

# MÉMOIRES DE LA S. M. F.

K. A. RIBET

## **Division fields of abelian varieties with complex multiplication**

*Mémoires de la S. M. F. 2<sup>e</sup> série*, tome 2 (1980), p. 75-94

[http://www.numdam.org/item?id=MSMF\\_1980\\_2\\_2\\_75\\_0](http://www.numdam.org/item?id=MSMF_1980_2_2_75_0)

© Mémoires de la S. M. F., 1980, tous droits réservés.

L'accès aux archives de la revue « Mémoires de la S. M. F. » (<http://smf.emath.fr/Publications/Memoires/Presentation.html>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

DIVISION FIELDS OF ABELIAN VARIETIES  
WITH COMPLEX MULTIPLICATION

by

K. A. RIBET

1. Let  $A$  be an abelian variety over a number field  $k$ . Suppose that  $A$  has complex multiplication over  $k$  in the sense that  $(\text{End}_k A) \otimes \mathbb{Q}$  contains a commutative semisimple algebra  $E$  over  $\mathbb{Q}$  of rank  $2 \cdot \dim A$ . For  $N \geq 1$ , let  $d(N)$  be the degree over  $k$  of the field  $k(A_N)$  obtained by adjoining to  $k$  the kernel  $A_N$  of multiplication by  $N$  on  $A$ . Let  $\alpha(N)$  be the number of (distinct) prime factors of  $N$ . The main purpose of this paper is to prove the following result :

THEOREM (1.1) : There exist positive constants  $C_1, C_2$  and an integer  $\nu > 0$  (depending on  $A, k$ , and  $E$ ) such that we have

$$C_1^{\alpha(N)} < \frac{d(N)}{N^\nu} < C_2^{\alpha(N)}$$

for all  $N \geq 1$ .

We prove (1.1) by writing the  $\ell$ -adic representations of  $A$  in the form given them by Serre-Tate [15] (using the theory of Shimura-Taniyama [17] and Weil [18]) and then exploiting some elementary facts about the "mod  $\ell^n$ " points

of tori over  $\mathbb{Q}$ . The possibility of doing this was suggested by a letter from Serre to Masser [13] concerning the special case where  $A$  is a product of several elliptic curves<sup>\*</sup>; to prove (1.1) we have followed Serre's arguments. It should be noted that a variant of (1.1) was proved by T. Kubota [4], who considered integers  $N$  of the form  $l^n$ ,  $l$  being a fixed prime. Also, in this volume, Masser [6] has treated prime numbers  $N$ , for abelian varieties of dimension 2.

In our proof of (1.1), the integer  $\nu$  arises as the dimension of a certain torus which is familiar in other contexts. Namely, it is the Hodge group of  $A$  (see [1] and §§ 3,4 of [14]), and so its dimension bounds the transcendence degree of the field generated by the periods of differentials on  $A$  [1]. For us, the torus is given as the image of a certain explicit map between tori; therefore, via the dictionary between tori and their character groups, computing  $\nu$  comes down to computing the rank of a certain matrix. In § 3 of this paper, we write down explicitly the matrix that intervenes, and this enables us to give the lower bound  $2 + \text{Log}_2 d$  for  $\nu$  in the case where  $A$  is absolutely simple,  $d$  denoting the dimension of  $A$ . There is also a trivial upper bound for  $\nu$ , namely the sum of 1 and the dimension of  $A$ . When  $\nu$  attains this upper bound, we say (following Kubota) that  $A$  is "non-degenerate." For  $A$  absolutely simple, it follows from our lower bound that  $A$  is always non-degenerate for  $d = 1, 2, 3$ . We give several examples (due, variously, to Shimura, Serre, and Lenstra) of absolutely simple  $A$  which do not have this property. The smallest example has  $d = 4$  and  $\nu = 4$ ; this is given by the CM type constructed by Mumford and described in [9].

For the reader who was present at the Conference, it might be pointed out that this article bears no relation to the author's talk, for which one can consult [10]. The material concerning the calculation of  $\nu$  is based on a manuscript written in 1977-78 after correspondence with Masser and discussions with Serre and Lenstra. It later formed the basis for a talk by the author at the Rennes conference on algebraic geometry in June, 1978.

---

\* For the convenience of the reader, the text of this letter (and a sequel) has been included as an appendix to this paper.

DIVISION FIELDS OF ABELIAN VARIETIES

2. Let  $T$  be a torus over  $\mathbb{Q}_\ell$ , and let

$$X(T) = \text{Hom}_{\mathbb{Q}_\ell} (T, \mathbb{G}_m)$$

be the character group of  $T$ . Using an idea of Ono [7, § 2], we define subgroups  $T(1 + \ell^n \mathbb{Z}_\ell)$  ( $n \geq 0$ ) of  $T(\mathbb{Q}_\ell)$  by the rule

$$T(1 + \ell^n \mathbb{Z}_\ell) = \{ t \in T(\mathbb{Q}_\ell) \mid \chi(t) \equiv 1 \pmod{\ell^n} \text{ for all } \chi \in X(T) \}.$$

We have  $\chi(t) \in \overline{\mathbb{Q}_\ell}^*$ , and the condition  $\chi(t) \equiv 1 \pmod{\ell^n}$  means that  $\text{ord}_\ell(\chi(t) - 1)$  is at least  $n$ . Thus  $T(\mathbb{Z}_\ell)$  is the maximal compact subgroup of  $T(\mathbb{Q}_\ell)$ , and the various  $T(1 + \ell^n \mathbb{Z}_\ell)$  define a filtration of  $T(\mathbb{Z}_\ell)$  by open subgroups. We further define

$$T(\mathbb{Z}/\ell^n \mathbb{Z}) = T(\mathbb{Z}_\ell) / T(1 + \ell^n \mathbb{Z}_\ell) \quad (n \geq 0).$$

