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Some functors related to polynomial theory. II

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SOME FUNCTORS RELATED TO POLYNOMIAL THEORY, II

by

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1. Introduction. We consider the following natural transformation :

$$T^m : \mathcal{P}^m_R(X,Y) \to \mathrm{Map}(X,Y) \ , \ T^m(f) = f_R$$

where R denotes a commutative ring with 1, X,Y - R-modules, and $\mathcal{P}^m_R(X,Y)$ is the R-module of all forms of degree m on the pair (X,Y) (in the sense of N.Roby [2]). An element of $\mathcal{P}^m_R(X,Y)$ is a system $f=(f_A)$ indexed by all commutative R-algebras A, where $f_A : X \otimes A \to Y \otimes A$ are mappings satisfying the following conditions :

(i) $(1 \otimes u)\circ f_A = f_B \circ (1 \otimes u)$ for any R-algebra homomorphism $u : A \to B$,

(ii) $f_A(\underline{x}a) = f_A(\underline{x})a^m$ for any R-algebra A, any $\underline{x} \in X \otimes A$ and $a \in A$.

It is proved in [1] that in the case $X = R^n$, $Y = R$ we obtain :

$$T^m : R[T_1,\ldots,T_n]_m \to \mathrm{Map}(R^n,R), \ T^m(F)(x_1,\ldots,x_n) = F(x_1,\ldots,x_n).$$

It is well-known that the above homomorphism is not always injective ; this is the starting point and the motivation of the following considerations.

It is known from [2] that the functor $\mathcal{P}^m_R(X,-)$ is represented by the m-th divided power $\Gamma^m_R(X)$ of the module X. Similarly, it is proved in [1] that $\tilde{\mathcal{P}}^m_R(X,-) = \mathrm{Ker}\, T^m$ is represented by $\tilde{\Gamma}^m_R(X)$ where :

$$\tilde{\Gamma}^m_R(X) = \Gamma^m_R(X) \ / \ R\{x^{(m)} ; x \in X\}.$$

The above module is generated by the classes of elements :

$$\gamma_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = x_1^{(m_1)} \ldots x_k^{(m_k)}, \ m_i \geq 0, \ m_1+\ldots+m_k = m, \ x_1,\ldots,x_k \in X,$$

which are denoted by $\tilde{\gamma}_{m_1,\ldots,m_k}(x_1,\ldots,x_k)$.

It is easy to see that $\tilde{\Gamma}^m_R$ is an endo-functor of the category R-Mod.
We recall the following results contained in [1] :

Lemma 1.1. $\tilde{\Gamma}^m_R$ commutes with direct limits.

Lemma 1.2. $\tilde{\Gamma}^m_R(X)$ is finitely generated if so is X.

Theorem 1.3. There exist the natural isomorphisms :

(1) $\tilde{\Gamma}^m_{R_S}(X_S) \approx \tilde{\Gamma}^m_R(X)_S$ *for any multiplicative set* S *in* R

(2) $\tilde{\Gamma}^m_{R/I}(X/IX) \approx \tilde{\Gamma}^m_R(X)/I\,\tilde{\Gamma}^m_R(X)$ *for any ideal* I *in* R

(3) $\tilde{\Gamma}^m_{R\times R'}(X\times X') \approx \tilde{\Gamma}^m_R(X) \times \tilde{\Gamma}^m_{R'}(X')$.

Theorem 1.4. For a finitely generated R-module X, *the following conditions are equivalent* :

(i) $\tilde{\Gamma}^m_R(X) = 0$

(ii) $\tilde{\Gamma}^m_{R/P}(X/PX) = 0$ for any $P \in \mathrm{Max}(R)$

(iii) For any $P \in \mathrm{Max}(R)$: either $\dim_{R/P}(X/PX) \leq 1$ or $m \leq |R/P|$.

In particular, $\tilde{\Gamma}^m_R = 0$ iff $m \leq d(R) : \inf\{|R/P| \; ; \; P \in \mathrm{Max}(R)\}$.

2. *The structure* of $\tilde{\Gamma}^m_R(X)$. We shall give some structural informations on $\tilde{\Gamma}^m_R(X)$ which generalize results contained in [1]. The first step is the following

Lemma 2.1. If $P \in \mathrm{Spec}(R) - \mathrm{Max}(R)$ *then* $\tilde{\Gamma}^m_R(X)_P = 0$ *for any R-module* X. *Moreover, is* X *is finitely generated then* $\mathrm{Ann}(\tilde{\Gamma}^m_R(X)) \not\subset P$.

