY 9.5.18. — *Let K be a totally complex number field which is not a CM-field.*

Then there exists a torus T with complex multiplication by K such that every meromorphic function on T is constant.

9.6. Endomorphism algebras of irreducible tori

For an abelian variety, the endomorphism algebra is always semisimple. The proof (see e.g. [**78**] or [**108**]) carries over to arbitrary semisimple tori. However, for non-semisimple tori the structure of the endomorphism algebra can be more intricate. In this section we study the structure of the endomorphism algebra of irreducible non-semisimple tori.

We will prove in particular that given a large commutative subalgebra in the endomorphism algebra of a torus the very existence of this subalgebra implies that the torus is semisimple. This is a key ingredient for our later proof that the Albanese of a nilmanifold is always semisimple unless the nilmanifold is degenerate in a certain sense.

To illustrate the phenomena occuring for non-semisimple tori, we first give an example.

EXAMPLE 9.6.1. — Let $C = \mathbb{C}/\langle 1,\tau\rangle_{\mathbb{Z}}$ be an elliptic curve with complex multiplication, *i.e.*, $\tau^2 = p\tau + q$ for some $p, q \in \mathbb{Q}$. Let $\alpha \in \mathbb{C}$, but $\alpha \notin \mathbb{Q}(\tau)$. Consider the torus T with period matrix

$$\Pi = \begin{pmatrix} 1 & \tau & 0 & \alpha \\ 0 & 0 & 1 & \tau \end{pmatrix}$$

Then T is irreducible with a unique subtorus T'. Both T' and T/T' are isomorphic to C. By explicit calculations, one can check that

$$\mathrm{End}_{\mathbb{Q}}(T) = \left\{ \begin{pmatrix} \bar{x} & t(x)\alpha + y \\ 0 & x \end{pmatrix} : x, y \in \mathbb{Q}(\tau) \right\}$$

with $t : \mathbb{Q}(\tau) \to \mathbb{Q}$ defined by $t(a\tau + b) = a$ for $a, b \in \mathbb{Q}$. There is a two-sided ideal

$$N = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} : y \in \mathbb{Q}(\tau) \right\},$$

which consists of all nilpotent elements in E. The algebra E is not commutative and not semisimple.

Now we start with the general theory for non-semisimple tori.

Let T be an irreducible torus. By definition, this means that there is a subtorus $\{e\} \subseteq T' \subsetneq T$ such that T' contains every proper subtorus of T. The quotient T/T' is simple. We represent T as a quotient of a complex vectorspace V by a lattice Λ. Then $T' = V'/\Lambda'$ with $\Lambda' = V' \cap \Lambda$ for some subvectorspace $V' \subset V$. Let $E = \mathrm{End}_{\mathbb{Q}}(T)$. Then we have the following basic facts.

LEMMA 9.6.2. — *The algebra E admits a two-sided ideal N such that the following are equivalent:*

1. $\phi \in N$,
2. $\phi(T) \subset T'$,
3. *ϕ is not an isogeny, i.e., not invertible.*
4. *ϕ is nilpotent.*

Proof. — We may define N by $N = \{\phi \in E : \phi(T) \subset T'\}$. Since T' contains every proper subtorus of T, it follows that $\phi \in N$ if and only if ϕ is not an isogeny. This implies that N is a two-sided ideal. To check nilpotency, choose $\phi \in N$ and let $A = \cap_{i\in\mathbb{N}}\phi^i(T)$. For dimension reasons, $A = \phi^n(T)$ for some $n \in \mathbb{N}$. Now $A = \phi^{2n}(T)$ implies that $\phi^n|_A$ is an isogeny. Hence $T = A + \ker(\phi^n)$. Since T is irreducible, and $A \neq T$, it follows that $T = \ker(\phi^n)$. Thus every $\phi \in N$ is nilpotent. □

LEMMA 9.6.3. — *There is an injective algebra homomorphism*

$$E/N \to \mathrm{End}_{\mathbb{Q}}(T/T')$$

and an injective homomorphism of additive groups $N \to \mathrm{Hom}(T, T')$.

Proof. — Both assertions follow immediately form the definition of N. □

LEMMA 9.6.4. — *Let $T = \mathbb{C}^g/\Lambda$ be an irreducible torus, T' the sum of all proper subtori, $\Lambda' \subset \Lambda$ the corresponding subgroup of the lattice.*

Let $E = \mathrm{End}_{\mathbb{Q}}(T)$ and $\gamma \in \Lambda$, but $\gamma \notin \Lambda'$. Then $E\gamma$ is a free E-submodule of $\Lambda \otimes \mathbb{Q}$.

Proof. — Assume that there is a $\phi \in E$ with $\phi\gamma = 0$. With $\gamma \notin \Lambda'$ this implies that $\ker\phi$ is not contained in T'. Hence $T = \ker\phi$, *i.e.*, $\phi = 0$. □

We are now approaching the main result of this section. It shows that the non-commutativity in our above example is not a coincidence.

PROPOSITION 9.6.5. — *Let $T = V/\Gamma$ be an irreducible torus and let E denote the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(T)$. Assume that E contains a commutative subalgebra F such that $\Lambda \otimes \mathbb{Q}$ is a principal F-module.*

Then T is simple, $F = E$ and F is a number field, i.e., T admits complex multiplication.

The previous example shows that commutativity of F is essential for the proposition.

The proof requires several steps. For brevity we make the following definition.

DEFINITION 9.6.6. — A torus $T = V/\Lambda$ has *weak CM* (weak complex multiplication) if the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(T)$ contains a commutative subalgebra F for which $\Lambda \otimes \mathbb{Q}$ is a principal F-module.

Thus our goal is to prove that a torus has weak complex multiplication (weak *CM*) if and only if it has complex multiplication.

LEMMA 9.6.7. — *Let S be a simple torus with weak CM.*

Then S has complex multiplication.

Proof. — Since S is simple, $E = \mathrm{End}_{\mathbb{Q}}(S)$ is a skewfield. By the assumption of weak *CM*, E admits a commutative $\mathbb{Q}$-subalgebra F such that $\Lambda \otimes \mathbb{Q}$ is a principal F-module. Thus a forteriori $\Lambda \otimes \mathbb{Q}$ is a one-dimensional E-left vectorspace. Now $E = F$ for dimension reasons, *i.e.*, E is commutative. □

LEMMA 9.6.8. — *Let T be an irreducible torus with weak CM. Then there exists a subtorus $A \subsetneq T'$ such that T/A likewise has weak CM and T'/A is the only (non-trivial) subtorus of T/A.*

Furthermore T/T' is isogenous to T'/A.

Proof. — The problem is to construct quotients of T in such a way that enough endomorphisms of T descend to endomorphisms of the quotient.

Let $\tau : T' \to T'_{ss}$ denote the semisimple reduction of T', and $C = \ker\tau$. Since T' and therefore C are canonically defined, it is clear that $\phi(C) \subset C$ for all $\phi \in \mathrm{End}_{\mathbb{Q}}(T)$.

Hence all endomorphisms of T can be pushed forward to endomorphisms of T/C. This implies that T/C has weak *CM* and we may assume that T' is semisimple.

Now let $\phi \in N$. Then $\phi(T)$ is an irreducible subtorus of T'. Since T' is semisimple, every irreducible subtorus is simple. It follows that $\phi(T)$ is a simple subtorus of T' and $\phi(T') = \{e\}$ for all $\phi \in N$.

Recall that E/N injects into $\mathrm{End}_{\mathbb{Q}}(T/T')$. Therefore the assumption that $\Lambda \otimes \mathbb{Q}$ is a principal F-module implies that $T' = \sum_{\phi \in N} \phi(T)$. It follows that every simple factor of T' is isogenous to T/T'. Furthermore this implies that for every simple subtorus $S \subset T'$ there is a $\phi \in N$ with $\phi(T) = S$. Dimension reasons imply $E = F$. Now $\alpha\phi = \phi\alpha$ for $\phi \in N$, $\alpha \in F$, clearly implies that every $\alpha \in F$ must stabilize the image of ϕ for all $\phi \in N$. It follows that every simple subtorus (hence: every subtorus) of T' is stabilized by all $\alpha \in E$. Consequently every quotient of T has weak *CM* as well. We may therefore assume that T' is simple and isogenous to T/T'. □

PROPOSITION 9.6.9. — *There is no irreducible torus T with weak CM such that the maximal non-trivial subtorus T' is simple and isogenous to T/T'.*

Proof. — We keep the above notation. Since E/N injects into $\mathrm{End}_{\mathbb{Q}}(T/T')$, it is clear that $S = T/T'$ is a simple torus with complex multiplication. Therefore there is a totally complex numberfield K and a type $\sigma : K \to \mathbb{C}^g$ such that S is isogenous to $\mathbb{C}^g/\Gamma$ with $\Gamma = \sigma(\mathcal{O}_K)$. There is an induced representation of K in $End_{\mathbb{C}}(\mathbb{C}^g)$ given by $x : (z_1, \dots, z_g) \mapsto (\sigma_1(x)z_1, \dots, \sigma_g(x)z_g)$. By abuse of language, this algebra homomorphism from K to $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^g)$ is again denoted by σ.

Replacing T be a suitable isogenous torus, we may represent T as a quotient of the vector space $V \oplus V$ with $V \simeq \mathbb{C}^g$ by a discrete subgroup Λ with $(v_1, v_2) \in \Lambda$ if and only if $v_1, v_2 - \zeta(v_1) \in \Gamma$ for some $\mathbb{Q}$-linear map $\zeta : V \to V$. (Note that the "coordinates" v_i are vectors in V and not numbers). Since only $\zeta|_\Gamma$ is relevant and $V \simeq_{\mathbb{R}} \Gamma \otimes \mathbb{R}$, we may assume that ζ is $\mathbb{R}$-linear. However, we can not assume that ζ is $\mathbb{C}$-linear. Now for every $A \in \mathrm{End}_{\mathbb{C}}(V)$ we have a $\mathbb{C}$-linear change of coordinates given by $v_1' = v_1$, $v_2' = v_2 + A(v_1)$ which transforms ζ into $A + \zeta$ with addition taking place in $\mathrm{End}_{\mathbb{R}}(V)$. We will keep this in mind.

All endomorphisms of T lift to $\mathbb{C}$-linear endomorphisms of $V \oplus V$ and therefore are representable in the algebra $M(2, \mathrm{End}_{\mathbb{C}}(V))$ of 2×2-matrices with entries in $\mathrm{End}_{\mathbb{C}}(V)$. Clearly E is a subalgebra of

$$\left\{ \begin{pmatrix} \sigma(x) & A \\ 0 & \sigma(y) \end{pmatrix} : x, y \in K; A \in \mathrm{End}_{\mathbb{C}}(V) \right\}.$$

Furthermore

$$N = \left\{ \begin{pmatrix} 0 & \sigma(z) \\ 0 & 0 \end{pmatrix} : z \in K \right\}.$$

Consider an element $\phi = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$. Commutativity of E implies that $\phi\eta = \eta\phi$ for all $\eta = \left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)$ with $Z \in \sigma(K)$. This reduces to $AZ = ZC$. Since $A, Z, C \in \sigma(K)$, which is commutative, this implies that $A = C$ for all $\phi \in E$.

The Zariski closure of $\sigma(K^*)$ in $GL_{\mathbb{C}}(V)$ is evidently the Cartan subgroup H consisting of all endomorphisms

$$(z_1, \ldots, z_g) \mapsto (\lambda_1 z_1, \ldots, \lambda_g z_g)$$

with $\lambda_i \in \mathbb{C}^*$. Now let G denote the complex linear algebraic subgroup of $GL_{\mathbb{C}}(V \oplus V)$ defined by

$$G = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} : A \in H, B \in \mathrm{End}_{\mathbb{C}}(V) \right\}.$$

Let E^* denote the multiplicative group of invertible elements of E. Then E^* is a subgroup of G. Let A denote the closure of E^* in G with respect to the (complex-algebraic) Zariski topology. A is commutative, because E^* is commutative. It follows that $A = A_s \times A_u$ with A_s reductive and A_u unipotent. Now

$$U = \left\{ \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} : B \in \mathrm{End}_{\mathbb{C}}(V) \right\}$$

is the maximal unipotent subgroup of the solvable algebraic group G. Hence $A_u \subset U$. Consider the projection $\pi : G \to G/U$. The quotient G/U is reductive and $\sigma(K^*)$ being dense in H implies that $\pi(E^*)$ (hence a forteriori $\pi(A)$) is dense in G/U. Therefore $\pi(A_s) = G/U$. This in turn implies that A_s is a Cartan subgroup of G. Now every two Cartan subgroups are conjugate. It follows that (by a suitable change of coordinates) we may assume that

$$A_s = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in H \right\}.$$

Now the centralizer of A_s in G is just

$$C_G(A_s) = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} : A \in H, B \in \bar{H} \right\}$$

where $\bar{H}$ denotes the closure of H in $\mathrm{End}_{\mathbb{C}}(V)$, *i.e.*, $\bar{H}$ consists of all endomorphisms given by

$$(z_1, \ldots, z_g) \mapsto (\lambda_1 z_1, \ldots, \lambda_g z_g)$$

with $\lambda_i \in \mathbb{C}$. Since A is commutative and contains A_s as well as

$$(1 + N) = \left\{ \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} : B \in \sigma(K) \right\},$$

it follows that $A = C_G(A_s)$.

Now let $x \in \mathcal{O}_K$, $x \neq 0$. Then $\gamma_0 = \sigma(x) \in \Gamma$. Since

$$\gamma_0 = (\sigma_1(x), \dots, \sigma_g(x))$$

and $\sigma_i(x) \neq 0$ for all i, there is a $B \in \bar{H} \subset \mathrm{End}_{\mathbb{C}}(V)$ such that $B\gamma_0 = -\zeta(\gamma_0)$ (with $\zeta \in \mathrm{End}_{\mathbb{R}}(V)$ defined as above). A change of coordinates $v_1' = v_1$, $v_2' = v_2 + Bv_1$ does not affect A, because $\left(\begin{smallmatrix} 1 & B \\ 0 & 1 \end{smallmatrix}\right) \in A$. Thus we may assume that $\zeta(\gamma_0) = 0$, *i.e.*, $(0, \gamma_0) \in \Lambda$.

