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REMARKS ON RANDOM WALKS ON SEMI SIMPLE LIE GROUPS

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The purpose of this note is to discuss two problems concerning random walks and harmonic functions on Lie groups, and there-by to complete and enlarge upon the results of A.RAUGI announced in [6] and proved in detail in [7] . Both problems have their origin in the work of H.FURSTENBERG [2,3] as further extended by R.AZENCOTT [1] and A.VIRTSER [8] . For the sake of convenience in notation and bookkeeping, the results of this paper are stated for connected semi simple groups , which are clearly the most important case , but one could easily extend them as in [7] to general almost connected Lie groups (those with only finitely many connected components) .

1. <u>μ-POSITIVE AND μ-NEGATIVE COCYCLES</u>

1.1.- Let G be a connected semisimple Lie group with finite center and with Iwasawa decomposition KAN . Let $\mathfrak{a}$ denote the Lie algebra of A , let $\mathfrak{a}^*$ be its real dual , and let $\mathfrak{a}^*_{\mathbb{C}}$ be the complexification of $\mathfrak{a}^*$. In the discussion that follows , the term <u>root</u> will always refer to an element of $\mathfrak{a}^*$ which is a restricted root of the pair (G , A) . The <u>Weyl group</u> is the finite group W of automorphisms of $\mathfrak{a}$ or $\mathfrak{a}^*$ induced by the conjugation action of the normalizer of A in K ; W permutes the roots . An ordering of the roots is canonically determined by the choice of N . For $x \in G$, write $x = kan$ with $k \in K$, $a \in A$, and $n \in N$, and let $H(x) = \log(a) \in \mathfrak{a}$. Then the functions ϕ on GxK of the form

$$(1) \qquad \phi(x , k) = \lambda(H(xk)) ,$$

where $\lambda \in \mathfrak{a}^*$, $x \in G$, and $k \in K$, are called <u>A-cocycles</u> (on K) . (See [3 , §6] -the present definition is not quite the same as Furstenberg's but is essentially equivalent) . Every such function is associated with an elementary spherical function Ψ on G defined by

$$(2) \qquad \Psi(x) = \int_K \exp(\phi(x , k))\, dk .$$

(Here dk denotes normalized Haar measure on the compact group K). In fact , all elementary spherical functions arise this way if we allow λ to be chosen from $\mathfrak{a}^*_{\mathbb{C}}$. For further discussion of spherical functions (about which there is an exten-

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sive literature) , the reader can consult [4] .

Now suppose μ is a probability measure on G . If f is a function on G , $x \in G$, and X_1 , X_2 , ... are independant G-valued random variables each with distribution μ , it is of interest to determine the asymptotic behavior of $f(X_n \ldots X_2X_1x)$ as $n \longrightarrow \infty$. This , of course , is the subject of "laws of large numbers" and "central limit theorems" for the left μ-walk on G . (One could analogously consider $f(xX_1X_2 \ldots X_n)$ and the right μ-walk on G) . The function f is at our disposal here , and it is convenient to have f related to the group-theoretic structure at hand . Furstenberg showed in [3 , §7] that if f is of the form $\phi(. , k)$ with ϕ an A-cocycle as above and $k \in K$, then $n^{-1}f(X_n \ldots X_2X_1x)$ converges with probability 1 to a limit $a_\mu(\phi)$ independent of x and k , provided that μ is regular enough . (Furstenberg takes μ of class B_1-absolutely continuous with a first moment-but it is clear from [1] that absolute continuity of μ can be weakened to the condition that μ be "étalé") . What is remarkable is that for certain choices of ϕ , this limit is strictly positive (or strictly negative) for all choices of μ . This was first shown by Furstenberg [3 , Theorem 7.6] under the additional restriction that μ be of class B_∞ (determined by a bounded L^1 function of compact support) and then extended by RAUGI [7 , Cor. (7.15)] to the conclusion that $a_\mu(\phi) > 0$ for all étalé μ with a first moment , whenever the $\lambda \in \mathfrak{a}^*$ that determines ϕ is a positive root . (The apparent change of sign from RAUGI's statement is due to the fact that RAUGI puts K on the right in the Iwasawa decomposition , whereas we are following Furstenberg and putting K on the left) . The purpose of this section is to point out that , with μ restricted to being of class B_∞ , one can deduce this fact directly from [3 , Theorem 7.4] and elementary facts about semisimple groups . This provides another proof of the conjecture on p. 417 of [3] .

