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THE MILNORRING OF A LOCAL RING

par

E. A. M. HORNIX

Let F be a field. Milnor defined a ring $k_*(F)$, and in the case that characteristic $(F) \neq 2$ he studied maps between $k_*(F)$, and groups or rings which play a role in the theory of quadratic forms. The aim of this talk is to extend some of his definitions and results to local rings. We do not suppose that 2 is a unit of the local ring. The only restriction for the local rings is, that the residue field has more than 3 elements. Sections 1,2,3 give a survey of [3], though the definitions of [3] are a bit generalized. In section 4, the analogue of Milnor's map s_* is given, and section 5 covers the example of a field of characteristic 2.

1. We repeat some of the definitions given by Milnor [6]. Let F be a field, denote $U(F) = \{x \in F \mid x \text{ is invertible}\}$. Let M be the $\mathbb{Z}$ -module $U(F)$, and denote $T(M)$ for the tensoralgebra of M . We write $\ell : M \rightarrow T(M)$ for the imbedding of M in $T(M)$. $K_*(F)$ is defined as $T(M) \text{ mod } I$, and I is the two-sided ideal of M , generated by $\{\ell(a)\ell(1-a) \mid a, 1-a \in U(F)\}$. Remark that $\langle -a, 1 \rangle \otimes \langle -(1-a), 1 \rangle \cong 2H$, as soon as $a, 1-a \in U(F)$ and $2 \neq 0 \in F$. $K_*(F) = \mathbb{Z} \oplus K_1(F) \oplus K_2(F) \oplus \dots$, and here $K_n(F) = \ell(M) \otimes \dots \otimes \ell(M) \text{ mod } \ell(M) \otimes \dots \otimes \ell(M) \cap I$. The elements of $K_n(F)$ are again denoted as sums of terms $\ell(a_1) \dots \ell(a_n)$. Finally, $k_*(F)$ is defined as $\mathbb{Z} \oplus K_1(F) / 2K_1(F) \oplus K_2(F) / 2K_2(F) \oplus \dots$ (1). We remark that for $a \in U(A)$ and $x \in k_*(F)$, the element $\overline{\ell}(a^2)x = 0 \in k_*(F)$. In fact, the defining relations for $k_*(F)$ are :

$$\begin{aligned} \overline{\ell}(ab) &= \overline{\ell}(a) + \overline{\ell}(b) & a \in U(A), b \in U(A) \\ \overline{\ell}(a)\overline{\ell}(1-a) &= 0 & a, 1-a \in U(A) \\ 2\overline{\ell}(a) &= 0 & a \in U(A) \end{aligned}$$

Suppose now that $\text{char}(F) \neq 2$. We write $\text{Quad}(F)$ for the Grothendieck monoid of finite-dimensional quadratic spaces over F . Milnor proved, that there exists a well-defined map

$$\begin{aligned} SW : \text{Quad}(F) &\rightarrow k_*(F) \text{ such that} \\ SW \langle a_1, \dots, a_n \rangle &= (1 + \overline{\ell}(a_1)) \dots (1 + \overline{\ell}(a_n)) \end{aligned}$$

(1) Write $\overline{\ell}(a)$ for the class of $\ell(a)$ in $K_1(F)$, etc.

We denote the Grothendieck-Writting of finite-dimensional quadratic spaces over F by $W(F)$, and we write $I(F) \subset W(F)$ for the kernel of the dimension map $W(F) \rightarrow \mathbb{Z}/2\mathbb{Z}$. Milnor proved also, that there exists a homomorphism

$$\begin{aligned} s_* : k_*(F) &\rightarrow \bigoplus_{n \geq 1} I^n(F) / I^{n+1}(F) \text{ such that} \\ s_n : K_n(F) / 2K_n(F) &\rightarrow I^n(F) / I^{n+1}(F) \text{ and} \\ s_n : \bar{\mathcal{L}}(a_1) \dots \bar{\mathcal{L}}(a_n) &= (\langle a_1 \rangle - 1) \dots (\langle a_n \rangle - 1) + I^{n+1}(F). \end{aligned}$$

2. Let A be a local ring with maximal ideal $\underline{m}$. Denote $U(A) = \{a \in A \mid a \text{ has inverse in } A\}$. If $2 \in \underline{m}$, then every nondegenerate quadratic form on A of finite dimension has even dimension.