The number field K is a finite extension of $\mathbb{Q}$. Hence there exists *primitive element* $\tau \in K$ such that $K = \mathbb{Q}(\tau)$. Then $\tau^{2g} = \sum_{k=0}^{2g-1} p_k \tau^k$ with $p_k \in \mathbb{Q}$. Let $\phi \in E$ be such that the natural projection $p : E \to E/N \simeq K$ maps ϕ to τ. Then E is generated (as a $\mathbb{Q}$-algebra) by N and ϕ. Furthermore $\phi \in A$. Hence $\phi = \left(\begin{smallmatrix} \theta & \xi \\ 0 & \theta \end{smallmatrix}\right)$ with $\theta = \sigma(\tau)$ and $\xi \in \bar{H}$. The $\mathbb{Q}$-vector space $\Lambda \otimes \mathbb{Q}$ is now generated by the elements $(\gamma, 0)$ with $\gamma \in \Gamma$ together with the elements $\phi^n(0, \gamma_0)$ with $n \in \mathbb{N}$. Recall that $\tau^{2g} = \sum_{k=0}^{2g-1} p_k \tau^k$. This implies that $\phi^{2g} - \sum_k p_k \phi^k \in N$. Hence

$$\left[\phi^{2g} - \sum_k p_k \phi^k\right] \begin{pmatrix} 0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} \widetilde{\gamma} \\ 0 \end{pmatrix}$$

for some $\widetilde{\gamma} \in \Gamma$. Observe that

$$\phi^m = \begin{pmatrix} \theta & \xi \\ 0 & \theta \end{pmatrix}^m = \begin{pmatrix} \theta^m & m\theta^{m-1}\xi \\ 0 & \theta^m \end{pmatrix}.$$

(Here as well as in the following calculations it is essential that θ and ξ commute.)

Thereby we obtain

$$\xi \left[2g\theta^{2g-1} - \sum_k p_k k \theta^{k-1}\right] \gamma_0 \in \Gamma \otimes \mathbb{Q}.$$

Now recall that τ is a primitive element for $[K : \mathbb{Q}]$. Therefore $\theta^0, \dots, \theta^{2g-1}$ are $\mathbb{Q}$-linearly independent, which implies that the expression $[2g\theta^{2g-1} + \cdots]$ in the formula above can not be zero. It follows that there is a number $y \in K^*$ such that $\xi\sigma(y) \in \Gamma \otimes \mathbb{Q} = \sigma(K)$. Thus $\xi\sigma(y) = \sigma(z)$ with $y \in K^*$, $z \in K$. Recall that $\xi \in \bar{H}$. Hence there are complex numbers $\lambda_1, \dots, \lambda_g$ such that

$$\xi : (z_1, \dots, z_g) \mapsto (\lambda_1 z_1, \dots, \lambda_g z_g).$$

It follows that $\lambda_i = \sigma_i(z)/\sigma_i(y) = \sigma_i(z/y)$ for all i. Thus $\xi \in \sigma(K)$. It follows that $\xi(\gamma) \in \Gamma \otimes \mathbb{Q}$ for all $\gamma \in \Gamma$. This furthermore implies that $\phi^n(0, \gamma_0) \in \Gamma \otimes \mathbb{Q}$ for all $n \in \mathbb{N}$. Finally this implies that $\Lambda \otimes \mathbb{Q} = \{(v_1, v_2) : v_i \in \Gamma \otimes \mathbb{Q}\}$. But this contradicts the assumption that T is irreducible, because it implies that T is isogenous to $V/\Gamma \times V/\Gamma$. □

9.7. Primitive elements

Let $T = \mathbb{C}^g/\Lambda$ be a torus. To each element λ in the lattice Λ we may associate a subtorus $\Theta(\lambda)$ in the following way. We consider the complex line $\langle\lambda\rangle_{\mathbb{C}}$ in $\mathbb{C}^g$ and let $\Theta(\lambda)$ be the smallest subtorus of T containing $\pi\left(\langle\lambda\rangle_{\mathbb{C}}\right)$, where $\pi : \mathbb{C}^g \to T$ denotes the natural projection. Equivalently, $\Theta(\lambda)$ may be defined as the smallest subtorus S of T with $\lambda \in Image\,(i_* : \pi_1(S) \to \pi_1(T))$, identifying Λ with $\pi_1(T)$.

DEFINITION 9.7.1. — Let $T = \mathbb{C}^g/\Lambda$ be a torus. An element $\lambda \in \Lambda$ is called *primitive* if the associated subtorus $\Theta(\lambda)$ (defined as above) is irreducible.

The set of all primitive elements in Λ is denoted by Λ^{prim}.

LEMMA 9.7.2. — *Let $f : T \to S$ denote a morphism of tori, $f_* : \Lambda \to \Gamma$ the associated group homomorphism between the lattices of T and S.*

Let $\lambda \in \Lambda$. Then $f\left(\Theta(\lambda)\right) = \Theta(f_*\lambda)$.

Furthermore $f_*(\Lambda^{\mathrm{prim}}) \subset \Gamma^{\mathrm{prim}}$.

Proof. — The inclusion $f\left(\Theta(\lambda)\right) \supset \Theta(f_*\lambda)$ is obvious as well as $\Theta(\lambda) \subset f^{-1}\left(\Theta(f_*\lambda)\right)$. This yields the first assertion. The second is an immediate consequence, since a quotient of an irreducible torus is again irreducible. □

LEMMA 9.7.3. — *Let $T = \mathbb{C}^g/\Lambda$ be an irreducible torus, T_0 a maximal proper subtorus, Λ_0 the corresponding sublattice.*

Then $\Theta(\lambda) = T$ for all $\lambda \in \Lambda \setminus \Lambda_0$.

Proof. — Obvious. □

COROLLARY 9.7.4. — *Let $T = \mathbb{C}^g/\Lambda$ be a torus.*

Then $\Lambda \otimes \mathbb{Q}$ is generated (as $\mathbb{Q}$-vectorspace) by Λ^{prim}.

Proof. — Thanks to the lemma, this is clearly true for irreducible tori. This implies the statement for arbitrary tori, because every torus equals the sum of it irreducible subtori. □

The $\mathbb{Z}$-module Λ is not necessarily generated by Λ^{prim}, e.g. let S be a simple torus, $\tau_0 \in \mathrm{End}(S)$ an involution, $\tau = (\tau_0, \tau_0)$ and $T = S \times S/\tau$.

LEMMA 9.7.5. — *Let $T = \mathbb{C}^g/\Lambda$ be an irreducible torus, $\lambda \in \Lambda^{\mathrm{prim}}$ with $\Theta(\lambda) = T$ and $E = \mathrm{End}_{\mathbb{Q}}(T)$.*

Then $E\lambda$ is a free E-submodule of the E-module $\Lambda \otimes \mathbb{Q}$.

Proof. — We have to show that $\phi\lambda = 0$ implies $\phi = 0$ for all $\phi \in E$. But this is evident, since $\phi\lambda = 0$ implies $\Theta\lambda \subset \ker\phi$. □

9.8. Bilinear forms on modules

Here we deduce an auxiliary algebraic result on bilinear forms on modules over possibly non-commutative rings. We will need this later on in order to deduce that all the tori in the tower of a non-degenerate nilmanifold admit weak complex multiplication.

PROPOSITION 9.8.1. — *Let F be a finite-dimensional associative algebra over a field, M a F-left module, $\lambda, \rho : M \to F$, $B : M \times M \to M$ maps such that*

$$B(x,y) = \lambda(x)y = \rho(y)x$$

for all $x, y \in M$. Furthermore assume that there are elements $m, n, p \in M$ such that Fp is a free F-module and $B(m,n) = p$.

Then F is commutative and M is a principal free F-module generated by each of the elements m, n and p, i.e., $M = Fm = Fn = Fp$.

Proof. — We start by evaluating

$$B(m,n) = \lambda(m)n = \rho(n)m = p.$$

Since Fp is free, this implies that right multiplication by $\lambda(m)$, $\rho(n)$ in F is injective. Now F is a finite-dimensional vector space over a field, thus injective linear maps are bijective. It follows that right multiplication by $\lambda(m)$ resp. $\rho(n)$ is a bijective self-map of F. Hence $\lambda(m)$, $\rho(n)$ admit left inverses. It follows that $Fm = Fn = Fp$. In particular Fm is a free F-module. Therefore $B(m,m) = \lambda(m)m = \rho(m)m$ implies $\lambda(m) = \rho(m)$. For the sake of brevity, let $\alpha = \lambda(m) = \rho(m)$. For arbitrary $x \in M$, consider

$$B(m,x) = \lambda(m)x = \alpha x = \rho(x)m.$$

Since α has an left-inverse, $x \in Fm$. Thus $M = Fm$, *i.e.*, M is a principal F-module. Now let $x = am$ with $a \in F$. Then $B(am,m) = \lambda(am)m = \alpha am$, hence $\lambda(am) = \alpha a$. Similarily one obtains $\rho(am) = \alpha a$. Therefore

$$B(am, bm) = \alpha abm = \alpha bam$$

for all $a, b \in F$. Since α has a left inverse and Fm is free, it follows that $ab = ba$ for all $a, b \in F$. □

9.9. Bilinear forms for tori

DEFINITION 9.9.1. — Let T_i $(i = 1,2,3)$ be tori with lattices Λ_i and universal coverings $\widetilde{T}_i$. A *bilinear form for the* T_i is a $\mathbb{C}$-bilinear map $B : \widetilde{T}_1 \times \widetilde{T}_2 \to \widetilde{T}_3$ such that $B(\Lambda_1 \times \Lambda_2) \subset \Lambda_3$.

Caveat: Such a bilinear form does *not* yield a map from $T_1 \times T_2 \to T_3$. However for $\lambda \in \Lambda_1$ one obtains a morphism of tori $B'(\lambda) : T_2 \to T_3$ induced by $B(\lambda, \cdot) : \widetilde{T}_2 \to \widetilde{T}_3$. The similar statements hold for $\lambda \in \Lambda_2$.

Using our results on primitive elements, it follows that $B(\Lambda_1^{\text{prim}} \times \Lambda_2^{\text{prim}}) \subset \Lambda_3^{\text{prim}}$.

Now, given such a bilinear form we have the left kernel

$$\widetilde{K}_1 = \{v \in \widetilde{T}_1 : B(v, w) = 0 \ \ \forall w \in \widetilde{T}_2\},$$

the right kernel $\widetilde{K}_2 \subset \widetilde{T}_2$ and the image $Image(B) \subset \widetilde{T}_3$. Observe that $v \in \widetilde{K}_1$ if and only if $B(v, \lambda) = 0$ for every $\lambda \in \Lambda_2$. This implies that $\widetilde{K}_1/\Lambda_1$ equals the intersection of the kernels of the morphisms of tori $T_1 \to T_3$ induced by the $B(\cdot, \lambda)$ with $\lambda \in \Lambda_2$. Therefore $K_1 = \widetilde{K}_1/(\widetilde{K}_1 \cap \Lambda_1)$ is a subtorus of T_1. Similarily for $K_2 = \widetilde{K}_2(\widetilde{K}_2 \cap \Lambda_2)$. Furthermore, if $\widetilde{S}$ denotes the $\mathbb{C}$-vector space spanned by the image of B, similar arguments yields that $S = \widetilde{S}/(\widetilde{S} \cap \Lambda_3)$ is a subtorus of T_3. We call these subtori K_i, S the subtori associated to the kernel resp. image of B.

We are interested in these bilinear forms, because they arise in a canonical way from nilmanifolds.

LEMMA 9.9.2. — *Let $X = G/\Gamma$ be a nilmanifold, G^k the descending and C_k the ascending central series.*

Then the commutator map $\zeta : (g, h) \mapsto ghg^{-1}h^{-1}$ induces bilinear forms for each of the triples $G/G'\Gamma$, $G^k/(G^{k+1}\Gamma \cap G^k)$, $G^{k+1}/(G^{k+2}\Gamma \cap G^{k+1})$ and G/G', $C_{k+2}/(C_{k+1}\Gamma \cap C_{k+2})$, $C_{k+1}/(C_k\Gamma \cap C_{k+1})$ for all $k \geq 0$. Let B^k resp. B_k denote these bilinear forms.

Then for B_k the right kernel is trivial, while for B^k the image generates everything.

Proof. — This is an immediate consequence of the results in the section on nilmanifolds. □

Furthermore such bilinear forms are related to the bilinear forms on modules discussed in the preceding section.

LEMMA 9.9.3. — *Let $T = \mathbb{C}^g/\Lambda$ be an irreducible torus, $\gamma_i \in \Lambda^{\text{prim}}$ with $T = \Theta(\gamma_i)$, B a bilinear form for T_1, T_2, T_3 with $T \simeq T_i$ for $i = 1, 2, 3$ such that $B(\gamma_1, \gamma_2) = \gamma_3$. Furthermore let $E = \operatorname{End}_{\mathbb{Q}}(T)$ and $M = \Lambda \otimes \mathbb{Q}$.*

Then B induces a map $B_0 : M \times M \to M$, such that there exist maps $\lambda : M \to E$, $\rho : M \to E$ with

$$B(x, y) = \lambda(x)y = \rho(y)x.$$

Moreover $E\lambda_3$ is a free E-submodule of M.

Proof. — Evidently B induces a map $\Lambda \otimes \mathbb{Q} \times \Lambda \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q}$ simply by restriction.

Since B induces a morphism of tori $B(\gamma, \cdot) : T \to T$, it is clear that for each $\gamma_1 \in \Lambda$ there exists an element $\lambda(\gamma) \in E$ such that $B(\gamma, y) = \lambda(\gamma)y$.

Finally, $E\gamma_3$ is a free E-module due to lemma 9.7.5. □

Now we will use these bilinear forms to deduce some properties of the tori which are involved.

LEMMA 9.9.4. — *Let B be a bilinear form for tori T_1, T_2, T_3 with lattices Λ_i, $\lambda_i \in \Lambda_i^{\text{prim}}$ with $B(\lambda_1, \lambda_2) = \lambda_3$ and let $\Theta_i = \Theta(\lambda_i)$ denote the irreducible subtori of T_i associated to λ_i.*

Then

$$B(\widetilde{\Theta}_1 \times \widetilde{\Theta}_2) \subset \widetilde{\Theta}_3$$

Moreover, Θ_3 is a simple torus with complex multiplication and there exist subtori $K_i \subset \Theta_i$ $(i = 1, 2)$ such that $\Theta_3 \simeq \Theta_i/K_i$.