1.2. Proposition. - *With notation as above , if* $\lambda \in \mathfrak{a}^*$ *is a positive root and* ϕ *is the* A-*cocycle defined by* (1) *, then* $a_\mu(\phi) > 0$ *for all probability measures* μ *of class* B_∞ .

Proof. - By [3 , Theorem 7.4] , we will have $a_\mu(\phi) < 0$ whenever the elementary spherical function Ψ defined by (2) satisfies $\|\Psi\|_{L^\infty(G)} \leqslant 1$. By the easy direction of the Helgason-Johnson Theorem [4 , p. 66] , this will be the case if $\lambda = i\nu - \rho$ with ρ half the sum of the positive roots and $\nu \in \mathfrak{a}^* + iC_\rho$, where C_ρ is the (closed) convex hull of the orbit of ρ under the Weyl group W . This is equivalent (since λ is real-valued) to having $\lambda \in -(\rho + C_\rho)$. But if $s_0 \in W$ is the element taking every positive root to a negative root , if α is a simple root ("with multiplicity") , and if $s_\alpha \in W$ is the reflection in the hyperplane normal to α , then $s_\alpha s_0 \rho = s_\alpha(-\rho) = \alpha - \rho$, and thus $\alpha = \rho + (\alpha - \rho) \in \rho + C_\rho$. This shows that each A-cocycle associated to the negative of a simple root is μ-negative for all μ of class B_∞ . Since the μ-positive and μ-negative cocycles constitute convex cones which are negatives of one another , and since each positive root is a sum of simple roots (possibly with repetitions) , the conclusion of the proposition follows .

1.3. Remark. - It is interesting to observe that the convex cone generated by $\rho + C_\rho$ is exactly the same as that generated by the positive roots (since for $s \in W$, $\rho + s\rho = \rho + (ss_0)(s_0\rho) = \rho - (ss_0)\rho$ is a sum of positive roots by dominance of ρ) . Furthermore , the Helgason-Johnson Theorem also shows that $\|\Psi\|_{L^\infty(G)} \leq 1$ only when $\lambda \in i\mathfrak{a}^* - (\rho + C_\rho)$. Thus the present method produces no more cocycles positive for all μ than does RAUGI's . If G is of real rank 1 , every non zero A-cocyle is either μ-positive for all μ or μ-negative for all μ , since every element of $\mathfrak{a}^*$ is a scalar multiple of a root . But when the real rank of G is > 1 , it is possible for a $\lambda \in \mathfrak{a}^*$ to be neither in the cone C^+ generated by the positive roots nor in the cone C^- generated by the negative roots . It seems likely that in this case the corresponding A-cocycle ϕ would be μ-positive for certain μ and μ-negative for certain other μ .

When μ is the class B_∞ probability measure defined by a K-biinvariant positive continuous function g of total mass 1 , it is easy to chech that

$$a_\mu(\phi) = \int_G g(x) \int_K \phi(x\,,\,k)dk\,dx$$

$$= \int_{\mathfrak{a}^+} g(\exp X)\,\lambda\,(T(X))\,\Delta\,(X)dx\,,$$

where $\mathfrak{a}^+ = \{X \in \mathfrak{a} : \alpha(X) > 0$ for all positive roots $\alpha\}$, Δ is a certain positive continuous function on $\mathfrak{a}^+$, dx is Lebesgue measure , and $T : \mathfrak{a} \longrightarrow \mathfrak{a}$ is given by

(3) $$T(X) = \int_K H((\exp X)k)dk.$$

Since $g|_{\mathfrak{a}^+}$ can have support in an arbitrarily small neighborhood of a given point $X \in \mathfrak{a}^+$, Proposition 1.2. implies that $\alpha(T(X)) > 0$ for all positive roots α . In other words , we have the following (purely group-theoretic) result :

1.4. Corollary. - *With* T *defined as in* (3) , $T(\mathfrak{a}^+) \subseteq \mathfrak{a}^+$. *If one could show that in fact* $T(\mathfrak{a}^+) = \mathfrak{a}^+$, *then for* $\lambda \notin (C^+ \cup C^-)$, $\lambda \circ T$ *would change sign in* $\mathfrak{a}^+$, *and one could choose* g *so that* $a_\mu(\phi)$ *was either positive or negative* .

