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FIXED POINT THEOREMS WITHOUT CONVEXITY

Jean-Paul PENOT

A large amount of fixed-point theorems without convexity conditions are available (and are useful) in analysis. We quote a few samples before delineating the purpose of this paper :

- the Banach-Cacciopoli-Picard theorem and its numerous generalizations (cf. for instance H. BREZIS (1965), M. EDELSTEIN (1961), (1962), R.D. HOLMES (1976), S. LEADER (1977), A. MEIR - E. KEELER (1969), V.M. SEHGAL (1969)...)
- the Lefschetz fixed point theorem (for recent and refined formulations see F.E. BROWDER (1965), (1975), R.F. BROWN (1965), (1971), G. FOURNIER (1975), L. GORNIEWICZ (1973), A. GRANAS (1967), J. LERAY (1972) and its connections with topological tools (as acyclic-valued multifunctions) see D.G. BOURGIN (1973), G. CONTI, P. NISTRI (1975), S. EILENBERG - D. MONTGOMERY (1946), J.M. LASRY - R. ROBERT (1976), M.J. POWERS (1970), (1972)...)
- fixed point theorem using an order structure (H. AMANN (1975), G. BIRKHOFF (1948), B. KNASTER (1928), L.A. TALMAN (1977), L. TARTAR (1974)...)
- fixed points using special structures as geometric structures (H. POINCARÉ, G. BIRKHOFF, A. WEINSTEIN (1968), (1977), or special topological assumptions (trees... c.f. T. VAN DER WALT (1967)) or particular analytical hypothesis (as holomorphy).

The purpose of this work is to derive fixed point theorems in some situations which are not too far from convexity (thus the title should be more modest). Moreover we focus our attention on boundary conditions. The connection between fixed point theory and differential equations revealed by G. VIDOSSICH, K. DEIMLING (1974), R.H. MARTIN (1973), S. REICH (1976) justifies this particular attention.

§.1. PRELIMINARIES - THE CONVEX CASE

The following definition will be used throughout this paper to express a boundary condition. It goes back to G. BOULIGAND and A. MARCHAUD and is extensively used in optimisation theory and invariance results for differential equations.

1.1. Definition

The tangent cone to a subset X of a t.v.s. E at $x \in X$ is the set $T_x X$ of $v \in E$ such that there exists a net $(t_i, x_i)_{i \in I}$ in $(0, +\infty) \times X$ with limit $(0, x)$ such that

$$v = \lim t_i^{-1}(x_i - x).$$

If E is metrizable $T_x X$ is equivalently the set of right derivatives of continuous curves $c : [0, 1] \rightarrow E$ (for which $\frac{d}{dt}|_{t=0_+} c(t)$ exists) with $c(0) = x$, 0 being an accumulation point of $c^{-1}(x)$. Another useful characterization of $T_x X$ is the following one: $v \in T_x X$ if and only if $\liminf_{t \searrow 0} t^{-1} d(x+tv, X) = 0$, with $d(y, X) = \inf\{d(y, x) \mid x \in X\}$.

When X is convex, $T_x X$ is easily seen to be the closure of the radial tangent cone to X at x , $T_x^r X$ given by $T_x^r X = \mathbb{R}_+(X-x)$.

Looking for a fixed point of $f : X \rightarrow E$, where X is a subset of a l.c.s. (locally convex topological vector space) E is a problem equivalent to the search of zeros of the associated vector field

$$v_f(x) = f(x) - x.$$

This point of view justifies the introduction of tangential conditions on v_f .

On the other hand, when E is a Hilbert space, this problem can be embodied in the more general problem of solving a variational inequality: instead of solving $v_f(x) = 0$ one looks for solutions of $v_f(x) \in N_x X$, where $N_x X = (T_x X)^\circ$ is the normal cone to X at x . When X is convex this inclusion is equivalent to $\langle v_f(x), y-x \rangle \leq 0$ for every $y \in X$. We quote a result of H. BREZIS (1967), th.2, as an illustration of this point of view.

1.2. Proposition

Suppose X is a compact convex subset of a Hilbert space E and $I - f$ is pseudo-monotone. Then there exists $\bar{x} \in X$ such that $v_f(\bar{x}) = f(\bar{x}) - \bar{x} \in N_{\bar{x}} X$. In particular, if for each $x \in X$ $f(x) - x \in T_x X$, f has a fixed point.

One can get rid of the compactness assumption on X provided

$$\liminf_{\substack{|x| \rightarrow +\infty \\ x \in X}} \langle x - f(x), x \rangle > 0.$$

We do not pursue this line, although more general results (involving multivalued mappings in more general Banach spaces) are likely to hold true.

The following result is a far reaching generalization of the classical Schauder-Tychonoff theorem. Simple proofs have been given by S. REICH (1976), V.M. SEHGAL (1977) and various generalizations considered by S. REICH (1972), J.P. AUBIN, B. CORNET

(1975), F.E. BROWDER (1968), KY FAN (1969), LASRY ROBERT (1976), V.W. PETRYSHYN - P.M. FITZPATRICK (1974).

1.3. Proposition (B. HALPERN (1965), (1970), B. HALPERN - G. BERGMAN (1968)).

Let X be a non empty convex compact subset of a $\ell.c.s.$ E and let $f : X \rightarrow E$ be continuous. If $f(x) - x \in T_x X$ for each $x \in X$ then f has a fixed point.

It is not known whether the result is still true if X is closed and convex and f is compact with $v_f = f - I$ tangent to X . However partial answers are known : see corollary 4.3. and proposition 1.4., 1.6. below.

1.4. Proposition (BREZIS (1965), corollary 21 for the singled valued case).

Let X be a nonempty closed convex subset of a $\ell.c.s.$ E and let $F : X \rightarrow E$ be a closed-convex valued multifunction with closed graph and $\overline{F(X)}$ compact. If $F(x) \subset X$ for each x in the boundary ∂X of X , then F has a fixed point.