Proof. — Let

$$C_1 := \ker[B(\cdot, \lambda_2) : \Theta_1 \to \Theta_3]$$

and

$$C_2 := \ker[B(\lambda_1, \cdot) : \Theta_2 \to \Theta_3].$$

Then $\Theta_i/C_i \simeq \Theta_3$ $(i = 1, 2)$. Now consider $\lambda \in \Lambda_1$ with $B(\lambda, \lambda_2) = 0$. Since Θ_3 is irreducible and equals the subtorus $\Theta(\lambda_2)$ associated to λ_2 (as defined in the section on primitive elements), it follows that $B(\lambda, v) = 0$ for all $v \in \widetilde{\Theta}_2$ as soon as $B(\lambda, \lambda_2) = 0$. Thus B induces a bilinear form for the triplet of tori Θ_1/C_1, Θ_2/C_2, Θ_3. Furthermore these tori are pairwise isomorphic. Now lemma 9.9.3, prop. 9.6.9 and prop. 9.5.13 imply that Θ_3 is a simple torus with complex multiplication. □

COROLLARY 9.9.5. — *Let B be a bilinear form for tori T_1, T_2, T_3, and let S denote the subtorus of T_3 associated to the image and $K_i \subset T_i$ $(i = 1, 2)$ the subtori associated to the kernel.*

Then S and both T_i/K_i are isogenous to products of simple tori with complex multiplication.

Proof. — Recall that for any torus $T = \mathbb{C}^g/\Lambda$, the set of primitive elements Λ^{prim} generates $\Lambda \otimes \mathbb{Q}$ as $\mathbb{Q}$-vector space. Therefore S is generated by its simple subtori with complex multiplication. It follows that S is isogenous to a direct product of simple tori with complex multiplication. Now K_i (for $i = 1, 2$) is the intersection of kernels of morphisms of tori to S. Thereby T_i/K_i is isomorphic to a subtorus of a S^{N_i} for $N_i \in \mathbb{N}$ sufficiently large. This implies that T_i/K_i is likewise isogenous to a product of simple tori with complex multiplication. □

This completes the proof of thm. 9.1.1.

9.10. A nilmanifold without non-constant meromorphic functions

Here we construct a non-trivial complex nilmanifold of algebraic dimension zero.

PROPOSITION 9.10.1. — *Let G_0 be the usual complex Heisenberg group, i.e.,*

$$\left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{C} \right\}.$$

Let $G = G_0 \times G_0$, $n \in \mathbb{Z} \setminus \{0\}$, $\xi \in \mathbb{C}$ such that $\xi^2 = n + i$ and Γ the set of all elements of the form

$$\left(\begin{pmatrix} 1 & a_1 + b_1\xi & a_3 + b_3\xi \\ & 1 & a_2 + b_2\xi \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & \bar{a}_1 - \bar{b}_1\bar{\xi} & \bar{a}_3 - \bar{b}_3\bar{\xi} \\ & 1 & \bar{a}_2 - \bar{b}_2\bar{\xi} \\ & & 1 \end{pmatrix} \right)$$

with $a_j, b_j \in \mathbb{Z} + i\mathbb{Z}$. Then Γ is a discrete subgroup of G and the quotient G/Γ is a compact nilmanifold without non-constant meromorphic functions.

We start with the construction of a certain number field. Fix $n \in \mathbb{Z} \setminus \{0\}$. Choose $\xi \in \mathbb{C}$ with $\xi^2 = n + i$. Let $K = \mathbb{Q}(i)(\xi)$. A general element of K is given by $a + b\xi$ with $a, b \in \mathbb{Q}(i)$. The four embeddings of K into $\mathbb{C}$ map $a + b\xi$ to $a + b\xi$, $a - b\xi$, $\bar{a} + \bar{b}\bar{\xi}$ and $\bar{a} - \bar{b}\bar{\xi}$ respectively. Thus an element $a + b\xi$ is totally real, if both a and $b\xi$ are real. However $b\xi \in \mathbb{R}$ implies $(b\xi)^2 \in \mathbb{R} \cap \mathbb{Q}(i) = \mathbb{Q}$, since $b \in \mathbb{Q}(i)$ and $\xi^2 = n + i$. Now $(b\xi)^2 = p \in \mathbb{Q}$ implies $N_{\mathbb{Q}(i)|\mathbb{Q}}(b^2) \cdot (n^2 + 1) = p^2$, which is impossible for $b \neq 0$, since $n^2 + 1$ is not a square in $\mathbb{Q}$. Thus K contains no totally real elements except the rational ones. In particular, K is not a *CM*-field. Furthermore, using lemma 9.5.16 it follows that $\mathbb{Q}(i)$ is the only *CM*-subfield of K.

LEMMA 9.10.2. — *Let $\xi \in \mathbb{C}$ with $\xi^2 = n + i$, $n \in \mathbb{Z} \setminus \{0\}$. Let $K = \mathbb{Q}(i)(\xi)$ with type σ given by $\sigma_1(z) = a + b\xi$ and $\sigma_2(z) = \bar{a} - \bar{b}\bar{\xi}$ for $z = a + b\xi$ $(a, b \in \mathbb{Q}(i))$. Let $R = \{a + b\xi : a, b \in \mathbb{Z} + i\mathbb{Z}\}$*

Then $T = \mathbb{C}^2/\sigma(R)$ is a simple two-dimensional complex torus with complex multiplication by K such that every meromorphic function on T is constant.

Proof. — Clearly $R = \{a + b\xi : a, b \in \mathbb{Z} + i\mathbb{Z}\} \subset \mathcal{O}_K$. Thus R is a subring of $\mathcal{O}_K$ with $\mathcal{O}_K/R$ finite (because $\mathrm{rank}_{\mathbb{Z}}(R) = 4 = [K : \mathbb{Q}]$). In particular $\mathbb{C}^2/\sigma(\mathcal{O}_K)$ is isogenous to $\mathbb{C}^2/\sigma(R)$ for every type σ. As we have seen above, K contains no *CM*-subfields except $\mathbb{Q}(i)$ and the type σ is clearly not lifted from $\mathbb{Q}(i)$. Therefore T is not algebraic. This implies the statement, since a torus with complex multiplication is either algebraic or does not admit any non-constant meromorphic function at all. □

Proof of the proposition. — First note that Γ is indeed a discrete subgroup. This is verified easily using the fact that $R = \{a + b\xi : a, b \in \mathbb{Z} + i\mathbb{Z}\}$ is a ring and

$\sigma_i|_R : R \to \mathbb{C}$ are ring-homomorphisms. Recall that for any nilmanifold X every meromorphic function is a lift from $\mathrm{Alb}(X)$. In our case this implies that G/Γ has no non-constant meromorphic function. □

9.11. An arithmetic method of construction

Let K be a totally complex numberfield, $[K : \mathbb{Q}] = 2g$. Let $\sigma = (\sigma_1, \ldots, \sigma_g)$ a type for K, *i.e.*, a choice of g mutually distinct and non-conjugate embeddings $K \to \mathbb{C}$.

Any field homomorphism $\phi : K \to K'$ yields a covariant functor from K-varieties to K'-varieties. For an affine variety $V \subset \mathbb{A}^n$ defined by polynomials $P_i \in K[X_1, \ldots, X_n]$ this functor simply takes V to the variety ${}^{\phi}V$ defined by the polynomials $\phi_* P_i \in K'[X_1, \ldots, X_n]$.

Thus a type $\sigma = (\sigma_1, \ldots, \sigma_g)$ on a totally complex number field K yields a covariant functor

$$^{\sigma} : V \mapsto {}^{\sigma}V = {}^{\sigma}_{1}V \times \cdots \times {}^{\sigma}_{g}V.$$

This functor takes n-dimensional K-varieties to gn-dimensional $\mathbb{C}$-varieties. It takes K-groups to algebraic groups defined over $\mathbb{C}$.

For an affine space $\mathbb{A}^n$, the map σ maps $\mathbb{A}^n(\mathcal{O}_K)$ onto a lattice in $\mathbb{A}^{ng}(\mathbb{C}) = \mathbb{C}^{ng}$.

We will apply these methods to construct nilmanifolds.

DEFINITION 9.11.1. — A nilpotent algebraic group defined over a field k is called a *nilpotent flag group*, if it is k-isomorphic to the affine space $\mathbb{A}^n$ endowed with a group structure defined by a k-morphism $m : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$ with $m = (m_1, \ldots, m_n)$ such that

$$m_i(x_1, \ldots, x_n, y_1 \ldots, y_n) = x_i + y_i + P_i(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1})$$

for some polynomials P_i.

Note. — If k is algebraically closed, then every unipotent k-group is a nilpotent flag group. For arbitrary fields this is not true.

LEMMA 9.11.2. — *Let* $\mathbf{g}$ *be a nilpotent Lie algebra over a field* k *of characteristic zero. Then there exists a nilpotent flag* k*-group* G *with* $\mathbf{g} = \mathcal{L}ie(G)$.

Proof. — By the Ado-Iwasawa theorem, there is an embedding $\rho : \mathbf{g} \to \mathcal{L}ie\, GL_n(k)$ such that all the $\rho(v)$ $(v \in \mathbf{g})$ are nilpotent. Thus the exponential power series $\exp(A) = \sum_k 1/k!A^k$ restricts to a polynomial isomorphism of varieties from $\rho(\mathbf{g})$ to a unipotent subgroup U of $GL_n(k)$. Now suitable linear coordinates on $\mathbf{g}$ induce the structure of a nilpotent flag group on U. □

LEMMA 9.11.3. — *Let* G *be a nilpotent flag group defined over a field* k *of characteristic zero. Let* $\mathcal{O}$ *be an additive subgroup of* k*, which generates* k *as a* $\mathbb{Q}$*-vector space.*

Then G is k-isomorphic to $\mathbb{A}^n$ (as k-variety) with the group structure defined by a polynomial map $m : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$ such that all coefficients of m are in $\mathcal{O}$.

Proof. — Note that for each $x \in k$ there is a natural number $q \in \mathbb{N}$ such that $qx \in \mathcal{O}$. Using this, one can prove easily by induction on $\dim(G)$ that by an application of a linear change of coordinates of the type

$$(z_1, \ldots, z_n) \mapsto (p_1 z_1, \ldots, p_n z_n) \quad (p_i \in \mathbb{N})$$

one can achieve that all the coefficients of the polynomial map defining group multiplication are in $\mathcal{O}$. □

These auxiliary results enable us to construct interesting nilmanifolds.

THEOREM 9.11.4. — *Let K be a totally complex numberfield, $\sigma : K \to \mathbb{C}^g$ a type and $\mathbf{g}$ a nilpotent K-Lie algebra.*

Then there exists a torus T with complex multiplication type (K, σ), a connected complex nilpotent Lie group G and a discrete cocompact subgroup $\Gamma \subset G$ such that

1. *$\mathcal{L}ie(G) \simeq \mathbf{g} \otimes_{\sigma_1} \mathbb{C} \times \cdots \mathbf{g} \otimes_{\sigma_g} \mathbb{C}$.*
2. *Each of the tori $G^k/(G^{k+1} \cap \Gamma)$ is isogenous to a direct product of copies of T.*

Proof. — Let G_K be a nilpotent flag group defined over K such that $\mathcal{L}ie\, G_K = \mathbf{g}$. Now G_K is isomorphic to the affine space $\mathbb{A}^n$ as a K-variety and due to the preceding lemma we may assume that the multiplication map is given by a polynomial map such that all the coefficients are contained in the ring of algebraic integers $\mathcal{O}_K$. This implies that $\mathbb{A}^n(\mathcal{O}_K) = \{(z_1, \ldots, z_n) : z_i \in \mathcal{O}_K\}$ is a subgroup of G_K. Now the type σ yields a complex unipotent group G given by $G = {}^{\sigma}G_K$ with a group homomorphism $\zeta : G_K \to G$ given by $x \mapsto (\sigma_1(x), \ldots, \sigma_g(x))$. Finally recall that $\sigma : K \to \mathbb{C}^g$ embedds $\mathcal{O}_K$ as a discrete cocompact subgroup in $(\mathbb{C}^g, +)$. Using induction, this implies that $\sigma\,(G_K(\mathcal{O}_K))$ is a discrete cocompact subgroup of G. Moreover $T = \mathbb{C}^g/\sigma(\mathcal{O}_K)$ is torus with complex multiplication type (K, σ) which yields the second assertion, again by induction. □

9.12. Examples of non-compact nilmanifolds

A variation of this procedure described above may be used to construct interesting non-compact nilmanifolds composed out of Cousin groups. (A Cousin group is a commutative complex Lie group without non-constant holomorphic functions.) The key point is to consider number fields which are not totally complex.

PROPOSITION 9.12.1. — *Let K be a number field with r real embeddings $K \to \mathbb{R}$ and s pairs of conjugate complex embeddings. Let $\sigma = (\sigma_1, \ldots, \sigma_{r+s})$ a choice of $r+s$ mutually distinct and non-conjugate embeddings such that σ_i is real for $i \leq r$. Let $\mathbf{g}$ be a nilpotent K-Lie algebra.*

Then there exists a Cousin group $H \simeq \mathbb{C}^{r+s}/\sigma(\mathcal{O}_K)$, a connected complex nilpotent Lie group G and a discrete subgroup $\Gamma \subset G$ such that

1. $\mathcal{L}ie(G) \simeq \mathbf{g} \otimes_{\sigma_1} \mathbb{C} \times \cdots \mathbf{g} \otimes_{\sigma_g} \mathbb{C}$.
2. $G^k\Gamma$ *is closed in* G *for all* k *and* $G^k/(G^{k+1} \cap \Gamma)$ *is isogenous to a direct product of copies of* H.
3. *If* $r > 0$, *then every compact analytic subset of* $X = G/\Gamma$ *is finite. If* $s > 0$, *then every holomorphic function on* X *is constant.*

Once one has derived the technical lemma below, the proof is an easy generalization of the proof of thm. 9.1.4. Therefore we only prove the subsequent lemma.

LEMMA 9.12.2. — *Let K be a number field with r real embeddings and s pairs of conjugate complex embeddings. Let $g = r + s$. Let $\mathcal{O}_K$ denote the ring of algebraic integers.*

Let $\sigma_1, \ldots, \sigma_g$ be a choice of distinct and pairwise non-conjugate embeddings $K \to \mathbb{C}$ such that $\sigma_1, \ldots, \sigma_r$ are real.