Example (2.1) : Let  $E = E_1 \times \dots \times E_m$  be a product of finite extensions of  $\mathbb{Q}_\ell$ , and let  $T$  be the torus obtained by viewing  $E^*$  as an algebraic group over  $\mathbb{Q}_\ell$ . Then we have

$$T(\mathbb{Z}_\ell) = R^*,$$

where  $R$  is the integer ring of  $E$ , namely the product of the integer rings of the  $E_i$ . Further, for  $n \geq 1$  we have

$$T(1 + \ell^n \mathbb{Z}_\ell) = \{ r \in R^* \mid r \in 1 + \ell^n R \},$$

and

$$T(\mathbb{Z}/\ell^n \mathbb{Z}) = (R/\ell^n R)^*.$$

It is easy to check that the cardinality of  $T(\mathbb{Z}/\ell^n \mathbb{Z})$  is given by the formula

$$\ell^{nv} = \prod_{i=1}^m \frac{q_i - 1}{q_i},$$

where  $v$  is the dimension of the torus  $T$  (i.e. the sum of the degrees of the  $E_i$  over  $\mathbb{Q}_\ell$ ) and  $q_i$  is the order of the residue field of  $R_i$ , for  $i = 1, \dots, m$ . (Cf. [12], Ch.IV, § 2, Prop. 6.) A special case occurs when  $E$  is given by  $F \otimes \mathbb{Q}_\ell$ , where  $F$  is a finite extension of  $\mathbb{Q}$ , or a product of such extensions. Then we have  $T = T_{F/\mathbb{Q}_\ell}$ , where  $T_F$  is the torus over  $\mathbb{Q}$  defined by  $F^*$ .

Remark (2.2) : If  $T$  has "good reduction" (i.e.  $T$  is split over a finite unramified extension of  $\mathbb{Q}_\ell$ ), then there is a commutative smooth group scheme  $T/\mathbb{Z}_\ell$  whose general fibre is the torus  $T$ . One may show that  $T(\mathbb{Z}/\ell^n\mathbb{Z})$  coincides with the group of  $\mathbb{Z}/\ell^n\mathbb{Z}$ -valued points of  $T/\mathbb{Z}_\ell$ , for  $n \geq 1$ . The special case  $n = 1$  is particularly easy to treat because it then suffices to identify  $T(\mathbb{Z}/\ell\mathbb{Z})$  with the group of rational points of the reduction  $T/\mathbb{F}_\ell$  of  $T$  (i.e. of  $T/\mathbb{Z}_\ell \pmod{\ell}$ ). For this see Prop. 2.3.1 of [7], where  $T/\mathbb{F}_\ell$  is defined directly by declaring its character group to be  $X(T)$ , viewed as a  $\text{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell)$ -module (which we may do since it is an unramified  $\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ -module by hypothesis).

We now consider a map  $\lambda : T \rightarrow T'$  between  $\mathbb{Q}_\ell$ -tori. For  $n \geq 0$ , it is clear that  $\lambda$  induces maps

$$\begin{aligned} T(1 + \ell^n\mathbb{Z}_\ell) &\longrightarrow T'(1 + \ell^n\mathbb{Z}_\ell) \\ T(\mathbb{Z}/\ell^n\mathbb{Z}) &\longrightarrow T'(\mathbb{Z}/\ell^n\mathbb{Z}) \end{aligned}$$

We denote the second map by  $\lambda_n$ .

Theorem (2.3) : If  $\lambda$  is surjective, then the order of the cokernel of  $\lambda_n$  is bounded independently of  $n$ . If  $\lambda$  is an isogeny, then both the kernel and the cokernel of  $\lambda_n$  have bounded order.

Proof : When  $\lambda$  is surjective, the map

$$\alpha : T(\mathbb{Z}_\ell) \rightarrow T'(\mathbb{Z}_\ell)$$

induced by  $\lambda$  has an open image, as one may see by viewing the two groups as  $\ell$ -adic Lie groups, since the surjectivity of  $\lambda$  just means that the map on Lie algebras induced by  $\alpha$  is surjective. (See Bourbaki, Groupes et Algèbres de Lie, Ch. III, Prop. 28 of § 3, n° 8 and Th. 2 of § 7, n° 1.) Since  $T'(\mathbb{Z}_\ell)$  is compact, the cokernel of  $\alpha$  is finite. Since the cokernel of  $\lambda_n$  is a quotient of this cokernel for each  $n$ , we get the first statement of the theorem.

Now assume that  $\lambda$  is an isogeny. Then the kernel of  $\alpha$  is finite, and the assertion to be proved reduces, via the snake lemma, to the assertion that the cokernel of the restriction

$$\alpha_n : T(1 + \ell^n \mathbb{Z}_\ell) \rightarrow T'(1 + \ell^n \mathbb{Z}_\ell)$$

of  $\alpha$  to  $T(1 + \ell^n \mathbb{Z}_\ell)$  has order which is bounded independently of  $n$ . For this, we consider the "transpose" isogeny  $\lambda^\wedge$  and define  $\alpha_n^-$  for  $n \geq 0$  as the map for  $\lambda^\wedge$  analogous to  $\alpha_n$ . Then  $\alpha_n \circ \alpha_n^-$  is just multiplication by the degree of  $\lambda$  on  $T'(1 + \ell^n \mathbb{Z}_\ell)$ . It is clear for  $n$  sufficiently large that  $T'(1 + \ell^n \mathbb{Z}_\ell)$  is isomorphic to the group  $\mathbb{Z}_\ell^\nu$  where  $\nu = \dim T = \dim T'$  is independent of  $n$ . Hence the cokernel of  $\alpha_n \circ \alpha_n^-$ , and therefore that of  $\alpha_n$ , has bounded order.

