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THE REDUCED WITTRING

by

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These notes give a brief account on a joint work of L. Brückner and the author. Detailed proofs will appear in the Journal of Algebra.

Let K be a real (= formally real) field, $X = X(K)$ the topological space of all orderings of K [5, p.63], and $W(K)$ the Witt ring of the nondegenerated bilinearforms over K . By $W_t(K)$ we denote the torsion-subgroup of $W(K)$, which is known also to be its nilradical [6].

Let $C(X, \mathbb{Z})$ be the ring of all continuous functions $X \rightarrow \mathbb{Z}$ ($\mathbb{Z}$ provided with the discrete topology). Then we get a homomorphism $\text{sign} : W(K) \rightarrow C(X, \mathbb{Z})$ defined by $(\text{sign}(\rho))(P) := \text{sign}_P(\rho) = \text{signature of } \rho \text{ at } P$. The following basic result is due to Pfister [6].

THEOREM 1 : The sequence $0 \rightarrow W_t(K) \rightarrow W(K) \xrightarrow{\text{sign}} C(X, \mathbb{Z})$ is exact.

So it must be considered as a main task in the theory of reduced Witt rings to characterize the elements of $\text{sign } W(K)$ among the functions $f \in C(X, \mathbb{Z})$. In order to state the main result the notion of a preorder has to be introduced. A subset $T \subset K$ is called a preorder of K iff the following conditions are satisfied :

- i) $T + T \subset K$, $T \cdot T \subset K$ ii) $K^2 \subset T$ iii) $T \cap -T = \{0\}$.

A preorder T is the intersection of all orderings in which it is contained. Given a preorder T , the subset $X_T := \{P \supset T \mid P \text{ ordering of } K\}$ is a closed subspace of X . Clearly, we have the restriction-homomorphism $\text{Res} : C(X, \mathbb{Z}) \rightarrow C(X_T, \mathbb{Z})$.

Denote by $W_T(K)$ the image of $\text{Res} \cdot \text{sign} : W(K) \rightarrow C(X_T, \mathbb{Z})$. Choose any ordering $P_0 \supset T$. Set $P_0^x = P_0 \setminus \{0\}$, $T^x = T \setminus \{0\}$; P_0^x and T^x are subgroups of K^x . As with $W(K)$, we find an epimorphism $\mathbb{Z}[P_0^x/T^x] \rightarrow W_T(K)$. Furthermore the mapping $X_T \rightarrow \text{char}(P_0^x/T^x)$, $P \mapsto (aT^x \rightarrow \text{sgn}_P(a))$ is a topological embedding of X_T into the (Pontrjagin-) character-group of P_0^x/T^x .

PROPOSITION. For a preorder T the following statements are equivalent :

- i) $\mathbb{Z}[P_0^x/T^x] \rightarrow W_T(K)$ is an isomorphism,
- ii) $X_T \rightarrow \text{char}(P_0^x/T^x)$ is a homeomorphism,
- iii) $T + Ta = T \cup Ta$ for all $a \in K$, such that $a \notin -T$.

A preorder which satisfies the equivalent conditions of the last proposition, is called a fan (in French : éventail). Fans turn out to be of great importance in other contexts, too [1], [3].

THEOREM 2. A function $f \in C(X, \mathbb{Z})$ lies in sign $W(K)$ iff

$$\sum_{P \in T} f(P) \equiv 0 \pmod{\frac{1}{2} (K^X : T^X)}$$

for all fans T with $(K^X : T^X) < \infty$.

The description of sign $W(K)$ in $C(X, \mathbb{Z})$ was also attacked by R. Brown [4] and settled for the case that K admits only finitely many real places. For the general case he was led to a conjecture which (in his terminology) states that all formally real fields are exact. From theorem 2 one can derive:

THEOREM 3. All formally real fields are exact.

The proof of theorem 2 heavily depends on two local-global principles for reduced quadratic forms, one of which has essentially been proved in [2]. Furthermore, the generalized theory of reduced Witt rings [1] is extensively used, i.e. $W(K)$ is factorized by forms $\langle 1, -t \rangle$, where $0 \neq t$ and t belongs to an arbitrary but fixed preorder T . This point of view turns out to be fundamental even for the study of ordinary reduced Witt rings.

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