Then $\sigma(\mathcal{O}_K) = \{(\sigma_1(x), \ldots, \sigma_g(x)) : x \in \mathcal{O}_K\}$ is a discrete subgroup of $\mathbb{C}^g$.

If $r > 0$, then $G = \mathbb{C}^g/\sigma(\mathcal{O}_K)$ has no non-finite compact analytic subset. If $s > 0$, then G admits no non-constant holomorphic functions.

Proof. — Every embedding of a number field K into $\mathbb{C}$ induces an absolute value on K by pulling-back the ordinary absolute values on $\mathbb{C}$. Moreover, two embeddings $\sigma, \tau : K \to \mathbb{C}$ induce independent absolute values unless $\sigma = \tau$ or $\sigma = \bar{\tau}$. Thus the Artin-Whaples Approximation theorem implies that for all $x = (x_1, \ldots, x_g) \in K^g$ and $\epsilon > 0$ there exists an element $z \in K$ such that $||\zeta(x) - \sigma(z)|| < \epsilon$ where $\zeta : K^g \to \mathbb{C}^g$ denotes the map given by $\zeta(x) = (\sigma_1(x_1), \ldots, \sigma_g(x_g))$. Now the image of $\zeta(K^g)$ is easily seen to be dense in $\mathbb{R}^r \oplus \mathbb{C}^s$. Thus $\sigma(K)$ is dense in $V = \mathbb{R}^r \oplus \mathbb{C}^s$. Since $K \simeq \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Z}$-module and $\dim_{\mathbb{R}}(V) = [K : \mathbb{Q}]$, it follows that $\sigma(\mathcal{O}_K)$ is a lattice in V. Now K acts by complex-linear endomorphisms on $\mathbb{C}^g$ stabilizing the discrete subgroup $\sigma(\mathcal{O}_k)$. Since $K \simeq \mathcal{O}_K \otimes \mathbb{Q}$, it follows that no subgroup of $\Lambda = \sigma(\mathcal{O}_K)$ can be stable under all endomorphisms of $G = \mathbb{C}^g/\Lambda$. It follows that for $r > 0$ the group G cannot contain a compact complex Lie subgroup. Since any connected compact analytic subset (containing 0) would generate a compact Lie subgroup, this implies that every compact analytic subset is finite (for $r > 0$).

We still have to show that there are no non-constant holomorphic functions on G for $s > 0$. Recall that for any complex Lie group G there exists a *holomorphic reduction*, *i.e.*, a holomorphic map $\pi : G \to H$ onto a Stein Lie group H such that all holomorphic functions on G are lifted from H **[96]**. Observe furthermore that a complex Stein manifold of *complex* dimension n is homotopy-equivalent to a *CW*-complex of *real* dimension n. Therefore a commutative complex Lie group $\mathbb{C}^g/\Gamma$ can not be Stein unless $\mathrm{rank}_{\mathbb{Z}}(\Gamma) \leq \dim_{\mathbb{C}}(\mathbb{C}^g) = g$. It follows that for $s > 0$ the group G is not Stein. Furthermore it is clear that, for any complex-linear projection $\pi : \mathbb{C}^g \to \mathbb{C}^d$

with $\mathbb{C}^d/\pi(\Lambda)$ Stein, there must be a non-trivial intersection $\ker \pi \cap \Lambda$. Since Λ has no non-trivial subgroups stable under all endomorphisms of G, this implies that there are no non-constant holomorphic functions on G. $\square$

9.13. Solvmanifolds

Our results obtained so far are specific to nilmanifolds. They do not apply to solvmanifolds.

EXAMPLE 9.13.1. — Let E be an elliptic curve (*i.e.*, a one-dimensional torus) without complex multiplication, $A \in SL_2(\mathbb{Z})$ with eigenvalues α, α^{-1} such that $|\alpha| < 1$. Let $X' = \mathbb{C}^* \times E \times E$ and $X = X'/\sim$ with $(x, y, z) \sim (\alpha x, A\binom{y}{z})$.

Then X is a three-dimensional compact complex solvmanifold, which is a $E \times E$-bundle over $\mathrm{Alb}(X) \simeq \mathbb{C}/\langle 2\pi i, \tau\rangle_{\mathbb{Z}}$ with $e^{2\pi i\tau} = \alpha$.

Neither E nor $\mathrm{Alb}(X)$ have complex multiplication. To check that $\mathrm{Alb}(X)$ has no complex multiplication, note that $\mathrm{Alb}(X) = \mathbb{C}/\langle 1, \tau\rangle$ with $e^{2\pi i\tau} = \alpha$. The theorem of Lindemann-Weierstraß implies that τ is not algebraic. But $\mathbb{C}/\langle 1, \tau\rangle$ has complex multiplication if and only if $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$.

We will now generalize this special result. First we need an auxiliary result in transcendental number theory. For this we recall Baker's theorem which is a generalization of the Lindemann-Weierstraß theorem ([**83**]).

THEOREM 9.13.2 (Baker [**8**]). — *Let $\alpha_1, \dots, \alpha_n$ be non-zero algebraic numbers. Fix a branch of the logarithm and assume that $1, \log\alpha_1, \dots, \log\alpha_n$ are linearly dependent over $\overline{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$).*

Then $\log\alpha_1, \dots, \log\alpha_n$ are already linearly dependent over $\mathbb{Q}$.

This result on linear forms of logarithms constitutes the base for the following auxiliary result in transcental number theory.

PROPOSITION 9.13.3. — *Let $V = (\mathbb{C}^n, +)$, $S = (\mathbb{C}^*)^d$. Let Λ be a discrete subgroup of V such that every holomorphic function on V/Λ is constant (e.g. a lattice). Let $\phi : V \to S$ be a complex Lie group homomorphism. Assume $\Lambda \subset \overline{\mathbb{Q}}^n$ and $\phi(\Lambda) \subset \left(\overline{\mathbb{Q}}^*\right)^d$.*

Then ϕ is constant.

Proof. — First note that there is no loss in generality in assuming $d = 1$.

The statement is trivially true for $d = 0$. Thus we may argue by induction. Now let Σ denote the kernel of $\phi|_\Lambda : \Lambda \to S$. Furthermore let $W = \langle\Sigma\rangle_{\mathbb{C}}$ denote the complex subvectorspace of V spanned by Σ. We claim: $\dim_{\mathbb{C}}(W) = \mathrm{rank}_{\mathbb{Z}}\Sigma$. To prove the claim, let $U = \langle\Sigma\rangle_{\mathbb{R}}$ and $K = U \cap iU$. We have to show that $K = \{0\}$. Now K is a complex subvector space of V which is defined over $\overline{\mathbb{Q}}$. It follows that $V' = V/K \simeq \mathbb{C}^m$ may be endowed with complex coordinates such that $\pi(\Lambda) \subset V'(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}^m$. Moreover there is a subgroup $\Lambda' \subset \pi(\Lambda)$ which is a lattice in V'. (A lattice has to be cocompact and *discrete* and $\pi(\Lambda)$ itself is possibly not discrete). Thus we may invoke induction, if we can show that $\phi : V \to S$ fibers through V/K. This is equivalent to $\phi|K \equiv 1$. Now $\phi|\Sigma \equiv 1$ and U/Σ is compact, hence $\phi(K) \subset \phi(U)$ is a compact subset of $S \simeq \mathbb{C}^*$. By the theorem of Liouville it follows that $\phi_K \equiv 1$. Thus invoking induction we may assume that $K = \{0\}$ which implies $\dim_{\mathbb{C}}(W) = \mathrm{rank}_{\mathbb{Z}}\Sigma$ as claimed.

There is a vector $a \in \mathbb{C}^n$ such that $\phi(z) = e^{a\cdot z}$ with $a \cdot z = \sum_i a_i z_i$. We may fix a basis $\lambda_1, \ldots, \lambda_n$ of $V \simeq \mathbb{C}^n$ in such a way that $\lambda_j \in \Lambda$ for all j and $\lambda_j \in \Sigma$ for $j \le k = \mathrm{rank}_{\mathbb{Z}}\Sigma$. Then $a_i \in i\pi\mathbb{Z}$ for $i \le k$. Moreover $i\pi, a_{k+1}, \ldots, a_n$ are $\mathbb{Q}$-linearly independent because $\ker\phi \cap \Lambda$ is contained in the $\mathbb{Q}$-vector space spanned by $a_1, \ldots, a_k$. Choose an element $\lambda = \sum_j \beta_j a_j$ of Λ. If $k < n$, we require $\beta_n \notin \mathbb{Q}$. (This is possible, because otherwise

$$\sum_j \beta_j \lambda_j \mapsto e^{N\beta_n}$$

will be a non-constant Λ-invariant holomorphic function on V for $N >> 0$.) Now $\phi(\lambda) = e^{\sum_j \beta_j a_j}$ and all the numbers $\phi(\lambda_j) = e^{a_j}$ are algebraic. Furthermore $\Lambda \subset \overline{\mathbb{Q}}^n$ implies that $\beta_j \in \overline{\mathbb{Q}}$, *i.e.*, $\log\phi(\lambda)$ is $\overline{\mathbb{Q}}$-linearly dependent on $i\pi, a_1, \ldots, a_n$. Thus Baker's theorem implies that $\sum_j \beta_j a_j$ is $\mathbb{Q}$-linearly dependent on $i\pi, a_1, \ldots, a_n$. If $k = \mathrm{rank}\Sigma < n$, then a_n is $\mathbb{Q}$-linearly dependent on $i\pi, a_1, \ldots, a_n$ and we arrive at a contradiction, since $\beta_n \notin \mathbb{Q}$. Thus we may assume $k = n$. In this case $\sum_j \beta_j a_j = \log(\phi(\lambda))$ is a $\mathbb{Q}$-multiple of $i\pi$. But this implies that $\phi(\lambda)$ is a root of unity. Recall that for $k = n$ the element $\lambda \in \Lambda$ was choosen arbitrarily. Hence $\phi(\lambda)$ is a root of unity for all $\lambda \in \Lambda$. Since Λ is a finitely generated group, it follows that $\phi(\Lambda) = E$ is a finite subgroup of $S \simeq \mathbb{C}^*$. Thus we obtain an induced Lie group homomorphism $\phi_0 : V/\Lambda \to (\mathbb{C}^*/E) \simeq \mathbb{C}^*$. Since V/Λ does not admit any non-constant holomorphic function, it follows that ϕ_0 and therefore ϕ are constant. □

LEMMA 9.13.4. — *Let G be a solvable complex Lie group, N its nilradical, i.e., the maximal connected nilpotent normal Lie subgroup.*

Then there exists a complex Lie group homomorphism $\rho \to (\mathbb{C}^*)^d$ ($d = \dim(G)$), such that

1. *The connected component of the kernel of ρ equals N.*
2. *For each $g \in G$, $\rho_1(g), \ldots, \rho_d(g) \in \mathbb{C}^*$ are the eigen-values for $\mathrm{Ad}(g) \in \mathcal{L}ie(G)$.*

Proof. — Let $\mathrm{Ad} : G \to GL(\mathcal{L}ie\, G)$ denote the adjoint representation. Since this is a complex linear representation of a connected solvable Lie group, there is a Borel group B of $GL(\mathcal{L}ie\, G)$ with $\mathrm{Ad}(G) \subset B$. Now $B' = [B, B]$ is nilpotent and conjugate to the group of all unipotent upper triangular matrices. Moreover B' is the set of all unipotent elements in B. In particular $\mathrm{Ad}(N) \subset B'$.

Conversely B' is nilpotent and $\mathrm{Ad}^{-1}(B')$ is a central extension of B', hence likewise nilpotent. It follows that N equals the connected component of $\mathrm{Ad}^{-1}(B')$. Finally, for each $b \in B \in GL(\mathcal{L}ie\, G)$ the eigenvalues of b depend only on the projection

$\pi(b) \in B/B'$, since B is conjugate to the group of upper triangular matrices. Thus $\rho = \pi \circ \mathrm{Ad} : G \to B/B'$ does the job. □

We want to apply this to groups with lattices. For this purpose we need the following fact.

LEMMA 9.13.5. — *Let V be a complex vectorspace, Λ a lattice, $\phi \in GL(V)$ with $\phi(\Lambda) \subset \Lambda$.*

Then all the eigen-values of ϕ are algebraic.

Proof. — This follows immediately from $V \simeq \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. □

PROPOSITION 9.13.6. — *Let G be a solvable complex connected Lie group, Γ a lattice, N the nilradical.*

Then $G/N\Gamma$ is a torus, hence isomorphic to a quotient V/Λ of a complex vector space V by a lattice Λ.

Furthermore there exists a complex Lie group homomorphism $\phi : V \to (\mathbb{C}^*)^n$ ($n = \dim_{\mathbb{C}}(G)$) such that $\ker \phi$ is discrete and for all $\lambda \in \Lambda$ all the $\phi_i(\lambda)$ are algebraic numbers.

Proof. — It is a result of Mostow (see thm. 3.5.3) that $N\Gamma$ is closed in G, which implies the first assertion. Moreover, a result of Malcev 2.2.3 implies that all the $N^k\Gamma$ are closed in $N\Gamma$, hence closed in G. Now the adjoint action clearly stabilizes all the $\mathcal{L}ie(N^k)$, furthermore induces the trivial action on $\mathcal{L}ie(G)/\mathcal{L}ie(N)$, because $G' \subset N$. It follows that all the eigenvalues of $\mathrm{Ad}(G)$ are eigenvalues of some action on N^k/N^{k+1} with $\mathrm{Ad}(\Lambda)$ stabilizing the lattice $\Lambda_k = (N^k \cap \Lambda)/(N^{k+1} \cap \Lambda)$. Hence all the eigen-values of $\mathrm{Ad}(\Lambda)$ are algebraic by the preceding lemma. With the help of lemma 9.13.4 this implies the last assertion of the proposition. □

COROLLARY 9.13.7. — *Let $X = G/\Gamma$ be a complex solvmanifold. Assume that $Z \subset G'$.*

Then G is nilpotent if and only if $\mathrm{Alb}(X)$ is isogenous to a product of simple tori with complex multiplication.