We now consider once again a surjection  $\lambda : T \rightarrow T'$  over  $\mathbb{Q}_\ell$ , and write  $\lambda^*$  for the corresponding inclusion

$$X(T') \hookrightarrow X(T)$$

of character groups. Let  $X''$  be the subgroup of  $X(T)$  given by

$$X'' = \{ \chi \in X(T) \mid n\chi \in \lambda^*(X(T')) \text{ for some } n \geq 1 \}.$$

Then  $(X'' : X(T'))$  is finite, and  $X(T)/(X'')$  is torsion free. If we let  $T''$  be the  $\mathbb{Q}_\ell$ -torus corresponding to  $X''$ , then the inclusion

$$X(T') \subseteq X''$$

corresponds to an isogeny

$$v : T'' \rightarrow T'$$

and the inclusion  $X(T'') \hookrightarrow X(T)$  corresponds to a map

$$\mu : T \rightarrow T''$$

whose kernel is connected (i.e. is a torus). We have

$$\lambda = v \circ \mu.$$

Theorem (2.4) : Suppose that  $X(T)$  is an unramified  $\text{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ -module, so that the tori  $T, T', T''$  have good reduction. Suppose also that  $\ell$  is prime to the degree  $N$  of the isogeny  $v$ . Then, with the notation as in (2.3), the order of the cokernel of  $\lambda_n$  is bounded by  $N$ . If, furthermore,  $\lambda$  is an isogeny (so that  $\lambda = v$ ), then the kernel of  $\lambda_n$  again has order bounded by  $N$ .

Proof : It is known that the map  $T(\mathbb{Z}_\ell) \rightarrow T''(\mathbb{Z}_\ell)$  induced by  $\mu$  is surjective because of the good reduction hypothesis ([8], § 4.2). It will therefore be enough to prove the statements when  $\lambda$  is an isogeny, which we now suppose to be the case. Under our hypotheses, the map

$$\alpha_1 : T(1 + \ell\mathbb{Z}_\ell) \rightarrow T'(1 + \ell\mathbb{Z}_\ell)$$

induced by  $\lambda$  is known to be an isomorphism [7, Prop. 2.2.2]. It follows formally from this that the maps

$$\alpha_n : T(1 + \ell^n\mathbb{Z}_\ell) \rightarrow T'(1 + \ell^n\mathbb{Z}_\ell)$$

DIVISION FIELDS OF ABELIAN VARIETIES

are isomorphisms for all  $n \geq 1$ . [Given  $t' \in T'(1 + \ell^n \mathbb{Z}_\ell)$ , suppose that  $t' = \lambda(t)$  for  $t \in T(1 + \ell \mathbb{Z}_\ell)$ . It is clear that we have

$$\begin{aligned} \chi(t) &= 1 \pmod{\ell} \\ \chi(t)^N &= 1 \pmod{\ell^n} \end{aligned}$$

for all  $\chi \in X(T)$ ; from this it follows that we have  $\chi(t) = 1 \pmod{\ell^n}$  for all  $\chi$ , giving  $t \in T(1 + \ell^n \mathbb{Z}_\ell)$ .]

We find that the cokernel and kernel of  $\lambda_n$  are independent of  $n$ , for  $n \geq 1$ . Taking  $n = 1$ , we see that  $\lambda_1$  is the map on  $\mathbb{F}_\ell$  - points induced by the reduction  $\lambda_{/\mathbb{F}_\ell} : T_{/\mathbb{F}_\ell} \rightarrow T'_{/\mathbb{F}_\ell}$  of  $\lambda$ . Thus the kernel of  $\lambda_1$  is the group of  $\mathbb{F}_\ell$  - rational points of the kernel of  $\lambda$ , so its order is in particular a divisor of  $N$ . On the other hand, it is well known that the kernel and cokernel of  $\lambda$ , have equal orders, because of Lang's Theorem ([5]; cf. [11] Ch.VI, n°6, Prop.5) and the triviality of the Herbrand quotient of a finite module. (Equivalently, the isogenous tori  $T_{/\mathbb{F}_\ell}$  and  $T'_{/\mathbb{F}_\ell}$  have the same number of rational points.) This completes the proof.

We next consider the situation where we are given tori over  $\mathbb{Q}$ . If  $T_{/\mathbb{Q}}$  is a torus, we define  $T(\mathbb{Z}/\ell^n \mathbb{Z})$  to be  $T_{/\mathbb{Q}_\ell}(\mathbb{Z}/\ell^n \mathbb{Z})$  for each prime  $\ell$  and all  $n \geq 1$ . Given a map  $\lambda: T \rightarrow T'$  between two tori, we now write  $\lambda_{\ell,n}$  for the induced maps  $T(\mathbb{Z}/\ell^n \mathbb{Z}) \rightarrow T'(\mathbb{Z}/\ell^n \mathbb{Z})$ .

Theorem (2.5) : Given a torus  $T$  over  $\mathbb{Q}$ , there are constants  $C, C' > 0$  such that we have

$$C \leq \ell^{-nv} \text{Card}(T(\mathbb{Z}/\ell^n \mathbb{Z})) \leq C'$$

for all  $n \geq 1$  and all primes  $\ell$ , where  $v$  is the dimension of  $T$ .

Theorem (2.6) : Let  $\lambda: T \rightarrow T'$  be a surjective map between  $\mathbb{Q}$ -tori. Then the order of coker  $\lambda_{\ell,n}$  is bounded independently of  $\ell$  and  $n$ . If moreover  $\lambda$  is an



isogeny, the order of the ker  $\lambda_{\ell,n}$  is similarly bounded.