We denote (a, b, c) for the form q which has a basis e, f satisfying $q(e) = a$, $q(f) = b$, $(e, f) = c$. The form (a, b, c) is nondegenerate if and only if $4ab - c^2 \in U(A)$. If $|A \bmod \underline{m}| > 3$ then we may choose $a, b, 1$ such that $a, b \in U(A)$. In that case $(a, b, 1) \cong a(1, ab, 1)$ and ab determines an invariant of $(a, b, 1)$ which we will describe now.

The following notions can be found in the notes of the 1968 Montpellier conference, Micali, Villamayor [4].

Let A be an arbitrary ring, define $A^0 = \{a \in A \mid 1-4a \in U(A)\}$. A^0 is a group under $\circ : A^0 \times A^0 \rightarrow A^0$, $a \circ b = a+b-4ab$.

The inverse of a in A^0 is the element $\frac{-a}{1-4a}$. Define $J(A) = \{x - x^2 \mid 1-2x \in U(A)\}$.

If $a \in A^0$, then $a \circ a \in J(A)$. $J(A)$ is a subgroup of A^0 , we denote $G(A) = A^0 \bmod J(A)$. There exist homomorphisms $\sigma : A^0 \rightarrow U(A)$, $\sigma(a) = 1-4a$, $\bar{\sigma} : G(A) \rightarrow U(A) \bmod U(A)^2$, $\bar{\sigma}(a \circ J) = (1-4a) U(A)^2$.

Examples. (1) If $2 \in U(A)$ then $\bar{\sigma}$ is an isomorphism..

(2) If $2 = 0$ then $A = A^0$ and $a \circ b = a+b$, $G(A) = A^+ \bmod \mathcal{P}(A)$.

Let A be a local ring. The quadratic form $a(1, d, 1)$ is nondegenerate if and only if $a \in U(A)$, $d \in A^0$. The class $d \circ J(A)$ is an invariant for the isometry class of $a(1, d, 1)$, for the proof see [3].

In general, we have the following result : Suppose that q is a nondegenerate quadratic form of dimension $2n$. Then

$$q \cong \bigoplus_{i=1}^n a_i(1, d_i, 1)$$

$a_i \in U(A)$, $d_i \in A^0$, $1 \leq i \leq n$ and $d_1 \circ d_2 \circ \dots \circ d_n \circ J(A)$ is an invariant for the isometry class of q .

Examples. (1) F is a field of characteristic $\neq 2$. $q \cong \bigoplus_{i=1}^n a_i(1, d_i, 1)$, then $\bar{\sigma}(d_1 \circ \dots \circ d_n \circ J(A))$ is the discriminant of q .

(2) F is a field of characteristic 2, q as before. Then $d_1 \circ \dots \circ d_n \circ J(A)$ is the Arf invariant of q .

3. Suppose again that A is a local ring. $|A \bmod \mathfrak{m}| > 3$. It is clear, that for the determination of the isometry class of $a(1, d, 1)$ a role is played by $d \circ J(A)$ and by $a \in U(A)$. So in the definition of the Milnorring of A , $G(A)$ and $U(A)$ should play a role. In the case of a fields of $\text{char} \neq 2$ it seemed important to remark that

$$\langle -a, 1 \rangle \otimes \langle -(1-a), 1 \rangle \cong 2H, \quad a, 1-a \in U(A).$$

We translate that remark :

if $a \in U(A) \cap A^\circ$ then $\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} \otimes (1, a, 1) \cong 2H$. Here $\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}$ denotes

a symmetric bilinear form, and we use the tensorproduct which is defined for symmetric bilinear forms and quadratic forms by H. Bass [1].

We now give a construction of the ring $g_*(A)$, which is almost equivalent to the construction of $k_*(A)$.

