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DISCRETIZED FEEDBACK FOR DIFFERENTIAL GAMES

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§1. Introduction

Let us recall a result of minimization for optimal control (see SENTIS [1]). For any initial condition (t, x) of $[0, T] \times \mathbb{R}^d$, we call $\mathcal{V}_{t,x}^*$ the set of the controls b of $L^\infty(0, T; \mathbb{R}^d)$ such that (1) admits a solution (which is denoted y_b):

$$(1) \quad \begin{aligned} y'(s) &= b(s) & b(s) &\in B(s, y(s)) & \text{a.e. } s \in [t, T] \\ y(t) &= x \end{aligned}$$

with the hypothesis:

$$(2) \quad \begin{aligned} &B \text{ is a Lipschitzian multivalued mapping from } [0, T] \times \mathbb{R}^d \\ &\text{with convex compact values in the sphere of radius } Q_0. \end{aligned}$$

For fixed (t, x) we will minimize on $\mathcal{V}_{t,x}^*$ the following cost

$$(3) \quad J_{t,x}(b) + F(y_b(T))$$

where F is Lipschitzian and has no propriety of convexity. This problem admits an optimal open-loop control, but we look for a feedback which approaches the optimum for any initial condition. For that purpose we discretize the interval of time defining:

$$(4) \quad \begin{cases} h_n = T/n \\ t_n^k = kh_n & \forall k \in \mathbb{N} \\ \theta_n^t \text{ the unique integer such that } t \in [t_n^{\theta_n^t}, t_n^{\theta_n^t+1}] \end{cases}$$

And there exist multivalued (m.v.) mappings $v_n^0, v_n^1, \dots, v_n^{n-1}$ from \mathbb{R}^d to \mathbb{R}^d such that

$$v_n^k(z) \subset B(t_n^k, z) \quad \forall z \in \mathbb{R}^d$$

and such that for any initial condition (t, x) , if we define a trajectory y_n (linear on any interval $[t_n^k, t_n^{k+1}]$) by $y_n(t_n^k) = x_n^k$: with

$$\begin{cases} x_n^{\theta t} = x \\ x_n^{k+1} \in x_n^k + v_n^k(x_n^k) h_n \end{cases} \quad k \geq \theta t$$

then any accumulation point y in $C^0(0, T; \mathbb{R}^d)$ of $(y_n)_n$ is a solution of (1) and is optimal, that is to say:

$$F(y(T)) = J_{t,x}(y') = \min_{b \in \mathcal{U}_{t,x}} J_{t,x}(b)$$

Let us now consider the following differential game. For any initial condition (t, x) the admissible trajectories are the solutions of

$$(7) \begin{cases} y'(s) \in A(s, y(s)) + B(s, y(s)) & \text{a.e. } s \in [t, T] \\ y(t) = x \end{cases}$$

where A and B satisfy (2). Let F be a Lipschitzian function on \mathbb{R}^d .

Heuristically, if u and v are two sections of A and B such that there exists a solution (denoted $y_{u,v}$) of:

$$(8) \begin{cases} y'(s) = u(s, y(s)) + v(s, y(s)) \\ y(t) = x \end{cases}$$

then we look for u^* and v^* , sections of A and B , such that

$$(9) \quad F(y_{u,v^*}(T)) \leq F(y_{u^*,v^*}(T)) \leq F(y_{u^*,v}(T))$$

for any u and v section of A and B .

In general, there do not exist sections u^* and v^* verifying (9) and such that u^* and v^* are continuous with respect to the state variable. (Obviously there do not exist open-loop controls u^* and v^* verifying (9).) The topic of this paper is to find a couple of strategies which is a saddle-point for the differential game in a certain class of strategies. For that purpose we must first define the class of admissible strategies (we use the notations (4) except $h_n = T/2^n$ and we write \bar{n} for 2^n)