Proof : If $\text{int } X = \emptyset$, $X = \partial X$ and the result is simply Ky Fan's theorem. If $\text{int } X \neq \emptyset$, we may suppose $0 \in \text{int } X$ so that corollary 4.3. below and the relation $X - x \subset T_x X$ give the result. \square

1.5. Theorem

Let X be a closed convex subset of a $\ell.c.s.$ E such that there exists a continuous inf-compact quasi-convex function $h : X \rightarrow \mathbb{R}$ (i.e. for each n , $X_n = h^{-1}((-\infty, n])$ is convex compact). If $f : X \rightarrow E$ is compact, and $f(x) - x \in T_x X$ for each $x \in X$, then f has a fixed point.

Let us note that the assumption on X is satisfied if E is a normed space and if X is boundedly compact (i.e. its intersection with any closed ball is compact).

Proof : Without any loss of generality we suppose $0 \in X$, $h(x) \geq h(0) = 0$ for any $x \in X$. We choose a continuous function $s : \mathbb{R} \rightarrow [0, 1]$ with $s(r) = 0$ for $r \leq 0$, $s(r) = 1$ for $r \geq 1$ and for $x \in X$ we set

$$s_n(x) = s(n-h(x)) \quad , \quad f_n(x) = s_n(x) f(x) \quad .$$

Let $x \in X_n$. If $h(x) < n$ we have $T_x X_n = T_x X$ as $X_n \cap U = X \cap U$ for a neighborhood U of x , thus

$$f_n(x) - x = s_n(x)(f(x) - x) + (1-s_n(x))(-x) \in T_x X_n$$

as $-x = 0 - x \in T_x X$ and $T_x X$ is convex. If $h(x) = n$, we have $f_n(x) - x = -x$

and $-x \in T_{x_n} X$ as $tx \in X_n$ for $t \in [0,1]$, since h is quasi-convex. We may apply proposition 1.3 : there exists $x_n \in X_n$ with $f_n(x_n) = x_n$. As (x_n) is contained in the compact set $[0,1] \overline{f(X)}$, (x_n) has a limit point x_∞ in X . We take $k > h(x_\infty)$ and we choose $n \geq k+1$ with x_n in the neighborhood $h^{-1}((-\infty, k))$ of x_∞ in X . Then $s_n(x_n) = 1$, thus $x_n = f_n(x_n) = f(x_n)$. \square

If X is a non void closed subset of a n.v.s. E , a vector $v \in E$ is said to be metrically normal to X at $x \in X$ if $v \neq 0$ and if $\liminf_{t \searrow 0} t^{-1} d(x+tv, X) = |v|$. Any vector normal to X at x in Bony's sense (BONY (1969)) is metrically normal to X at x . Moreover it is easily seen that if $v \in T_x X$ then v is not metrically normal to X at x .

1.6. Proposition (S. REICH (1976))

Let X be a non void closed convex subset of a Banach space E . If $f : X \rightarrow E$ is continuous and such that, for each $x \in \partial X$, $f(x) - x$ is not metrically normal to X at x . Then one of the following conditions implies that f has a fixed point

- a) X is compact.
- b) f is compact and E is uniformly convex.
- c) f is non expansive ($|f(x) - f(y)| \leq |x - y|$ for each $(x, y) \in X \times X$), X is bounded and E is uniformly convex.
- d) f is a contraction (for some $k \in (0, 1)$ one has $|f(x) - f(y)| \leq k|x - y|$ for each $(x, y) \in X \times X$).

The proof consists in showing that $P \circ f$ has a fixed point, P being the metric projection on X , and observing that if $x \in P(f(x))$ then $f(x) - x$ is metrically normal to X at x . Thus the result is valid if $f : X \rightarrow E$ is nonexpansive, if X has the fixed point property for nonexpansive maps and if X is a nonexpansive retract of some subset Y of E containing $f(X)$. Convex sets which are nonexpansive retracts of the whole space have been recently characterized by B. BEAUZAMY ; note that if the unit ball or if every closed affine subspace of E is a nonexpansive retract of E , then E is a Hilbert space (H. FAKHOURY (1972)).

Before leaving the convex case, we give a rather special result.

1.7. Proposition (compare with C. MAYER (1965), v.2.2.)

Let X be a convex compact subset of a l.c.s. E and let $F : X \rightarrow E$ be a

| multifunction with closed convex graph. If $F(X) \supset X$ then F has a fixed point.

Proof : Let $H = F^{-1}|X$: for $y \in X$ $H(y) = \{x \in X | y \in F(x)\}$. Its graph $G(H)$ is the symmetric of $G(F) \cap X \times X$, hence is closed and convex. Hence its values are closed and convex. Ky Fan's theorem implies that there exists $x \in X$ such that $x \in H(x)$; thus $x \in F(x)$. \square

§.2. CONTRACTIONS ON NONCONVEX SETS

Our main result relies on the following striking fixed point theorem due to J. CARISTI. The original proof being quite involved, for the reader's convenience, we reproduce here with the kind authorization of its author a simple elementary proof (which does not use transfinite induction). See also H. BREZIS - F.E. BROWDER (1976), J. EISENFELD - V. LAKSHMIKANTHAM (1976), W.A. KIRK (1976), J. SIEGEL (1977).

2.1. Proposition (J. CARISTI (1975))

Let (M, d) be a complete metric space, $f : M \rightarrow M$ be any (not necessarily continuous) map, $g : M \rightarrow \mathbb{R}_+$ be lower-semi-continuous such that for each $x \in M$

$$d(x, f(x)) \leq g(x) - g(f(x)).$$

Then f has a fixed point.

Proof : (J.P. PENOT (1976)). We order M by setting $x \geq y$ iff $d(x, y) \leq g(y) - g(x)$. We set $M(x) = \{y \in M | y \geq x\}$. We define inductively an increasing sequence (x_n) in the following way. We choose x_0 arbitrarily and when x_0, \dots, x_n are given, we choose $x_{n+1} \in M(x_n)$ with $g(x_{n+1}) < \inf g(M(x_n)) + 1/n$. Thus $x_{n+1} \geq x_n$ and for each $x \in M(x_{n+1}) \subset M(x_n)$ we have

$$g(x) \geq \inf g(M(x_n)) > g(x_{n+1}) - 1/n$$

$$d(x, x_{n+1}) \leq g(x_{n+1}) - g(x)$$

so that the diameter of $M(x_{n+1})$ is not larger than $2/n$. As (M, d) is complete the intersection of the decreasing sequence of closed subsets $\{M(x_n)\}_{n \geq 0}$ contains exactly one point x_∞ . This point is a maximal element of (M, \leq) as $x \geq x_\infty$ implies $x \in M(x_n)$ for all n . As $f(x_\infty) \geq x_\infty$, we must have $f(x_\infty) = x_\infty$. \square

2.2. Remark : It would be interesting to know if the same result holds true for a multifunction $F : M \rightarrow M$ with non void closed values such that

$$d(x, F(x)) \leq g(x) - \inf g(F(x)), \quad x \in M.$$

The answer is positive in the special case $g(x) = (1-k)^{-1}d(x, F(x))$ with $k \in (0, 1)$

and F having a closed graph (this case includes the case F is k -lipschitzian, due to S.B. NADLER (1969)). \square

The following notion will dispense us with a compactness assumption.