We start with the $\mathbb{Z}$ -module $M = A^\circ \oplus U(A)$, and we denote $\omega(a) = (a, 0)$ for $a \in A^\circ$ and $\gamma(a) = (0, a)$ for $a \in U(A)$. $T(M)$ is again the tensor algebra of M . $\mathcal{J}$ is the two-sided ideal of $T(M)$ generated by

$$\{\omega(a) \gamma(a) \mid a \in A^\circ \cap U(A)\} \cup \{\gamma(a) \omega(a) \mid a \in A^\circ \cap U(A)\} \cup \{\omega(a) \mid a \in J(A)\}.$$

$g_*(A) = T(M) \bmod \mathcal{J}$, $g_*(A)$ is isomorphic with $\mathbb{Z} \oplus g_1(A) \oplus g_2(A) \oplus \dots$,

$$g_i(A) = M \otimes \dots \otimes M / M \otimes \dots \otimes M \cap \mathcal{J}.$$

We denote $\bar{g}(a)$ for the image of $\gamma(a)$ ($a \in U(A)$) in $g_*(A)$. We write $\bar{O}(A)$ for the image of $\omega(a)$ ($a \in A^\circ$) in $g_*(A)$. In fact, $g_*(A)$ satisfies the following defining relations :

$$\bar{g}(ab) = \bar{g}(a) + \bar{g}(b), \quad a \in U(A), b \in U(A)$$

$$\bar{O}(a \circ b) = \bar{O}(a) + \bar{O}(b), \quad a, b \in A^\circ$$

$$\bar{g}(a) \bar{O}(a) = \bar{O}(a) \bar{g}(a) = 0, \quad a \in A^\circ \cap U(A)$$

$$\bar{O}(a) = 0, \quad a \in J(A).$$

We would like to define a map

$$SW : \text{Quad}(A) \rightarrow g_*(A).$$

The analogue of Milnor's definition is for even dimensional forms :

$$\begin{aligned} (\text{DEF}) : SW(a_1(1, d_1, 1) \oplus a_2(1, d_2, 1) \oplus \dots \oplus a_n(1, d_n, 1)) = \\ = (1 + \bar{g}(-1) + \bar{O}(d_1) + \bar{g}(a_1) \bar{O}(d_1)) \dots (1 + \bar{g}(-1) + \bar{O}(d_n) + \\ + \bar{g}(a_n) \bar{O}(d_n)). \end{aligned}$$

This definition works for $n = 1$: if $a(1, d, 1) \cong a_1(1, d_1, 1)$ then $d \circ J(A) = d_1 \circ J(A)$, so $\bar{O}(d) = \bar{O}(d_1)$, and it can easily be proved that $\bar{g}(a) \bar{O}(d) = \bar{g}(a_1) \bar{O}(d_1)$.

For the proof that the definition works for $n = 2$, we have to impose some extra conditions. Some of these come from the commutativity of $\text{Quad}(A)$. The more important conditions are :

$$W_2 : \overline{g}(1-4a) \overline{O}(b) - \overline{O}(a) \overline{O}(b) \text{ should be equal to } 0, \text{ as soon as } a \in A^{\circ}, \\ b \in U(A) \cap A^{\circ}.$$

$$W_7 : \overline{g}(a) \overline{g}(a) \overline{O}(b) \overline{O}(d) - \overline{g}(a) \overline{O}(b) \overline{O}(d) \overline{O}(d) \text{ should be equal to } 0 \text{ for} \\ a \in U(A), b \in A^{\circ} \cap U(A), d \in A^{\circ}.$$

So we consider the ring $g_*(A) \bmod Cg_*(A)$, $Cg_*(A)$ being the ideal in $g_*(A)$ generated by the elements mentioned in W_2, W_7 and by some more elements. For an explicit and precise definition see [3].

Let us denote $\overline{g}(a)$ for $\overline{g}(a) + Cg_*(A)$, $\overline{O}(a)$ for $\overline{O}(a) + Cg_*(A)$.

Suppose that $2 \in \underline{m}$. Then one can prove that the map $SW : \text{Quad}(A) \rightarrow g_*(A) \bmod Cg_*(A)$ as proposed in (DEF), is well-defined.