Proofs : The second theorem is an obvious consequence of (2.3), (2.4). To prove (2.5), we will use (2.6) and a general philosophy due to Ono. First, we observe that (2.5) is visibly correct in the special case where  $T = T_E$  is the torus attached to a finite extension  $E$  of  $\mathbb{Q}$ , cf. (2.1). Next we notice, by Brauer's theorem on induced characters, that there are finite extensions  $K_1, \dots, K_n$ ;  $L_1, \dots, L_m$  of  $\mathbb{Q}$  and an integer  $r > 0$  such that the two tori

$$\left\{ \begin{array}{l} T^r \times (T_{K_1} \times \dots \times T_{K_n}) \\ T_{L_1} \times \dots \times T_{L_m} \end{array} \right.$$

are isogenous. (See [7], Th. 1.5.1.) Since (2.5) holds for the second of the two, it holds for the first by (2.6). It thus holds for  $T^r$  (using again the case  $T = T_E$ ) and thus for  $T$ .

Corollary (2.7) : Let  $\lambda : T \rightarrow T'$  be a homomorphism of  $\mathbb{Q}$ -tori. There exist constants  $C, C' > 0$  such that we have

$$C < \frac{\text{Card}(\text{Im}(\lambda_{\ell,n}))}{\ell^{nv}} < C'$$

for all  $\ell$  and  $n$ , where  $v$  is the dimension of the image of  $\lambda$ .

Proof : Without loss of generality, we may suppose  $\lambda$  surjective. Then it is clear that our result follows from (2.5) and (2.6).

3. We now wish to deduce (1.1) from the above corollary. Before beginning to do so, we make some preliminary simplifications. It is clear that to prove (1.1) for a given  $A/k$ , we may replace  $k$  by a finite extension of  $k$  and  $A$  by an abelian variety which is isogenous to it. We thus introduce the hypothesis that  $A$  has everywhere good reduction, as we have a right to do by a well known theorem of Serre-Tate ([15], Th. 7). Secondly, after replacing  $A$  by a variety isogenous to it, we may suppose that  $\text{End}_k A$  contains the "integer ring" of  $E$ . Namely, if we write  $E$  as a product  $E_1 \times \dots \times E_t$  of fields, we suppose that  $\text{End}_k A$  contains the product  $\mathcal{O}$  of the integer rings  $\mathcal{O}_i$  of the  $E_i$ . In particular, this assumption enables us to write  $A$  as a product

$$A = A_1 \times \dots \times A_i$$

where  $A_i$  has complex multiplication by  $\mathcal{O}_i$ . (It is clear that  $[E_i : \mathbb{Q}] = 2 \dim A_i$  for each  $i$  because the first member is known to be a divisor of the second [17, Prop.2, p.39] and because of the assumption  $[E : \mathbb{Q}] = 2 \dim A$ .)

It is well known (see ch. II, §.5.1 of [17]) that each  $A_i$  is isogenous over  $\bar{k}$  to a power of an (absolutely) simple abelian variety  $B_i$  of CM type. In proving (1.1), we may replace each  $A_i$  by the corresponding  $B_i$  (making a finite extension of  $k$  at the same time). This enables us to assume that each of the  $A_i$  occurring above is in fact absolutely simple. This means that each of the fields  $E_i$  is a CM field and that the "CM type" attached to each  $A_i$  is simple in a sense which will be presently recalled. This assumption, and the previous ones will be in force for the remainder of this paper. To summarize, we assume :

- i) that  $A$  has everywhere good reduction over  $k$  ;
- ii) that  $A$  is given as a product  $A_1 \times \dots \times A_i$  of absolutely simple abelian varieties, with each  $A_i$  having complex multiplication by the integer ring of a CM field  $E_i$ .

Proving (1.1) under these assumptions will give a proof in general.

In order to discuss the individual factors with a minimum of notation, we now temporarily suppose

iii) that  $t = 1$ ,

i.e. that  $A$  is already absolutely simple. This assumption will be in force for the remainder of this paragraph.

To discuss the "CM-type" attached to  $A$ , and the "dual" (or reflex) CM type derived from  $A$ , we embed  $k$  and  $E$  into the complex field  $\mathbb{C}$ . Let  $L$  be the Galois closure of  $E$  in  $\mathbb{C}$ , and let

$$G = \text{Gal}(L/\mathbb{Q}), \quad H = \text{Gal}(L/E).$$

We introduce the convention that  $G$  acts on  $E$  on the right. Thus for example, we may view the set  $\text{Hom}(E, \mathbb{C})$  of embeddings of  $E$  into  $\mathbb{C}$  as the coset space  $H \backslash G$ . We write  $c$  for the complex conjugation of  $\mathbb{C}$ , or any of its restrictions.

As in [17], the data  $(A/k, E)$  define a CM-type  $S \subseteq \text{Hom}(E, \mathbb{C})$ . This is a subset of  $H \backslash G$  such that  $H \backslash G$  is the disjoint union of  $S$  and  $S^c$ . Put

$$\tilde{S} = \{g \in G \mid Hg \in S\}.$$

The absolute simplicity of  $A$  translates into the equality ([17], Prop. 26 )

$$H = \{g \in G \mid g\tilde{S} = \tilde{S}\}.$$

We say that the CM type  $(E, S)$  is simple. We symmetrically introduce

$$\begin{aligned} H' &= \{g \in G \mid \tilde{S}g = \tilde{S}\} \\ &= \{g \in G \mid g\tilde{R} = \tilde{R}\}, \end{aligned}$$

where  $\tilde{R}$  is the set  $\tilde{S}^{-1}$  of inverses of elements of  $\tilde{S}$ . Let  $K$  be the fixed field of  $H'$ , and let  $R \subseteq \text{Hom}(K, \mathbb{C})$  be the image of  $\tilde{R}$  in  $H' \backslash G$ .