2. SEMISIMPLE GROUPS WITH INFINITE CENTER

2.1. The purpose of this section is to examine random walks on semisimple Lie groups with infinite center , and in particular to extend some of the results of [8] . Our principal result , Theorem 2.10 , is exactly what is needed to make it possible to drop the condition on the center of G_o/R in the proof of theorem (8.4) of [7] . Although semisimple groups with infinite center may not often arise in practice,, it is nice to have a unified treatment of random walks and harmonic functions valid for all connected Lie groups (without side conditions on the center) .

Let G be a connected semisimple Lie group with center Z , let μ be an étalé probability measure on G , and let π_μ be the Poisson space of μ . Then if Z is infinite , it may happen [5] that G is not transitive on π_μ . More precisely , let T_μ be the closed semigroup generated by the support of μ and let S_μ be the open semigroup of [1 , Définition IV.2] . Then by [1 , Proposition IV.5] , G is transitive on π_μ if and only if $[Z : Z \cap S_\mu S_\mu^{-1}] < \infty$. Since π_μ depends only on T_μ by [1 , Théorème II.4] , one expects to be able to phrase this condition in terms of T_μ , and in fact , $Z \cap S_\mu S_\mu^{-1} = Z \cap T_\mu T_\mu^{-1}$:

2.2. Lemma. - *Let* G *be a group with center* Z *, and let* S *and* T *be non-empty sub-semigroups of* G *satisfying* $S \subseteq T$ *and* $TS \subseteq S$ *. Then* $Z \cap SS^{-1} = Z \cap TT^{-1}$

Proof. - Since $S \subseteq T$, $Z \cap SS^{-1} \subseteq Z \cap TT^{-1}$. But if $z \in Z \cap TT^{-1}$, we can write $z = t_1 t_2^{-1}$ with t_1 , $t_2 \in T$. Let $s \in S$. Then $z = (t_1 s)^{-1} z (t_1 s)$ (since z is central) $= s^{-1} t_1^{-1} t_1 t_2^{-1} t_1 s = (t_2 s)^{-1} (t_1 s) \in (TS)^{-1}(TS) \subseteq S^{-1}S$, so $Z \cap TT^{-1} \subseteq Z \cap S^{-1}S = Z \cap SS^{-1}$.

Corollary. - *In the setting of* 2.1 *above ,* $Z \cap S_\mu S_\mu^{-1} = Z \cap T_\mu T_\mu^{-1}$.

Proof. - By [1 , p. 76] , we can take $S = S_\mu$ and $T = T_\mu$ in the lemma .

2.3.- Let KAN be an Iwasawa decomposition of G and let M be the centralizer of A in K . We begin by analyzing the asymptotic behavior of the left μ-walk on $G/AN \cong K$ (equivalently , we could deal with the right μ-walk on $AN \backslash G$) in the case where G is transitive on π_μ . The results are basically as in the finite center case , except for the fact that since K is no longer assumed compact , it is necessary to consider infinite measures .

2.4. Lemma. - *Let* $G = KAN$ *and* M *be as in* 2.3 *, and let* M_1 *be an open subgroup of finite index in* M*. Then* MAN *leaves fixed on* $G/AN \cong K$ *exactly one Radon measure* m *(up to scalar multiples) , namely , the image in* K *of a Haar measure on* M *.* M_1AN *leaves fixed on* $G/AN \cong K$ *exactly (up to scalar multiples) the convex combinations of the restrictions of this measure to the right* M_1*-cosets in* M *. In particular , the cone of* M_1AN*-invariant (positive) Radon measures on* G/AN *is generated by its extremal rays , and there are exactly* $[M : M_1]$ *of these .*

Proof. - $G/ZAN = K/Z$ is compact , so every Radon measure on it is a multiple of a probability measure . By [3 , Theorem 2.6] , there is exactly one probability measure λ on G/ZAN which is MAN/Z-invariant , and λ is obviously normalized Haar measure on the image of M/Z .