Suppose $2 \notin \underline{m}$. If A is a field, then $g_*(A)$ and $k_*(A)$ are not isomorphic. We should have identified A° and $U(A)$. More precisely, choose

$M = U(A) \oplus A^{\circ} \bmod \{\gamma(1-4a) - \omega(a) \mid a \in A^{\circ}\}$ and repeat the definition of $T(M) \bmod \mathcal{I}$, hence the defining relations for $T(M) \bmod \mathcal{I}$ are

$$\begin{aligned} \overline{g}(1-4a) &= \overline{O}(a) , & a \in A^{\circ} \\ \overline{g}(ab) &= \overline{g}(a) + \overline{g}(b) , & a, b \in U(A) \\ \overline{g}(a) \overline{O}(a) &= \overline{O}(a) \overline{g}(a) , & a \in A^{\circ} \cap U(A) \\ \overline{g}(a) &= 0 , & a \in U(A)^2 \end{aligned}$$

In fact, this was the definition, proposed in [3] for any local ring A with 2 unit in A .

It is then easily proved that $Cg_*(A) = 0$, and that SW is defined on all of $\text{Quad}(A)$, such that

$$(*) : SW \langle a_1, \dots, a_n \rangle = (1 + \overline{g}(a_1)) \dots (1 + \overline{g}(a_n)) .$$

For isometry classes of even dimension, the definitions (*) and (DEF) coincide.

There are situations in which we have that $2 \in U(A)$ and that we want to restrict ourselves to isometry classes of even-dimensional forms. It is possible to define $g_*(A)$ based on $M = U(A) \oplus A^{\circ}$. The map SW can be defined as proposed in (DEF). For proving this, the proofs in [3] can completely be repeated. The map SW , as proposed in (*) cannot be defined, since $2 \overline{g}(a)$ ($a \in U(A)$) is not necessarily equal to 0.

4. We give now the analogue for the map s_* . For convenience, we work with rings $g_*(A)$, based on $M = U(A) \oplus A^{\circ}$.

$W_q(A)$ is the Witt-group of free finite-dimensional nondegenerate quadratic forms on A . $W(A)$ is the Witt-ring of free finite-dimensional nondegenerate symmetric bilinear forms on A , $I(A) \subset W(A)$ is the ideal of forms of even dimension. We denote the class of a form in $W_q(A)$, $W(A)$ by square brackets. $W_q^O(A) \subset W_q(A)$ is the Witt-group of forms of even dimension.

It is well known that $W_q(A)$ can be considered as an $W(A)$ -module. According to definitions given by Micali + Villamayor [5], we give $W_q(A)$ a structure of ring by defining :

$$q_1 \cdot q_2 = (,)_{q_1} \otimes q_2 .$$

This definition induces a structure of ring on $\bigoplus_{n \geq 0} I^n(A) W_q^O(A) \bmod I^{n+1}(A) W_q^O(A)$.

In analogy with Milnor's definition, we would like to define a homomorphism of rings

$$s_* : g_*(A) \bmod Cg_*(A) \rightarrow \bigoplus_{i \geq 0} I^i(A) W_q^O(A) \bmod I^{i+1}(A) W_q^O(A) \oplus \bigoplus_{n \geq 1} I^n(A) \bmod I^{n+1}(A)$$

For $a \in A^O$, we propose to define $s_1 \bar{O}(a) = [-1, -a, 1] + I(A) W_q^O(A)$.

If $a \in U(A)$ we would like to define

$$s_1 \bar{g}(a) = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} + I^2(A) .$$

The map s_1 can be extended to a homomorphism of rings, if the image of s_1 satisfies the defining relations of $g_*(A) \bmod Cg_*(A)$. It is clear that the following results hold :

$$4.1. \quad \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & b \end{bmatrix} \in \begin{bmatrix} -1 & 0 \\ 0 & ab \end{bmatrix} + I^2(A) , \quad a, b \in U(A)$$

$$4.2. \quad [-1, -a, 1] + [-1, -b, 1] \in [-1, -a \cdot b, 1] + I(A) W_q^O(A) , \quad a, b \in A^O$$

$$4.3. \quad \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} \cdot [-1, -a, 1] = 0 , \quad a \in U(A) \cap A^O .$$

For proving the other relations, we derive some formulas.

