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YANG-MILLS-HIGGS FIELDS
IN THREE SPACE TIME DIMENSIONS

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INTRODUCTION.

The global existence on Minkovski space-time M_{n+1} of solutions of the Yang-Mills equations coupled with the Higgs equations for a scalar multiplet has been proved for n = 1 or 2 by Ginibre and Velo (1981), for n = 3 by Eardley and Moncrief (1981) on the one hand, by Choquet-Bruhat and Christodoulou (1981) on the other hand (the global existence in this article is proved only for small Cauchy data, but includes also spinor sources and the corresponding gauge covariant Dirac equation).

The proof of Ginibre and Velo rests on the local existence theorem obtained by Segal(1978) using the temporal gauge and semi-group theory, and on a priori estimates in this temporal gauge deduced from energy conservation and "higher" energy non blow up for n = 1 or 2 . These estimates are not sufficient to complete the proof in the case n = 3 . The complete proof of Eardley and Moncrief uses, in addition, estimates of the L^∞ norms of the fields and potential through the use of another gauge (the Cronström gauge) and the properties of the solutions of the usual d'Alembert equation with a source on M_4 . Choquet-Bruhat and Christodoulou use the conformal transformation of M_4 onto an open bounded set of the Einstein cylinder $S^3 \times \mathbb{R}$. The proof of Eardley-Moncrief does not seem to extend in any easy way from M_4 to another lorentzian manifold. The proof of Choquet-Bruhat and Christodoulou extends only to lorentzian manifold which are asymptotically minkovskian at infinity. On the opposite, we shall show in this article that the proof of Ginibre and Velo on M_2 or M_3 extends to a general globally hyperbolic manifold of dimension 2 or 3 , even if the Yang-Mills bundle is not trivial, when we use the local existence theorem proved on such manifolds in temporal gauge by Choquet-Bruhat, Paneitz and Segal (1983).

1. FIELDS.

A space time (V_{n+1}, g) is a C^∞ manifold endowed with a lorentzian metric, that is a pseudo riemannian metric of signature $(-, +, +, \ldots)$.

We denote by P a C^∞ principal bundle with base V_{n+1} and group a Lie group G. We suppose that G admits a non-degenerate bi-invariant metric, for instance is the product of abelian and semi-simple groups. The Lie algebra $\mathcal{G}$ admits then an Ad invariant non degenerate scalar product, denoted by a dot, which enjoys the property :

$$(1.1) \qquad a.[b, c] = [a, b].c \quad .$$

When we shall prove global existence, we shall suppose moreover that this scalar product is positive. A Yang-Mills connection (or potential) is usually defined as a 1-form ω on P with values in $\mathcal{G}$, enjoying various properties. Its representant in a local trivialization of P over $U \subset V_{n+1}$,

$$\varphi : p \mapsto (x, a), \quad p \in P, \quad x \in U, \quad a \in G,$$

is the 1-form $s^* \omega$ on U, where s is the local section of P corresponding canonically to the local trivialization,

$$s(x) = \varphi^{-1}(x, e) \quad ,$$

called a gauge in the physics litterature. Let $A_{(I)}$ and $A_{(J)}$ be representants of ω in gauges s_I and s_J over U_I and U_J, then in $U_I \cap U_J$:

$$(1.2) \qquad A_{(I)} = \mathrm{Ad}(u_{IJ}^{-1}) A_{(J)} + u_{IJ}^* \theta_{Mc} \quad ,$$

where θ_{Mc} is the Maurer-Cartan form on G, and

$$u_{IJ} : U_I \cap U_J \to G$$

is the transition function between the two local trivializations :

$$s_I = R_{u_{IJ}} s_J, \quad R_{u_{IJ}} \text{ right translation on } P \text{ by } u_{IJ}$$

The property (1.2) leads to the following definition, equivalent to the usual one.

Définition 1 : Given the principal bundle $P \to V_{n+1}$, a Yang-Mills potential A on V_{n+1} is a section of the (fibered) tensor product :

$$T^* V_{n+1} \underset{V_{n+1}}{\otimes} P_{Aff,\mathcal{G}} ,$$

where $P_{Aff,\mathcal{G}}$ is the affine bundle with base V_{n+1} and typical fiber $\mathcal{G}$ associated to P via the relation (1.2).

Note : Let $\check{A}$ be a given Yang-Mills potential on V_{n+1}, then $A - \check{A}$ is a section of the tensor product of vector bundles :

$$T^* V_{n+1} \underset{V_{n+1}}{\otimes} P_{Ad,\mathcal{G}} ,$$

where $P_{Ad,\mathcal{G}} = P \underset{Ad}{\times} \mathcal{G}$ is the vector bundle associated to P by the adjoint representation of G on $\mathcal{G}$.

Remark : There is a scalar product in the fibers of $P_{Ad,\mathcal{G}}$, deduced from the Ad invariant scalar product on $\mathcal{G}$.

The curvature Ω of the connection ω considered as a 1-form on P is a $\mathcal{G}$-valued 2-form on P. Its representant in a gauge where ω is represented by $A_{(i)}$ is given by

$$F_{(i)} = dA_{(i)} + \frac{1}{2}[A_{(i)} , A_{(i)}] , \tag{1.3}$$

and the relation between two representants is

$$F_{(i)} = Ad(u_{ij}^{-1}) F_{(j)} \quad \text{in } U_i \cap U_j . \tag{1.4}$$

We have therefore for the Yang-Mills field equivalent to the curvature :

Définition 2 : The Yang-Mills field is a section of the vector bundle $\Lambda^2 T^* V_{n+1} \underset{V_{n+1}}{\otimes} P_{Ad,\mathcal{G}}$ given by

$$F = dA + \frac{1}{2}[A , A] , \tag{1.5}$$

where (1.5) means that (1.3) is satisfied in each local

trivialization.
We also say that F is a 2-form on V_{n+1} of type $(Ad\ ,\ \mathcal{G})$.
In local coordinates on V_{n+1} , and for a choice of a basis (ε_a) of $\mathcal{G}$ a representative of F has components $F^a_{\lambda\mu}$ given by

$$F^a_{\lambda\mu} = \partial_\lambda A^a_\mu - \partial_\mu A^a_\lambda + C^a_{bc} A^b_\lambda A^c_\mu \quad ,$$

where C^a_{bc} are the structure constants of G .
In addition to the Yang-Mills field, many physical theories consider a scalar multiplet or "Higgs field".
Définition 3 :A Higgs field Φ is a section of a vector bundle $P_{r,\mathbb{C}^N}$ over V_{n+1} with typical fiber $\mathbb{C}^N$ (or $\mathbb{R}^N$) associated to P via a unitary (or orthogonal) representation r of G : the representants $\Phi_{(I)}$ and $\Phi_{(J)}$ are linked by

$$\Phi_{(I)} = r(u_{IJ})\ \Phi_{(J)} \qquad \text{in } U_I \cap U_J \quad .$$

A particular case when G is itself a unitary group $U(N)$ in matrix representation is

$$r(u_{IJ})\ \Phi_{(J)} = u_{IJ}\ \Phi_{(J)} \ , \qquad \text{action of } u_{IJ} \text{ on } \Phi_{(J)} \in \mathbb{C}^N \quad .$$

In all cases, we have a scalar product in the fibers of $P_{r,\mathbb{C}^N}$ deduced from the hermitian scalar product in $\mathbb{C}^N$, invariant under the unitary group.