Let m be an MAN-invariant Radon measure on $G/AN \cong K$. Since $Z \subseteq M$, m is Z-invariant and projects to an MAN/Z-invariant measure on K/M . Hence m is of the form

$$\int_K f \, dm = \int_K \int_Z f(zk) dz \, d\lambda(\dot{k})$$

for some Haar measure dz on Z , and this proves the first assertion .

Now if $\gamma \in M$, $(M_1AN)(M_1 \gamma AN) = AN(M_1 \gamma AN)$ (since M_1 normalizes AN) = $M_1 \gamma AN$ (since $M_1\gamma$ normalizes AN) , so the restrictions of m to right M_1-cosets in M (or rather , their images in G/AN) are M_1AN-invariant . Conversely , suppose σ is an M_1AN-invariant measure on G/AN . Let $r = [M : M_1]$ and choose representatives $\gamma_1 , \dots , \gamma_r$ for M/M_1 in M. Then $\Sigma\gamma_i . \sigma$ is MAN-invariant , hence is a multiple of m , and is therefore supported on MAN . It follows that σ is supported on MAN , or on M if we identify G/AN with K . But an M_1-invariant measure on M is a sum of translates of Haar measure on M_1 . This proves the lemma .

2.5. Lemma . - *Let* $G = KAN$ *and* M *be as in* 2.3 *. Let* μ *be an étalé probability measure on* G *such that* G *is transitive on* π_μ *. We take* $\pi_\mu = G/M_\mu AN$ *with* M_μ *an open subgroup of* M *; since* π_μ *is compact ,* $[M : M_\mu] < \infty$ *. Let* ν *denote the Poisson kernel for* μ *on* π_μ *, and let* X *be a locally compact (left)* G*-space . Then if* λ *is an* $M_\mu AN$*-invariant Radon measure on* X *, we can define a* μ*-stationary measure* $\nu * \lambda$ *on* X *by*

$$\int_X f \, d(\nu * \lambda) = \int_X \int_{\pi_\mu} f(xy) \, d\nu(\dot{x}) \, d\lambda(y)$$

(the integral converges since ν *has compact support). Conversely , every* μ*-stationary Radon measure on* X *is of the form* $\nu * \lambda$ *with* λ *as above .*

Proof. - This follows from trivial modifications in the proof of [3 , Lemma 2.1].

2.6. Theorem. - *Assume the hypotheses of* 2.5.. *Then* $G/AN \cong K$ *admits (up to scalar multiples) exactly* $[M : M_\mu]$ *ergodic* μ*-stationary Radon measures , and every* μ*-stationary measure on* K *is (up to scalar multiples) a convex combination of these . With probability* 1 *, any element of* K *enters the support of one of the ergodic measures after finitely many steps of the left* μ*-walk on* G/AN *, and then stays there thereafter .*

Proof. - Note that ergodic μ-stationary measures on K , considered up to scalar multiples , are the same as extremal rays in the cone of μ-stationary measures on K . By Lemmas 2.4 and 2.5 , this cone is generated by its extremal rays , and there are exactly $[M : M_\mu]$ of these . Also , we see that the ergodic measures have disjoint supports whose union E is the inverse image in K (under the natural map $p : K \longrightarrow K/M_\mu$) of the support of ν . Hence , if $k \in K$ and $X_1 , \ldots , X_n , \ldots$ are independent G-valued random variables with distribution μ , $X_n \ldots X_1 kAN$ eventually lands in E if and only if $X_n \ldots X_1 p_1(k)AN$ eventually lands in $p_1(\mathrm{supp}\ \nu)$, where p_1 denotes projection onto K/M . But we know that this last assertion is true either by the ergodic theorem (cf . [3 , p. 397]) or else by the Doeblin condition [8 , Lemma 1] applied to the unique stationary μ-process on the maximal boundary $G/MAN \cong K/M$ of G .

Now we go on to the general case (in which G is not necessarily transitive on π_μ) .