Definition 1. An admissible strategy for the player U [or V] is a sequence $(u_n)_n$ [or $(v_n)_n$] of elements u_n [or v_n] (which are called discretized feedbacks) with:

$$(10) \left\{ \begin{array}{l} u_n = \{u_n^0, u_n^1, u_n^2, \dots, u_n^{\bar{n}-1}\} \in \prod_{k=0}^{\bar{n}-1} \mathcal{U}_n^k \\ \text{where } \mathcal{U}_n^k \text{ is the set of the m.v. mappings } u \text{ on } \mathbb{R}^d \text{ verifying :} \\ u(z) \subset A(t_n^k, z) \end{array} \right.$$

and :

$$(10') \left\{ \begin{array}{l} v_n = \{v_n^0, v_n^1, v_n^2, \dots, v_n^{\bar{n}-1}\} \in \prod_{k=0}^{\bar{n}-1} \mathcal{V}_n^k \\ \text{where } \mathcal{V}_n^k \text{ is the set of the m.v. mappings } v \text{ on } \mathbb{R}^d \text{ verifying :} \\ v(z) \subset B(t_n^k, z) \quad \forall z \in \mathbb{R}^d \end{array} \right.$$

In §2, we exhibit particular discretized feedbacks associated to each h_n and in §3, let n go to infinity, to show that the sequences of such discretized feedbacks constitutes a saddle point in the class of admissible strategies (for detailed proofs, see SENTIS [2]).

§2. Definition of the discretized feedbacks \tilde{u}_n and \tilde{v}_n .

The following proposition justifies the term admissible in definition 1.

Proposition 1. Let us fix (t, x) . If (u_n) and (v_n) are admissible strategies and if we define a trajectory y_n linear on each interval $[t_n^k, t_n^{k+1}]$ by $y_n(t_n^k) = x_n^k$ and x_n^k given by (11)

$$(11) \left\{ \begin{array}{l} x_n^0 = x \\ x_n^{k+1} \in x_n^k + h_n(u_n^k(x_n^k) + v_n^k(x_n^k)) \end{array} \right. \quad k \geq 0, t_n^k \leq t$$

then any accumulation point y in C^0 verifies (7).

Now let us give two definitions for the cost of a game with initial conditions (t, x) .

Definition 2. The cost of the game for the two discretized feedbacks u_n and v_n is the subset of \mathbb{R} defined by :

$$J_{t,x}(u_n, v_n) = \{F(x_n^n) \text{ such that there exists } (x_n^k)_k \text{ verifying (11)}\}$$

Definition 3. The cost of the game for the two admissible strategies (u_n) and (v_n) is the subset of R denoted by $J_{t,x}((u_n), (v_n))$ and containing the accumulation points of all the sequences (a_n) verifying $a_n \in J_{t,x}((u_n), (v_n))$. Let us yet define the lower and upper optimal cost-functions \bar{W}_n^k and \hat{W}_n^k as FRIEDMAN [1] by decreasing induction :

$$(12) \quad \begin{cases} \bar{W}_n^k(x) = F(x) \\ \bar{W}_n^k(x) = \max_{u \in A(t_n^k, x)} \bar{Z}_n^k(x, u) \text{ and } \bar{Z}_n^k(x, u) = \min_{v \in B(t_n^k, x)} \bar{W}_n^{k+1}(x + (u+v)h_n) \end{cases}$$

and :

$$(12') \quad \begin{cases} \hat{W}_n^k(x) = F(x) \\ \hat{W}_n^k(x) = \min_{v \in B(t_n^k, x)} \hat{Z}_n^k(x, v) \text{ and } \hat{Z}_n^k(x, v) = \max_{u \in A(t_n^k, x)} \hat{W}_n^{k+1}(x + (u+v)h_n) \end{cases}$$

Now we can exhibit the m.v. mappings \hat{u}_n^k and \hat{v}_n^k , which do not depend on the initial conditions.

$$(13) \quad \begin{cases} \hat{u}_n^k(x) = \text{Arg Max}_{u \in A(t_n^k, x)} \hat{Z}_n^k(x, u) \\ \hat{v}_n^k(x) = \text{Arg Min}_{v \in B(t_n^k, x)} \hat{Z}_n^k(x, v) \end{cases}$$

We can prove easily by induction the following :

Proposition 2. All the mappings \bar{W}_n^k , \bar{Z}_n^k , \hat{W}_n^k , \hat{Z}_n^k are Lipschitzian (with respect to x) with constant K (independent of n and k).