2. COVARIANT DERIVATIVES.

We call P-tensor a differentiable section of a vector bundle

$$E = \otimes^p T^* V_{n+1} \underset{V_{n+1}}{\otimes} P_\rho \quad ,$$

where P_ρ is a vector bundle associated to P by the representation ρ of G . If V_{n+1} is endowed with a metric g , the vector bundle $\otimes^p T^* V_{n+1}$ has a natural connection, deduced from the riemannian connection of g , while P_ρ has a connection deduced from the connection A of P , with representative $SA_{(I)}$ if A is represented by $A_{(I)} \in \mathcal{G}$, S being the mapping from $\mathcal{G}$ into the Lie algebra of

$\rho(G)$ deduced from ρ. The covariant derivative of a P-tensor f is defined by using these two connections : if $f = h \otimes k$ we define for any tangent vector v to V_{n+1} :

$$(2.1) \qquad \hat{\nabla}_v f = \nabla_v h \otimes k + h \otimes \hat{\nabla}_v k \quad ,$$

where ∇ is the riemannian covariant derivative, and $\hat{\nabla}_v k$ the usual gauge covariant derivative. The derivative $\hat{\nabla}_v f$ is extended by linearity to all sections of E. It is also a section of E, and depends linearly on v ; we define $\hat{\nabla} f$ as the section of

$$T^* V_{n+1} \otimes^p T^* V_{n+1} \otimes P_\rho$$

obtained through this linearity. If $A_{(i)}$ is the representative of the Yang-Mills potential in a local trivialization of P_ρ, and $f_{(i)}$ the representative of f, a p-tensor with values in a fixed vector space, we have :

$$\hat{\nabla} f_{(i)} = \nabla f_{(i)} + S A_{(i)} f_{(i)} \quad ,$$

which we often write, omitting the index (i) :

$$\hat{\nabla} f = \nabla f + S A f \quad .$$

The same reasonning applies to sections $\otimes^p T V_{n+1} \underset{V_{n+1}}{\otimes} P_\rho$.

<u>Examples</u> :

1° The Yang-Mills field F is a section of $\otimes^2 T^* V_{n+1} \underset{V_{n+1}}{\otimes} P_{(Ad,\mathcal{G})}$
Its covariant derivative is :

$$\hat{\nabla} F = \nabla F + [A \; , \; F] \quad .$$

2° If G is a unitary group $U(N)$, and the Higgs field is a section of the vector bundle $P_{(Id,\mathbb{C}^N)}$ with fiber $\mathbb{C}^N$ the representation space of G, then :

$$\hat{\nabla} \Phi = \nabla \Phi + A \Phi \quad .$$

3. EQUATIONS AND IDENTITIES.

The simplest way to obtain intrinsic equations for the fields is to derive them from an intrinsic lagrangian. The physical lagrangian densities, leading to second order equations for the potential A and the field Φ , are of the form :

$$L \equiv \frac{1}{2} F.F + \hat{\nabla} \Phi.\hat{\nabla} \Phi + V(|\Phi|^2) \quad , \tag{3.1}$$

where the dot denotes the scalar product in the fibers of the relevant vector bundles and

$$|\Phi|^2 = \Phi.\Phi \quad ; \tag{3.2}$$

V is some smooth function (self interaction potential).

We shall moreover suppose that $V(0) = 0$. The case $V(0) \neq 0$ (for instance the cosmic strings) can be treated by alterations of the present method, or by working directly in local spaces.

The stationnary points (A , Φ) of the lagrangian of density (3.1), with respect to arbitrary variations $(\delta A , \delta\Phi)$, with compact support in V_{n+1} , are the solutions of the following intrinsic equations.

1° Yang-Mills equations, $(Ad , \mathcal{G})$ valued vector equation on V_{n+1} :

$$\hat{\delta} F = J \;, \qquad \text{i. e.} \qquad \hat{\nabla}_\lambda F^{\lambda\mu} = J^\mu \quad , \tag{3.3}$$

where the current J^μ is the $(Ad , \mathcal{G})$ valued vector with components :

$$J^\mu = (S\Phi.\ \hat{\nabla}^\mu \Phi + \hat{\nabla}^\mu \Phi.S\Phi) \quad . \tag{3.4}$$

2° Higgs equation, $(r , \mathbb{C}^N)$ valued scalar equation on V_{n+1} :

$$\hat{\nabla}^\lambda \hat{\nabla}_\lambda \Phi = V'(|\Phi|^2)\, \Phi \quad , \tag{3.5}$$

where V' is the derivative of V .

It is well known that the Yang-Mills field, curvature of the Yang-Mills potential A , satisfies the Bianchi identity :

$$\hat{d} F \equiv 0 \quad , \tag{3.6}$$

where $\hat{d}$ is the totally antisymetrized covariant derivative, that is, in coordinates on V_{n+1} :

$$(3.6) \qquad \hat{\nabla}_\alpha F_{\beta\gamma} + \hat{\nabla}_\gamma F_{\alpha\beta} + \hat{\nabla}_\beta F_{\gamma\alpha} \equiv 0 \quad .$$

It also satisfies the identity :

$$(3.7) \qquad \hat{\delta}^2 F \equiv 0 \ , \quad \text{i. e.} \quad \hat{\nabla}_\alpha \hat{\nabla}_\beta F^{\alpha\beta} \equiv 0 \quad .$$

The equation (3.5) implies that the current J satisfies the equation :

$$(3.8) \qquad \hat{\delta} J = 0 \ , \quad \text{i. e.} \quad \hat{\nabla}_\mu J^\mu = 0 \quad ,$$

hence the system (3.3),(3.5) is coherent.

Remark : The intrinsic lagrangian (3.1) is invariant under diffeomorphisms of V_{n+1} (with induced transformations on $\otimes^p T^* V_{n+1}$ and g) and automorphisms of P (with induced transformations on the associated bundles). The "conservation" (3.8) of the current J is a consequence of the second invariance. The first invariance leads to the conservation of the stress energy tensor which we shall use later, for a priori estimates.

4. CAUCHY PROBLEM.

We suppose, in this article, that the manifold V_{n+1} is of the type $S \times \mathbb{R}$, with $S_t = S \times \{t\}$ space like for g , and $x \times \{\mathbb{R}\}$ time like. We denote by X the tangent to $x \times \{\mathbb{R}\}$, and by n the normal to S_t . Adapted local coordinates will be $x^0 = t$, and (x^i) , $i = 1,\dots,n$, local coordinates in S .

