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UNDECIDABLE THEORIES OF VALUATED ABELIAN GROUPS

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INTRODUCTION

Since their first appearence in [5] valuated abelian groups have quickly developed into a popular and promising area of research in abelian group theory. For information on the goals and achievments of this theory we refer to the survey articles [4] and [2]. All we need about valuated abelian groups for the purpose of this paper will be explained in section 1 below.

We are interested in a modeltheoretic investigation of the class of valuated abelian groups. Ideally we would wish to obtain a complete classification upto elementary equivalence. Experience has shown that this problem can be attacked with hope for success only if the theory under consideration is decidable. (It is ofcourse possible to construct theories with a complete system of elementary invariants, where the question , which finite combinations of these are consistent is undecidable; but this situation is unlikely to occur for the "natural" theories arising from mathematical practise) Consequently the first step in the pursuit of our ideal goal is to ask: Is the theory of valuated abelian groups decidable ?

We consider valuated abelian groups as two-sorted structures and restrict attention to abelian groups with a p-valuation for just one prime p. The main results are:

Theorem: *The theory of p-valuated abelian groups is hereditarily undecidable.*

We will even show that the class of all p-valuated abelian groups, where the underlying group is a direct sum of copies of $\mathbb{Z}(p^9)$ is hereditarily undecidable.

Theorem: *The theory of p-valuated torsionfree abelian groups is hereditarily undecidable.*

It is possible to trace back the reasons for undecidability and arrive at classes of valuated p-groups and valuated torsionfree groups respectively for which a relative quantifier elimination procedure can be obtained (i.e. quantifiers over

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group elements are eliminated in favor of quantifiers over the linearly ordered set of values). These results together with the accompanying decidability results will appear elsewhere.

We assume that the reader is familiar with the basic facts about undecidability, abelian groups and ordinal arithmetic. All groups considered are assumed to be abelian.

§1 P-VALUATED GROUPS

Let G be a group, p a prime.

Definition: A p-valuation on G is a mapping v from G onto a successor ordinal $\alpha+1$ satisfying the following axioms:

(V1) $v(g-h) \geq \min\{v(g), v(h)\}$

(V2) $v(pg) > v(g)$ if $v(g) < \alpha$.

(V3) $v(g) = \alpha$ iff $g = 0$

We will follow established notation and write ∞ for α, the greatest possible value. Axiom (V3) is usually not counted among the axioms for a p-valuation, but including it here gives stronger undecidability results.

A p-valuated group is a group G together with a p-valuation. A valuated group is a group with a p-valutation for every prime p.

Lemma 1.1: *Every p-valuated group (G,v) satisfies for all $g,h \in G$:*

(i) if $v(g) < v(h)$ then $v(g+h) = v(g)$

(ii) if $m \in \mathbb{Z}$ is not divisible by p then $v(mg) = v(g)$.

Proof: Easy.

Definition: A p-filtration on G is a sequence G_β, $\beta\leq\alpha$ of subgroups of G such that: (F0) $G_0 = G$

(F1) $G_\beta \supset G_\gamma$ for $\beta < \gamma \leq \alpha$

(F2) $pG_\beta \subseteq G_{\beta+1}$

(F3) $G_\alpha = \{0\}$

There is a one-one correspondence between p-filtrations and p-valuations on G.

Lemma 1.2:

(i) *If* $v:G \longrightarrow \alpha+1$ *is a p-valuation then* $G_\beta = \{g\in G : v(g)\geq\beta\}$ *defines a p-filtration on* G.

(ii) *If* G_β , $\beta\leq\alpha$ *is a p-filtration then*

$$v(g) = \begin{cases} \text{the smallest } \beta<\alpha \text{ with } g \notin G_{\beta+1}, & \text{if there exists one} \\ \infty & \text{otherwise} \end{cases}$$

defines a p-valuation .

Proof: Obvious.

Definition: The direct product (sum) of a family (G_i, v_i) , $i\in I$ of p-valuated groups consists of the direct product $\prod(G_i : i\in I)$ (resp. direct sum $\Sigma(G_i : i\in I)$) of the underlying groups with the valuation v given in both cases by $v(g) = \min\{ v_i(g(i)) : i \in I\}$.

Definition : For given p-valuation v on G and integer $s \geq 1$ we denote by $v_{p,s}$ the function given by :

$$v_{p,s}(g) = \min\{ \beta : \text{there is no } h \in G \text{ such that } v(g+p^s h)\geq \beta \}$$

To make this definition work also for $g \in p^sG$ we add a new element ∞^+ on top of ∞. We thus have by definition for all $g \in G$: $g \in p^sG$ iff $v_{p,s}(g) = \infty^+$.