2.3. Definition

Let A be a non void closed subset of a n.v.s. E , let $x \in E \setminus A$ and let $m \in [0, 1]$. A point $a \in A$ is a point of m -attraction of x in A (written $a \in A_m(x)$) if there exists $\epsilon > 0$ such that for every $t \in [0, \epsilon]$

$$d(x_t, A) \leq d(x, A) - md(x_t, x)$$

with $x_t = x + t(a-x)$. A is said to be m -attractive for x if $A_m(x) \neq \emptyset$;

A is said to be m -attractive if A is m -attractive for every $x \in E \setminus A$.

It is clear that $A_k(x) \supset A_m(x)$ if $k \leq m$. If $g(t) = d(x+t(a-x)/|a-x|, A)$, then for $a \in A_m(x)$ we have $D^+g(0) \leq -m$, and conversely, if $D^+g(0) < -m$, then $a \in A_m(x)$, D^+g being the right-upper Dini derivative of g . In particular, if $e'_A = d(\cdot, A)$ is differentiable at x with $e'_A(x)(a-x) < -m|a-x|$, then $a \in A_m(x)$ if $a \in A$.

2.4. Lemma

If a is a closest point to x in A , then $a \in A_1(x)$.

This follows from the fact that, for each $t \in [0, 1]$ $|x_t - a| = (1-t)|x-a|$
 $= d(x_t, A) - |x_t - x|$. \square

Combining this lemma and a result of M. EDELSTEIN we get the following corollary.

2.5. Corollary

If A is any closed non void subset of a uniformly convex Banach space E there exists a dense subset D of $E \setminus A$ such A is m -attractive for each $x \in D$ and each $m \in [0, 1]$.

The following examples are immediate consequences of the lemma or the definition.

- A is 1-attractive if A is boundedly weakly compact (i.e. for each closed ball B , $A \cap B$ is weakly compact)
- In particular A is 1-attractive if A is weakly closed and E is a reflexive Banach space (or more generally, if E is a dual space and A is weak*-closed).
- Any closed ball of a n.v.s. is 1-attractive.
- A finite union of m -attractive subsets is m -attractive.

Some more examples are given in the following lemmas. The first one shows that any bounded closed subset is attractive for any point which is remote enough.

2.6. Lemma

Let A be a bounded closed subset with diameter α in a Hilbert space E and let $m \in (0,1)$. For any $x \in E$ with $d(x,A) > \alpha(1+m)(1-m)^{-1}$ any point of A is a point of m -attraction for x in A .

Proof : Without loss of generality we may show that if $0 \in A$ then $0 \in A_m(x)$ if $d(x,A) > \alpha(1+m)(1-m)^{-1}$. For such a point x , we have $m < (r-\alpha)(r+\alpha)^{-1}$ with $r = |x|$. Thus for each $a \in A$ we obtain $(x-a|x) \geq r(r-|a|) \geq r(r-\alpha) > mr(r+\alpha) \geq mr|x-a|$. For $t > 0$ small enough we get

$$((1-t)x-a|x) \geq mr|(1-t)x-a|$$

thus, for $x_t = (1-t)x$, $(x_t-a|x) \geq mr|x_t-a|$ and

$$|x-a|^2 = |x-x_t|^2 + |x_t-a|^2 + 2t(x_t-a|x) \geq (m|x-x_t| + |x_t-a|)^2.$$

Taking the infimum on $a \in A$ we get $d(x,A) \geq md(x,x_t) + d(x_t,A)$, for t small enough, thus $0 \in A_m(x)$. \square

2.7. Lemma

If A is a closed affine subspace of a n.v.s. E , then for each $m \in (0,1)$ A is m -attractive.

Proof : Let $x \in E \setminus A$. Without loss of generality we may suppose A is a linear subspace of E . We choose $a \in A$ with $|x-a| < m^{-1}d(x,A)$. For every $y \in A$ and $t \in [0,1)$ we have, with $x_t = x + t(a-x)$

$$|x+t(a-x)-y| = (1-t)|x-(1-t)^{-1}(y-ta)|$$

hence

$$d(x_t,A) = (1-t)d(x,A)$$

as $(1-t)^{-1}(y-ta)$ runs over A as y runs over A . We get

$$d(x_t,A) \leq d(x,A) - tm|x-a| = d(x,A) - m|x_t-x|. \quad \square$$

2.8. Corollary

Any half-subspace (or union of half-subspaces) is m -attractive for all $m \in (0,1)$.

This follows from the fact that A is m -attractive whenever its boundary ∂A is m -attractive.

2.9. Lemma

Every polyhedral convex subset of a n.v.s. E is m -attractive for every $m \in (0,1)$.

Proof: Suppose first that F is a closed linear subspace of E , that $p: E \rightarrow \dot{E} = E/F$ is the canonical projection and that $A = p^{-1}(\dot{A})$ for some k -attractive subset \dot{A} of \dot{E} (endowed with the quotient norm). Then for every $m \in (0,k)$ A is m -attractive as for each $x \in E \setminus A$ we have $d(x,A) = d(\dot{x},\dot{A})$ for $\dot{x} = p(x)$ and we can find $a \in A$ with $m|x-a| \leq k|\dot{x}-\dot{a}|$, $p(a) = \dot{a}$ given in $\dot{A}_k(\dot{x})$ which implies that for $x_t = x + t(x-a)$, $\dot{x}_t = p(x_t)$

$$d(x_t, A) = d(\dot{x}_t, \dot{A}) \leq d(\dot{x}, \dot{A}) - kt|\dot{x}-\dot{a}| \leq d(x, A) - mt|x-a| = d(x, A) - md(x_t, x).$$

If A is a polyhedral subset of E , then there is a closed subspace F with finite codimension, and a polyhedral convex subset \dot{A} of \dot{E}/F such that $A = p^{-1}(\dot{A})$, p being the canonical projection. The result follows from the first part of the proof and examples a) or b) above as E/F is finite dimensional. \square

In the following statements we use the classical Hausdorff distance to define lipschitzian multifunctions.