The Cauchy problem for a Y.M.H. system is the determination of a solution from data on the submanifold $S_0 \equiv S \times \{0\}$. These data are Cauchy data.

1° A section a over S_0 of the bundle $T^* S_0 \otimes P_{Afr,\mathcal{G}}$;

2° A section a_0 over S_0 of the bundle $P_{Afr,\mathcal{G}}$;

3° A section $\bar{E}$ over S_0 of the bundle $T V_{n+1} \otimes P_{Ad,\mathcal{G}}$, i. e. a tangent vector to S_0 of type $(Ad\ ,\ \mathcal{G})$;

4° A section φ and a section $\dot{\varphi}$ over S_0 of the bundle $P_{r,\mathbb{C}^N}$, i. e.

scalar multiplets of type $(r\ ,\ \mathbb{C}^N)$.

A solution $(A\ ,\ F\ ,\ \Phi)$ of the Y.M.H. system is said to take these initial data if :

1°

(4.1) $$i^* A = a \ ,$$

where i^* is the pull back $T^* V \to T^* S_0$ deduced from the inclusion mapping $S_0 \to V_{n+1}$;

2°

(4.2) $$A.X|_{S_0} = a_0$$

(where the dot denotes the interior product between $T^* V$ and TV defined by the metric g) ;

3°

$$F.n|^{\#}_{S_0} = \bar{E}$$

(i. e. $F^{\lambda\mu}\ n_\lambda|_{S_0} = \bar{E}^\mu$, we have $\bar{E}^0 = 0$, $\bar{E}$ is tangent to S_0) ;

4°

$$\Phi|_{S_0} = \varphi\ , \qquad X.\hat{\nabla}\ \Phi|_{S_0} = \dot{\varphi}$$

5. CONSTRAINT.

It is easy, to see, in coordinates adapted to the slicing $V = S \times \mathbb{R}$, that the equation

(5.1) $$\hat{\nabla}_\lambda\ F^{\lambda 0} = J^0 \qquad \text{restricted to } S_0$$

depends only on the Cauchy data. It is therefore a constraint to be satisfied by these data. It reads in local coordinates and gauge, since F is antisymmetric :

$$\frac{1}{N\sqrt{\bar{g}}}\,\partial_j\ (F^{j0}\ N\sqrt{\bar{g}}\ ) + [A_j\ ,\ F^{j0}] = J^0 \qquad \text{on } S_0 \quad ,$$

with $\bar{g}$ the metric induced by g on S_0 and, in our signature :

$$|g^{00}|^{-1/2} = |g(X\ ,\ n)| = N = -\ n_0 \quad ,$$

hence

(5.2) $$F^{J0}\ N|_{S_0} = -\ \bar{E}^J \quad ,$$

and the constraint reads ($\bar{\nabla}$ metric derivative in the metric $\bar{g}$) :

$$\bar{\nabla}_J\ \bar{E}^J + [a_J\ ,\ \bar{E}^J] = J^0\ n_0|_{S_0} \quad ,$$

which can be written intrinsically :

(5.3) $$\hat{\bar{\text{div}}}\ \bar{E} = q\ , \qquad q = J.n|_{S_0} \quad ;$$

J is given by (3.4), hence q is a quadratic polynomial in the Cauchy data φ , $\dot{\varphi}$.

It can be proved that the operator $\hat{\bar{\text{div}}}$ is the L^2-adjoint of the operator $\hat{\text{grad}}$ mapping scalars on V_{n+1} of type (Ad , $\mathcal{G}$) into 1-forms on V_{n+1} of type (Ad , $\mathcal{G}$) , given in any representation by

$$f \mapsto \hat{\text{grad}}\ f = df + [a\ ,\ f] \quad ,$$

and that the operator $\hat{\Delta} = \hat{\bar{\text{div}}}\ \hat{\text{grad}}$ is an elliptic operator on V_{n+1}. In appropriate functional spaces depending on S , we shall have a L^2-orthogonal splitting saying that the constraint (5.3) has solutions $\bar{E}$ for any q orthogonal to the kernel of the operator $\hat{\text{grad}}$. In particular, if this kernel is empty (we then say that the potential a is "generic"), the equation (5.3) has solutions $\bar{E}$ for arbitrary q . Examples of generic potentials on a compact manifold S are given in Choquet-Bruhat and Christodoulou (1981). For an asymptotically euclidean (S , $\bar{g}$) , and in appropriate functional spaces capturing the asymptotically zero character of the fields :

$$\hat{\text{grad}}\ u = 0 \qquad \text{implies} \qquad u = 0$$

(because $\hat{\text{grad}}\ u = 0$ implies $\text{grad}\ |u|^2 = 0$), and the constraint (5.3) has solutions for arbitrary q in appropriate functional space (for instance, a weighted Sobolev space).

6. TEMPORAL GAUGE.

To solve the evolution problem of a Y.M.H. field from Cauchy data one chooses a gauge, that is one imposes an extra condition on the potential such that the Y.M.H. truncated by using this extra condition becomes a hyperbolic system with domain of dependance determined by the null cone of the metric, in order to satisfy the relativistic causality requirement. Since we are interested in non trivial bundles P, we shall adopt the active view point for gauge transformations, that is we consider them as automorphisms of P. The temporal gauge for a potential A will be defined with respect to some given smooth potential $\check{A}$ on V_{n+1}.

Définition : The potential A is said to be in $\check{A}$-temporal gauge, if the vector $(A - \check{A})^{\#}$ of type $(Ad, \mathcal{G})$ is orthogonal to the time line, that is :

$$A_0 - \check{A}_0 = 0 \quad . \tag{6.1}$$

In the case where P is trivial, it is possible to work with representations globally defined on V_{n+1}, and to choose $\check{A}$ such that its representative is zero.

Lemma : For an arbitrary potential A, there is an automorphism of P, such that its transform by this automorphism is in $\check{A}$-temporal gauge.

Proof : We want to find a mapping $u : V_{n+1} \to G$ such that the $(Ad, \mathcal{G})$ valued scalars A_0 and $\check{A}_0$ are linked by, in physicist notation :

$$u^{-1} A_0 u + u^{-1} \partial_0 u = \check{A}_0 \quad ; \tag{6.2}$$

this is a differential equation for u which can be solved from initial data on S, for instance $u|_S = \mathbb{1}$, the unit of G (cf. I. Segal (1979) for the case $V_{n+1} = M^{n+1}$, $\check{A}_0 = 0$).