2.7. Lemma. - *Let* G *be a connected semisimple Lie group with (not necessarily finite) center* Z *, and let* S *be a non-empty open sub-semigroup of* G *. Then one can choose an Iwasawa decomposition* KAN *of* G *so that* S *intersects* ZA *and so that the projection of* $S \cap ZA$ *onto* A *contains a set of the form* $C \cap B^c$ *, where* C *is an open subcone of the (open) positive Weyl chamber in* A *and* B^c *is the complement of some relatively compact neighborhood* B *of the identity element* e .

Proof. - Since the set of regular elements in G is dense , and since S is non-empty and open , S meets some Cartan subgroup H of G . Choose a Cartan decomposition $\mathfrak{k} + \mathfrak{p}$ of the Lie algebra of G so that $H = H_K \cdot H_{\mathfrak{p}}$, where H_K lies in the connected subgroup K of G with Lie algebra $\mathfrak{k}$, and where $H_{\mathfrak{p}}$ is a vector group contained in $\exp(\mathfrak{p})$. We have $Z \subseteq H_K$, and H_K/Z is compact . (For all this , see [9 , §1.4]) . Hence the projection onto H_K/Z of the image in H/Z of $S \cap H$ is a non-empty sub-semigroup of a compact group , and is therefore [10] a group . In particular , the identity element of H_K/Z lies in the image of $S \cap H$, which means that S meets $ZH_{\mathfrak{p}}$. Now we can choose an Iwasawa decomposition KAN of G with K as above and with $H_{\mathcal{G}} \subseteq A$, so S meets ZA . Since $S \cap ZA$ is open , it must contain an element of the form za with $z \in Z$ and with a regular (that is , with a contained in one of the open Weyl chambers of A) , and assuming a suitable ordering of the roots (and thus a suitable choice of N) , a **will** be in the positive Weyl chamber . The last statement now follows from the structure of open sub-semigroups of vector groups .

2.8. Lemma. - *Let* G , Z , *and* S *be as in the hypotheses of* 2.7 , *and let* K , A , *and* N *be chosen as in that lemma . Let* M *be the centralizer of* A *in* K , *and let* Z' *be a subgroup of* Z *such that* $Z' \cap S^{-1}S = \{e\}$. *Let* $p : G \longrightarrow G/Z'$ *be the canonical map . Then for any* $m \in M$, p *is injective on* SmAN .

Proof. - Assume this is false . Then for some $m \in M$, $z \neq e$, in Z' , a_1 and a_2 in A , n_1 and n_2 in N , and s_1 and s_2 in S , we have $zs_1ma_1n_1 = s_2ma_2n_2$; or $s_1^{-1}s_2 = (zma_1n_1)(ma_2n_2)^{-1} = zm(a_1n_1n_2^{-1}a_2^{-1})m^{-1}$, and $zan \in S^{-1}S$ for suitable $a \in A$, $n \in N$ (recall M normalizes AN) . Let C and B^c be as in Lemma 2.7 , and choose $h_1 \in C \cap B^c$. Also choose $z_1 \in Z$ with $z_1h_1 \in S$ (this is possible by the conclusion of 2.7) . Then for any integer q , we have $z_1^{-q}h_1^{-q} \in S^{-1}$ and $z_1^{a}h_1^{q} \in S$, so that

$$(z_1^{-q}h_1^{-q})(zan)(z_1^{q}h_1^{q}) = zah_1^{-q}nh_1^{q} \in S^{-1}S .$$

(Note that z, z_1 , and a commute with h_1) . But as $q \longrightarrow \infty$, $h_1^{-q}nh_1^{q} \longrightarrow e$ (cf. [1 , Lemme III.1]) , so $za \in S^{-1}S$. One can now choose $h_2 \in C \cap B^c$ so that $h_3 = ah_2 \in C \cap B^c$, and if z_2 , $z_3 \in Z$ are such that z_2h_2 , $z_3h_3 \in S$, we get $(z_3h_3)^{-1}(za)(z_2h_2) = z_3^{-1}z_2z \in S^{-1}S$. In fact , it is easy to see that we can take $z_2 = z_3$ if h_2 is chosen far enough away from the identity element of A , so $z \in S^{-1}S$. This actually implies that $z \in S^{-1}S$, for we can choose a neighborhood U of e in G and an element $s \in S$ with $Us \subseteq S$ (since S is open) and with $zu \in S^{-1}S$ for some $u \in U^{-1}$, and then $z \in z(s^{-1}U^{-1}uUs) = (Us)^{-1}(zu)(Us) \subseteq S^{-1}S$. But this is a contradiction , since we assumed $z \neq e$ and $Z' \cap S^{-1}S = \{e\}$.