2.10 Theorem

Let X be a non void complete subset of a normed vector space E , and let $F: X \rightarrow E$ be a k -lipschitzian multifunction with closed values ($k \in [0,1)$). If there exists $m > k$ such that for every $x \in X$ the set $F_m(x)$ of points of m -attraction of x in $F(x)$ has a non void intersection with $x + T_x X$, then F has a fixed point.

2.11 Corollary

Let X be a non void complete subset of a normed vector space E , and let $F: X \rightarrow E$ be a k -lipschitzian multifunction ($k \in [0,1)$). The following two conditions ensure that F has a fixed point:

- a) for every $x \in X$ $F(x) - x \subset T_x X$
- b) for some $m \in (k,1)$, for every $x \in X$, $F(x)$ is m -attractive for x .

These results extends theorem 3.4. in Reich: X is not supposed to be convex, and F is not supposed to be compact-valued.

Proof: We suppose F has no fixed points and we are going to built an auxiliary mapping $f: X \rightarrow X$ which violates Caristi's fixed point theorem.

We choose $q > 0$ small enough so that $r = \frac{m-q}{1+q} - k > 0$ and we set $g(x) = r^{-1}d(x, F(x))$, a continuous function. For a given $x \in X$ we choose

$z \in F_m(x) \cap (x + T_x X)$ and $\epsilon(x) > 0$ such that

$$d(x_t, F(x)) \leq d(x, F(x)) - m|x - x_t|$$

for every $t \in [0, \epsilon(x)]$, with $x_t = x + t(z - x)$. As $z - x \in T_x X$, we can find $t \in (0, \min(q, \epsilon(x)))$ with

$$d(x_t, X) < tq|z - x|,$$

hence we can also find $y \in X$ with

$$d(x_t, y) < tq|z - x| = q|x_t - x|.$$

Thus $|x_t - x| > (1+q)^{-1}|x - y|$. For each $w \in F(x)$ we have

$$d(y, F(y)) \leq d(y, x_t) + d(x_t, w) + d(w, F(y))$$

hence $d(y, F(y)) \leq d(y, x_t) + d(x_t, F(x)) + d(F(x), F(y))$.

It follows that

$$\begin{aligned} d(y, F(y)) &\leq |y - x_t| + d(x, F(x)) - m|x - x_t| + k|x - y| \\ &\leq d(x, F(x)) - (m - q)|x - x_t| + k|x - y| \\ &< d(x, F(x)) - [(m - q)(1 + q^{-1}) - k]|x - y|. \end{aligned}$$

Setting $y = f(x)$ and dividing by r we obtain

$$g(f(x)) < g(x) - d(x, f(x)).$$

Thus the self map $f : X \rightarrow X$ has no fixed point, contradicting Caristi's fixed point theorem. \square

Now we relax the assumption in theorem 2.10 that F is a contraction. We suppose F is pseudo-strongly-contractive in the following sense : for all $r > 0$ there exists $c_r > 1$ such that for all $x, y \in X$, $x' \in F(x)$, $y' \in F(y)$

$$|(1+r)(x-y) - r(x'-y')| \geq c_r|x-y|.$$

This condition strengthens the classical definition of pseudo-contractive mappings (F.E. BROWDER (1967)). If F is a single-valued contraction (with Lipschitz constant $k \in [0, 1)$) then F is pseudo-strongly contractive as

$$|(1+r)(x-y) - r(F(x)-F(y))| \geq (1+r)|x-y| - r|F(x)-F(y)| \geq (1+r(1-k))|x-y|.$$

Moreover we have the following lemma in which J denotes the duality mapping of the n.v.s. E .

2.12. Lemma

F is pseudo-strongly contractive if $D = I - F$ is strongly accretive in the following sense : there exists $c > 0$ such that for all $x, y \in X$,

$| \begin{array}{l} x' \in D(x), y' \in D(y) \text{ there exists } z \in J(x-y) \text{ such that} \\ \langle x'-y', z \rangle \geq c|x-y|^2. \end{array} \right.$

Proof : Suppose D is strongly accretive. Then for $x \neq y$ in X , $x' \in F(x)$, $y' \in F(y)$ we have $x-x' \in D(x)$, $y-y' \in D(y)$ thus for some $z \in J(x-y)$ we get
 $\langle (1+r)(x-y) - r(x'-y'), z \rangle = \langle x-y, z \rangle + r \langle (x-x') - (y-y'), z \rangle \geq |x-y|^2 + cr|x-y|^2$
hence

$$|(1+r)(x-y) - r(x'-y')| \geq (1+rc) |x-y|. \quad \square$$

2.13. Remark : Pseudo-strongly contractive mappings associated with constants $c_r > 1$ such that $\liminf_{r \rightarrow 0_+} r^{-1}(c_r - 1) > 0$ are characterized by the fact that

$D = I - F$ is strongly accretive. The preceding proof shows that this condition is sufficient. Conversely, suppose we are given a function $r \mapsto c_r$ on $(0, +\infty)$ with $\liminf_{r \rightarrow 0_+} r^{-1}(c_r - 1) \geq c > 0$ in such a way that for all $x, y \in X$, $r > 0$ $x' \in D(x)$, $y' \in D(y)$ we have

$$|x-y+r(x'-y')| \geq c_r |x-y|.$$

We set $u = x - y$, $v = x' - y'$ and we take $z_r \in J(u+rv)$.