The Y.M. equations truncated by the $\check{A}$-temporal gauge do not appear at first sight as a hyperbolic system. We set :

$$B = A - \check{A} \quad , \quad (Ad, \mathcal{G}) \text{ valued 1-form on } V_{n+1} \quad .$$

We have, by a straightforward computation :

(6.3) $$F_{\lambda\mu} \equiv \check{F}_{\lambda\mu} + \check{\nabla}_\lambda B_\mu - \check{\nabla}_\mu B_\lambda + [B_\lambda , B_\mu] \quad ,$$

where $\check{\nabla}$ is the riemannian, and $\check{A}$ covariant derivative, and $\check{F}$ the curvature of $\check{A}$; then :

(6.4) $$\hat{\nabla}_\lambda F^{\lambda\mu} \equiv \check{\nabla}_\lambda F^{\lambda\mu} + [B_\lambda , F^{\lambda\mu}] \quad .$$

In $\check{A}$ temporal gauge, $B_0 = 0$, and the Yang-Mills equations with unknown B split into the constraint :

(6.5) $$\hat{\nabla}^\lambda F_{\lambda 0} \equiv \check{\nabla}^\lambda \check{\nabla}_\lambda B_0 - \check{\nabla}^\lambda \check{\nabla}_0 B_\lambda + \check{\nabla}^\lambda [B_\lambda , B_0] + \check{\nabla}^\lambda \check{F}_{\lambda 0} + [B^\lambda , F_{\lambda 0}] = J_0$$

(we have left B_0 in the formula because the riemannian part of its covariant derivative does not necessarily vanish, it may contain B_i, but not its derivative) :this equation does not contain second derivatives of B_i transversal to the S_t , and the evolution equations :

(6.6) $$\hat{\nabla}^\lambda F_{\lambda i} \equiv \check{\nabla}^\lambda \check{\nabla}_\lambda B_i - \check{\nabla}^\lambda \check{\nabla}_i B_\lambda + \check{\nabla}^\lambda [B_\lambda , B_i] + [B^\lambda , F_{\lambda i}] = J_i \quad ;$$

the operator on the B_i' s in these equations is not hyperbolic, it is a non diagonal operator with multiple characteristics : its characteristic cone at a point of V_{n+1} is $(n - 1)$ copies of the null cone of the metric g and two copies of the tangent to the time line.
We obtain a hyperbolic operator for B_i by combining equations (6.5) and (6.6). If (6.5) and (6.6) are satisfied, B_i satisfies a system of the form :

(6.7) $$\check{\nabla}_0 J_i - \check{\nabla}_i J_0 = \check{\nabla}_0 \check{\nabla}^\lambda \check{\nabla}_\lambda B_i + h_i \quad ,$$

where the h_i depend on the B_i' s , and their derivatives only up to second order.
Expressed in terms of the given $\check{A}$ and the unknown B , the Higgs equation reads, since

(6.8) $$\hat{\nabla} \Phi = \check{\nabla} \Phi + S B \Phi \quad ,$$

(6.9) $$\hat{\nabla}^\lambda \hat{\nabla}_\lambda \Phi \equiv \check{\nabla}^\lambda \check{\nabla}_\lambda \Phi + H = V'(|\Phi|^2) \Phi \quad ,$$

where H depends on Φ and B and their first order derivatives. The equations (6.7) together with the equations for the Higgs field form at each point of V_{n+1}, and in any representation a hyperbolic system in Leray's sense, with characteristic cone one copy of the null one and one copy of the tangent to the time line, interior to the null cone by hypothesis, the domain of dependance is therefore determined by that cone.

7. REGULARLY HYPERBOLIC MANIFOLDS.

The Leray theory of hyperbolic systems is formulated for sets of numerical valued functions over a manifold V_{n+1}. Such a system is globally hyperbolic, if it is hyperbolic at each point, and the set of time like paths (i. e. with future tangent interior to the future characteristic cone, the manifold is supposed to be "time oriented") is empty or compact, in the set of paths. When the system is semi linear, the characteristics, hence the global hyperbolicity, does not depend on the solutions. When moreover the outer sheet of the characteristic cone is the null cone of g, the hyperbolic system is globally hyperbolic if, and only if, the manifold (V_{n+1}, g) is globally hyperbolic. It is known (Geroch, 1969) that V_{n+1} is then a product $S \times \mathbb{R}$ with $S_t = S \times \{t\}$ space like and $\{x\} \times \mathbb{R}$ time like. We shall make somewhat stronger hypothesis on (V_{n+1}, g), which we shall call "regular hyperbolicity".

A manifold (V_{n+1}, g) will be said to be regularly hyperbolic (it is then globally hyperbolic), if :

1° $V_{n+1} = S \times \mathbb{R}$ is the direct product of a smooth manifold S of dimension n and $\mathbb{R}$.

2° The metric g is of signature $(-, +, +, \ldots)$. The submanifolds $S_t = S \times \{t\}$ are space like ; their unit future directed normal is denoted by n, $g(n, n) = -1$. The curves $\{x\} \times \mathbb{R}$ are time like. Their tangent vector is denoted by X, $g(X, X) < 0$. We suppose that:

(a) There exist numbers $\alpha > 0$ and $\beta > 0$ such that, on V_{n+1} :

$$(5.1) \qquad \alpha \leqslant |g(X, X)|^{1/2} \leqslant |g(X, n)| \leqslant \beta .$$

Remark : On a lorentzian manifold with our signature hypothesis, we always have, since X and n are time like and future directed (increasing t) :

$$(5.2) \qquad g(X , n) < 0 , \quad |g(X , n)| \geqslant |g(X , X)|^{1/2} \, |g(n , n)| \quad .$$

We set $N = - g(X , n)$ (lapse function).

(b) The properly riemannian metrics $\bar{g}_t$, induced on each S_t by the metric g, are all uniformly equivalent to some smooth riemannian metric $\bar{g}$: there exists $k_1 > 0$ and $k_2 > 0$ such that :

$$(5.3) \qquad k_1 \, \bar{g} \, (\xi , \eta) \leqslant g_t(\xi , \eta) \leqslant k_2 \, \bar{g} \, (\xi , \eta) , \quad \forall \, r \in \mathbb{R} , \quad \xi , \eta \in TS$$

We suppose the metric $\bar{g}$ has a non zero injectivity radius (hence is complete).

We have :

Lemma : On a regularly hyperbolic manifold, there exists a smooth properly riemannian metric e defined by the contravariant tensor (recall $g(X , n) = - N$) :

$$(5.4) \qquad e^{\#} = g^{\#} + N(X \otimes n + n \otimes X) \quad .$$

8. GAUGE INVARIANT SOBOLEV SPACES.

Local existence theorems for the solutions of the Cauchy problem for sections of bundles over V_{n+1} of the type considered in previous paragraphs can be obtained by working with representatives in open sets $U \subset V_{n+1}$ over which P is trivialized, for which the usual theorems with ordinary Sobolev spaces apply, and using uniqueness theorems to glue together these solutions. However, it is more in the spirit of the theory to work with gauge invariant objects, and it becomes fundamental for the proof of global existence theorems. We first define the Sobolev space W^p_s for tensors of type $(Ad , \mathcal{G})$ or $(r , \mathbb{C}^N)$ on $(S , \bar{g})$, given some smooth Yang-Mills potential $\check{a}$ on S . We now suppose the Ad-invariant scalar product in $\mathcal{G}$ positive definite.