REMARK 9.13.8. — Due to a result of Barth and Otte (see cor. 3.11.3) the commutator group G' has closed orbits in G/Γ. Therefore $\mathrm{Alb}(X) = G/G'\Gamma$.

Proof. — The nilradical N of G contains the commutator group G', hence $G/N\Gamma$ is a torus. It follows that $G/N\Gamma$ is a quotient of the Albanese $\mathrm{Alb}(X)$. Thus $G/N\Gamma$ is likewise isogenous to a product of simple tori with complex multiplication. It follows that $G/N\Gamma \simeq \mathbb{C}^g/\Lambda$ for a lattice Λ with $\Lambda \subset \overline{\mathbb{Q}}^g$. Now prop. 9.6.9 and prop. 9.8.1 together imply that $G/N\Gamma$ is a point, *i.e.*, $G = N$. □

CHAPTER 10

HOLOMORPHIC FUNCTIONS ON AN ALGEBRAIC GROUP INVARIANT UNDER A ZARISKI DENSE SUBGROUP

10.1. Survey

Let G be a reductive complex linear algebraic group, Γ a subgroup, $\mathcal{O}(G)^\Gamma$ the algebra of Γ-invariant holomorphic functions on G. It is known **[11]** that $\mathcal{O}(G)^\Gamma = \mathbb{C}$ if Γ is dense in G with respect to the algebraic Zariski topology. This can be proved in different ways. Barth and Otte indicated one method based on studying the representation of a maximal compact subgroup of G acting on the Fréchet space of holomorphic functions on G/Γ and another method based on Fourier series. We will use yet another method where the theorem of Liouville plays a key rôle.

We are interested in similar results for non-reductive groups. If G is a complex linear algebraic group with G/G' non-reductive, then there exists a surjective group morphism $\tau : G \to (\mathbb{C}, +)$ and $\Gamma = \tau^{-1}(\mathbb{Z})$ is a Zariski dense subgroup of G with $\mathcal{O}(G)^\Gamma \simeq \mathcal{O}(\mathbb{C}^*) \neq \mathbb{C}$. Hence we are led to the question whether, given a connected complex linear algebraic group G, the following two properties are equivalent:

(i) G/G' is reductive.
(ii) $\mathcal{O}(G)^\Gamma = \mathbb{C}$ for every Zariski dense subgroup Γ.

The above argument gave us (ii) $\implies$ (i) and the result of Barth and Otte **[11]** implies the equivalence of (i) and (ii) for G reductive.

We will prove that (i) and (ii) are likewise equivalent in the following two cases:

(a) G is solvable.
(b) The adjoint representation of S on $\mathcal{L}ie(U)$ has no zero weight, where S denotes a maximal connected semisimple subgroup of G and U the unipotent radical of G.

Case (b) is equivalent to each of the following two conditions

1. G/G' is reductive and the semisimple elements are dense in G'.
2. G/G' is reductive and $N_{G'}(T)/T$ is finite, where T is a maximal torus in G' and $N_{G'}(T)$ denotes the normalizer of T in G'.

For instance, if we take G to be a semi-direct product $SL_2(\mathbb{C}) \ltimes_\rho (\mathbb{C}^n, +)$ with $\rho : SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$ irreducible, then G fulfills the condition of case (b) if and only if n is an even number.

The proof for case (a) is based on the usual solvable group methods and the structure theorem on holomorphically separable solvmanifolds by Huckleberry and E. Oeljeklaus [**60**].

The proof for case (b) relies on the discussion of semisimple elements of infinite order in such a Γ. For this reason we conclude our investigation with an example of Margulis which implies that $G = SL_2(\mathbb{C}) \ltimes_\rho (\mathbb{C}^3, +)$ (ρ irreducible) admits a Zariski dense discrete subgroup Γ such that no element of Γ is semisimple. Thus condition (b) is really needed in order to find semisimple elements in Zariski dense subgroups. We underline that, although our method does not work for the example of Margulis, we have no knowledge whether there actually exist non-constant holomorphic functions in this case.

Finally we discuss invariant meromorphic and plurisubharmonic functions on certain groups.

Most of the material covered in this chapter has been published in [**154**].

10.2. Commutative groups

For the convenience of the reader we provide a rather elementary proof for the commutative case.

LEMMA 10.2.1. — *Let V be a complex vector space, Γ a discrete subgroup and $V_{\mathbb{R}}$ the real vector space spanned by Γ.*

Then $X = V/\Gamma$ is holomorphically separable if and only if $V_{\mathbb{R}}$ is totally real, i.e., $V_{\mathbb{R}} \cap iV_{\mathbb{R}} = \{0\}$.

Proof. — If $V_{\mathbb{R}}$ is tolally real, then a $\mathbb{Z}$-basis of Γ can be extended to a $\mathbb{C}$-basis of V and consequently $V/\Gamma \simeq (\mathbb{C}^*)^k \times (\mathbb{C})^{n-k}$ (with $k = \mathrm{rank}_{\mathbb{Z}}\Gamma$ and $n = \dim_{\mathbb{C}}(V)$).

Conversely, if $V_{\mathbb{R}} \cap iV_{\mathbb{R}} \neq \{0\}$, there is a complex line contained in $V_{\mathbb{R}}$. It follows that there is a non-constant holomorphic map ϕ from $\mathbb{C}$ to $X = V/\Gamma$ such that $\phi(\mathbb{C})$ is contained in the compact subset $V_{\mathbb{R}}/\Gamma$ of X. However, this implies that every holomorphic function on X is bounded on $\phi(\mathbb{C})$. By Liouville's theorem it follows that $f \circ \phi$ is constant for every holomorphic function on X. Hence X is not holomorphically separable. □

LEMMA 10.2.2. — *Let $G = (\mathbb{C}^*)^k$, Γ a subgroup and $\overline{\Gamma}$ the closure of Γ in G with respect to the (algebraic) Zariski topology.*

Then every Γ-invariant holomorphic function on G is already invariant under $\overline{\Gamma}$.

Proof. — Let H be the set of all $g \in G$ such that $f(g) = f(e)$ for all $f \in \mathcal{O}(G)^{\Gamma}$. Then H is a closed complex Lie subgroup of G and by construction G/H is holomorphically separable.

Observe that every holomorphic Lie group homomorphism from $(\mathbb{C}^*, \cdot)$ to $(\mathbb{C}, +)$ is constant. It follows that $G/H \simeq (\mathbb{C}^*)^m$ for some $m \in \mathbb{N}$. Observe furthermore that every holomorphic Lie group homomorphism from $\mathbb{C}^*$ to $\mathbb{C}^*$ is given by $z \mapsto z^k$ for some $k \in \mathbb{Z}$ and therefore is algebraic. This implies that the Lie group homomorphism $G \to G/H$ is an morphism of algebraic groups. Since $\Gamma \subset H$, it follows that $\overline{\Gamma} \subset H$. □

10.3. Solvable groups

Here we will discuss solvable groups. First we will develope some auxiliary lemmata.

LEMMA 10.3.1. — *Let G be a connected complex linear algebraic group such that G/G' is reductive.*

Then $[G, G'] = G'$.

Proof. — By taking the appropriate quotient, we may assume $[G, G'] = \{e\}$. We have to show that this implies $G' = \{e\}$. Now $[G, G'] = \{e\}$ means that G' is central, hence $\mathrm{Ad}(G)$ factors through G/G'. But G/G' is reductive and acts trivially (by conjugation) on both G/G' and G'. Due to complete reducibility of representations of reductive groups it follows that $\mathrm{Ad}(G)$ is trivial, *i.e.*, G is abelian, *i.e.*, $G' = \{e\}$. □

LEMMA 10.3.2. — *Let G be a connected complex linear algebraic group, $H \subset G'$ a connected complex Lie subgroup which is normal in G'.*

Then H is algebraic.

Proof. — Let U denote the unipotent radical of G'. Then G'/U is semisimple. Now $A = (H \cap U)^0$ is algebraic, because every connected complex Lie subgroup of a unipotent group is algebraic. Normality of H implies that $H/(H \cap U)$ is semisimple. Hence H/A is semisimple, too. Since H/A is an algebraic subgroup of G'/A, it follows that H is algebraic as well. □

LEMMA 10.3.3. — *Let G be a connected topological group, H a normal subgroup such that $H \cap G'$ is totally disconnected.*

Then H is central.

Proof. — For each $h \in H$ the set $S_h = \{ghg^{-1}h^{-1} : g \in G\}$ is both totally disconnected and connected and therefore reduces to $\{e\}$. □

LEMMA 10.3.4. — *Let G be a connected complex linear algebraic group and $A \subset G'$ a complex Lie subgroup which is normal in G and Zariski dense in G'. Assume moreover that $[G, G'] = G'$.*

Then $A = G'$.

Proof. — The connected component A^0 of A is algebraic (lemma 10.3.2). Thus G/A^0 is again algebraic. Moreover G'/A^0 is the commutator group of G/A^0. Therefore, by replacing G with G/A^0 we may assume that A is totally disconnected. But totally disconnected normal subgroups of connected Lie groups are central (lemma 10.3.3) Since A is Zariski dense in G', and $[G, G'] = G'$, this may occur only for $A = G' = \{e\}$. □

THEOREM 10.3.5. — *Let G be a connected solvable complex linear algebraic group, Γ a subgroup which is dense in the algebraic Zariski topology. Assume that G/G' is reductive.*

Then $\mathcal{O}(G)^\Gamma = \mathbb{C}$.

Proof. — Let $G/\Gamma \to G/H$ denote the holomorphic reduction, *i.e.*,

$$H = \{g \in G : f(g) = f(e) \quad \forall f \in \mathcal{O}(G)^\Gamma\}.$$

Now $\Gamma \subset H$ normalizes H^0. Since Γ is Zariski dense in G and the normalizer of a connected Lie subgroup is necessarily algebraic, it follows that H^0 is normal in G. Let $A = H^0 \cap G'$. This is again a closed normal subgroup in G. By a result of Huckleberry and E. Oeljeklaus **[60]** H/H^0 is almost nilpotent (*i.e.*, admits a subgroup of finite index which is nilpotent). Let Γ_0 be a subgroup of finite index in Γ with $\Gamma_0/(\Gamma_0 \cap H^0)$ nilpotent. By definition this means there exists a number k such that $C^k\Gamma_0 \subset H^0$ where C^k denotes the central series. Now $[G, G'] = G'$ implies $C^kG = G'$ for all $k \geq 1$. Therefore $C^k\Gamma_0$ is Zariski dense in G'. It follows that $A = H^0 \cap G'$ is a closed normal Lie subgroup of G which is Zariski dense in G'. By the preceding lemma it follows that $A = G'$, *i.e.*, $G' \subset H$. Now G/G' is assumed to be reductive. Thus the statement of the theorem now follows from the result for reductive groups (**[11]**). □

10.4. Groups with many semisimple elements

Here we will prove the following theorem.

THEOREM 10.4.1. — *Let G be a connected complex linear algebraic group. Assume that G/G' is reductive and that furthermore one (hence all) of the following equivalent conditions is fulfilled.*

1. *G' contains a dense open subset Ω such that each element in Ω is semisimple.*
2. *For any maximal torus T in G' the quotient $N_{G'}(T)/T$ is finite.*

3. *Let S denote a maximal connected semisimple subgroup of G and U the unipotent radical of G. Let $\rho : S \to GL(\mathcal{L}ie\, U)$ denote the representation obtained by restriction from the adjoint representation* $\mathrm{Ad} : G \to GL(\mathcal{L}ie\, G)$*. The condition is that all weights of ρ are non-zero.*

Under these assumptions $\mathcal{O}(G)^\Gamma = \mathbb{C}$ for any Zariski dense subgroup $\Gamma \subset G$.

EXAMPLE 10.4.2

- Let G be a reductive group. Then G/G' is reductive and G' semisimple, hence $N_{G'}(T)/T$ finite for any maximal torus $T \subset G'$. Therefore this theorem is a generalization of the result of Barth and Otte **[11]** on redutive groups.
- Let G be a parabolic subgroup of a semisimple group S.
  G/G' is obviously reductive. Furthermore a maximal torus T in G is already a maximal torus in S. Hence $N_S(T)/T$ is finite. Consequently $N_G(T)/T$ is finite and G fulfills the assumptions of the theorem.
- Let G be a semi-direct product of $SL_2(\mathbb{C})$ with a unipotent group $U \simeq \mathbb{C}^n$ induced by an irreducible representation $\xi : SL_2(\mathbb{C}) \to GL(U)$. Then G fulfills the assumptions of the theorem if and only if n is even.

Now we will demonstrate that (1), (2) and (3) are indeed equivalent. The equivalence of (2) and (3) is rather obvious from standard results on algebraic groups. For the equivalence of (1) and (2) we need some elementary facts on semisimple elements in a connected algebraic group G. Let G_s denote the set of all semisimple elements in G and T be a maximal torus in G. Now $g \in G_s$ iff g is conjugate to an element in T. It follows that G_s is the image of the map $\zeta : G \times T \to G_s$ given by $\zeta(g,t) = gtg^{-1}$. In particular G_s is a constructible set. Now a torus contains only countably many algebraic subgroups, hence a generic element $h \in T$ generates a Zariski dense subgroup of T. It follows that for a generic element $h \in T$ the assumption $g \in G$ with $ghg^{-1} \in T$ implies $gTg^{-1} = T$. From this it follows that a generic fiber of ζ has the dimension $\dim N_G(T)$. Therefore the dimension of $G_s = \mathrm{Image}(\zeta)$ equals $\dim G - \dim N_G(T)$. Thus we obtained the following lemma, which implies the equivalence of (1) and (2).

LEMMA 10.4.3. — *Let G be a connected linear algebraic group, T a maximal torus and G_s the set of semisimple elements in T.*

Then G_s is dense in G if and only if $\dim N_G(T) = \dim T$.

Next we state some simple consequences of the assumptions of theorem 10.4.1.

LEMMA 10.4.4. — *Let G be an algebraic group fulfilling the assumptions of theorem 10.4.1 and $\tau : G \to H$ a surjective morphism of algebraic groups.*

Then H likewise fulfills the assumptions of thm. 10.4.1.