As we may suppose $u \neq 0$, z_r is non null ($|z_r| = |u+rv| \geq c_r |u|$) and we can introduce $w_r = z_r / |z_r|$. We have

$$c_r |u| \leq |u+rv| = \langle w_r, u+rv \rangle \leq |u| + r \langle w_r, v \rangle,$$

hence if w is a weak* limit point of (w_r) as $r \rightarrow 0$ we have ,

$$\langle w, v \rangle \geq \liminf_{r \rightarrow 0_+} \frac{(c_r - 1)}{r} |u| \geq c |u|.$$

Moreover $|w| \leq 1$ as $|w_r| \leq 1$,

$$\langle w, u \rangle \geq \liminf_{r \rightarrow 0_+} \langle w_r, u+rv \rangle \geq |u|.$$

Hence $z = |u|w \in J(u)$ and $\langle z, v \rangle \geq c|u|^2$. Thus

$$\langle z, x'-y' \rangle \geq c|x-y|^2. \quad \square$$

A subset X of a t.v.s. E is said to be tangentially convex if $T_x X$ is convex for each $x \in X$; it is said to be pseudo-convex if $X - x \subset \overline{\text{co}(T_x X)}$ for each $x \in X$, where $\overline{\text{co}(A)}$ is the closed convex hull of A .

2.14. Theorem

Let X be a tangentially convex and pseudo-convex complete subset of a normed space E and let $F : X \rightarrow E$ be a lipschitzian pseudo-strongly-contractive multifunction with weakly compact values such that $F(x) \subset x + T_x X$ for each $x \in X$. Then F has a fixed point in X .

If X is weakly compact or if E is a reflexive Banach space and X is weakly closed then the assumptions imply that X is convex (J. BORWEIN, R. O'BRIEN (1976)). Thus we are not very far from the convex case. For a related result see K. DEIMLING (1974).

Proof: Choose $r > 0$ so small that rF is a contraction mapping. Observe that $G = [I + r(I-F)]^{-1}$ is a single-valued contraction mapping: for $u = x + r(x-x')$, $v = y + r(y-y')$ with $x, y \in X$, $x' \in F(x)$, $y' \in F(y)$ we have

$$|u-v| \geq c_r |x-y|$$

and $c_r^{-1} < 1$. We will show that the domain of G contains X . Given $a \in X$ we define $H : X \rightarrow E$ by

$$H(x) = k F(x) + (1-k)a$$

with $k = r(r+1)^{-1}$. H is easily seen to be a contraction mapping. Moreover, for every $x \in X$ we have $H(x) - x \subset T_x X$ as $T_x X$ is convex and $F(x) - x \subset T_x X$, $a - x \in T_x X$. Let b be a fixed point of H (corollary 2.11). We have

$$b \in r(r+1)^{-1} F(b) + (r+1)^{-1} a,$$

hence for some $c \in F(b)$

$$a = (r+1)b - rc \in b + r(I-F)(b).$$

This shows that a belongs to the range of $I + r(I-F)$ which is the domain of G . The restriction of G to X is a contraction which must have a fixed point as X is complete. This fixed point is also a fixed point for F . \square

The following result generalizes th.1 in J. REINERMAN - R. SCHONEBERG (1976) and results due to W.G. DOTSON Jr (1972), P. KUHFITTING (1977).

2.15 Theorem

Let $F : X \rightarrow E$ be a nonexpansive multifunction with weakly closed values, where E is a Hilbert space and X is a closed subset of E such that for some $x_0 \in X$ either (a) X is starshaped w.r.t. x_0 (i.e. $x_0 + t(x-x_0) \in X$ for $t \in (0,1)$, $x \in X$) or (b) X is tangentially convex and $x_0 - x \in T_x X$ for each $x \in X$.

Then if $F(x) - x \in T_x X$ for each $x \in \partial X$ and if X is bounded or contains an invariant bounded subset S , then F has a fixed point.

Proof : Using the method of J. REINERMANN and R. SCHONEBERG (1976) which relies on a result of M.G. CRANDALL and A. PAZY, it suffices to show that for each $\lambda \in (0,1)$ λF has a fixed point. As λF is a contraction, we are reduced to show that for each $x \in \partial X$ $\lambda F(x) - x \in T_x X$ (corollary 2.11). We may suppose $x_0 = 0$. Let $y \in F(x)$ hence $y - x \in T_x X$. In case (b) we have $\lambda y - x = \lambda(y - x) + (1 - \lambda)(-x) \in T_x X$ as $-x \in T_x X$ and $T_x X$ is convex. In case (a) we can find sequences $(t_n) \subset \mathbb{R}_+$, $(x_n) \subset X$ with $\lim t_n = +\infty$, $\lim x_n = x$, $\lim t_n(x_n - x) = y - x$. Then $\lambda y - x = \lim s_n(y_n - x)$ with $s_n = \lambda t_n + 1 - \lambda$, $y_n = (\lambda t_n + 1 - \lambda)^{-1} \lambda t_n x_n \in X$, hence $\lambda y - x \in T_x X$. \square

§.3 - NONEXPANSIVE MAPPINGS

Let (M, d) be a metric space, and let A be a subset of M with diameter $\delta(A) = \sup_{a, b \in A} d(a, b)$. A point a of A is said to be diametral if $\sup_{x \in A} d(a, x) = \delta(A)$.

3.1. Definition

A class \mathcal{C} of subsets of (M, d) is said to be normal if each non void member A of \mathcal{C} with $\delta(A) > 0$ contains a point $a \in A$ which is not diametral for A .

Recall (c.f. P.A. MEYER (1966)) that a class \mathcal{C} of subsets of a set X is said to be compact if each subfamily of \mathcal{C} with the finite intersection property has a non void intersection. A topology on X is not needed in this definition. For instance the class of finite subsets is compact.

3.2. Theorem

Let (M, d) be a bounded metric space, let $f : M \rightarrow M$ be a non expansive map.

Suppose \mathcal{C} is a class of subsets of M such that

- a) \mathcal{C} is compact.
- b) \mathcal{C} is stable under (finite or infinite) intersections.
- c) \mathcal{C} is normal.
- d) \mathcal{C} contains the closed balls of (M, d) .

Then f has a fixed point.

The proof we give is a simple abstraction of Kirk's proof.