Let L be the two-sorted first-order language with one sort of variables denoted by $x,y,z,..$, the group variables, and the other sort of variables denoted by $\alpha,\beta,\gamma,...$, the value variables; furthermore L contains a symbol for the group operations $+,-$, a constant symbol 0 , a symbol for the order relation $\leq$ between values, a constant symbol ∞ and a symbol v for the valuation.

It is straightforward how p-valuated groups are regarded as L-structures.

Let $TV(p)$ denote the L-theory of the class of all p-valuated groups. There will certainly be models (M,v) of $TV(p)$ where the ordered set $Im(v)$ of values, while still a model of the theory of well-orderings is not a well-ordered set. These generalised p-valuated groups as we might call them will play no particular rôle in the following.

§2 THE UNDECIDABILITY RESULTS

Theorem 2.1: *TV(p) is hereditarily undecidable.*

This theorem is an obvious corollary to the following result:

Theorem 2.2: *The L-theory $T(p^9)$ of the class of p-valuated groups (G,v) with:*

(i) *G is a direct sum of copies of $\mathbb{Z}(p^9)$*

(ii) *$card(Im(v)) \leq 28$*

is hereditarily undecidable.

In the proof of theorem 2.2. we will use the following lemma :

Lemma 2.3: *The class of all groups G with two distinguished subgroups C_1, C_2 such that :*

(1) $C_2 \subsetneq C_1 \subseteq G$

(2) *G is a direct sum of copies of $\mathbb{Z}(p^9)$*

is hereditarily undecidable.

This lemma is obtained in turn from the following:

Lemma 2.4: *The class of all groups G satisfying $p^9 G = \{0\}$ with one distinguished subgroup C is hereditarily undecidable.*

To derive lemma 2.3. from lemma 2.4. we note that any pair (G,C) with $p^9 G = \{0\}$ can be interpreted as $(G/C_2, C_1/C_2)$ using a triple (G,C_1,C_2) subject to the conditions of lemma 2.3. Lemma 2.4. itself was proved in [6] with 12 in place of 9 . This latter improvement is due to W.Baur , [1] .

It seems to be an open question wether 9 is the best possible exponent in lemma 2.4.

Proof of Theorem 2.2.

Let L^* be obtained from L by adding two constant symbols γ_1, γ_2 for values and let $T^* = T(p^9) + \gamma_2 \geq \gamma_1$ .Because of $T^* \vdash \varphi(\gamma_1, \gamma_2)$ iff $T \vdash \forall\alpha,\beta(\alpha \leq \beta \longrightarrow \varphi(\alpha,\beta))$ it suffices to show that T^* is hereditarily undecidable. To achieve this we have to construct for every given triple (G,C_1,C_2) subject to the conditions of lemma 2.3. a p-valuation v on G such that

$C_j = \{g \in G : v(g) \geq \gamma_j\}$ for $j=1,2$.

Consider the following sequence H_n of subgroups of G :

$$H_n = p^nG + C_1 \quad \text{for } 0 \leq n < 9$$

$$H_{9+n} = p^nC_1 + C_2 \quad \text{for } 0 \leq n < 9$$

$$H_{18+n} = p^nC_2 \quad \text{for } 0 \leq n < 9$$

We get a p-filtration H'_n from H_n by dropping repetitions. Finally γ_1, γ_2 are chosen such that $H'_{\gamma_1} = H_9$ and $H'_{\gamma_2} = H_{18}$.

The undecidability theorem 2.2. did not use the full strenth of the language L; quantifiers over values were not used. This will change when we now consider the torsionfree case.

Theorem 2.5: *The theory T_{tf} of p-valuated torsionfree groups is hereditarily undecidable.*

We will prove the following stronger result:

Theorem 2.6: *The L-theory T^1_{tf} of the class of all p-valuated torsionfree groups (G, v) satisfying: (i) and (ii) is hereditarily undecidable .*

(i) G is divisible by any prime q , $q \neq p$.

(ii) for all $g \in G$, $g \neq 0$: $v(pg) = v(g) + 1$.

Proof: We will interpret in T^1_{tf} the theory of two equivalence relations which by [3 , p.295] is hereditarily undecidable (even finitely inseperable).