4.4. Suppose $2 \in \underline{m}$. Let $1-pq \in U(A)$, $d \in U(A) \cap A^O$. Then

$$\begin{bmatrix} p & 1 \\ 1 & q \end{bmatrix} \cdot [-1, -d, 1] = \begin{bmatrix} d(pq-1) & 0 \\ 0 & 1 \end{bmatrix} \cdot \left[\frac{pd}{1-4d}, \frac{q}{1-pq}, 1 \right] .$$

Proof. Let e, f be a basis of V , let $(,)$ be a symmetric bilinear form on V such that $(e, e) = p$, $(f, f) = q$, $(e, f) = 1$. Let x, y be a basis of W , $q : W \rightarrow A$ a quadratic form and $q(x) = -1$, $q(y) = -d$, $(x, y) = 1$.

The bilinear form and the quadratic form are nondegenerate. Choose $X = e \otimes (2dx + y)$, $Y = (-qe + f) \otimes x$, $S = f \otimes y$, $T = (-e + pf) \otimes (x + 2y)$. Since $2 \in \underline{m}$, we have that X, Y, S, T is a basis of $V \otimes W$. Moreover, $\langle X \rangle + \langle Y \rangle \perp \langle S \rangle + \langle T \rangle$. It is

clear that $\langle X \rangle + \langle Y \rangle \cong (\frac{pd}{1-4d}, \frac{q}{1-pq}, 1)$ and that

$$\langle S \rangle + \langle T \rangle \cong (-qd, \frac{-p}{(1-4d)(1-pq)}, 1).$$

4.5. Lemma. $I(A) W_q^O(A)$ is generated as an additive group by elements of the form

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} [1, d, 1], \quad a \in A^O, d \in A^O \cap U(A).$$

4.6. Let $a \in A^O, d \in U(A) \cap A^O$. Then we have that

$$\begin{bmatrix} 1-4a & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1, -d, 1] = \begin{bmatrix} -2 & 1 \\ 1 & -2a \end{bmatrix} \cdot [-1, -d, 1] \in I^2(A) W_q^O(A).$$

Proof. If $2 \notin \underline{m}$ then this statement is easily proved. So suppose $2 \in \underline{m}$. Applying (4.4.) we find that

$$\begin{aligned} \begin{bmatrix} -2 & 1 \\ 1 & -2a \end{bmatrix} \cdot [-1, -d, 1] &= \begin{bmatrix} d(4a-1) & 0 \\ 0 & 1 \end{bmatrix} \cdot \left[\frac{-2d}{1-4d}, \frac{-2a}{1-4a}, 1 \right] = \\ &= [\rho] \cdot \begin{bmatrix} d(4a-1) & 0 \\ 0 & 1 \end{bmatrix} \cdot [-1, \frac{-4ad}{(1-4d)(1-4a)}, 1] \quad \text{for certain } \rho \in U(A) \end{aligned}$$

Now we consider the form $\begin{bmatrix} 1-4a & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1, -d, 1]$.

Let e, f be a basis of V , and let $(,)$ be a symmetric bilinear form satisfying $(e, e) = 1-4a, (f, f) = -1, (e, f) = 0$.

Let x, y be a basis of W , and let q be a quadratic form such that $q(x) = -1, q(y) = -d, (x, y) = 1$. Denote $A = e \otimes y, B = (e+f) \otimes (x+2y), C = f \otimes x, D = (e + (1-4a)f) \otimes (2dx+y)$.