Proof : Let \mathcal{F} be the class of non void invariant members of \mathcal{C} :

$$\mathcal{F} = \{C \in \mathcal{C} \mid f(C) \subset C\}.$$

We have $M \in \mathcal{F}$. Using Zorn's lemma, we first show that (\mathcal{F}, \subset) has a minimal element. If \mathcal{G} is a totally ordered subfamily of \mathcal{F} then $F = \bigcap_{G \in \mathcal{G}} G$ belongs to \mathcal{C} , is invariant, and non void by a). Hence F is a lower bound for \mathcal{G} and \mathcal{F} is inductive.

Let $N \in \mathcal{F}$ be a minimal element. We want to show that N is reduced to one element, hence consists of a fixed point of f . If this is not the case, i.e. if the diameter δ of N is strictly positive, there exist $a \in N$ and $r \in (0, \delta)$ with $N \subset \overline{B}(a, r)$, the closed ball with center a and radius r .

Let

$$P = \{x \in N \mid N \subset \overline{B}(x, r)\} = N \cap \bigcap_{y \in N} \overline{B}(y, r).$$

Then $P \neq \emptyset$, P belongs to \mathcal{C} by b) and d). Moreover P is different from N as its diameter is $\pi r < \delta$. If we show that P is invariant, we get a contradiction.

We first observe that the smallest element $C \in \mathcal{C}$ which contains $f(N)$ (which exists by b)) is included in N and thus is invariant. Hence C coincides with N .

Then we choose $a \in P$ and observe that for every $x \in N$

$$d(f(x), f(a)) \leq d(x, a) \leq r.$$

Thus $f(N) \subset \overline{B}(f(a), r)$. As N is the smallest element of \mathcal{C} which contains $f(N)$ we also have $N \subset \overline{B}(f(a), r)$ by d). Hence $f(a) \in P$ and P is invariant. \square

3.3. Remark : Suppose the closed balls of (M, d) are compact for a topology σ on M which is coarser than the metric topology. Then the class \mathcal{C} of compact sets in (M, σ) satisfies b) (and obviously a) and d)). Hence if c) is satisfied, the result holds.

This occurs for the weak^{*}-topology of a dual Banach space when M is a weak^{*}-closed bounded convex subset with a normal structure (BROWDER - GÖHDE - KIRK theorem).

In the same spirit we give now an adaptation of a result of W.A. KIRK (1970) which does not use normal structure. Given a non expansive mapping f of a metric space (M, d) into itself, for $x \in M$ we set

$$\rho(x) = \delta(O(x)) = \sup_{n \geq 1} d(x, f^n(x))$$

where $\delta(A)$ is the diameter of A and $O(x)$ is the orbit of x :

$$O(x) = \{x, f(x), f^2(x), \dots\}.$$

As the sequence $\rho(f^n(x))$ is non increasing, it has a limit $r(x)$ (possibly infi-

nite) called the limiting orbital diameter of f at x . If for each $x \in M$ $\rho(x)$ is finite and $r(x) < \rho(x)$ whenever $\rho(x) > 0$, f is said to have diminishing orbital diameters (KIRK (1970)).

3.4. Theorem

Let (M, d) be a metric space, let $f : M \rightarrow M$ be a non expansive mapping which has diminishing orbital diameters. Suppose a class \mathcal{C} of subsets of M is given verifying:

- \mathcal{C} is compact.
- \mathcal{C} is stable under arbitrary intersections.
- Any closed member of the family \mathcal{C}' of countable intersections of unions of increasing sequences of members of \mathcal{C} belongs to \mathcal{C} .
- \mathcal{C} contains M and the closed balls of M .

Then f has a fixed point.

Proof : The class of non void invariant members of \mathcal{C} has a minimal element N by Zorn's lemma, a), b) and d). Again we show that we get a contradiction if we suppose the diameter δ of N is strictly positive. Given $a \in N$ we have $\rho(a) > 0$ by minimality of N , thus for some iterate $b = f^i(a)$ of a we have $\rho(b) < \rho(a)$. For each $k \geq 1$ let

$$F_k = \{x \in M \mid \exists m \in \mathbb{N} \quad \forall n \geq m \quad d(x, f^n(a)) \leq \rho(b) + 1/k\}$$

$$= \bigcup_{m \geq 1} \bigcap_{n \geq m} \overline{B}(f^n(a), \rho(b) + 1/k)$$

and let $F = \bigcap_{k \geq 1} F_k$.

Let us show that F is closed. Suppose $(z_p) \subset F$ and $(z_p) \rightarrow z$. Given $k \geq 1$ there exists p with $d(z_p, z) < 1/2k$ and $m \in \mathbb{N}$ such that $d(z_p, f^n(a)) \leq \rho(b) + 1/2k$ for $n \geq m$. Thus for $n \geq m$ we have $d(z, f^n(a)) \leq \rho(b) + 1/k$. We conclude $z \in F_k$ for each $k \geq 1$ and thus $z \in F$; so F is closed.

Assumption c) implies that $F \in \mathcal{C}$. As $F \cap N \in \mathcal{C}$ and $f(N \cap F) \subset N \cap F$, by minimality of N and the fact that $b \in F$ we obtain $N = N \cap F$.

Given $k \geq 1$ the closed ball $\overline{B}(x, \rho(b) + 1/k) \cap N$ with radius $\rho(b) + 1/k$ and center x in $N \subset F \subset F_k$ contains the orbit of some iterate $f^m(a)$ of a . Hence the balls $\overline{B}(x, \rho(b) + 1/k) \cap N$, $x \in N$ have the finite intersection property. Assumptions a) and d) imply that

$$B_k = \bigcap_{x \in N} \overline{B}(x, \rho(b) + 1/k) \cap N$$

is non void as is $B = \bigcap_{k \geq 1} B_k$. For each $x \in B$ we have $N \subset \overline{B}(x, \rho(b))$. This means that each point of B is a non diametral point of N . Introducing P as in the proof of theorem 3.2., the same argument shows that $\delta > 0$ leads to a contradiction. \square

Assumptions a) and d) are usually satisfied by endowing M with another topology σ for which the closed balls of (M, d) are compact. This occurs in the following corollary which contains a classical result of BROWDER (1965), GOHDE (1965), KIRK (1965) (\mathcal{C} is taken to be the family of weak*-closed convex subsets).

3.5. Corollary

If $f : M \rightarrow M$ is a non expansive mapping in a weak*-closed convex bounded subset of a dual Banach space and if the class \mathcal{C} of weak*-closed convex subset of M is normal, or if f has diminishing orbital diameters, then f has a fixed point.