Proof. — Surjectivity of τ gives a surjective morphism of algebraic groups from G/G' onto H/H'. Therefore H/H' is reductive. The surjectivity of τ furthermore implies

$\tau(G') = H'$. Since morphisms of algebraic groups map semisimple elements to semisimple elements, it follows that H fulfills condition (1). $\square$

LEMMA 10.4.5. — *Let G be an algebraic group fulfilling the assumptions of theorem 10.4.1. Then the center Z of G must be reductive.*

Proof. — Condition (2) implies that $(Z \cap G')^0$ is contained in a maximal torus of G'. Since G/G' is reductive, this implies that Z is reductive. $\square$

The following lemma illuminates why semisimple elements are important for our purposes.

LEMMA 10.4.6. — *Let G be a complex linear algebraic group, $g \in G$ an element of infinite order, Γ the subgroup generated by g and H the Zariski closure of Γ.*

Then $Z = H/\Gamma$ is a Cousin group (hence in particular $\mathcal{O}(Z) = \mathbb{C}$) if g is semisimple; but Z is biholomorphic to some $(\mathbb{C}^*)^n$ (hence holomorphically separable) if g is not semisimple.

Proof. — Note that $\bar{\Gamma} = H$ implies $H = H^0\Gamma$. Hence $H/\Gamma = H^0/(H^0 \cap \Gamma)$ is connected. If g is semisimple, the Zariski closure of Γ is reductive and the statement follows from **[11]**. If g is not semisimple then $H \simeq (\mathbb{C}^*)^{n-1} \times \mathbb{C}$ for some $n \geq 1$ and g is not contained in the maximal torus of H. This implies $H/\Gamma \simeq (\mathbb{C}^*)^n$. $\square$

LEMMA 10.4.7. — *Let G be a connected real Lie group, Γ a subgroup such that each element $\gamma \in \Gamma$ is of finite order.*

Then Γ is almost abelian and relatively compact in G.

Proof. — If G is abelian, then $G \simeq \mathbb{R}^k \times (S^1)^n$. In this case $\Gamma \subset (S^1)^n$ and the statement is immediate.

Now let us assume that G may be embedded into a complex linear algebraic group $\widetilde{G}$. Let H denote the (complex-algebraic) Zariski closure of Γ in $\widetilde{G}$. By the theorem of Tits **[145]** Γ is almost solvable, hence H^0 is solvable. Now the commutator group of H^0 is unipotent and therefore contains no non-trivial element of finite order. Hence $\Gamma \cap H^0$ is abelian, which completes the proof for this case, since we discussed already the abelian case.

Finally let us discuss the general case. By the above considerations $\mathrm{Ad}(\Gamma_0)$ is contained in an abelian connected compact subgroup K of $\mathrm{Ad}(G)$ for some subgroup Γ_0 of finite index in Γ. Now $N = (\mathrm{Ad})^{-1}(K)$ is a central extension $1 \to Z \to N \to K \to 1$. (where Z is the center of G). But complete reducibility of the representations of compact groups implies that this sequence splits on the Lie algebra level. Hence N is abelian and we can complete the proof as before. $\square$

Thm. 10.4.1 follows by induction on $\dim(G)$ using the following lemma.

LEMMA 10.4.8. — *Let G be a positive-dimensional complex linear algebraic group fulfilling the assumptions of thm. 10.4.1 and Γ be a Zariski dense subgroup.*

Then there exists a positive-dimensional normal algebraic subgroup A with $\mathcal{O}(G)^\Gamma \subset \mathcal{O}(G)^A$.

Proof. — If G is abelian, the assumptions imply that G is reductive and $\mathcal{O}(G)^\Gamma = \mathbb{C}$.

Otherwise let $H = \{g : f(g) = f(e)\ \forall f \in \mathcal{O}(G)^\Gamma\}$. Now $\Gamma \subset H$, hence Γ normalizes H^0. The normalizer of a connected Lie subgroup is algebraic, thus H^0 is a normal subgroup of G. It follows that $(H \cap G')^0$ is a normal algebraic subgroup (lemma 10.3.2). This completes the proof unless $H \cap G'$ is discrete.

If $(H \cap G')$ is discrete, then lemma 10.3.3 implies that H^0 is contained in the center Z of G. Let A denote the Zariski closure of H^0. The center is reductive (lemma 10.4.5). It follows that for each Z-orbit every H^0-invariant function is already A-invariant.

Therefore we can restrict to the case where H is discrete. Now Γ is discrete and contains a subgroup Γ_0 which is finitely generated and whose Zariski closure contains G'. By a result of Selberg Γ_0 contains a subgroup of finite index Γ_1 which is torsion-free (see prop. 1.7.2). Now let $\Gamma_2 = \Gamma_1 \cap G'$. Then being Zariski dense, Γ_2 must contain a semisimple element of infinite order. Using lemma 10.4.6, this yields a contradiction to the assumption that H is discrete. □

10.5. An example

At a first glance it would seem to be obvious that a Zariski dense subgroup should contain enough elements of infinite order to generate a subgroup which is still Zariski dense. However, one has to careful.

LEMMA 10.5.1. — *Let $G = \mathbb{C}^* \ltimes \mathbb{C}$ with group law $(\lambda, z) \cdot (\mu, w) = (\lambda\mu, z + \lambda w)$ and Γ the subgroup generated by the elements $a_n = (e^{2\pi i/n}, 0)$ $(n \in \mathbb{N})$ and $a_0 = (1, 1)$. Then $\gamma \in G' = \{(1, x) : x \in \mathbb{C}\}$ for any element $\gamma \in \Gamma$ of infinite order, although Γ is Zariski dense in G.*

Proof. — It is clear that Γ is Zariski dense in G. The other assertion follows from the fact that an arbitrary element in G is either unipotent or semisimple. Hence every element $g \in G \setminus G'$ is conjugate to an element in $\mathbb{C}^* \times \{0\}$. □

10.6. Margulis' example

We will use an example of Margulis to demonstrate the following.

PROPOSITION 10.6.1. — *There exists a discrete Zariski dense subgroup Γ in $G = SL_2(\mathbb{C}) \ltimes_\rho (\mathbb{C}^3, +)$ with ρ irreducible such that Γ contains no semisimple element.*

Thus the condition G/G' reductive is not sufficient to guarantee the existence of semisimple elements in Zariski dense subgroups.

Margulis **[90, 91]** constructed his example in order to prove that there exist free non-commutative groups acting on $\mathbb{R}^n$ properly discontinuous and by affin-linear transformations, thereby contradicting a conjecture of Milnor **[98]**.

We will now start with the description of Margulis' example. Let B denote the bilinear form on $\mathbb{R}^3$ given by $B(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$, $W = \{x \in \mathbb{R}^3 : B(x, x) = 0\}$ the zero cone and $W^+ = \{x \in W : x_3 > 0\}$ the positive part. Let $S = \{x \in W^+ : |x| = 1\}$. Let H be the connected component of the isometry group $O(2,1)$ of B. (As a Lie group H is isomorphic to $PSL_2(\mathbb{R})$.) Let $G_{\mathbb{R}} = H \ltimes (\mathbb{R}^3, +)$ the group of affine-linear transformations on $\mathbb{R}^3$ whose linear part is in H.

The following is easy to verify: Let x^+, x^- to different vectors in S. Then there exists a unique vector x^0 such that $B(x^0, x^0) = 1$, $B(x^+, x^0) = 0 = B(x^-, x^0)$ and x^0, x^-, x^+ form a positively oriented basis of the vector space $\mathbb{R}^3$. Furthermore for any $\lambda \in]0,1[$ there is an element $g \in H$ (depending on $x^+, x^- \in S$ and λ) defined as follows:

$$g : ax^0 + bx^- + cx^+ \mapsto ax^0 + \frac{b}{\lambda}x^- + \lambda cx^+$$

Conversely any non-trivial diagonalizable element $g \in H$ is given in such a way and x^+, x^- and λ are uniquely determined by g.

The result of Margulis is the following:

Let $x^+, x^-, \widetilde{x}^+, \widetilde{x}^-$ be four different points in S and let $\lambda, \widetilde{\lambda} \in]0,1[$ and $v, \widetilde{v} \in \mathbb{R}^3$ be such that v, x^-, x^+ resp. $\widetilde{v}, \widetilde{x}^-, \widetilde{x}^+$ forms a positively oriented basis of $\mathbb{R}^3$. Let $h, \widetilde{h} \in H$ be the elements corresponding to x^-, x^+, λ resp. $\widetilde{x}^-, \widetilde{x}^+, \widetilde{\lambda}$ and $g, \widetilde{g} \in G_{\mathbb{R}} = H \ltimes \mathbb{R}^3$ given by $g = (h, v)$, $\widetilde{g} = (\widetilde{h}, \widetilde{v})$.

Then there exists a number $N = N(g, \widetilde{g})$ such that the elements g^N, $\widetilde{g}^N$ generate a (non-commutative) free discrete subgroup $\Gamma \subset G_{\mathbb{R}}$ such that the action on $\mathbb{R}^3$ is properly discontinuous and free.

Now an element $g \in G_{\mathbb{R}}$ is conjugate to an element in H if and only if $g(w) = w$ for some $w \in \mathbb{R}^3$. Hence no element in Γ is conjugate to an element in H. In particular no element in semisimple. Furthermore it is clear that Γ is Zariski dense in the complexification $G = SL_2(\mathbb{C}) \ltimes \mathbb{C}^3$ of $G_{\mathbb{R}}$.

10.7. Meromorphic and plurisubharmonic functions

PROPOSITION 10.7.1. — *Let G be a connected complex linear algebraic group with $G = G'$ and suppose that G contains an open subset Ω such that each element in Ω is semisimple. Let Γ be a Zariski dense subgroup.*

Then any Γ-invariant plurisubharmonic or meromorphic function is constant and there exist no Γ-invariant analytic hypersurface.

Proof. — We may assume that Γ is closed (in the Hausdorff topology). Since $G = G'$, it follows that H^0 is a normal algebraic subgroup for each Zariski dense subgroup H. Therefore we may assume that Γ is discrete and furthermore it suffices (by induction on $\dim(G)$) to demonstrate that the functions resp. hypersurfaces are invariant under a positive-dimensional subgroup. Now $G = G'$ implies that Γ admits a finitely generated subgroup Γ_0 which is still Zariski dense. By the theorem of Selberg Γ_0 admits a subgroup of finite index Γ_1 which is torsion-free. Thus Γ_1 contains a semisimple element of infinite order γ which generates a subgroup I whose Zariski closure $\bar{I}$ is a torus. $G = G'$ implies that this torus is contained in a connected semisimple subgroup S of G. Now known results on subgroups in semisimple groups [**58**][**12**] imply that the functions resp. hypersurfaces are invariant under $\bar{I}$, which is positive-dimensional. □

For this result it is essential to require $G = G'$ and not only G/G' reductive.

LEMMA 10.7.2. — *Let $G = \mathbb{C}^* \times \mathbb{C}^*$ and $\Gamma \simeq \mathbb{Z}$ a (possibly Zariski dense) discrete subgroup.*

Then G admits Γ-invariant non-constant plurisubharmonic and meromorphic functions.

Proof. — $G/\Gamma \simeq \mathbb{C}^2/\Lambda$ with $\Lambda \simeq \mathbb{Z}^3$. Let $V = \langle\Lambda\rangle_{\mathbb{R}}$ the real subvector space of $\mathbb{C}^2$ spanned by Λ and $t : \mathbb{C}^2 \to \mathbb{C}^2/V \simeq \mathbb{R}$ a $\mathbb{R}$-linear map. Then t^2 yields a Γ-invariant plurisubharmonic function on G.

Let $L = V \cap iV$ and $\gamma \in \Lambda \setminus L$. Let $H = \langle\gamma\rangle_{\mathbb{C}}$. Then $H \neq L$, hence $H + L = \mathbb{C}^2$. It follows that the H-orbits in G/Γ are closed and induce a fibration $G/\Gamma \to G/H\Gamma$ onto a one-dimensional torus. One-dimensional tori are projective and therefore admit non-constant meromorphic functions. □

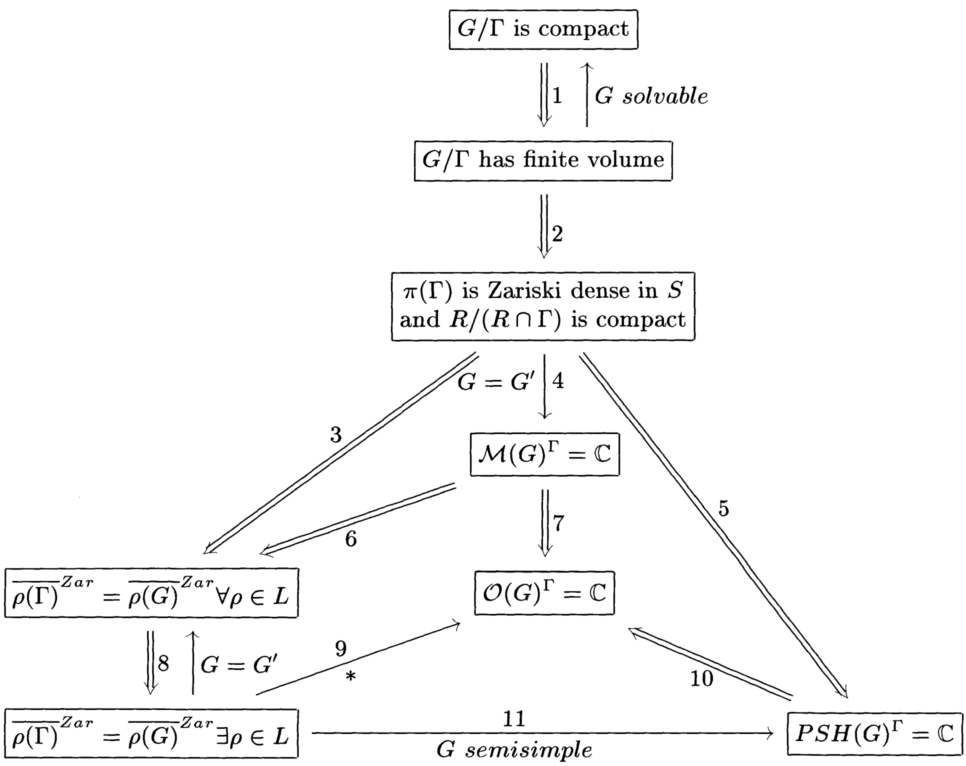
CHAPTER 11

DENSITY PROPERTIES: OVERVIEW

Here we summarize the interdependence of various density properties for discrete subgroups.