<u>Definition</u> : The space W^p_s of tensors of some given order and type over S is the completion of the space C^∞_0 of C^∞ such tensors with compact support with respect to the norm :

$$\|f\|_{W^p_s} = \{ \sum_{0 \leqslant k \leqslant s} \int_S |D^k f|^p d\bar{\mu}\}^{1/p} \quad ,$$

$1 \leqslant p < \infty$, s non negative integer, $d\bar{\mu}$ volume element of $\bar{g}$, Df $\bar{g}$ riemannian and $\check{a}$ gauge derivative, $|\ |$ norm at a point corresponding to the scalar products deduced from $\bar{g}$ and the Ad-invariant scalar product in $\mathcal{G}$.

We set $H_s = W^2_s$. It is a Hilbert space, W^p_s is a Banach space.

It can be proved that the usual Sobolev inequalities and multiplication theorems are valid for these spaces W^p_s , as well as the Gagliardo-Nirenberg inequality :

$$\|f\|_{L^q} \leqslant C\|f\|^{1-\sigma}_{L^r} \|Df\|^{\sigma}_{L^p} \quad ,$$

$$\frac{1}{q} = \frac{1-\sigma}{r} + \sigma\left(\frac{1}{p} - \frac{1}{n}\right) \quad , \qquad 0 \leqslant \sigma < 1 \quad ,$$

$$1 \leqslant p \leqslant r \ , \quad q < q_1 \ , \quad q_1 = +\infty \ \text{ if } \ n \leqslant p \ ,$$

$$q_1 = \frac{np}{n-p} \ \text{ if } \ n > p \quad ,$$

C a constant depending only on $(S , \bar{g})$.

Let now a be another, non necessarily smooth Yang-Mills potential.

Lemma : If $a - \check{a} \in W^p_s$, $f \in W^p_s$ and $s > \frac{n}{p}$, then :

$$\|f\|_{\hat{W}^p_s} = \left\{ \sum_{0 \leqslant k \leqslant s} \|\hat{\nabla}^k f\|^p_{L^p} \right\}^{1/p}$$

is finite ; moreover there exists a constant C such that :

$$\|f\|_{W^p_s} \leqslant C(1 + \|a - \check{a}\|_{\hat{W}^p_s} \|f\|_{\hat{W}^p_s}) \quad .$$

Proof : Uses the relation between $\hat{\nabla}$ and $\check{\nabla}$ and the Sobolev multiplication theorem.

Moreover, it can be proved that the Sobolev and Nirenberg-Gagliardo inequalities are valid when $\| \ \|_{W^p_s}$ is replaced by $\| \ \|_{\hat{W}^p_s}$.

We denote by $V_{n+1}(T)$ the manifold $S \times (-T, T)$, and by $E_s(T)$ a space of P-tensors of some given type on $V_{n+1}(T)$ which is the closure of the space of C_0^∞ such tensors with respect to the norm :

$$\|f\|_{E_s(T)} = \sup_{|t|<T} \left\{ \int_{S_t} \sum_{0\leqslant k\leqslant s} |\check{\nabla}^k f|^2 \, d\mu_t \right\} .$$

$E_s(T)$ is a Banach space.

Remark : $V_{n+1} = S \times \mathbb{R}$, hence admits an atlas with domains of maps of the type $U_{(I)} \times \mathbb{R}$, $U_{(I)}$ homeomorphic to $\mathbb{R}^n$. The principal fiber bundle P can therefore be trivialized over $U_{(I)} \times \mathbb{R}$, the transition functions are of the form :

$$u_{(IJ)} : (U_I \cap U_J) \times \mathbb{R} \to G .$$

If we suppose, to simplify our work, that the bundle structure is time independant, that is that there exists a family of local trivializations covering P, called admissible, such that the transition functions do not depend on t. There exists then Yang-Mills potentials $\check{A}$ whoserepresentation in every admissible local trivialization is such that $\check{A}_0 = 0$ and $\check{a} = i_t^* \check{A}$ does not depend on t, where i_t is the embedding of S_t in V_{n+1}. By strengthening moreover regular hyperbolicity to boundedness of curvature and an appropriate number of its derivations (cf. Choquet-Bruhat, Christodoulou and Francaviglia, Cagnac and Choquet-Bruhat, and for full details and proofs, Cagnac), we can show that :

$$E_s(V_T) = \bigcap_{0\leqslant k\leqslant s} C^k([-T, T], H_{s-k}(S)) .$$

9. LOCAL EXISTENCE.

The usual functional machinery can be used together with the definitions of § 8 to prove the existence of the solutions of the Cauchy problem on a small enough time interval I.

Theorem 1 (local existence in temporal gauge) : *The Y.M.H. system in $\check{A}$ temporal gauge admits a solution in $E_s(T)$ on $V_T = S \times (-T, T)$ taking the Cauchy data on S :*

$$b\ ,\ \varphi \in H_s\ ;\qquad \dot b\ ,\ \dot\varphi \in H_{s-1}\ ,\qquad s \geq 2\ ,$$

if $s > \frac{n}{2}$ *and* T *is small enough.*
T depends only on the $H_{s_0} \times H_{s_0 - 1}$ norm of the Cauchy data, with s_0 the smallest integer such that $s_0 > \frac{n}{2}$, $s_0 \geq 2$.
The solution depends continuously on the Cauchy data. The support of the solution is in the future of the Cauchy data ; its trace on S_t, $|t| < T$, is compact if the support of the Cauchy data is compact. Hence a solution in $E_s(T)$ is a limit of solutions in $C_0^\infty(T)$.

<u>Remark</u> : The proof that we have given is valid only for $s \geq 2$; it is no further restriction than $s > \frac{n}{2}$ if $n > 1$. For $n = 1$, one can prove the following corollary :

<u>Corollary</u> : In the case $n = 1$, the solution exists on V_T, T small enough if :

$$b\ ,\ \varphi\ ,\ \bar E - \check{\bar E} \in H_1\ ,\qquad \dot\varphi \in L^2\ .$$

The solution is in $E_s(T)$, if the data are respectively in H_s and H_{s-1}.

Theorem 2 (local existence for the original system) : *The solution in* $E_s(V_T)$ *of the Y.M.H. equations in* $\check A$*-temporal gauge satisfies the original Y.M.H. equations, if the Cauchy data satisfy the constraint.*

<u>Proof</u> : Denote by $f_\mu \equiv \hat\nabla_\lambda F^{\lambda\mu} - J^\mu = 0$ the original Y.M. equations. The equations we solved are, in addition to the Higgs equation (3.5):

$$(9.1)\qquad \check\nabla_0 f_i - \check\nabla_i f_0 = 0\ .$$

It can be proved, using (9.1), (3.5), and the identities (3.7), that the f_μ satisfy a linear homogeneous system which take zero Cauchy data if the original Cauchy data satisfy the constraint.