The following criterion (whose proof is similar to a result of M.S. BRODSKII and D.P. MILMAN (1948)) shows that a kind of convexity condition is helpful for verifying the normality of a class.

3.6. Lemma

If each member C of a class \mathcal{C} of compact subsets of a metric space (M, d) is such that for any finite family a_1, \dots, a_n of distinct points of C there exists $a \in C$ with

$$d(a, x) \leq 1/n(d(a_1, x) + \dots + d(a_n, x))$$

for each $x \in C$, then \mathcal{C} is normal.

The same result holds if the members of \mathcal{C} are non compact but if their measures of noncompactness are strictly less than their diameters (the proof is left to the reader).

The problem of characterizing Banach spaces with a normal structure (BRODSKII - MILMAN (1948), GOSSEZ-LAMI-DOZO (1969), (1972)...) can be broaden to the problem of characterizing riemannian manifolds (or more generally finslerian manifolds with a connection) which possess a normal structure.

For instance, it is easily shown that any convex (in the riemannian sense) subset of the unit sphere of a Hilbert space with diameter less than 1 has a normal structure.

§.4. AN ELEMENTARY HOMOTOPY ARGUMENT

In this section we develop as the result of a joint work with M. LASSONDE an elementary argument following the lines of A. GRANAS (1976) showing that a fixed-point result on a nonconvex domain is a consequence of a fixed point theorem on a closed convex domain. This argument replaces the use of degree theory ; it relies on the use of special homotopies. As we wish to apply this method to various classes of mappings we follow an axiomatic approach. Our setting takes place in a fixed convex subset of a t.v.s., which is in general either the whole space or a convex cone.

We consider a t.v.s. E , a convex subset D of E with $0 \in D$, a class \mathcal{C} of subsets of D and for each subset X of D a class $M(X,D)$ of multifunctions from X into D with closed graph in $X \times D$ and bounded range. We adopt the following assumptions :

- (G1) For each $C \in \mathcal{C}$ and each $F \in M(X,D)$ with $F(C) \subset C$, F has a fixed point.
- (G2) Each bounded subset of D is contained in some member of \mathcal{C} .
- (G3) If $F \in M(C,D)$ with $C \in \mathcal{C}$, and if X is a closed subset of C , then $F|_X \in M(X,D)$.
- (G4) If X is a closed subset of $C \in \mathcal{C}$, and if $F \in M(X,D)$ is such that $F(x) = \{0\}$ for each x in the relative boundary $\partial_C X$ of X in C , then the extension \tilde{F} of F to C given by $\tilde{F}(x) = \{0\}$ for $x \in C \setminus X$ belongs to $M(C,D)$.
- (G5) If X is a closed subset of D and if $F \in M(X,D)$, then for each continuous function $s : X \rightarrow [0,1]$, the multifunction G given by $G(x) = s(x) F(x)$ belongs to $M(X,D)$.

We first give an easy consequence of the preceding axioms.

4.1. Lemma

Let X be a closed subset of some $C \in \mathcal{C}$ with $0 \in X$, let $F \in M(X,C)$ with $F(X) \subset C$ and $F(x) = \{0\}$ for each x in the relative boundary $\partial_C X$ of X in C . Then F has a fixed point in X .

Proof : Let $\tilde{F} \in M(C,C)$ be the extension of F by $\{0\}$ on $C \setminus X$ (G4).

As $\tilde{F}(C) \subset C$, \tilde{F} has a fixed point \tilde{x} in C (G1). As $\tilde{F}(C \setminus X) = \{0\}$ and $0 \in X$, we must have $\tilde{x} \in X$, thus $\tilde{x} \in F(\tilde{x})$. \square

4.2. Theorem

Suppose X is a closed normal subset of D , with 0 in the relative interior

$X \setminus \partial_D X$ of X in D . If $F \in M(X, D)$ is such that $x \notin tF(x)$ for each $(t, x) \in (0, 1) \times \partial_D X$, then F has a fixed point.

Proof : We consider the nontrivial case where $F|_{\partial_D X}$ is fixed-point free. Then $x \notin tF(x)$ for each $(t, x) \in [0, 1] \times \partial_D X$. Let $C \in \mathcal{C}$ with $\{0\} \cup F(X) \subset C$ (G2), and let $Y = X \cap C$. Then $\partial_C Y \subset \partial_D X$. Let

$$B_0 = \{x \in X \mid \exists t \in [0, 1], x \in tF(x)\}, \quad B = B_0 \cup \{0\}.$$

We show that B is closed in X . Let $x \in X$ be the limit of a net $(x_i)_{i \in I}$ of B . Using a subnet if necessary, we can find a net $(t_i)_{i \in I}$ in $[0, 1]$ with $x_i \in t_i F(x_i)$ and suppose that $(t_i)_{i \in I}$ has a limit t in $[0, 1]$. If $t = 0$, we have $x = 0$ as $F(X)$ is bounded. If $t > 0$, we have $x/t \in F(x)$, as F has a closed graph. Hence $x \in B$ in each case.

As X is normal and $A = \partial_D X$ is disjoint from B , we can find a continuous function $s : X \rightarrow [0, 1]$ with $s|_A = 0$, $s|_B = 1$. Let $G : X \rightarrow D$ be given by

$$G(x) = s(x) F(x).$$

Then $G \in M(X, D)$ by (G5) and $G(x) = \{0\}$ for each $x \in A = \partial_D X$, hence also for each $x \in \partial_C Y$. Moreover $G(Y) \subset C$ as C is convex. As $0 \in Y = X \cap C$, G has a fixed point $y \in Y$. Then $y \in s(y) F(y)$, thus $y \in B$ and $s(y) = 1$. Thus y is a fixed point of F . \square

4.3. Corollary

Suppose X is a closed normal subset of D with 0 in the relative interior $X \setminus \partial_D X$ of X in D and $x \in T_X X$ for each $x \in \partial_D X$. If $F \in M(X, D)$ is such that $F(x) - x \subset T_X X$ for each $x \in \partial_D X$ then F has a fixed point.