A, B, C, D is a basis for $V \otimes W$ and $\langle A \rangle + \langle B \rangle \perp \langle C \rangle + \langle D \rangle$.

$$\langle A \rangle + \langle B \rangle \cong (\frac{-d}{1-4a}, \frac{-4a}{1-4d}, 1) \cong -d(4a-1)(-1, \frac{-4ad}{(1-4a)(1-4d)}, 1)$$

$$\langle C \rangle + \langle D \rangle \cong (1, \frac{4ad}{(1-4a)(1-4d)}, 1) \cong -(-1, \frac{-4ad}{(1-4a)(1-4d)}, 1)$$

$$\text{Hence } V \otimes W \cong (-1) \begin{pmatrix} d(4a-1) & 0 \\ 0 & 1 \end{pmatrix} \cdot (-1, \frac{-4ad}{(1-4a)(1-4d)}, 1).$$

Now it is easily proved that

$$\begin{aligned} \begin{bmatrix} 1-4a & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1, -d, 1] &= \begin{bmatrix} -2 & 1 \\ 1 & -2a \end{bmatrix} \cdot [-1, -d, 1] = \\ &= \begin{bmatrix} \rho & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} d(4a-1) & 0 \\ 0 & 1 \end{bmatrix} \cdot [-1, \frac{-4ad}{(1-4a)(1-4d)}, 1]. \end{aligned}$$

$$\begin{aligned}
 4.7. \quad & \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot [-1, -c, 1] - \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \\
 & \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot [-1, -c, 1] \in I(A)^4 W_q^O(A), \quad a \in U(A), \quad b \in U(A) \cap A^O, \\
 & c \in A^O.
 \end{aligned}$$

Proof. Since $b \in U(A) \cap A^O$ we have that

$$\begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot [-1, -c, 1] = \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot [-1, -b, 1] \in \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1, -b, 1] + \\
 + I^2(A) W_q^O(A).$$

$$\text{It is clear that } \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\begin{aligned}
 \text{Now we calculate : } & \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot [-1, -c, 1] - \\
 & - \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot [-1, -c, 1] \in \\
 & \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot [-1, -b, 1] - \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \\
 & \cdot \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1, -b, 1] + I^4(A) W_q^O(A) = (\text{applying (4.4.)}) = \\
 & = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1, -b, 1] - \\
 & - \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1, -b, 1] + I^4(A) W_q^O(A) = \\
 & = I^4(A) W_q^O(A).
 \end{aligned}$$

The relations (4.1), (4.2), (4.3), (4.6), (4.7) are translations of relations, which have been mentioned explicitly in the definition of $g_*(A) \bmod Cg_*(A)$.

The other relations have to do with commutativity. Now,

$\bigoplus_{n \geq 0} I^n(A) W_q^O(A) \bmod I^{n+1}(A) W_q^O(A) \oplus \bigoplus_{n \geq 1} I^n(A) \bmod I^{n+1}(A)$ is commutative with respect to multiplication. So we have verified that the defining relations for $g_*(A) \bmod Cg_*(A)$ also hold for the image of s_1 . Hence the following theorem is proved :

4.8. Theorem. There exists a well-defined homomorphism of rings

$$s_* : g_*(A) \bmod Cg_*(A) \rightarrow \bigoplus_{n \geq 0} I^n(A) W_q^O(A) \bmod I^{n+1}(A) W_q^O(A) \oplus$$

$$\bigoplus_{n \geq 1} I^n(A) \bmod I^{n+1}(A), \text{ such that}$$

$$s_1 \bar{g}(a) = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} + I^2(A), \quad a \in U(A)$$

$$s_1 \bar{O}(a) = [-1, -a, 1] + I(A) W_q^O(A), \quad a \in A^O$$

We denote s_n for the restriction of s_* to $g_n(A) \bmod Cg_*(A) \cap g_n(A)$.

We denote $\mathcal{O}(A) \subset g_*(A) \bmod Cg_*(A)$ for the two-sided ideal, generated by $\{\bar{O}(a) | a \in A^O\}$. Let us write $\mathcal{O}_n(A)$ for the intersection of $\mathcal{O}(A)$ with $g_n(A) \bmod Cg_*(A) \cap g_n(A)$.

Denote s_n for the restriction of s_* to $\mathcal{O}_n(A)$, and denote the restriction of s_* to $\mathcal{O}(A)$ by s_* .