§3. Saddle point theorem

Proposition 3 We have when n goes to infinity:

$$\bar{W}_n^{\theta_n t}(x) \rightarrow W^-(t, x) \quad \hat{W}_n^{\theta_n t}(x) \rightarrow W^+(t, x)$$

moreover:

$$W^-(t, x) \leq W^+(t, x)$$

principle of the proof.

First we show by decreasing induction on k that

$$\bar{W}_n^k(x) - \bar{W}_{n+1}^{2k}(x) \leq (n-k) C_0 (h_n)^2 \quad \forall k$$

And as $\theta_{n+1}t$ is equal to $(2\theta_n t)$ or $(2\theta_n t + 1)$ we have according to proposition 2:

$$(14) \quad \bar{W}_n^{\theta_n t}(x) - \bar{W}_n^{\theta_{n+1} t}(x) \leq C_1 h_n \quad \text{with } C_1 = C_0 T + 2KQ_0$$

Hence if we denote:

$$W^-(t, x) = \lim_n \sup \bar{W}_n^{\theta_n t}(x)$$

we can show easily according to (14) that $\bar{W}_n^{\theta_n t}(x) \rightarrow W^-(t, x)$. We show exactly the same way that $\hat{W}_n^{\theta_n t}(x) \rightarrow W^+(t, x)$. The end of the proposition is a consequence of the following fact:

$$\bar{W}_n^k(x) \leq \hat{W}_n^k(x) \quad \forall x, n, k \quad \text{Q.E.D.}$$

The following proposition is fundamental and is proved in FRIEDMAN [1], using the m.v. mappings:

$$\text{Arg Min}_{v \in B(t_n^k, x)} \bar{W}_n^{k+1}(x + (u+v)h_n) \quad \text{and} \quad \text{Arg Max}_{u \in A(t_n^k, x)} \hat{W}_n^{k+1}(x + (u+v)h_n)$$

Proposition 4We have

$$W^-(t, x) = W^+(t, x)$$

We write thus $W(t, x)$ instead of $W^-(t, x)$. This number is called the value of the game.

Proposition 5

For any $u_n \in \prod_{k=0}^{\bar{n}-1} \mathcal{U}_n^k$, we have

$$(15) \quad J_{t,x}(u_n, \tilde{v}_n) \leq \hat{W}_n^{\theta_n t}(x)$$

(This means that any element of the left-hand side is smaller than the right-hand side.)

Proof

Using the notations (11) (changing v_n^k into \tilde{v}_n^k), we note that there exist $q_n^k \in \tilde{v}_n(x_n^k)$ such that

$$x_n^{k+1} \in x_n^k + h_n(u_n^k(x_n^k) + q_n^k) \quad \forall k \geq \theta_n t$$

Thus we have:

$$\hat{W}_n^k(x_n^k) = \hat{Z}_n^k(x_n^k, q_n^k) \geq \hat{W}_n^{k+1}(x_n^{k+1})$$

Rewriting this inequality for k from $\theta_n t$ to \bar{n} , we obtain (15). Q.E.D.

We have evidently also:

$$(16) \quad J_{t,x}(\tilde{u}_n, v_n) \geq \bar{W}_n^{\theta_n t}(x)$$

Let n go to infinity in (15) and (16), we deduce immediately from the propositions 3 and 4 the following:

Theorem

For any admissible strategy (u_n) and (v_n) , we have:

$$J_{t,x}((u_n), (\tilde{v}_n)) \leq W(t,x) \leq J_{t,x}((\tilde{u}_n), (v_n))$$

Thus we have:

$$W(t,x) = \min_{(v_n)} \max_{(u_n)} J_{t,x}((u_n), (v_n)) = \max_{(u_n)} \min_{(v_n)} J_{t,x}((u_n), (v_n))$$

And if y is an accumulation point in C^0 of trajectories y_n associated to \tilde{u}_n and \tilde{v}_n we have

$$W(t,x) = J_{t,x}((\tilde{u}_n), (\tilde{v}_n)) = F(y(T))$$

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