We first list the formulas needed in this interpretation. Let $s \geq 2$ be an integer fixed for the remainder of this proof.

$$\varphi_0(\alpha) = \exists x(v_{p,s}(x)=\alpha) \ \& \ \text{"}\alpha=\omega n \text{ for some } n, 0 < n<\omega\text{"}$$

$$\chi_1(\alpha,\gamma) = \varphi_0(\alpha) \ \& \ \text{"}\gamma > \omega^2 \cdot 2\text{"} \ \& \ \exists x(v_{p,s}(x)=\alpha \ \& \ v_{p,s}(px)=\gamma) \ \& \ \& \ \forall x(v_{p,s}(x)=\alpha \rightarrow v_{p,s}(px) \leq \gamma)$$

$$\varphi_1(\alpha,\beta) = \varphi_0(\alpha) \ \& \ \varphi_0(\beta) \ \&[\exists\gamma(\chi_1(\alpha,\gamma) \ \& \ \chi_1(\beta,\gamma)) \vee \alpha=\beta].$$

$$\chi_2(\alpha,\gamma) = \varphi_0(\alpha) \ \& \ \text{"}\omega^2 < \gamma < \omega^2 \cdot 2\text{"} \ \& \ \exists x(v_{p,s}(px)=\alpha \ \& \ v_{p,s}(p^2x)=\gamma) \ \& \ \& \ \forall x(v_{p,s}(px)=\alpha \rightarrow v_{p,s}(p^2x) \leq \gamma)$$

$$\varphi_2(\alpha,\beta) = \varphi_0(\alpha) \,\&\, \varphi_0(\beta) \,\&\, [\exists\gamma(\chi_2(\alpha,\gamma) \,\&\, \chi_2(\beta,\gamma)) \vee \alpha = \beta\,]$$

By definition of χ_i there can be for every α at most one γ with $\chi_i(\alpha,\gamma)$. Thus we see that forevery model (G,v) of T^1_{tf} φ_i^G defines an equivalence relation on φ_0^G for $i=1,2$.

Now let V be a countable set and E_1,E_2 equivalence relations on V. We shall construct a p-valuated torsionfree group (G,v) satisfying conditions (i),(ii) such that $(\varphi_0^G,\varphi_1^G,\varphi_2^G) \simeq (V,E_1,E_2)$.

For this purpose let $f:V \to \omega-\{0\}$ be an injection and $\{C_{m,i} : 1 \le m < k_i\}$ enumerations of all E_i-equivalence classes, $i = 1,2$; $k_i \le \omega$.

As a preparation we introduce groups (G_α,v_α) for all α, $0 \le \alpha \le \omega^2\cdot 3$ by

$G_\alpha \simeq \mathbb{Z}_p$ = the subgroup of the rationals consisting of all fractions z_0/z_1 with z_1 prime to p.

and

$$v_\alpha(z) = \begin{cases} \infty = \omega^2\cdot 3 & \text{if } z = 0 \\ \alpha + k & \text{if } z = p^k z_0/z_1 \text{ with } (p,z_0) = 1 . \end{cases}$$

Let $(G^*,v^*) = \prod_\alpha (G_\alpha,v_\alpha)$ and $(G^o,v^o) = \sum_\alpha (G_\alpha,v_\alpha)$

We observe the following easy facts:

(o) for $g \in G^*$, $g \neq 0$: $v^*(pg) = v^*(g) + 1$

(1) for $g \in G^o$ $v^o_{p,s}(g)$ is never a limit

(2) if for $g \in G^*$ $v^*(g) \ge \alpha$ and α is a limit, then for all $\gamma<\alpha$ $g(\gamma)=0$.

(3) if for $g \in G^*$ $v^*_{p,s}(g) \ge \alpha$ and α is a limit ordinal , then for all $\gamma < \alpha$ $g(\gamma) \in p^s Z_p$.

Fix $x \in V$.

Let $C_{m_i,i}$ be the E_i-equivalence class of x. We define elements $a_{x,i}, b_{x,i}$ of G^* as follows:

$$a_{x,1}(\gamma) = a_{x,2}(\gamma) = \begin{cases} p & \omega\cdot(f(x)-1) \le \gamma < \omega\cdot f(x) \\ 0 & \text{otherwise} \end{cases}$$

$$b_{x,1}(\gamma) = \begin{cases} p^{s-1} & \omega^2\cdot 2 + \omega(m_1-1) \le \gamma < \omega^2\cdot 2 + \omega\cdot m_1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_{x,2}(\gamma) = \begin{cases} p^{s-2} & \text{if } \omega^2 + \omega(m_2-1) \leq \gamma < \omega^2 + \omega\cdot m_2 \\ 0 & \text{otherwise} \end{cases}$$

Let $G_{(x,i)}$ be the $\mathbb{Z}_p$-submodule of G^* generated by $G^0 \cup \{a_{x,i}, b_{x,i}\}$ and $v^{(x,i)}$ the restriction of v^* to $G_{(x,i)}$.