4.9. Theorem. $s_* : \mathcal{O}(A) \rightarrow \bigoplus_{n \geq 0} I^n(A) W_q^O(A) \bmod I^{n+1}(A) W_q^O(A)$ is a surjective homomorphism of rings.

Proof. The elements of $\mathcal{O}(A)$ are of the form $\sum_{i=1}^n x_i \bar{O}(a_i) y_i$, with $a_i \in A^O$,

$x_i, y_i \in g_*(A) \bmod Cg_*(A)$.

So $s_* \mathcal{O}(A) \subset \bigoplus_{n \geq 0} I^n(A) W_q^O(A) \bmod I^{n+1}(A) W_q^O(A)$.

Lemma (4.5) proves that s_* maps $\mathcal{O}(A)$ surjectively on

$$\bigoplus_{n \geq 0} I^n(A) W_q^O(A) \bmod I^{n+1}(A) W_q^O(A)$$

We will now prove, that s_1 is an injective map on $\mathcal{O}_1(A)$.

4.10. There exists a homomorphism of groups

$$\text{discr} : W_q^O(A) \rightarrow G(A), \text{ satisfying}$$

$$\text{discr} [a] [1, d, 1] = d \circ J(A), \quad a \in U(A), d \in A^O.$$

The following sequence is exact :

$$1 \rightarrow I(A) W_q^O(A) \rightarrow W_q^O(A) \xrightarrow{\text{discr}} G(A) \rightarrow 1.$$

Proof. The existence of the homomorphism discr follows from what is said in section 2. The map discr is surjective since $\text{discr} [1, d, 1] = d \circ J(A)$, $d \in A^O$. Lemma (4.5) shows that $I(A) W_q^O(A)$ is generated by elements of the form

$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \cdot [1, d, 1]$, $a \in U(A)$, $d \in A^O$. Hence $I(A) W_q^O(A) \subset \ker(\text{discr})$. Suppose that

$\bigoplus_{i=1}^n [a_i] [1, d_i, 1] \in W_q^O(A)$ and that $d_1 \circ \dots \circ d_n \circ J(A) = J(A)$. Then we have that

$\bigoplus_{i=1}^n [a_i] [1, d_i, 1] = \bigoplus_{i=1}^{n-1} [a_i] [1, d_i, 1] \oplus [a_n] [1, d_1 \circ \dots \circ d_{n-1}]$. Applying (4.2) we

find that $\bigoplus_{i=1}^n [a_i] [1, d_i, 1] \in \bigoplus_{i=1}^{n-1} [a_i] [1, d_i, 1] \oplus \bigoplus_{i=1}^{n-1} [a_n] [1, d_i, 1] + I(A) W_q^O(A) = I(A) W_q^O(A)$.

Remark. Compare Knebusch [2], (7.10).

4.11. Theorem. $s_1 : \mathcal{O}_1(A) \rightarrow W_q^O(A) \bmod I(A)W_q^O(A)$ is an isomorphism of additive groups.

Remark. We cannot repeat Milnor's proof for the injectivity of s_2 , since we do not work with 1-dimensional quadratic forms.

5. Example. F is a field of characteristic 2, $F \neq \mathbb{F}_2$.

We have $U(F) = \{a \in F \mid a \neq 0\}$, $F^O = F$.

The most important defining relations for $g_*(F) \bmod Cg_*(F)$ are

$$\bar{g}(ab) = \bar{g}(a) + \bar{g}(b) \quad , \quad a, b \neq 0$$

$$\bar{O}(a+b) = \bar{O}(a) + \bar{O}(b) \quad ,$$

$$\bar{g}(a) \bar{O}(a) = 0 \quad , \quad a \neq 0$$

$$\bar{O}(a) \bar{O}(b) = 0 \quad .$$

The elements of $g_n(F) \bmod Cg_*(F) \cap g_n(F)$ can be written as sums of elements of the type

$$\bar{g}(a_1) \dots \bar{g}(a_n) \quad , \quad \bar{g}(a_1) \dots \bar{g}(a_{n-1}) \bar{O}(b) \quad .$$

The elements of $\mathcal{O}_n(F) \bmod Cg_*(F) \cap \mathcal{O}_n(F)$ are sums of terms $\bar{g}(a_1) \dots \bar{g}(a_{n-1}) \bar{O}(b)$.