2.9. Theorem. - *Let* G *be a connected semisimple Lie group with possibly infinite center* Z , *and let* μ *be an étalé probability measure on* G . *Then we can choose an Iwasawa decomposition* KAN *of* G *such that with respect to the left* μ-*walk on* $G/AN \cong K$, K *decomposes into a transient set and a countable disjoint union of ergodic invariant sets . With probability* 1 , *any element of* K *enters one of the ergodic sets after finitely many steps of the left* μ -*walk on* G/AN .

Proof. - If $[Z : Z \cap S_\mu S_\mu^{-1}] < \infty$, Theorem 2.6 applies . If not , it follows from the structure theorem for finitely generated abelian groups (as applied to Z) that we can choose a subgroup Z' of Z with $Z' \cap S_\mu S_\mu^{-1} = \{e\}$ and $[Z : Z'.(Z \cap S_\mu S_\mu^{-1})] < \infty$. Apply 2.7 with $S = S_\mu$ to get an Iwasawa decomposition of G , and let M be the centralizer of A in K . The set $S_\mu MAN$ is open and invariant for the left μ-walk on $G/AN \cong K$, and with probability 1 , any element of K enters $S_\mu MAN$ after finitely many steps of the μ-walk on G/AN because of the ergodicity of the μ-walk on G/MAN (the maximal boundary of G). (The μ-walk on G/MAN may be viewed as a walk for the image of μ in G/Z on the same space , so [8 , p.671] and [3] apply .) Hence it is enough to decompose $S_\mu MAN$ into a transient set and countably many ergodic sets . Let m_1, m_2 , ... be a sequence dense in M.Then $S_\mu MAN$ is the (not necessarily disjoint) union of the invariant sets $S_\mu m_i AN$. Let $p : G \longrightarrow G/Z'$ be the canonical map . Then p is clearly equivariant for the μ-walk on G/AN and the $p(\mu)$ -walk on G/Z'AN , and by 2.8 , p is injective on each $S_\mu m_i AN$. But by construction of Z' , the $p(\mu)$-walk on G/Z'AN satisfies the hypotheses of Theorem 2.6 . Therefore each $S_\mu m_i AN$ decomposes into a transient set and finitely many ergodic sets . Since ergodic sets (up to sets of measure zero) either coincide or are disjoint , we conclude that $S_\mu MAN$ can be decomposed into a transient set (the union of the transient parts of the $S_\mu m_i AN$) and countably many ergodic sets. This proves the theorem .

2.10. Theorem. - *Let* G *be a connected semisimple Lie group with Iwasawa decomposition* KAN *and with possibly infinite center* Z *, and let* μ *be an étalé probability measure on* G . *Then with respect to the left* μ*-walk on* G/AN = K , K *decomposes into a transient set and a countable disjoint union of ergodic invariant sets . With probability* 1 *, any element of* K *enters one of the ergodic sets after finitely many steps of the left* μ*-walk on* G/AN .

Proof. - (kindly suggested by A.RAUGI) . The only difference between this theorem and 2.9 is that here our Iwasawa decomposition is specified in advance . Let $G = K_1A_1N_1$ be a decomposition as provided by 2.9 . Then for some $k \in K$, $A_1N_1 = kANk^{-1}$. Let $G/A_1N_1 = T \cup E_i$ be a decomposition of G/A_1N_1 into a transient set T and countably many ergodic sets E_i . Then Tk and the E_ik are evidently invariant for the left μ-walk and give a partition of G/AN ; they are also clearly transient and ergodic , respectively . Finally , if $x \in K$, and if X_1, X_2 , ... are independent G-valued random variables each with distribution μ , then by 2.9 we know that with probability 1 , $X_n \ldots X_1(xk^{-1})A_1N_1 \in E_i$ for some i and some n . This says that $X_n \ldots X_1 xAN \in E_ik$, so that x enters one of the ergodic sets for the μ-walk on G/AN after finitely many steps .

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