The following properties of these groups are easily verified:

(4) If for $g \in G_{(x,1)}$ $v_{p,s}^{(x,1)}(g)$ is a limit ordinal, then it is equal to $\omega\cdot f(x)$ or $\omega^2\cdot 2 + \omega\cdot m_1$

(5) $v_{p,s}^{(x,1)}(p^{s-1}a_{x,1} + b_{x,1}) = \omega\cdot f(x)$

$v_{p,s}^{(x,1)}(p^s a_{x,1} + pb_{x,1}) = v_{p,s}^{(x,1)}(pb_{x,1}) = \omega^2\cdot 2 + \omega\cdot m_1$

(6) If for $g \in pG_{(x,1)}$ $v_{p,s}^{(x,1)}(g)$ is a limit , then $v_{p,s}^{(x,1)}(g) = \omega^2\cdot 2 + \omega\cdot m_1$

(7) If for $g \in G_{(x,2)}$ $v_{p,s}^{(x,2)}(g)$ is a limit ordinal then it is equal to $\omega\cdot f(x)$ or $\omega^2 + \omega\cdot m_2$.

(8) $v_{p,s}^{(x,2)}(p^{s-1}a_{x,2} + pb_{x,2}) = \omega\cdot f(x)$

$v_{p,s}^{(x,2)}(p^s a_{x,2} + p^2 b_{x,2}) = v_{p,s}^{(x,2)}(p^2 b_{x,2}) = \omega^2 + \omega\cdot m_2$.

Finally we set : $(G,v) = \oplus \sum_x [(G_{(x,1)}, v^{(x,1)}) \oplus (G_{(x,2)}, v^{(x,2)})]$

By definition we have :

(9) $v_{p,s}(g) = \min\{ v_{p,s}^{(x,i)}(g(x,i)) : x \in V , i=1,2 \}$

From this :

(10) $\varphi_0^G = \{\omega\cdot f(x) : x \in V \}$

We claim for all $x \in V$:

(11) If for $g \in G$ $v_{p,s}(g) = \omega\cdot f(x)$ and $V_{p,s}(pg)$ is a limit $< \infty$,then $v_{p,s}(pg) \leq \omega^2\cdot 2 + \omega\cdot m_1$ where $C_{m_1,1}$ is the E_1-equivalence class of x.

Let $g = \sum_{y\in V} \sum_{i=1,2} g(y,i)$ with $g(y,i) \in G_{(y,i)}$. By (10) $v_{p,s}(g) = \omega\cdot f(x)$ implies $v_{p,s}^{(x,i)}(g(x,i)) = \omega\cdot f(x)$ for i=1 or i=2 . Now the claim follows from (7) and (4) .

By (11) and (5) we get for $x,y \in V$:

(12) If xE_1y then $(G,V) \models \varphi_1(\omega\cdot f(x), \omega\cdot f(y))$

Furthermore we claim for all $x \in V$:

(13) If for $g \in G$ $v_{p,s}(pg) = \omega\cdot f(x)$ and $v_{p,s}(p^2g)$ is a limit $< \infty$,then $v_{p,s}(pg) \leq \omega^2 + \omega\cdot m_2$ where $C_{m_2,2}$ is the E_2-equivalence class of x.

To see this let g again be given in the form $\sum_{y\in V} \sum_{i=1,2} g(y,i)$. By (10) and (6) we must have $v_{p,s}^{(x,2)}(g(x,2)) = \omega\cdot f(x)$ which yields the desired result by (12)

By (13) and (8) we get for all $x,y \in V$:

(14) If xE_2y then $(G,v) \models \varphi_2(\omega\cdot f(x), \omega\cdot f(y))$

The reverse implications of (12) and (14) follow simply from the fact that $\chi_1(\omega\cdot f(x),\gamma)$ (resp. $\chi_2(\omega\cdot f(x),\gamma)$) implies $\gamma = \omega^2\cdot 2 + \omega\cdot m_1$ ($\gamma = \omega^2 + \omega\cdot m_2$)

Complementary to theorem 2.6. we have the following undecidability result:

Theorem 2.7. *The L-theory T^2_{tf} of the class of all p-valuated torsionfree groups (G,v) satisfying :*

(i) for all $s \geq 1$ and all $g \in G$ $v_{p,s}(g)$ is not a limit number

(ii) for all $g \in G, g \neq 0$ $v(pg) = v(g) + 1$.

is hereditarily undecidable.

