

MÉMOIRES DE LA S. M. F.

HERBERT GROSS

Quadratic forms and sesquilinear forms in infinite dimensional spaces. Witt type theorems in spaces of denumerably infinite dimension

Mémoires de la S. M. F., tome 48 (1976), p. 21-33

http://www.numdam.org/item?id=MSMF_1976__48__21_0

© Mémoires de la S. M. F., 1976, tous droits réservés.

L'accès aux archives de la revue « Mémoires de la S. M. F. » (<http://smf.emath.fr/Publications/Memoires/Presentation.html>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Col. sur les Formes Quadratiques (1975, Montpellier)
 Bull. Soc. Math. France
 Mémoire 48, 1976, p. 21-33

QUADRATIC FORMS AND SESQUILINEAR FORMS IN INFINITE DIMENSIONAL SPACES

WITT TYPE THEOREMS IN SPACES OF DENUMERABLY INFINITE DIMENSION

par

H. GROSS

O. - Introduction

All forms considered here are forms over divisionrings.

If $\phi : E \times E \rightarrow k$ is a sesquilinear form on the vector-space E over the divisionring k (with antiautomorphism $\alpha \rightarrow \alpha^*$) then we shall always tacitly assume ϕ to be orthosymmetric, i.e. $\phi(x,y) = 0$ if and only if $\phi(y,x) = 0$. Provided that $\dim E/E^\perp$ is at least 2 the automorphism $\alpha \rightarrow \alpha^{**}$ is then inner and there exist nonzero $\gamma \in k$ such that $\phi\gamma$ is hermitean or antihermitean (with respect to an involution of k). An ε -hermitean form ϕ (i.e. $\phi(y,x) = \varepsilon\phi(x,y)^*$ for some $\varepsilon \in \text{center}(k)$ and all x,y) is said to be tracevalued iff for every $x \in E$ there is $\alpha \in k$ such that $\phi(x,x) = \alpha + \varepsilon\alpha^*$. An arbitrary form ψ on a space F with $\dim F/F^\perp \geq 2$ is said to be tracevalued iff some (and hence every) ε -hermitean multiple $\psi\gamma$, $\gamma \neq 0$, is tracevalued.

If Φ is a form on the space E then $\|\phi\|$ or $\|E\|$ is the set $\{\phi(x,x) | x \in E\}$; an alternate form has $\|\phi\| = \{0\}$.

Here I shall mainly be concerned with Witt-type theorems. The celebrated theorem of Witt states that an isometry $T_0 : F \rightarrow \overline{F}$ between finite dimensional subspaces of a non degenerate tracevalued sesquilinear space E always extends to a metric automorphism on all of E ([3], p. 71). The classical theory of forms and its associated groups pivots on this theorem; it is therefore not necessary to discuss the importance of our matter.

It is easy to discover that the theorem as stated above is false when $\dim F$ is infinite ([9], chap. 3). When trying to describe the state of affairs in this case it is first of all necessary to distinguish between two problems of a rather different nature :

Problem 1. Given isometric subspaces F, \overline{F} of a sesquilinear space E when does there exist a metric automorphism T of E with $TF = \overline{F}$? In other words, when will there be at least some isometry $T_0 : F \rightarrow \overline{F}$ which extends to all of E ?

Problem 2. Describe conditions which are sufficient for a given isometry $T_0 : F \rightarrow \bar{F}$ to admit an extension to all of E .

Theorems 1, 2, 10 below concern Problem 1, theorems 6, 7, 11 and Remark 4 concern Problem 2.

We shall consider two extreme situations here. On one hand we shall discuss forms which admit "many" isotropic vectors ; on the other hand we shall discuss definite forms over ordered fields. The differences as regards the answers to Problem 1 and Problem 2 are astonishingly different for the two classes of sesquilinear spaces (see e.g. Remark 6 below).