Let $\bigoplus_{i=1}^n a_i(1, d_i, 1)$ be a quadratic form.

$$SW(\bigoplus_{i=1}^n a_i(1, d_i, 1)) = 1 + \bar{O}(d_1 \circ \dots \circ d_n) + \sum_{i=1}^n \bar{g}(a_i) \bar{O}(d_i) \quad .$$

Hence, $SW(H) = 0$, and we can extend SW to a map

$$SW : W_q(F) \rightarrow \mathcal{O}(F) \bmod Cg_*(F) \cap \mathcal{O}(F) \quad .$$

We calculate the action of SW on $I^n(F)W_q(F)$.

$$SW[1, d, 1] = 1 + \bar{O}(d)$$

$$SW \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} [1, d, 1] = 1 + \bar{g}(a) \bar{O}(d) \quad .$$

$$SW \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} [1, d, 1] = 0$$

Hence, SW acts trivially on $I^2(F)W_q(F)$.

We calculate $s_* : \mathcal{O}(F) \rightarrow \bigoplus_{n \geq 0} I^n(F)W_q(F) \bmod I^{n+1}(F)W_q(F)$.

$$s_1 \bar{O}(a) = [1, a, 1] + I(F)W_q(F) \quad .$$

$$s_2(\bigoplus_{i=1}^n \bar{g}(a_i) \bar{O}(d_i)) = \bigoplus_{i=1}^n \begin{bmatrix} a_i & 0 \\ 0 & 1 \end{bmatrix} [1, d_i, 1] + I^2(F)W_q(F) \quad .$$

It is easy to see, that :

$$SW \circ s_2(x) = 1 + x, \quad x \in \mathcal{O}_2(F)/Cg_*(F) \cap \mathcal{O}_2(F).$$

This proves that s_2 is a monomorphism.

There are no results about the injectivity of s_i , $i \geq 3$.

$$s_2 : \mathcal{O}_2(F) \bmod \mathcal{O}_2(F) \cap Cg_*(F) \rightarrow I(F)W_q(F) \bmod I^2(F)W_q(F)$$

is an isomorphism of additive groups.

We refer to another description of $I(F)W_q(F) \bmod I^2(F)W_q(F)$ by C.H. Sah, [7].

Let $Cl[M, q]$ denote the class of the Clifford algebra of (M, q) in the ungraded Brauer group of F . $Cl[M, q]$ is an element of ${}_2Br(F)$, the subgroup generated by the elements of order 2 of $Br(F)$. Cl induces a split exact sequence :

$$0 \rightarrow I^2(F)W_q(F) \rightarrow I(F)W_q(F) \xrightarrow{Cl} {}_2Br(F) \rightarrow 0$$

Hence, Cl induces an isomorphism

$$\overline{Cl} : I(F)W_q(F) \bmod I^2(F)W_q(F) \rightarrow {}_2Br(F).$$

In proving this theorem, C.H. Sah uses the following result :

Denote $(a, d]$ for the F -algebra H with F -basis $1, u, v, uv$ and with relations $u^2 = a \neq 0$, $v^2 + v = d$, $uv + vu = 1$.

H is a quaternion algebra with norm form

$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot (1, d, 1)$. The class of the Clifford algebra of $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot (1, d, 1)$ is equal to the class $[H]$ of H in the Brauer group.

Combining these results, we find that

$$\overline{Cl} \circ s_2 : \mathcal{O}_2(F) \bmod \mathcal{O}_2(F) \cap Cg_*(F) \rightarrow {}_2Br(F)$$

is an isomorphism of groups.

$\overline{Cl} \circ s_2 \left(\bigoplus_{i=1}^n \overline{g}(a_i) \overline{O}(d_i) \right) = \bigotimes_{i=1}^n [(a_i, d_i)]$, since tensor product induces multiplication in ${}_2Br(F)$.

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