The proof of Theorem 2.7. follows along the very same lines as that of the previous theorem. So we will only give a sketch.

Fix a prime number q, $q \neq p$ and an integer $s \geq 2$. Again we will interpret the theory of two equivalence relations in T^2_{tf} , this time using $v_{q,s}$ rather than $v_{p,s}$ .Since $v_{q,s}(g)$ can never by a successor ordinal $\neq \infty^+$, we have to consider higher powers of ω. We use the following formulas :

$$\varphi_0(\alpha) = \exists x(v_{q,s}(x)=\alpha) \ \& \ \text{"}\alpha = \omega^2\cdot n \text{ for some } n \text{ , } 0 < n < \omega\text{"}$$

$$\chi_1(\alpha,\gamma) = \varphi_0(\alpha) \ \& \ \text{"}\gamma > \omega^3\cdot 2\text{"} \ \& \ \exists x(v_{q,s}(x)=\alpha \ \& \ v_{q,s}(qx)=\gamma) \ \& \ \& \ \forall x(v_{q,s}(x)=\alpha \to v_{q,s}(qx) \leq \gamma))$$

$$\chi_2(\alpha,\gamma) = \varphi_0(\alpha) \ \& \ \text{"}\omega^3 < \gamma < \omega^3\cdot 2\text{"} \ \& \ \exists x(v_{q,s}(qx)=\alpha \ \& \ v_{q,s}(q^2x)= \gamma) \ \& \ \& \ \forall x(v_{q,s}(qx)=\alpha \to v_{q,s}(q^2x)\leq \gamma))$$

φ_1,φ_2 arise from φ_0, χ_1, χ_2 as in the proof of theorem 2.6.

Given two equivalence relations E_1,E_2 on a countable set V we construct a p-valuated torsionfree group satisfying (i),(ii) such that $(\varphi_0^G,\varphi_1^G,\varphi_2^G) \simeq (V,E_1,E_2)$. For α, $0 \leq \alpha < \omega^3\cdot 3$ we define p-valuated groups (G_α,v_α) by :

$G_\alpha \simeq \mathbb{Z}$ for all α

$$v_\alpha(z) = \begin{cases} \infty = \omega^3\cdot 3 & \text{if } z = 0 \\ \alpha+k & \text{if } z = p^k z_0 \text{ with } (p,z_0) = 1 . \end{cases}$$

(G^*,v^*) , (G^0,v^0) denote the direct product , direct sum of the family (G_α,v_α) $0 \leq \alpha < \omega^3\cdot 3$ .We observe :

(1) Let $g \in G^*$, $g \in q^sG^*$, $\alpha = \min\{\gamma: g(\gamma) \in q^s Z\}$ and β the smallest limit ordinal $> \alpha$, then $v^*_{q,s}(g) = \beta$.

Fix $x \in V$ and let $m_1.m_2$ be defined as in the proof of theorem 2.6. . We define elements $a_{x,i}$,$b_{x,i}$ of G^* by :

$$a_{x,1}(\gamma) = a_{x,2}(\gamma) = \begin{cases} q & \text{if } \omega^2(f(x)-1) \leq \gamma < \omega^2\cdot f(x) \\ 0 & \text{otherwise} \end{cases}$$

$$b_{x,1}(\gamma) = \begin{cases} q^{s-1} & \text{if } \omega^3\cdot 2 + \omega^2(m_1-1) \leq \gamma < \omega^3\cdot 2 + \omega^2\cdot m_1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_{x,2}(\gamma) = \begin{cases} q^{s-2} & \text{if } \omega^3 + \omega^2(m_2-1) \leq \gamma < \omega^3 + \omega^2\cdot m_2 \\ 0 & \text{otherwise} \end{cases}$$

From this data we obtain $(G_{(x,i)},v^{(x,i)})$ and (G,v) as before. The verification that $x \to \omega^2\cdot f(x)$ is an isomorphism from (V,E_1,E_2) onto $(\varphi_0^G,\varphi_1^G,\varphi_2^G)$ now parallels the corresponding argument in the proof of theorem 2.6.

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