Proof : Observe that R/P is an infinite domain (it is not a field!) and hence $d(R_P) = \infty$. It follows from Theorem 1.3 and 1.4 that $\tilde{\Gamma}^m_R(X)_P = \tilde{\Gamma}^m_{R_P}(X_P) = 0$. Then the second part of the lemma follows from Lemma 1.2.

Corollary 2.2. If $\dim(R) > 0$ *then* :

(1) $\tilde{\Gamma}^m_R(X)$ *are torsion modules*.

(2) $\tilde{\Gamma}^m_R(X)$ *is free iff it is zero*.

If $\dim(R_P) > 0$ *for any* $P \in \mathrm{Max}(R)$ *then* :

(3) $\tilde{\Gamma}^m_R(X)$ *is projective iff it is zero*.

Now we explain the structure of $\tilde{\Gamma}^m(X)$ over Noetherian rings.

Theorem 2.3. Let R *be a Noetherian ring and let* X *be a finitely generated R-module. Then there exists a natural R-isomorphism* :

$$\tilde{\Gamma}^m_R(X) \approx \bigoplus_{P\in \mathrm{Max}(R)} \tilde{\Gamma}^m_{R/P^{k_P}}(X/P^{k_P}X)$$

induced by $X \to X/P^{k_P}X$, *for all sufficiently large* k_P.

Proof : We can assume that $\mathrm{Ann}(\tilde{\Gamma}^m_R(X)) \neq R$. Let $\mathrm{Ann}(\tilde{\Gamma}^m_R(X)) = Q_1 \cap \ldots \cap Q_s$ be a primary decomposition, and let $P_i = \mathrm{rad}(Q_i)$. Observe that $P_i^{k_i} \subset Q_i$ for all sufficiently large k_i. Denote $I = P_1^{k_1} \ldots P_s^{k_s} \subset \mathrm{Ann}(\tilde{\Gamma}^m_R(X))$. Since $P_1,\ldots,P_s \in \mathrm{Max}(R)$ by Lemma 2.1, it follows that $R/I \approx \prod_{i=1}^{s} R/P_i^{k_i}$ and hence :

$$\tilde{\Gamma}^m_R(X) = \tilde{\Gamma}^m_R(X) \,/\, I\tilde{\Gamma}^m_R(X) \approx \tilde{\Gamma}^m_{R/I}(X/IX) \approx \bigoplus_{i=1}^{s} \tilde{\Gamma}^m_{R/P_i^{k_i}}(X/P_i^{k_i}X).$$

If $P \in \mathrm{Max}(R) - \{P_1,\dots,P_s\}$ then $I + P^k = R$ for each natural k, and hence : $\tilde{\Gamma}^m_{R/P^k}(X/P^kX) = \tilde{\Gamma}^m_R(X)/P^k\,\tilde{\Gamma}^m_R(X) = 0$.

This completes the proof.

Corollary 2.4. If R is a Noetherian ring then there exists a natural R-isomorphism

$$\tilde{\Gamma}^m_R(X) \approx \bigoplus_{P\in \mathrm{Max}(R)} \tilde{\Gamma}^m_{R_P}(X_P)$$

induced by $X \to X_P$.

Proof: Compare the decompositions from Theorem 2.3 for X and X_P in the case if X is finitely generated. Next apply Lemma 1.1.

The same argument prove the following

Corollary 2.5. If R is a local Noetherian ring then there exists a natural R-isomorphism :

$$\tilde{\Gamma}^m_R(X) \approx \tilde{\Gamma}^m_{\hat{R}}(X \otimes \hat{R})$$

induced by $X \to X \otimes \hat{R}$.

Observe that the above two corollaries reduce the computation of $\tilde{\Gamma}^m_R(X)$ for Noetherian R to the case when R is local and complete. Theorem 2.3 reduces this problem (for finitely generated X) to the case when R is local Artinian. This case will be studied in the next section.

3. The Artinian case. Let (R,P) be an Artinian local ring. Then $P^k = 0$ for some natural k. Observe that $r^2 = 0$ for any $r \in P^{k-1}$ (if $k > 1$). This is the motivation of the following

Proposition 3.1. If $r^2=0$ in R and $m \leq 5$ then $r\tilde{\Gamma}^m_R(X) = 0$ for any R-module X.