10. ENERGY ESTIMATE.

Standard reasonning shows that the solution will exist on V_{n+1} if we can find a continuous function C on $\mathbb{R}$ such that any local solution on a manifold V_T satisfies the a priori estimate, with $f = (B , \Phi)$:

$$\|f\|_{E_s(T)} \leq C(T) , \qquad s > \frac{n}{2} .$$

The backbone for the obtention of such estimates is the energy inequality, energy conservation in the case of a stationnary space-time. It is sufficient to prove the estimate for solutions in $C_0^\infty(T)$, since any solution in $E_s(T)$ is limit of such solutions. The lemma of § 8 proves moreover that it is sufficient to obtain these estimates for the $\hat{E}_s(T)$ norms.

<u>Definition</u> : The stress energy tensor of a Y.M.H. field with self interaction potential V is the usual 2-tensor on V_{n+1} (dots are products in the scalar product deduced from the Ad-invariant one, we have written indices to explicit the g scalar products) :

$$\text{(10.1)} \quad T_{\alpha\beta} = F^{\lambda}_{\alpha}.F_{\beta\lambda} - \frac{1}{4} g_{\alpha\beta} F_{\lambda\mu}.F^{\lambda\mu} + \frac{1}{2} \{\hat{\nabla}_\alpha \Phi.\hat{\nabla}_\beta \Phi + \hat{\nabla}_\beta \Phi.\hat{\nabla}_\alpha \Phi - g_{\alpha\beta}(\hat{\nabla}_\lambda \Phi.\hat{\nabla}^\lambda \Phi + V(|\Phi|^2))\} .$$

Its covariant divergence $\nabla_\alpha T^{\alpha\beta} = 0$ modulo the Y.M.H. equations, as could be foreseen (cf. § 3).

The energy momentum vector relative to X is :

$$\text{(10.2)} \qquad P^\alpha = T^\alpha_\beta X^\beta ,$$

and the energy density relative to S_t is found to be :

$$\text{(10.3)} \qquad \varepsilon \equiv T_{\alpha\beta} n^\alpha X^\beta = \frac{N^{-1}}{4} |F|^2 + |\hat{\nabla}\Phi|^2 + \frac{1}{2} V(|\Phi|^2) ,$$

where the norms at a point of V_{n+1} of a P-tensor is taken in the positive definite scalar products deduced from the Ad-invariant product in $\mathcal{G}$ and from the properly riemannian metric (5.4) on V_{n+1} .

The energy density is positive if :

$$\text{(10.4)} \qquad V(\zeta) \geq 0 \quad \text{for} \quad \zeta \geq 0 ,$$

which we shall suppose from now on.

The energy at time t exists, for $|t| < T$, if the fields F and $\hat{\nabla}\Phi$ as well as $V(|\Phi|^2)$ are in $E_0(T)$, and is given by :

$$y(t) = \int_{S_t} \varepsilon \, d\mu_t \quad .$$

We suppose from now on that :

$$V(0) = 0 \quad . \tag{10.5}$$

The energy is then defined for fields whose support has a compact trace in S_t. For solutions of the Y.M.H. system in $C_0^\infty(T)$, we deduce from the equality :

$$\nabla_\alpha P^\alpha = T_{\alpha\beta}(\nabla^\alpha X^\beta + \nabla^\beta X^\alpha) = T.LX \quad ,$$

the energy equality :

$$y(t) = y(0) + \int_0^t \int_{S_\tau} T.LX \, d\mu_\tau \, d\tau \quad , \tag{10.6}$$

hence the energy inequality, with C_0 some constant depending only on (V_{n+1}, g) and X :

$$y(t) \leqslant y(0) + C_0 \int_0^t y(\tau) \, d\tau \quad . \tag{10.7}$$

If X is a Killing field of (V_{n+1}, g), then $C_0 = 0$; the energy is conserved. If $C_0 \neq 0$, the inequality implies that the continuous function y satisfies on V_T (Gronwall lemma) :

$$y(t) \leqslant C(t) \qquad \text{with} \quad C(t) = y(0) \exp(C_0 \, t) \quad . \tag{10.8}$$

This estimate bounds the E_0 norm of F and $\hat{\nabla}\Phi$, hence the L^2 norms of $B = A - \check{A}$ in $\check{A}$-temporal gauge and the norm of Φ in such a gauge, by using the relations

$$\check{\nabla}_0 B_i + \Gamma^j_{0i} B_j = F_{0i} - \check{F}_{0i} \quad , \qquad \check{\nabla}_0 \Phi = \hat{\nabla}_0 \Phi \quad ,$$

which are first order differential operators along the time line when

F_{0i} and $\hat{\nabla}_0 \Phi$ are given and lead to the inequalities :

$$\|B\|_t^2 \leqslant \|B\|_0^2 + C \int_0^t \{\|B\|_\tau^2 + \|B\|_\tau (\|F\|_\tau + \|\check{F}\|_\tau)\} \, d\tau \quad ,$$

where C depends only on (V_{n+1}, g) , and $\| \ \|_t$ stands for $\| \ \|_{L^2(S_t)}$, and :

$$\|\Phi\|_t^2 \leqslant \|\Phi\|_0^2 + 2 \int_0^t \|\Phi\|_\tau \, \|\hat{\nabla}_0 \Phi\|_\tau \, d\tau \quad .$$

These inequalities lead to estimates :

$$\|\Phi\|_t \leqslant C(t) \ , \qquad \|B\|_t \leqslant C(t) \quad ,$$

when such estimates are known for F and $\hat{\nabla}\Phi$.
The estimates in E_0 -norm of B , F , Φ , $\hat{\nabla}\Phi$ is not sufficient to obtain an estimate of $\check{\nabla}\Phi$, and is not sufficient to have global existence even for $n = 1$.

11. SECOND ENERGY ESTIMATE.

To bound the derivatives $\hat{\nabla}F$ and $\hat{\nabla}^2 \Phi$, one considers the 2-tensor

$$T^{(1)}_{\alpha\beta} = e^{\lambda\mu}\{\hat{\nabla}_\lambda F_{\alpha\rho} . \hat{\nabla}_\mu F_\beta{}^\rho - \frac{1}{4} \hat{\nabla}_\lambda F_{\rho\sigma} . \hat{\nabla}_\mu F^{\rho\sigma}\} + \mathrm{Re}(\hat{\nabla}_\lambda \hat{\nabla}_\alpha \Phi . \nabla_\mu \hat{\nabla}_\beta \Phi - \frac{1}{2} g_{\alpha\beta} \hat{\nabla}_\lambda \hat{\nabla}_\rho \Phi . \hat{\nabla}_\mu \nabla^\rho \Phi) \quad .$$