DIVISION FIELDS OF ABELIAN VARIETIES

Then  $(K,R)$  is again a simple CM type, that dual to  $(E,S)$ . Because  $(E,S)$  is simple, it is its own double dual (i.e. the dual of  $(K,R)$ ), and it is known that  $k$  contains  $K$ . (See [17], Props. 28 and 30.)

We may view  $G$  as acting (on the right) on the set of CM types for  $E$ , and  $H'$  is then the stabilizer of the CM type  $(E,S)$ . Since the number of CM types for  $E$  is  $2^d$  where  $d = [E : \mathbb{Q}]/2$  is the dimension of  $A$ , we clearly have

$$(3.1) \quad [K : \mathbb{Q}] = (G : H') < 2^d.$$

If we put  $d' = [K : \mathbb{Q}]/2$ , then by (3.1) and the symmetry we have

$$(3.2) \quad 1 + \log_2 d < d' < 2^{d-1}.$$

It is known that for each  $d \geq 1$ , we may find a CM type  $(E,S)$  with this  $d$  such that  $d' = 2^{d-1}$ . (For a more precise statement, see ([16], 1.10).)

Associated to the pair of CM types  $(E,S)$ ,  $(K,R)$  is a homomorphism

$$\phi : T_K \rightarrow T_E,$$

which is most easily described by giving the corresponding homomorphism  $\phi^*$  of character groups of these tori. For  $F$  a finite extension of  $\mathbb{Q}$ , we write  $X_F$  for the character group of the torus  $T_F$ ; this is the right  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module consisting of integral linear combinations  $\sum n_\sigma [\sigma]$  with  $\sigma \in \text{Hom}(F, \mathbb{C})$ . (We take  $\bar{\mathbb{Q}}$  to be the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ .) This applies especially when  $F = K$  or  $E$ , in which case for  $g \in G$  we write  $[g]$  for the embedding of  $F$  into  $\mathbb{C}$  induced by  $g$ .

We define  $\phi^*: X_E \rightarrow X_K$  by the formula

$$[g] \mapsto \sum_{r \in R} [rg].$$

This makes sense because replacing  $g$  by  $hg$  ( $h \in H$ ) has the effect of permuting the various terms  $[rg]$ . The map  $\phi^*$  is visibly  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -equivariant.

(Cf. [17], Prop. 29.)

Following T. Kubota [4], we refer to the dimension of the image of  $\phi: T_K \rightarrow T_E$  as a rank, the rank of the CM type  $(E, S)$ . This integer may be expressed as the rank of the  $\mathbb{Z}$ -submodule  $\phi^*(X_E)$  of  $X_K$ ; using the natural bases for  $X_K$  and  $X_E$ , we then see the rank as the rank of the matrix

$$\begin{aligned} & (i(\tau, \sigma))_{\substack{\sigma \in H \setminus G \\ \tau \in H' \setminus G}} \end{aligned}$$

defined by

$$i(\tau, \sigma) = \begin{cases} 1 & \text{if } \tau \sigma^{-1} \in \tilde{R} \\ 0 & \text{if not} \end{cases}$$

(For  $\tau, \sigma \in G$ , and for  $h \in H, h' \in H'$ , we have  $\tau \sigma^{-1} \in \tilde{R}$  if and only if  $(h' \tau)(h \sigma)^{-1} \in \tilde{R}$ .) It is obvious that if we exchange the roles of  $(E, S)$  and  $(K, R)$ , we replace  $(i(\tau, \sigma))$  by its transpose. Hence we find

Proposition (3.3) : The rank of a CM type  $(E, S)$  is equal to the rank of its dual.

It is easy to see that the rank of  $(E, S)$  satisfies the inequality

$$\text{rank}(E, S) \leq d + 1.$$

For example, we have  $\text{Im } \phi \subset T$ , where  $T$  is the  $(d+1)$ -dimensional torus such that

$$T(A) = \{x \in T_E(A) = (E \otimes_{\mathbb{Q}} A)^* \mid xx^c \in A^*\}$$

for  $\mathbb{Q}$ -algebras  $A$ . By the symmetry (i.e. by (3.3)), we have

$$(3.4) \quad \text{rank}(E, S) \leq \min(d+1, d'+1).$$

DIVISION FIELDS OF ABELIAN VARIETIES

These facts were all pointed out by Kubota [4], who calls a CM type non-degenerate if its rank is equal to  $d+1$ .

It is amusing to note that there is a lower bound for the rank :

$$(3.5) \quad \max(2 + \text{Log}_2 d, 2 + \text{Log}_2 d') \leq \text{rank}(E, S).$$

To prove (3.5), it suffices by the symmetry to prove that  $\text{rank}(E, S)$  is at least  $2 + \text{Log}_2 d = \text{Log}_2(4d)$ . We note that the image of  $\phi^*$  contains the vectors

$$\begin{cases} \begin{matrix} \Sigma [rg] \\ r \end{matrix} & g \in H \setminus G \\ \begin{matrix} \Sigma ([r] + [rg]) \\ r \end{matrix} & g \in H \setminus G \end{cases},$$

which we easily check to have pairwise distinct images in  $X_K/2X_K$ . [The only tricky point is to check that no vector in the first group is congruent mod 2 to a vector in the second. Write all vectors in the form  $\sum_{g \in H \setminus G} n_g [g]$ . For vectors in the first group, we have  $n_g + n_{gc} = 1$  for all  $g$ ; for those in the second, we have  $n_g + n_{gc} = 2$  for all  $g$ .] It follows (as Lenstra pointed out) that the image of  $\phi^*$  generates an  $\mathbb{F}_2$ -subspace of  $X_K/2X_K$  of dimension at least  $\text{Log}_2 4d$ . This implies in particular the assertion about the rank.