Proof : To start with, we give some general formulas. It follows from [1] that :

$$\sum_{m_i > 0} \tilde{\gamma}_{m_1,\dots,m_n}(x_1,\dots,x_n) = 0 \quad \text{for any } x_1,\dots,x_n \in X.$$

Denote $/m_1,\dots,m_n/ = \tilde{\gamma}_{m_1,\dots,m_n}(x_1,\dots,x_n)$ for $m_i > 0$, $\Sigma\, m_i = m$. We must prove that r annihilates all this generators. We have :

(1) $\Sigma/m_1,\dots,m_n/ = 0$.

Replacing x_1 by rx_1 and $(1+r)x_1$ we get :

(2) $r\Sigma/1,m_2,\dots,m_n/ = 0$

(2') $\Sigma\,(1+rm_1)/m_1,\dots,m_n/ = 0$

since $r^2 = 0$ and $(1+r)^k = 1+kr$. In view of (1) and (2) we get from (2') :

(3) $$r \sum_{k=3}^{m-n+1} (k-2)\, \Sigma/k,m_2,\dots,m_n/ = 0.$$

In particular, it follows that :

(a) $/1,\dots,1/ = 0$ by (1) $(n=m)$
(b) $r/2,1,\dots,1/ = 0$ by (1) and (2) $(n=m-1)$
(c) $r/3,1,\dots,1/ = 0$ by (3) $(n=m-2)$
(d) $r/1,m-1/ = 0$ by (2) $(n=2)$.

For $m \le 2$ there is nothing to prove. For $m=3$ we utilize (a),(b). For $m=4$ we get $r/3,1/ = r/1,3/ = r/2,1,1/ = r/1,2,1/ = r/1,1,2/ = r/1,1,1,1/ = 0$. Hence also $r/2,2/ = 0$ by (1). For $m=5$ we have $r/1,4/ = r/3,1,1/ = r/2,1,1,1/ = r/1,1,1,1,1/ = 0$ and analogously for any permutation. Then (2) and (3) get us $r/1,2,2/ = r/3,2/ = 0$. This completes the proof.

Remark 3.2. Using the same formulas (when we also replace x_2 by $-x_2$) we can prove the above proposition for $m \le 7$ with the assumption that 2 is invertible in R.

Corollary 3.3. *Let R be a Noetherian ring and $m \le 5$ (or $m \le 7$ and 2 is invertible in R). Then there exists a natural R-isomorphism :*

$$\tilde{\Gamma}^m_R(X) \approx \bigoplus_{P \in \mathrm{Max}(R)} \tilde{\Gamma}^m_{R/P}(X/PX)$$

induced by $X \to X/PX$.

Proof : It can be assumed that X is finitely generated. In view of Theorem 2.3, it suffices to prove that $\tilde{\Gamma}^m_R(X) \approx \tilde{\Gamma}^m_{R/P}(X/PX)$ for any Artinian local (R,P). If $P^k = 0$, $P^{k-1} \neq 0$ and $k > 1$ (i.e. R is not a field) then :

$$\tilde{\Gamma}^m_R(X) = \tilde{\Gamma}^m_R(X) \,/\, P^{k-1}\tilde{\Gamma}^m_R(X) \approx \tilde{\Gamma}^m_{R/P^{k-1}}(X/P^{k-1}X)$$

by Proposition 3.1 and Remark 3.2. Induction on k completes the proof.

Remark 3.4. The assumptions of the above corollary are necessary. In fact, it can be computed that :

$$\tilde{\Gamma}^6_{Z_4}(Z_4^2) = Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_4, \quad \tilde{\Gamma}^8_{Z_9}(Z_9^2) = Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_9.$$

Remark 3.5. Since the dimensions of $\tilde{\Gamma}^m(X)$ over fields are known (see [1]), Corollary 3.3 solves the problem of computation of $\tilde{\Gamma}^m(X)$ over Noetherian rings for small m. For example, it can be proved that :

$$\tilde{\Gamma}^3_Z(Z^n) = \binom{n}{2} Z_2$$
$$\tilde{\Gamma}^4_Z(Z^n) = 2\binom{n+1}{3} Z_2 \oplus \binom{n}{2} Z_3$$

$$\tilde{\Gamma}^5_Z(Z^n) = (3\binom{n}{2} + 5\binom{n}{3} + 3\binom{n}{4})Z_2 \oplus 2\binom{n+1}{3}Z_3$$

where $\binom{n}{k} = 0$ for $n < k$. However, the problem is open for large m.

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