Let G be a simply connected complex Lie group and Γ a discrete subgroup. Furthermore let R denote the radical of G, and let $\pi : G \to S = G/R$ denote the natural projection. As a semisimple complex Lie group S carries the structure of an algebraic group in a natural way. Let L denote the set of all holomorphic group homomorphisms from G into some $GL_N(\mathbb{C})$ with $n \in \mathbb{N}$.

Then one has the following implications given in the diagram below, where $\Rightarrow$ denotes an unconditional implication, and $\to$ a conditional one. Condition $(*)$ means: $G = G'$ and the set of semisimple elements of G contains a Zariski open subset of G (see also §10.1).



Proofs, explanations and counterexamples for the converse directions

1. $\Rightarrow$: lemma 1.5.4

   $\Leftarrow$: this holds for solvable groups (cor. 3.6.3).

   $\not\Leftarrow$: Every semisimple complex Lie group contains a non-cocompact lattice (thm. 2.3.1).
2. $\Rightarrow$: Combine cor. 3.6.3 with thm. 3.5.3 and thm. 3.4.1.

   $\not\Leftarrow$: The results in **[156]** (see also **[145]**) imply that every semisimple complex Lie group contains a Zariski dense discrete subgroup which is a free group with infinitely many generators. Since lattices are necessarily finitely generated (thm. 1.8.1), these subgroups can not be of finite covolume.
3. $\Rightarrow$: Follows from thm. 3.4.1.

   $\not\Leftarrow$: For $n \geq 3$ the commutative Lie group $(\mathbb{C}^n, +)$ contains a discrete subgroup Γ which is not cocompact such that every meromorphic function on G/Γ is constant (**[120]**, p. 132, see also **[29]**).
4. $\Rightarrow$: thm. 3.15.4 combined with **[58]**.

   $\not\Leftarrow$ same counterexample as above.
5. $\Rightarrow$ Combine prop. 3.7.2 with the fact that plurisubharmonic functions are constant on a compact complex manifold due to the maximum principle.

   $\not\Leftarrow$ Consider the usual action of $GL_2(\mathbb{C})$ on $\mathbb{C}^2$ and let H denote the Zariski closure of $\{A^n : n \in \mathbb{Z}\}$ where $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Let

$$G = H \ltimes (\mathbb{C}^2, +) \quad \text{and} \quad \Gamma = \{A^n : n \in \mathbb{Z}\} \ltimes (\mathbb{Z}^2, +).$$

   Then Γ is a discrete subgroup of G, not cocompact, but Zariski dense. Let $I = \{g \in G : f(g) = f(e)\ \forall f \in PSH(G)^\Gamma\}$. Then I contains H, since $H/(H \cap \Gamma)$ is a compact complex one-dimensional torus. Evidently $\Gamma \subset I$ holds, too. It follows that the connected component I^0 of I contains H and is normalized by Γ. Since Γ is Zariski dense in G, it follows that I^0 is normal in G. Thus $I^0 = I = G$, since G does not contain any proper normal connected Lie subgroup containing H. Thus every plurisubharmonic function on G/Γ is constant.
6. $\Rightarrow$: Obvious.

   $\not\Leftarrow$: Abelian varieties are counterexamples for the converse direction.
7. $\Rightarrow$: Trivial.

   $\not\Leftarrow$: Abelian varieties.
8. $\Rightarrow$: Trivial.

   $\Leftarrow$: For $G = G'$ this is true due to prop. 1.9.1.

   $\not\Leftarrow$: The subgroup $\Gamma = (2\pi i\mathbb{Z}, +)$ is Zariski dense in $\mathbb{C}$, but $\exp(\Gamma) = \{1\}$ is not Zariski dense in $\exp(\mathbb{C}) = \mathbb{C}^{(*)}$.
9. $\Rightarrow$: See ch. 10

   $\not\Leftarrow$ See counterexample to 11. below.
10. $\Rightarrow$: Trivial.

$\not\leftarrow$: There are non-compact quotients (called *Cousin groups*) $(\mathbb{C}^d,+)/\Gamma$ of $(\mathbb{C}^d,+)$ by a discrete subgroup Γ such that every holomorphic function on G is constant. However, every non-compact such quotient carries non-constant plurisubharmonic functions: Simply take a real linear map $\zeta : \mathbb{C}^d \Rightarrow \mathbb{R}$ with $\Gamma \subset \ker \zeta$.

11. $\rightarrow$: prop. 3.7.2

$\not\leftarrow$ Let $G = SL_3(\mathbb{C})$ and let P be a maximal parabolic subgroup. Then $P = L \cdot S \ltimes V$ with $L \simeq \mathbb{C}^*$, $S \simeq SL_2(\mathbb{C})$ and $V \simeq (\mathbb{C}^2,+)$. Using the methods of **[156]** one can construct a Zariski dense discrete subgroup $\Gamma \subset P$ such that $S \cap \Gamma$ is Zariski dense in S and Γ contains an element $(\lambda, s, v) \in L \cdot S \ltimes V$ with $|\lambda| > 1$. By prop. 3.7.2 every Γ-invariant plurisubharmonic function f on G is S-invariant. An argumentation parallel to the reasoning as in 5. above yields that every such f is invariant under $H = S \ltimes V$. Now G/P and $P/H\Gamma$ are both compact and on compact complex manifolds every plurisubharmonic function is constant. Therefore $PSH(G)^\Gamma = \mathbb{R}$.

BIBLIOGRAPHY

[1] Akhiezer, D.N.: Invariant meromorphic functions on complex semisimple Lie groups. *Invent. Math.* **65** , no. 3, 325–329 (1981/82)

[2] Akhiezer, D.N.: Lie Groups actions in complex analysis. Aspects of Mathematics. Vieweg 1995

[3] Akhiezer, D.N.: Group actions on the Dolbeault cohomology of homogeneous manifolds. *Math. Z.* (1996)

[4] Atiyah, M.: On the Krull-Schmidt theorem with applications to sheaves. *Bull. Soc. Math. France* **84**, 307–317 (1956)

[5] Atiyah, M.: Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.* **85**, 181–207 (1957)

[6] Atiyah, M.: K-Theory. Benjamin 1967

[7] Atiyah, M.; Rees, F.: Vector Bundles on Projective 3-space. *Invent. math.* **36**, 131–153 (1976)

[8] Baker, A.: Transcendental Number Theory. Cambridge University Press 1975

[9] Barlet, D.: Familles analytiques de cycles paramétrées par un espace analytique réduit. *LN 481*, 1–158. Springer 1975

[10] Barth, W.; Otte, M.: Über fast-uniforme Untergruppen komplexer Liegruppen und auflösbare komplexe Mannigfaltigkeiten. *Comm. Math. Helv.* **44**, 269–281 (1969)

[11] Barth, W.; Otte, M.: Invariante Holomorphe Funktionen auf reduktiven Liegruppen. *Math. Ann.* **201**, 91–112 (1973)

[12] Berteloot, F.; Oeljeklaus, K.: Invariant Plurisubharmonic Functions and Hypersurfaces on Semisimple Complex Lie Groups. *Math. Ann.* **281**, 513–530 (1988)

[13] Bierstone, E.; Milman, P.D.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.* **128**, 207–302 (1997)

[14] Blanchard, A.: Sur les variétés analytiques complexes. *Ann. Sci. ecole norm. sup.* **73**, 157–202 (1956)

[15] Bochner, S.; Montgomery, D.: Groups on analytic manifolds. *Ann. of Math.* **48** (2), 659-669 (1947).

[16] Boothby, W.; Wang, H.: On the finite Subgroups of Connected Lie Groups. *Comm. Math. Helv.* **39**, 281–294 (1964)

[17] Borel, A.: Density properties of certain subgroups of semisimple groups without compact components. *Ann. of Math.* **72**, 179–188 (1960)

[18] Borel, A.: Compact Clifford Klein forms of symmetric spaces. *Topology* **2**, 111–122 (1963)

[19] Borel, A.: Introduction aux groupes arithmetiques. Hermann 1969

[20] Borel, A.: Linear algebraic groups. Second Enlarged Edition. Springer 1991

[21] Borel, A.; Harish-Chandra: Arithmetic subgroups of algebraic groups. *Ann. Math.* **75**, 485–535 (1962)

[22] Borel, A.; Remmert, R.: Über kompakte homogene Kählersche Mannigfaltigkeiten, *Math. Ann.* **145**, 429–439 (1962)

[23] Borel, A.; Tits, J.: Groupes Reductifs. *IHES* **27**, 55–152 (1965)

[24] Bourbaki, N.: Intégration. Ch. VII. Hermann, Paris. 1960

[25] Brody, R.: Compact manifolds and hyperbolicity. *T.A.M.S.* **235**, 213–219 (1978)

[26] Campana, F.: On Twistor Spaces of the class $\mathcal{C}$. *J. Diff. Geo.* **33**, 541–549 (1991)

[27] Campana, F.: Remarques sur le revêtement universel des variétés Kählériennes compactes. *Bull. Soc. math. France.* **122**, 255–284 (1994)

[28] Campana, F.: Connexité abélienne des variétés kählériennes compactes. *C.R. Acad. Sci. Paris*, t. 325, Série I, 755-758 (1997)

[29] Capocasa, F.; Catanese, F.: Periodic meromorphic functions. *Acta Math.* **166**, 27–68 (1991)

[30] Cartan, E.: La topologie des groupes de Lie. *Enseign. math.* **35**, 177–200 (1936)

[31] Cernousov, V.I.: On the Hasse-principle for groups of type E_8. *Dokl. Akad. Nauk. SSSR* **306**, 1059–1063 (1989) Translation: *Sov. Math. Dokl.* **39**, 592–596 (1989)

[32] Cordero, L.; Fernandez, M.; Gray, A.: La suite spectrale de Frölicher et les nilvariétés complexes compactes. *C.R. Acad. Sci. Paris* **305** , 753–756 (1987)

[33] Cordero, L.; Fernandez, M.; Gray, A.: The Froehlicher spectral sequence for compact nilmanifolds. *Ill. J. Math.* **35**, no.1, 56–67 (1991)

[34] Corlette, K.: Archimedean superrigidity and hyperbolic geometry. *Ann. of Math.* **135**, 165–182 (1992)

[35] Curtis, C.; Reiner, I.: Representation Theory of Finite Groups and associative algebras. Interscience 1962. (Reprinted 1988 in the *Wiley Classics Library* series.)

[36] Dixmier, J.; Lister, W.G.: Derivations of nilpotent Lie algebras *Proc. A.M.S.* **8**, 155–158 (1957)

[37] Dold, A.: Lectures on Algebraic Topology. Springer Berlin Heidelberg New York 1972

[38] Dynkin, E.B.: Semisimple subalgebras of semisimple Lie algebras. *Mat. Sbornik N.S.* **30** (72), 349-462, Moskva (1952)

[39] Dynkin, E.B.: Maximal subgroups of classical groups. *Trudy Moskov. Mat. Obsc.1*, **39-66**, Moskva (1952)

[40] Forster, O.; Knorr, K.: Über die Deformationen von Vektorraumbündeln auf kompakten komplexen Räumen. *Math. Ann.* **209** , 291–346 (1974)

[41] Frölicher: Relations between the cohomology groups of Dolbeault and topological invariants. *Proc. Nat. Acad. Sci. U.S.A.* **41**, 641–644 (1955)

[42] Ghys, E.: Déformations des structures complexes sur les espaces homogènes de $SL(2,\mathbb{C})$. J. Reine Angew. Math. **468**, 113–138 (1995)

[43] Goto, M.: On an arcwise connected subgroup of a Lie group. *Proc. A.M.S.* **20**, 157–162 (1969)

[44] Grauert, H.: Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen. *Math. Ann.* **133**, 450–472 (1957)

[45] Grauert, H.; Remmert, R.: Über kompakte homogene komplexe Mannigfaltigkeiten. *Arch. Math.* **13**, 498–507 (1962)

[46] Grauert, H., Remmert, R.: Coherent Analytic Sheaves. Springer 1984

[47] Green, M.; Griffiths, P.: Two Applications of Algebraic Geometry to Entire Holomorphic Mappings. *The Chern Symposium 1979, Springer, New-York Berlin* (1980)

[48] Griffiths, P.; Harris, J.: Principles of Algebraic Geometry. *Pure and Applied Mathematics, Wiley-Interscience, New-York* (1978)

[49] Gromov, M.; Schoen, R.: Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Publ. IHES* **76**, 165–246 (1992)

[50] Gunning, R.: Introduction to Holomorphic Functions of several variables. 3 volumes. Wadsworth 1990

[51] Harder, G.: Bericht über neuere Resultate der Galoiskohomologie halbeinfacher Gruppen. *Jahresber. DMV*, **70,** 182–216 (1968)

[52] Hartshorne, R.: Varieties of small codimension in projective space. *Bull. A.M.S.* **80** , 1017-1032 (1974)

[53] Hartshorne, R.: Algebraic Geometry. Springer 1977

[54] Higman, G.: A finitely generated infinite simple group. *J. London Math. Soc.* **26**, 61–64 (1951)

[55] Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. I,II. *Ann. of Math. (2)*, **79**, 109–326 (1964)

[56] Hironaka, H.: Bimeromorphic smoothing of a complex–analytic space. Math. Inst. Warwick Univ. 1971

[57] Hochschild, G.: The Structure of Lie groups. Holden Day Inc 1965

[58] Huckleberry, A.T.; Margulis, G.A.: Invariant analytic hypersurfaces. *Invent. Math.* **71**, 235–240 (1983)

[59] Huckleberry, A.T.; Oeljeklaus, E.: Classification theorems for almost homogeneous spaces. *Revue De l'Institut Elie Cartan*, Numéro 9, Janvier 1984

[60] Huckleberry, A.T.; Oeljeklaus, E.: On holomorphically separable complex solvmanifolds. *Ann. Inst. Fourier* **XXXVI 3**, 57–65 (1986)

[61] Huckleberry, A.T.: Actions of Groups of Holomorphic Transformations. Encyclopaedia of Mathematical Sciences. Volume **69** Several Complex Variables VI. (1990)

[62] Huckleberry, A.T.; Snow, D.: Pseudoconcave homogeneous manifolds. *Ann. Scuola Norm. Sup. Pisa, (4)*, **7**, 29–54 (1980)