Corollary (3.6) : Suppose that we have  $d' = 2^{d-1}$ . Then  $(E, S)$  is non-degenerate : we have  $\text{rank}(E, S) = d+1$ .

Proof : We have under the hypothesis, by (3.5) and (3.4),

$$d+1 = 2 + \text{Log}_2 d' \leq \text{rank}(E, S) \leq d+1.$$

Examples (3.7) : If  $d = 1, 2, 3$ , then the inequalities

$$2 + \text{Log}_2 d \leq \text{rank}(E, S) \leq d+1$$

show that  $(E, S)$  is always non-degenerate. If  $d = 4$ , then we find

$$4 < \text{rank}(E,S) \leq 5 ,$$

and both possibilities may occur. Indeed, if  $d' = 8$ , then the rank is 5 by (3.6) ; we may specify examples with  $d' = 8$  as remarked after (3.2). Similarly, if  $d' = 3$  (note that  $4 = 2^{3-1}$ ), we have  $\text{rank}(E,S) = 4$  by (3.6), which we apply after switching the roles of  $(E,S)$  and its dual. (Incidentally, the CM type constructed by Mumford and described in Pohlmann [9] in connection with Hodge classes on abelian varieties gives a specific example where  $d = 4$  but  $d' = 3$ .) A case-by-case analysis once performed by the author showed that  $\text{rank}(E,S)$  is 5 in all cases where  $d = 4$  and  $d' > 4$ . The tedious proof of this fact has been mislaid.

Before moving to the special case where  $E$  is an abelian extension of  $\mathbb{Q}$ , we mention an alternate interpretation of  $\phi$ , or rather of the composite of  $\phi$  and the norm map  $N_{k/K} : T_k \rightarrow T_K$ . We let  $t_{A/k}$  be the tangent space to  $A/k$  at the origin, so that  $t_{A/k}$  is a  $k$ -vector space of rank  $d$  on which  $E$  acts. It is alternately an  $E$ -vector space on which  $k$  acts. For  $\alpha \in k^*$  we let  $\psi(\alpha) \in E^*$  be the determinant of the  $E$ -linear map "multiplication by  $\alpha$ " on  $t_{A/k}$ . The map  $\psi : k^* \rightarrow E^*$  is obviously induced by an algebraic map  $T_k \rightarrow T_E$ , which we again denote by  $\psi$ , (cf. [15], p.511).

Proposition (3.8) : The map  $\psi$  is the composition of the norm map  $N_{k/K} : T_k \rightarrow T_K$  and the map  $\phi : T_K \rightarrow T_E$ .

This is well known, and is used implicitly in [15], § 7. For a proof, see [16], § 1.3.

Corollary (3.9) : The rank of  $(E,S)$  is equal to the dimension of the image of  $\psi$ .

Proof : The map  $N_{k/K}$  is surjective .

In addition to assumptions (i), (ii), (iii) introduced above, we suppose now and for the remainder of this §, that  $E$  is an abelian extension of  $\mathbb{Q}$ . Then  $L = K = E$ , and  $G = \text{Gal}(E/\mathbb{Q})$ . We may view  $\phi^*$  as an endomorphism of  $X_E = X_K$ .

DIVISION FIELDS OF ABELIAN VARIETIES

We calculate the effect of  $\phi^*$  on the basis vectors  $v_\chi = \sum_{g \in G} \chi(g)[g]$  of

$X_E \otimes \mathbb{C}$ , where  $\chi$  runs over the set of  $\mathbb{C}^*$ -valued characters of the abelian group  $G$ . We find

$$\phi^*(v_\chi) = \left( \sum_{s \in S} \chi(s) \right) v_\chi$$

for each  $\chi$ . This gives

Proposition (3.10) (cf. [4], lemma 2) : The rank of  $(E, S)$  is the number of characters  $\chi$  for which the sum

$$\chi(S) = \sum_{s \in S} \chi(s)$$

is non zero.

Note that when  $\chi$  is an even character (i.e.  $\chi(c) = +1$ ), we have  $\chi(S) \neq 0$  if and only if  $\chi$  is non trivial. The rank is thus one plus the number of odd characters  $\chi$  for which  $\chi(S)$  is non zero.

We close this paragraph with some examples.

(3.11) Let  $p \geq 5$  be a prime, and let  $E$  be the field  $\mathbb{Q}(\mu_p)$  of  $p^{\text{th}}$  roots of unity. We identify  $G$  with  $(\mathbb{Z}/p\mathbb{Z})^*$  in the usual way. For  $g \in G$ , write  $\langle g \rangle$  for the integer between 1 and  $p-1$  which represents  $g \pmod p$ . Let  $a$  be an integer satisfying  $1 < a < p-2$ . The set

$$S = \{g \in G : \langle g \rangle + \langle ag \rangle < p\}$$

is readily seen to be a CM type for  $E$ . It is simple if and only if  $a$  is of order  $\neq 3$  in  $(\mathbb{Z}/p\mathbb{Z})^*$ , which we suppose to be the case. (See, e.g., [3], Th. 2.) For  $\chi$  odd, one finds

$$\chi(S) = L(0, \chi) (1 + \chi^{-1}(a) - \chi^{-1}(1+a)),$$



by a computation generalizing that of [4] , lemma 3. Thus  $(E,S)$  is "degenerate" if and only if there is an odd character  $\chi$  satisfying the unlikely equality

$$\chi(1+a) = \chi(1) + \chi(a) .$$

Thus  $S$  is non-degenerate, for example, if  $a = 1$ . Greenberg [2] found that  $S$  is degenerate for  $p = 67$  and  $a = 10, 19, 47, 56, 60$ . For sufficiently large primes  $p \equiv 7 \pmod{12}$ , Lenstra and Stark noticed that there is always an  $a$  for which  $S$  is degenerate (loc. cit.).