We have :

$$T^{(1)}_{\alpha\beta} X^\alpha n^\beta = \frac{1}{4} N^{-1} |\hat{\nabla}F|^2 + |\hat{\nabla}^2 \Phi|^2 \quad ,$$

and we show, by using the Ricci identity, that, when the Y.M.H. field equations are satisfied, $\nabla_\alpha T^{\alpha\beta}_{(1)}$ is a sum of terms of the form, where juxtaposition denotes scalar product, Lie bracket or action of the constant linear operator S :

$$f \, f \, \hat{\nabla} f \ , \qquad f = F \text{ or } \hat{\nabla} \Phi \quad ,$$

and, if g is non flat or $\nabla n \neq 0$, terms $f\,\hat{\nabla} f$, and moreover, if $V \not\equiv 0$:

$$V'(|\Phi|^2)\ \hat{\nabla}\Phi\ \hat{\nabla}^2\Phi \qquad \text{and} \qquad V''(|\Phi|^2)\ \Phi\ \Phi\ \hat{\nabla}\Phi\ \hat{\nabla}^2\Phi \ .$$

The integration of the relation :

$$\nabla_\alpha(T^{\alpha\beta}_{(1)}\ X_\beta) = X_\beta\ \nabla_\alpha\ T^{\alpha\beta}_{(1)} + \frac{1}{2}\ T_{(1)}\ .LX$$

leads to an inequality of the form, with C, C_1, C_2 constants depending only on (V_{n+1}, g, P) (remark : $C_i' = 0$ if $(V_{n+1}, g) = M_{n+1}$) :

$$(11.1) \quad y_1(t) \leq y_1(0) + C\int_0^t y_1(\tau)\,d\tau + C_1\int_0^t \int_{S_\tau} |f|^2\ |\hat{\nabla}f|\ d\mu_\tau\,d\tau$$

$$+ C_1'\int_0^t\int_{S_\tau} |f|\ |\hat{\nabla}f|\ d\mu_\tau\,d\tau + C_2\int_0^t\int_{S_\tau}\{|V'(|\Phi|)^2|\ |\hat{\nabla}\Phi|\ |\hat{\nabla}^2\Phi|$$

$$+ |V''(|\Phi|^2)|\ |\Phi|^2\ |\hat{\nabla}\Phi|\ |\hat{\nabla}^2\Phi|\}\ d\mu_\tau\,d\tau$$

where

$$(11.2) \qquad y_1(t) = \int_{S_t} |\hat{\nabla}f|^2\ d\mu_t \ ,$$

with

$$(11.3) \qquad |\hat{\nabla}f|^2 = \frac{1}{4}\ N^{-1}\ |\hat{\nabla}F|^2 + |\hat{\nabla}^2\ \Phi|^2 \ .$$

We have :

$$(11.4) \qquad \int_{S_\tau} |f|\ |\hat{\nabla}f|\ d\mu_\tau \leq \|f\|_\tau\ \|\hat{\nabla}f\|_\tau \ .$$

To estimate

$$(11.5) \qquad \int_{S_\tau} |f|^2\ |\hat{\nabla}f|\ d\mu_\tau \leq \|f\|^2_{L^4(S_\tau)}\ \|\hat{\nabla}f\|_\tau$$

through the estimate of $\|f\|_\tau$ obtained in the previous paragraph, we use the Gagliardo-Nirenberg inequality <u>when</u> $n < 4$:

$$(11.6) \qquad \|f\|_{L^4} \leq C\ \|f\|^{1-\sigma}_{L^2}\ \|\hat{\nabla}f\|^{\sigma}_{L^2} \ ,$$

with

$$\frac{1}{4} = \frac{1-\sigma}{2} + \sigma\left(\frac{1}{2} - \frac{1}{n}\right), \qquad \text{i. e.}\quad \sigma = \frac{n}{4}, \qquad n < 4 \ ,$$

which gives :

$$\int_{S_\tau} |f|^2 \ |\hat{\nabla}f| \ d\mu_\tau \leqslant C \ \|f\|^{(4-n)/4}_{L^2(S_\tau)} \ \|\hat{\nabla}f\|^{n/2+1}_{L^2(S_\tau)} \quad .$$

This inequality, inserted in (11.1), will lead, when the estimate of $y(t)$ from the previous paragraph is used, to an inequality containing no power $\geqslant 1$ of $y_1(\tau)$ if :

$$\frac{n}{2} + 1 \leqslant 2 \ , \qquad \text{i. e.} \quad n \leqslant 2 \quad .$$

Therefore :

Lemma : If $n \leqslant 2$ and $V \equiv 0$, the function $y_1(t)$ satisfies an estimate :

$$y_1(t) \leqslant C_1(t) \quad ,$$

where the continuous function $C_1 : \mathbb{R} \to \mathbb{R}$ depends only on (S,g,P) and, continuously, on $y_1(0)$.

12. GLOBAL EXISTENCE THEOREM FOR $n = 1$.

When $n = 1$, the local existence theorem is valid for b , $\bar{E}$, $\varphi \in H_1$, $\dot{\varphi} \in L^2$. An a priori estimate of the $E_1(T)$ norm of the solution for F and $\hat{\nabla}\Phi$ is sufficient to obtain the global existence. The previous paragraph leads to this a priori estimate if $V \equiv 0$ .No further restriction on V than $V \in C^2$ with $V \geqslant 0$ and $V(0) = 0$ supposed in previous paragraphs is necessary to obtain the a priori estimate of $y_1(t)$ if $n = 1$. Indeed, the estimate of $y(t)$ and a Sobolev inequality shows that, for our $C_0^\infty(T)$ fields, the L^∞ norm of Φ admits an estimate :

$$\|\Phi\|_{L^\infty(S_t)} \leqslant C(t) \quad .$$

The function $V(|\Phi|^2)$, $V'(|\Phi|)$ and $V''(|\Phi|)$, admits estimates of the same type, if V is C^2 .

Theorem : *The Y.M.H. equations with regular bundle P over a V_{1+1} regularly hyperbolic manifold admit a global solution, if :*

(a) The potential V *is* C^2 *, non negative, and* $V(0) = 0$ *;*
(b) The Cauchy data b *,* φ *are in* H_2 *,* $\bar{E}$ *,* $\dot{\varphi}$ *are in* H_1 *, and satisfy the constraint.*