Proof : Suppose $x \notin tF(x)$ for some $(t, x) \in (0, 1) \times \partial_D X$. Then $(t^{-1}-1)x \in F(x) - x \subset T_X X$, and this is a contraction to $x \notin T_X X$ as $t^{-1}-1 > 0$ and $T_X X$ is a cone. \square

When the class M is invariant by translation, any point a in the relative interior of X can replace 0 .

The assumption $x \in T_X X$ is satisfied if X is a closed proper convex subset of E with $0 \in \text{int}(X)$: $-x \in \text{int}(T_X X)$ hence belongs to a half-space $h^{-1}((0, \infty))$ containing $\text{int}(T_X X)$ (with $h \in E'$) whereas x belongs to the opposite half-space $h^{-1}((-\infty, 0))$.

It is also satisfied for some nonconvex subsets ; for instance the epigraph $X = \{(r, s) \in \mathbb{R}^2 \mid s \geq h(r)\}$ of $h : r \mapsto -(|r| + 1)$ in $E = \mathbb{R}^2$.

It is easy to show that the axioms we gave subsume several cases : compact multi-functions (A. GRANAS (1976)), condensing mappings (C.J. HIMMELBERG, J.R. POTTER,

F.S. VAN VLECK (1969)), Φ -condensing mappings (V.W. PETRYSHYN, P.M. FITZPATRICK (1974) ...). The details will be given elsewhere. More general boundary conditions as in M. MARTELLI - A. VIGNOLI (1974) could be considered with slightly stronger axioms.

§.5. CRITICAL POINTS AND FIXED POINTS

The present section was suggested by the reading of J-P. AUBIN (1977) which presents related results obtained recently by J.P. AUBIN and F.H. CLARKE .

The problem of looking for fixed points of a map is embodied in a broader setting : one seeks sufficient conditions for a map $f : X \rightarrow F$, where E and F are t.v.s., X is a subset of E , to have a critical point, i.e. a point $a \in X$ with $f(a) = 0$.

We begin with a slight generalization of a classical notion.

5.1. Definition

A function $h : E \rightarrow \mathbb{R}^* = \mathbb{R} \cup \{+\infty\}$ is said to be pseudo-convex at a point $a \in \text{dom } h$ if $h(x) \geq h(a)$ whenever

$$\bar{d}h(a, x-a) = : \limsup_{t \downarrow 0, v \rightarrow x-a} t^{-1} [h(a+tv) - h(a)]$$

is non negative.

h is said to be pseudo-convex if h is pseudo-convex at each point of its domain. It is easy to verify that any continuous convex function is pseudo-convex.

5.2. Theorem

Let E and F be t.v.s., let X be a subset of E , $f : X \rightarrow F$ be any map such that

a) $-f(x) \in T_{f(x)} f(X)$.

If there exists a pseudo-convex map $h : F \rightarrow \mathbb{R}^*$ such that

b) $\text{dom } h \supset f(X)$,

c) $h(0) = 0$, $h(y) > 0$ for every $y \in Y = f(X)$, $y \neq 0$,

d) h attains its minimum on Y ,

then f has a critical point a in X .

Usually h is taken to be a norm on F .

Proof : Let $b = f(a)$ be such that $h(b) = \min_{y \in Y} h(y)$. Then for every $w \in T_b Y$ we have

$$\bar{d}h(b, w) \geq 0$$

as the definitions show immediately. In particular, we have

$$\bar{d}h(b, -f(a)) \geq 0.$$

As h is pseudo-convex at b we obtain $h(0) - h(b) \geq 0$. Hence $h(b)=0$, $b=0$. \square

5.3. Corollary

Let E be a t.v.s., F be a normed vector space, X be a subset of E , $f : X \rightarrow F$ be a map with a boundedly compact image $Y = f(X)$. Then if $-f(x) \in T_{f(x)} Y$ for every $x \in X$ there exists $a \in X$ with $f(a) = 0$.

Proof : Take for h the norm of F . \square

With the same choice for h we obtain the following variant of this corollary using the weak* lower semi-continuity of h .

5.4. Corollary

Let E be a t.v.s., F be a dual Banach space, X be a subset of E , $f : X \rightarrow F$ be a map with a weak*-closed image $Y = f(X)$. Then if $-f(x) \in T_{f(x)} Y$ for every $x \in X$ there exists $a \in X$ with $f(a) = 0$.

Now we add a differentiability assumption on f .

5.5. Definition

Let E, F be t.v.s. and let $X \subset F$. A map $f : X \rightarrow F$ is said to be M -differentiable (MICHAL-BASTIANI differentiable) at a point $a \in X$ if there exists a continuous linear map $u \in L(E, F)$ such that

$$\lim_{\substack{t \downarrow 0 \\ v \rightarrow x \\ x+tv \in X}} t^{-1} [f(a+tv) - f(a) - u(x)] = 0.$$

This notion is weaker than Fréchet differentiability but stronger than Gâteaux differentiability. It plays a convenient role in optimization theory and optimal control theory.

Note that although $f'(a)$ is not uniquely defined by the preceding definition if a is not interior to X , the value $u(x)$ of u on $x \in T_a X$ is independant of the choice of u and will be denoted by $f'(a).x$.

It is easy to show that if $f : X \rightarrow F$ is M -differentiable at $a \in X$ with derivative $f'(a)$ then $f'(a)(T_a X) \subset T_{f(a)} Y$ with $Y = f(X)$. Hence the condition $-f(x) \in T_{f(x)} f(X)$ is implied by the condition $-f(x) \in f'(x)(T_x X)$.

Let us remark that the condition used by J.P. AUBIN and F.H. CLARKE is still stronger as they deal with the tangent cone as defined by F.H. CLARKE (or peritangent cone) which is smaller than the usual tangent cone. For instance when $E = \mathbb{R}^2$, $X = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$, the peritangent cone at the origin is $\{0\}$ whereas $T_0 X = X$.

Let us state a fixed point theorem as an application of the preceding results.

5.6. Corollary

Let E be a dual Banach space (resp. a Banach space), X be a subset of E , $g : X \rightarrow E$ be a M -differentiable mapping such that

- a) $(I-g)(X)$ is a weakly* closed (resp. weakly boundedly compact) in E
- b) $g(x) - x \in (I-g'(x))(T_x X)$ for each $x \in \partial X$

Then g has a fixed point in X .

Proof : Take $f = I - g$. \square

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