(3.12) Let  $E$  be the field of  $32^{\text{nd}}$  roots of unity. As usual, we identify  $G$  with  $(\mathbb{Z}/32\mathbb{Z})^*$  . Let  $S$  be the subset  $\{1, 7, 13, 15, 21, 23, 27, 29\}$  of  $G$ . Then  $S$  is a "simple" CM type, and  $\chi(S)$  vanishes when  $\chi$  is the character

$$x \mapsto \begin{cases} -1 & \text{if } x \equiv 3 \pmod{4} \\ +1 & \text{if } x \equiv 1 \pmod{4} . \end{cases}$$

Thus  $S$  is degenerate . (This example was found by Lenstra.) Similarly, if we take  $S' = \{1, 7, 9, 11, 13, 15, 27, 29\}$  , then  $S'$  is again a simple CM type such that  $\chi(S') = 0$  for both odd characters  $\chi'$  of order 2.

(3.13) Take  $E$  this time to be the field of  $19^{\text{th}}$  roots of unity, so that  $G$  is  $(\mathbb{Z}/19\mathbb{Z})^*$  . Let  $S = \{1,3,4,5,6,7,8,10,17\}$  . Then  $S$  is a simple CM type such that  $\chi(S) = 0$  for both (odd) characters  $\chi$  of order 6. (This example was provided by Serre in response to a question of Masser.)

(3.14) Let  $p,q,r$  be distinct odd primes, and let  $G$  be the cyclic group  $\mathbb{Z}/2pqr\mathbb{Z}$  . Let  $E$  be a Galois extension of  $\mathbb{Q}$  with  $\text{Gal}(E/\mathbb{Q}) \simeq G$ . Let  $S$  be the subset of  $G$  consisting of those elements having order 1,  $pqr$ ,  $2p$ ,  $2q$ ,  $2r$ ,  $2pq$ ,  $2pr$ , or  $2qr$ . It is evident that  $(E,S)$  is a CM type, and it is simple because  $S$  contains the identity element of  $G$  but no non-trivial subgroup of  $G$ . A calculation shows that, for  $\chi$  odd, we have  $\chi(S) = 0$  if and only if  $\chi$  has order  $2pqr$ . Hence we have

DIVISION FIELDS OF ABELIAN VARIETIES

$$\text{rank}(E,S) = 1 + pqr - (p-1)(q-1)(r-1).$$

This example, recently constructed by Lenstra, shows that the rank may be quite small relative to  $[E : \mathbb{Q}]$ , even in the case where the CM field  $E$  is abelian.

4. We now return to the situation outlined at the beginning of § 3, where the abelian variety  $A/k$  satisfies conditions (i) and (ii), but we no longer assume that  $E$  is a single field. We wish to prove (1.1) for  $A$ .

For  $N \geq 1$ , let  $A_N$  be the  $\text{Gal}(\bar{k}/k)$ -module of  $N$ -division points on  $A$ , and let  $G_N$  be the image of the representation

$$\rho_N : \text{Gal}(\bar{k}/k) \rightarrow \text{Aut } A_N$$

giving the action of  $\text{Gal}(\bar{k}/k)$  on  $A_N$ . Thus  $G_N$  is the Galois group over  $k$  of the division field  $k(A_N)$ , and the order of  $G_N$  is the degree  $d(N)$  of this field. Since  $A$  has everywhere good reduction over  $k$ ,  $k(A_N)/k$  is ramified only at primes of  $k$  dividing  $N$ . Thus, if  $N$  and  $M$  are relatively prime,  $k(A_N) \cap k(A_M)$  is contained in the Hilbert class field of  $k$ . After replacing  $k$  by its Hilbert class field, we thus find that the function

$$N \longmapsto d(N)$$

is "multiplicative" in the usual arithmetic sense. Thus to prove (1.1) it suffices to obtain for each prime  $\ell$  and each integer  $n \geq 1$  an inequality

$$(4.1) \quad c < \frac{d(\ell^n)}{\ell^{n\nu}} < c'$$

in which  $C$  and  $C'$  are constants depending on  $A$ ,  $k$ ,  $E$ , and where  $\nu \geq 0$  is an integer.

We recall now the decomposition  $A = A_1 \times \dots \times A_t$ . For each  $i$ , let

$\psi_i : T_k \rightarrow T_{E_i}$  be the map  $\psi$  of (3.8) made with the abelian variety  $A$  of § 3 taken to be  $A_i$ . Let

$$\psi : T_k \rightarrow T_E = T_{E_1} \times \dots \times T_{E_t}$$

be the product of the  $\psi_i$ . We will prove that (4.1) holds with  $v$  equal to the dimension of the image of  $\psi$ . Thus if  $A$  is simple (i.e.  $t = 1$ ), then  $v$  is the rank of the CM type attached to  $A$ , in the sense of § 3.