13. GLOBAL EXISTENCE THEOREM FOR $n = 2$.

For $n = 2$, the local existence is valid for b , $\varphi \in H_2$, $\bar{E}$, $\dot{\varphi} \in H_1$. However, since we have no way of finding directly a priori estimates for B , but have to use the relation giving B in terms of F , we need to find a priori estimates for F in E_2 . This estimate is obtained by considering the tensor :

$$T^{(2)}_{\alpha\beta} = e^{\lambda_1\mu_1} e^{\lambda_2\mu_2} (\hat{\nabla}_{\lambda_1} \hat{\nabla}_{\lambda_2} \hat{F}^{\rho}_{\alpha} \cdot \hat{\nabla}_{\mu_1} \hat{\nabla}_{\mu_2} F_{\beta\rho} - \frac{1}{4} \hat{\nabla}_{\lambda_1} \hat{\nabla}_{\lambda_2} F_{\rho\sigma} \cdot \hat{\nabla}_{\mu_1} \hat{\nabla}_{\mu_2} F^{\rho\sigma}$$

$$+ 2\rho e \, \hat{\nabla}_{\lambda_1} \hat{\nabla}_{\lambda_2} \hat{\nabla}_{\alpha} \Phi \cdot \hat{\nabla}_{\mu_1} \hat{\nabla}_{\mu_2} \hat{\nabla}_{\beta} \Phi - g_{\alpha\beta} \hat{\nabla}_{\lambda_1} \hat{\nabla}_{\lambda_2} \hat{\nabla}_{\rho} \Phi \cdot \hat{\nabla}_{\mu_1} \hat{\nabla}_{\mu_2} \hat{\nabla}_{\rho} \Phi) ,$$

then :

$$T^{(2)}_{\alpha\beta} X^{\alpha} n^{\beta} = \frac{1}{4} N^{-1} |\hat{\nabla}^2 F|^2 + |\hat{\nabla}^3 \Phi|^2 ,$$

while $\nabla_{\alpha} T^{\alpha\beta}_{(2)}$ is found to be, modulo the Y.M.H. equations, a sum of terms of the form, with $f = F$ or $\hat{\nabla}\Phi$:

$$f \, \hat{\nabla} f \, \hat{\nabla}^2 f ,$$

and, if g is non flat :

$$\mathrm{Riem}(g) \, \hat{\nabla} f \, \hat{\nabla}^2 f , \qquad \nabla \mathrm{Riem}(g) \, f \, \hat{\nabla}^2 f .$$

We set, for $C^{\infty}_0(T)$ fields :

$$y_2(t) = \int_{S_t} |\hat{\nabla}^2 f|^2 \, d\mu_t , \qquad (12.1)$$

and we find, by the same method as in previous paragraphs, that, if $V \equiv 0$:

(12.2) $y_2(t) \leqslant y_2(0) + C_1 \int_0^t y_2^{1/2}(\tau) \{\int_{S_\tau} |\hat{\nabla} f|^2 \ |f|^2 \ d\mu_\tau\}^{1/2} d\tau$

$+ C_2 \int_0^t (y^{1/2}(\tau) + y_1^{1/2}(\tau))\ y_2^{1/2}(\ \tau) \quad d\tau + C_3 \int_0^t \int_{S_\tau} \{|V'(|\Phi|^2)|\ |\hat{\nabla}^2 \ \Phi|$

$+ V''(|\Phi|^2)|\Phi|(|\hat{\nabla}\Phi|^2 + |\Phi||\hat{\nabla}^2\Phi|) + V'''(|\Phi|^2)|\Phi|^3 \ |\hat{\nabla}\Phi|^2\} \ |\hat{\nabla}^3 \ \Phi| \ d\mu_\tau d\tau$

We use again the Cauchy-Schwarz and the Gagliardo-Nirenberg inequalities to obtain, if $n < 4$, here with $n = 2$, L^q standing for $L^q(S_\tau)$:

$$\{\int_{S_\tau} |\hat{\nabla} f|^2 \ |f|^2 \ d\mu_\tau\}^{1/2} \leqslant \|f\|_{L^4} \ \|\hat{\nabla} f\|_{L^4} \leqslant C \ \|f\|^{1/2}_{L^2} \ \|\hat{\nabla} f\|_{L^2} \ \|\hat{\nabla}^2 \ f\|^{1/2}_{L^2}$$

which leads to an integral inequality containing only powers $\frac{1}{2}$ and 1 of y_2, when the estimates previously found for y and y_1 are used.

Note that the use of the more general Gagliardo-Nirenberg inequality:

$$\|D^j \ u\|_{L^p} \leqslant C \ \|D^m \ u\|^a_{L^2} \ \|u\|^{1-a}_{L^2} , \qquad a = \frac{j}{m} + \frac{n}{2m} - \frac{n}{pm} , \qquad \frac{j}{m} \leqslant a < 1$$

does not lead either to an estimate of $y_1 + y_2$ when $n \geqslant 2$.

Lemma 2 : If $n = 2$ and $V \equiv 0$, the function y_2 satisfies an estimate :

$$y_2(t) \leqslant C_2(t)$$

with $C_2 : \mathbb{R} \to \mathbb{R}$ a continuous function depending only on (V_{n+1}, g, P) and, continuously, on $y(0)$, $y_1(0)$ and $y_2(0)$.

We now consider the case $V \not\equiv 0$. We deduce from the estimate of $y(t)$ and a Sobolev inequality that, for a $C_0^\infty(T)$ field we have, if $n = 2$

(12.3) $$\|\Phi\|_{L^q} \leqslant C(t) , \qquad 2 \leqslant q < \infty ,$$

hence the coefficient of C_2 in (11.1) is bounded by :

$$\int_0^t \{\|(|V'(|\Phi|^2)| + |V''(|\Phi|^2)||\Phi|^2)|\hat{\nabla}\Phi|\|_\tau \|\hat{\nabla}^2 \Phi\|_\tau\} d\tau \quad , \tag{12.4}$$

which will lead to a term at most linear in $\|\hat{\nabla}^2 \Phi\|_\tau^2$, hence linear in $y_1(\tau)$, through the use of the Gagliardo-Nirenberg estimate for $\hat{\nabla}\Phi$ and (12.3) if V is C^2 and its derivatives V' and V'' have at most a power growth, that is, there exist constants $C > 0$ and numbers p , $1 \leqslant p < \infty$, such that :

$$|V'(\rho)| \ , \ |V''(\rho)| \leqslant C\rho^p \qquad \text{for } \rho \geqslant 1 \quad . \tag{12.5}$$

(Note that V' and V'' are bounded for $\rho \leqslant 1$, since continuous.)
The inequality (12.2) will lead to an inequality at most linear in $y_2(\tau)$, if V is C^3 , and satisfies in addition to (12.4) :

$$|V'''(\rho)| \leqslant C\rho^p \qquad \text{for } \rho \geqslant 1 \quad . \tag{12.6}$$

From these results, and inspection of $y(0)$, $y_1(0)$, $y_2(0)$, we conclude :

Theorem : *The Yang-Mills-Higgs equations with regular bundle P over a regularly hyperbolic manifold (V_{2+1}, g) with bounded curvature and curvature gradient admit a global solution, if :*

1° The Cauchy data b , φ are in H_3 and $\bar{E}$, φ are in H_2 , and satisfy the constraint.

2° The interaction potential is C^3 , non negative, $V(0) = 0$, and V satisfies (12.5), (12.6).

CONCLUSION.

The proof of global existence of solutions of the Y.M.H. system, or even of the sourceless Y.M. equations on a general regularly hyperbolic manifold of dimension 4 does not follow from the standard use of Sobolev and Gagliardo-Nirenberg inequalities in view of obtaining the required a priori estimates.

The Eardley-Moncrief method for proving L^∞ estimates of F and Φ does not seem to extend to non conformally flat space-times. The problem is therefore open to know if the Y.M. equations admit, or do not admit, global solutions on a general regularly hyperbolic manifold

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