[63] Huckleberry, A.T.; Winkelmann, J.: Subvarieties of parallelizable manifolds. *Math. Ann.* **295**, 469–483 (1993)

[64] Humphries, J.: Linear algebraic groups. GTM **21** Springer 1975

[65] Humphries, J.: Arithmetic Groups. LN **789** Springer 1980

[66] Husemoller, Dale: Fibre bundles. 3. ed. Springer 1994

[67] Iwamoto, T.: Density properties of complex Lie groups. *Osaka J.Math.* **23**, 859–865 (1986)

[68] Jordan, C.: Mémoire sur les équations différent linéaires à intégrale algébriques. *J. Reine Angew. Math.* **84**, 89–215 (1878)

[69] Jørgensen, T.: Compact 3–manifolds of constant negative curvature fibering over the circle. *Ann. Math.* **106**, 61–72 (1977)

[70] Jost, J.; Yau, S.T: Harmonic maps and superrigidity. . *Proc. Symp. Pure Math.* **54**, 245–280 (1993)

[71] Kaup, B.; Kaup, L.: Holomorphic Functions of Several Variables. De Gruyter 1983

[72] Každan, D.A.: Connection of the dual space of a group with the structure of its closed subgroups. *Funct. Anal. Appl.* **1**, 63–65 (1967)

[73] Každan, D.A.; Margulis, G.A.: A proof of Selbergs hypothesis. *Math. Sbornik* **75** , 162–168 (1968)

[74] Knapp, A.: Lie Groups Beyond an Introduction. Birkhäuser 1996

[75] Kobayashi, S.: Hyperbolic manifolds and holomorphic mappings. Dekker Inc. 1970

[76] Koblitz, N.; Rohrlich, D.: Simple factors in the Jacobian of a Fermat curve. *Canad. J. Math.* **31**, 1183–1205 (1978)

[77] Kraft, H.: Geometrische Methoden in der Invariantentheorie. Aspekte der Mathematik, Band D1. Vieweg 1984

[78] Lang, S.: Abelian Varieties. Interscience 1959

[79] Lang, S.: Complex Multiplication. Springer 1983

[80] Lang, S.: Introduction to Complex Hyperbolic Spaces. Springer 1987

[81] Lescure, F.: Action non triviale sur le premier groupe de comologie de Dolbeault. *Comptes Rendus Acad. Sci. Paris* **316**, 823–825 (1993)

[82] Lescure, F.: Cohomologie totale et courants de Dolbeault invariants. *J. Reine Angew. Math.* **475**, 103-136 (1996)

[83] Lindemann: Über die Zahl π. *Math. Ann.* **20**, 213–225 (1882)

[84] Lubotzky, A.: Free Quotients and the first Betti number of some hyperbolic manifolds. *Transformation Groups.* **1**, 71–82 (1996)

[85] Lundell, A.; Weingram, S.: The topology of CW-complexes. Van Nostrand 1969

[86] Makarov, V.S.: On a certain class of discrete groups of Lobachevsky space having an infinite fundamental region of finite measure. *Soviet Math. Dokl.* **7**, 328–331 (1966)

[87] Malcev, A.: On isomorphism matrix representations of infinite groups. *Mat. Sb.* **8**, 405-422 (1940)

[88] Malcev, A.: Commutative subalgebras of semisimple Lie algebras. *Izv. Nauk USSR* **9**, 291–300 (1945)

[89] Malcev, A.: On a class of homogeneous spaces. *Izvestiya Akad. Nauk SSSR Ser. Math.* **13** (1949)/ *AMS Transl. no.* **39** (1951)

[90] Margulis, G.A.: Free totally discontinuous groups of affine transformations. *Soviet Math. Doklady* **28**, 435–439 (1983)

[91] Margulis, G.A.: Complete affine locally flat manifolds with free fundamental group. *Zapiski Nauen. Sem. Leningrad Otd. Mat. Inst. Steklov* **134**, 190-205 (1984) [in Russian]

[92] Margulis, G.: Discrete subgroups of semi-simple Lie groups. *Erg. Math., Springer-Verlag* 1990

[93] Matsushima, Y.: On the discrete subgroups and homogeneous spaces of nilpotent Lie groups. *Nagoya Math. J.* **2**, 95–110 (1951)

[94] Matsushima, Y.: Fibrés holomorphes sur un tore complexe. *Nagoya Math. J.* **14**, 1–24 (1959)

[95] Matsushima, Y.: Espaces homogènes de Stein des groupes de Lie complexes I. *Nagoya Math. J.* **16**, 205–218 (1960)

[96] Matsushima, Y.; Morimoto, A.: Sur certaines espaces fibrés holomorphes sur une variété de Stein. *Bull. Soc. Math. France* **88**, 137-155 (1960)

[97] Millson, J.J.: On the first Betti number of a compact constant negatively curved manifold. *Ann. of Math.* **104**, 235–247 (1976)

[98] Milnor, J.: On fundamental groups of completely affinely flat manifolds. *Adv. Math.* **25**, 178–187 (1977)

[99] Mok, N.; Siu, Y.T.; Yeung, S.K.: Geometric superrigidity. *Invent. math.* **113**, 57–83 (1993)

[100] Montgomery, D.; Zippin, L.: Topological Transformation Groups. Interscience 1955

[101] Morimoto, A.: Sur le groupe d'automorphismes d'un espace fibré principal analytique complexe. *Nagoya Math. J.* **13**, 157–168 (1959)

[102] Moskowitz, M.: On the density theorems of Borel and Furstenberg. *Ark. Mat.* **16**, 11–27 (1978)

[103] Mostow, G.D.: Factor spaces of solvable groups. Ann. of Math. **60**, 1–27 (1954)

[104] Mostow, G.D.: Homogeneous spaces of finite invariant measures *Ann. of Math.* **75**, 17–37 (1962)

[105] Mostow, G.D.: Intersection of discrete subgroups with Cartan subgroups. *J. Ind. Math. Soc.* **34**, 203–214 (1970)

[106] Mostow, G.D.: Arithmetic subgroups of groups with radical. *Ann. Math.* **93**, 409–438 (1971)

[107] Mostow, G.D.; Tamagawa, T.: On the compactness of the arithmetically defined homogeneous spaces. *Ann. Math.* **76**, 440-463 (1962)

[108] Mumford, D.: Abelian varieties. Oxford University Press 1970

[109] Murakami, S.: Sur certains espaces fibrés principaux holomorphes admettant des connexions holomorphes. *Osaka Math. J.* **11**, 43-62 (1959)

[110] Nakamura, I.: Complex parallisable manifolds and their small deformations. *J. Diff. Geo.* **10**, 85–112 (1975)

[111] Neukirch: Algebraische Zahlentheorie. Springer 1992

[112] Oeljeklaus, K.: Hyperflächen und Geradenbündel auf homogenen komplexen Mannnigfaltigkeiten. *Schriftenreihe Math. Inst. Univ. Münster (2)* **36** (1985)

[113] Oeljeklaus, K.; Winkelmann, J.: Some Remarks on Parallelizable Manifolds. *Publ. IRMA, Lille* **27**, 1992

[114] Okonek, Ch.; Schneider, M.; and Spindler, H.: Vector Bundles on Complex Projective Spaces. *Progress in Math.* **3**, Birkhäuser Boston, 1980

[115] Onishchik, A.; Vinberg, E.: Lie groups and Algebraic groups. Springer 1990

[116] Otte, M.: Über homogene kompakte komplexe Mannigfaltigkeiten. Habilitationsschrift. Münster 1972

[117] Otte, M.; Potters, J.: Beispiele homogener Mannigfaltigkeiten. *Manu. math.* **10**, 117–127 (1973)

[118] Pansu, P.: Sous-groupes discrets des groupes de Lie: rigidité, arithmeticité. *Séminaire Bourbaki.* **46**ème année (1993/94), exposé 778

[119] Parshin: Generalized Jacobians. *Transl. A.M.S.* **84** (1968)

[120] Pothering, G.: Meromorphic function fields of non-compact $\mathbb{C}/\Gamma$. Ph.D. thesis. Notre Dame University, Indiana 1977

[121] Raghunathan, M.S.: On the first cohomology of discrete subgroups of semisimple Lie groups. *Amer. J. Math.* **87** , 103–139 (1965)

[122] Raghunathan, M.S.: Vanishing Theorems for Cohomology Groups Associated To Discrete Subgroups of Semisimple Lie Groups. *Osaka J. Math.* **3** , 243-256 (1966)

[123] Raghunathan, M.S.: Discrete subgroups of Lie groups. *Erg. Math. Grenzgeb.* **68**, Springer (1972)

[124] Rajan, C.S.: Deformations of complex structure $\Gamma\backslash SL_2(\mathbb{C})$. *Proc. Indian Acad. Sci. Math. Sci.* **104**, no. 2, 389–395 (1994).

[125] Remmert, R.: Meromorphe Funktionen in Kompakten Komplexen Räumen. *Math. Ann.* **132**, 277–288 (1956)

[126] Remmert, R.; van de Ven, T.: Zur Funktionentheorie homogener komplexer Mannigfaltigkeiten. *Topology* **2** , 137–157 (1963)

[127] Rosay, J.P.; Rudin, W.: Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ *Trans. A.M.S.* **310**, 47–86 (1988)

[128] Rosenlicht, M.: On quotient varieties and the affine embedding of certain homogeneous spaces. *Trans. A.M.S.* **101**, 211–223 (1961)

[129] Sakane, Y.: On Compact Complex parallelisable Solvmanifolds. *Osaka J. Math.* **13**, 187-219 (1976)

[130] Scharlau, W.: Quadratic and hermitian forms. Grundlehren der mathematischen Wissenschaften, 270. Springer 1985

[131] Schur, I.: Über Gruppen periodischer Substitutionen. *Sitzber. Preuss. Akad. Wiss.*, 619–627 (1911)

[132] Schur, I.: Zur Theorie vertauschbarer Matrizen. *J. Reine Angew. Math.* **30**, 66-76 (1905)

[133] Serre, J.P.: Galois Cohomology. Translated from the French by Patrick Ion and revised by the author. Springer 1997

[134] Serre, J.P.: Cohomologie galoisienne. Lecture Notes in Mathematics, Springer 1994

[135] Shimura, G.; Taniyama, Y.: Complex multiplications of abelian varieties and its applications to number theory. *Publ. Math.Soc. Japan* **6** (1961)

[136] Selberg, A.: On discontinuous groups in higher-dimensional symmetric spaces. *in* Contributions to function theory. Bombay 1960, p. 147–164

[137] Serre, J.P.: Sur les modules projectifs. *Sem. Dubreil-Pisot* exposé 2. (1960/61)

[138] Snow, D.; Winkelmann, J.: Homogeneous Manifolds with Large Automorphism Groups. *Invent. math.* (1998)

[139] Spanier, E.: Algebraic Topology McGraw-Hill 1966. Corrected Reprint: Springer 1981

[140] Taubes, C.H.: The existence of self-dual conformal structures. *J. Diff. Geom.* **36**, 163–253 (1992)

[141] Thurston, W.: Threedimensional manifolds, Kleinian groups and Hyperbolic Geometry. *Bull. A.M.S.* **6**, N.S. 357–381 (1982)

[142] Thurston, W: Three-dimensional geometry and topology. Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, **35**. Princeton University Press 1997

[143] Tits, J.: Espaces homogènes complexes compacts. *Comm. Math. Helv.* **37** (1962), 111–120

[144] Tits, J.: Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen. Springer LN 40. 1967

[145] Tits, J.: Free subgroups in linear groups. *J. Algebra* **20**, 250–270 (1972)

[146] Ueno, K.: Classification Theory of Algebraic Varieties and Compact Complex Spaces. *LN 439 Springer* 1975

[147] Varouchas, J.: Kähler spaces and proper open morphisms. *Math. Ann.* **283**, 13–52 (1989)

[148] Vinberg, E.B.: Discrete groups generated by reflections in Lobachevsky spaces. *Math. USSR Sbornik* **1**, 429–444 (1967)

[149] Wang, H.: Complex parallelisable manifolds. *Proc. A.M.S.* **5**, 771–776 (1954)

[150] Weil, A.: On discrete subgroups of Lie groups I, II. *Ann. Math.* **72**, 369–384 (1960) and **75**, 578–602 (1962)

[151] Weil, A.: Remarks on the cohomology of groups. *Ann. of Math.* **80**, 149–157 (1964)

[152] Winkelmann, J.: Every compact complex manifold admits a holomorphic vector bundle. *Revue Roum. Math. Pures et Appl.* **38**, 743-744 (1993)

[153] Winkelmann, J.: Complex-Analytic Geometry of Complex Parallelizable Manifolds. Habilitationsschrift. Ruhr-Universität Bochum. *Schriftenreihe des Graduiertenkolleg*, Heft **13** (1995)

[154] Holomorphic Functions on Algebraic Groups Invariant under a Zariski dense subgroup. 519–529 in *Proc. Complex Analysis and Geometry. Lecture Notes in Pure and Applied Math., Dekker Inc. 1995*

[155] Winkelmann, J.: Complex Parallelizable Manifolds. *Proc. Geometric Complex Analysis. 667-678 ed. by J. Noguchi et. al. World Scientific Publishing. Singapur* 1996

[156] Winkelmann, J.: On Discrete Zariski Dense Subgroups of Algebraic Groups. *Math. Nachr.* **186**, 285–302 (1997)

[157] Winkelmann, J.: Only Countably Many Simply-Connected Lie Groups Admit Lattices. *Complex Analysis and Geometry.* Pitman Research Notes in Mathematics Series. V. Ancona, E. Ballico, R. Miro-Roig, A. Silva eds.

[158] Winkelmann, J.: Holomorphic self-maps of Parallelizable Manifolds. *Transformation Groups.* (1998)

[159] Winkelmann, J.: Homogeneous Vector bundles on Parallelizable Manifolds. (to appear)

[160] Winkelmann, J.: The Albanese Variety of a Complex Parallelizable Manifold. (to appear)

[161] Winkelmann, J.: Arithmeticity of complex nilmanifolds. (to appear)

[162] Wu, T.S.: A Note on a Theorem on lattices in Lie Groups. *Canad. Math. Bull.* **31**(2), (1988)

[163] Zimmer, R.: Ergodic Theory and Semisimple Groups. Birkhäuser 1984

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