For a prime  $\ell$ , let  $\rho_{\ell^\infty}$  be the  $\ell$ -adic representation of  $\text{Gal}(\bar{k}/k)$  attached to  $A$ , i.e. the projective limit of the  $\rho_{\ell^n}$  ( $n \geq 1$ ). A priori,  $\rho_{\ell^\infty}$  takes values in the group of automorphisms of the Tate module  $\varprojlim_{\ell^n} A_n$  of  $A$ , but it is well known that the values of  $\rho_{\ell^\infty}$  lie in the subgroup  $(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)^*$  of  $\text{Aut}(\varprojlim_{\ell^n} A_n)$ , cf. [15, § 4, Cor. 2]. Hence  $\rho_{\ell^\infty}$  is abelian and may be viewed as a map

$$I_k \longrightarrow (\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)^*,$$

where  $I_k$  is the group of idèles of  $k$ . For (4.1), no harm is done in replacing  $I_k$  by the product  $\prod_v U_v$  of the groups of units at the non-archimedean completions of  $k$ , since this product has finite index in the abelianized Galois group of  $k$ . (In fact, this replacement is the replacement of  $k$  by its Hilbert class field which we discussed above.) Now  $\rho_{\ell^\infty}$  kills the group  $U_v$  if  $v$  is not of residue characteristic  $\ell$ . Hence, we will view  $\rho_{\ell^\infty}$  as a map

$$\prod_{v|\ell} U_v \longrightarrow (\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)^*$$

i.e., as a map

$$T_k(\mathbb{Z}_\ell) \rightarrow T_E(\mathbb{Z}_\ell),$$

cf. (2.1).

Let  $\lambda : T_k \rightarrow T_E$  be the "inverse" of  $\psi$ , i.e., the map  $x \mapsto \psi(x^{-1})$ . Then  $\rho_{\ell^\infty}$  is just the map on  $\mathbb{Z}_\ell$ -points induced by  $\lambda$ , in view of Theorem 11 of [15] and its corollaries. Now  $\rho_{\ell^n}$  for  $n \geq 1$  is the composition of the map

DIVISION FIELDS OF ABELIAN VARIETIES

$\rho_{\ell^\infty}$  with the reduction map  $(\mathcal{O} \oplus \mathbb{Z}_\ell)^* \rightarrow (\mathcal{O}/\ell^n \mathcal{O})^*$ , i.e.,  $T_E(\mathbb{Z}_\ell) \rightarrow T_E(\mathbb{Z}/\ell^n \mathbb{Z})$ .

We have an evident commutative diagram

$$\begin{array}{ccc}
 T_K(\mathbb{Z}_\ell) & \longrightarrow & T_E(\mathbb{Z}_\ell) \\
 \downarrow & & \downarrow \\
 T_K(\mathbb{Z}/\ell \mathbb{Z}) & \longrightarrow & T_E(\mathbb{Z}/\ell \mathbb{Z})
 \end{array}$$

in which the two vertical maps are reduction maps, the top horizontal map is  $\rho_{\ell^\infty}$  and the lower horizontal map is the map denoted  $\lambda_{\ell,n}$  toward the end of § 1. Hence  $G_{\ell^n}$  is just the image of  $\lambda_{\ell,n}$  (for  $n \geq 1$ ), and  $d(\ell^n)$  is the order of  $\text{Im}(\lambda_{\ell,n})$ . Thus (1.1) is a special case of (2.7). Finally, again by (2.7), the integer  $\nu$  of (1.1) is the dimension of the image of  $\lambda$ , or in other words the dimension of the image of  $\psi$ . This establishes (4.1) and the claim concerning the value of  $\nu$ .

---

**BIBLIOGRAPHY**

- [ 1 ] Deligne, P. , Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes, rédigé par J. L. Brylinski. This volume.
- [ 2 ] Greenberg, R. , On the Jacobian variety of some algebraic curves. Preprint, 1978.
- [ 3 ] Koblitz, N. and Rohrlich, N. , Simple factors in the Jacobian of a Fermat curve. Canadian J. Math. 30, 1183-1205 (1978).
- [ 4 ] Kubota, T. , On the field extension by complex multiplication. Trans. AMS 118, n° 6, 113-122 (1965).
- [ 5 ] Lang, S. , Algebraic groups over finite fields. Am. J. Math 78 , 555-563 (1956).
- [ 6 ] Masser, D. W. , On quasi-periods of abelian functions with complex multiplication. This volume.
- [ 7 ] Ono, T. , Arithmetic of algebraic tori. Ann. of Math. 74, 101-139 (1961).

- [8] Ono, T. , On the Tamagawa number of algebraic tori. Ann. of Math. 78 , 47-73 (1963).
- [9] Pohlmann, H. , Algebraic cycles on abelian varieties of complex multiplication type. Ann. of Math. 88, 161-180 (1968).
- [10] Ribet, K. A. , Kummer theory on extensions of abelian varieties by tori. Duke Math. J. 46, 745-761 (1979).
- [11] Serre, J. P. , Groupes Algébriques et Corps de Classes. Hermann, Paris, 1959.
- [12] Serre, J. P. , Corps Locaux. Deuxième édition revue et corrigée. Hermann, Paris, 1968.
- [13] Serre, J. P. , Letter to D. Masser, November, 1975.
- [14] Serre, J. P. , Représentations  $\ell$ -adiques. In Algebraic Number Theory (Int. Symp., Kyoto, 1976), Japan Society for the Promotion of Science, Tokyo, 1977.
- [15] Serre, J. P. and Tate, J. , Good reduction of abelian varieties. Ann. of Math. 88, 492-517 (1968).
- [16] Shimura, G. , Arithmetic quotients of bounded symmetric domains. Ann. of Math. 91, 144-222 (1970).
- [17] Shimura, G. and Taniyama, Y. , Complex Multiplication of Abelian Varieties and its Applications to Number Theory. Publ. Math. Soc. Japan n°6, Tokyo, 1961.
- [18] Weil, A. , On a certain type of characters of the idèle-class group of an algebraic number-field. Proc. International Symp. on Algebraic Number Theory, Tokyo-Nikko, 1-7 (1955) = Collected Papers [1955c].

Ecole Polytechnique  
Centre de Mathématiques  
91128 Palaiseau Cedex (France)