I. - Witt type theorems in the case of many isotropic vectors

I. 1. The Main Theorem

Let E be a non degenerate sesquilinear space of dimension N_0 and $L(E)$ the lattice of all subspaces of E . Consider sublattices $\mathcal{V}, \mathcal{V}^\tau$ of E that are stable under the operation \perp (taking the orthogonal). We are interested in situations where lattice isomorphisms $\tau : \mathcal{V} \rightarrow \mathcal{V}^\tau$ must be induced by metric automorphisms of the sesquilinear space E . For this to be the case there are many obvious conditions ; we mention two of them

$$(0) \quad (x^\perp)^\tau = (x^\tau)^\perp, \quad x \in \mathcal{V}$$

$$(1) \quad \dim X/\Sigma \{Y \in \mathcal{V} \mid Y \subsetneq X\} = \dim X^\tau/\Sigma \{Y^\tau \mid Y \subsetneq X\}, \quad X \in \mathcal{V}$$

Notice that \mathcal{V} is not assumed to be complete, so $\Sigma \{Y \in \mathcal{V} \mid Y \subsetneq X\}$ is a subspace of E which need not be an element of \mathcal{V} .

In the proof of the main theorem the elements $X \in \mathcal{V}$ with

$$(2) \quad X \neq \Sigma \{Y \in \mathcal{V} \mid Y \subsetneq X\}$$

are of primary importance (they might be called " Σ -inaccessible" elements and must not be confused with the join-inaccessible elements of [2]). We shall impose the following condition on the lattices \mathcal{V} .

(3) For all $X \in \mathcal{V}$ satisfying (2) the principal filter generated by X in \mathcal{V} is prime.

Examples where (3) always holds are provided by the distributive lattices : for, every $X \in \mathcal{V}$ with (2) is a join-irreducible element of \mathcal{V} and in a distributive lattice a principal filter is prime if and only if the generator is join-irreducible.

A condition on τ which is quite obvious is that τ preserve indices, i.e. dimensions of quotients of neighbouring elements in \mathcal{V} .

The easiest spaces to work with when discussing Witt-type theorems are the alternate spaces (since E is non degenerate the involution must be the iden-

tity and thus k commutative). For non alternate spaces the following imposes a restriction (cf. [8], p. 159) :

(4) The set $||E|| = \{\phi(x, x) | x \in E\}$ is an additive subgroup of k . If W is any degenerate infinite dimensional subspace of D , $D \in \mathcal{U}$, then for every $\alpha \in ||E||$ there is a $w \in W$ with $\phi(w, w) = \alpha$.

For tracevalued forms ϕ condition (4) may be formulated more conveniently by postulating that every W contain a totally isotropic subspace of infinite dimension. However, it turns out that in the tracevalued case this condition would be unnecessarily severe ; it will be sufficient to require

(4_{tr}) If $\dim D/D^\perp \cap D = \aleph_0$, $D \not\subset D^\perp$, then there exists an infinite dimensional totally isotropic subspace $Y \subset D$ with $(D^\perp \cap D) \cap Y = (0)$ or else $D = D^{\perp\perp}$ and the principal ideal generated by D in the lattice \mathcal{U} is $\{D, D^\perp, (0)\}$.

For the sake of easier formulation of our results we put down one more condition (cf. remark 1).

(5) If the form is not alternate then $\dim D/D^\perp \cap D \in \{0, \aleph_0\}$ for all $D \in \mathcal{U}$.

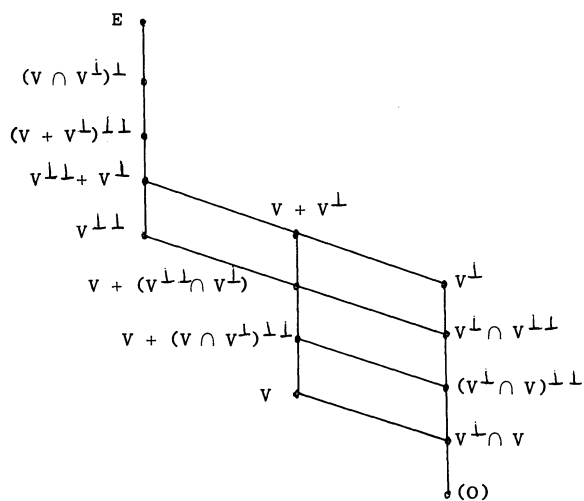
Theorem 1 ("Main theorem"). Let E be a non degenerate sesquilinear space of dimension \aleph_0 . Let \mathcal{U} and \mathcal{V} be lattices of subspaces of E which contain the spaces (0) , E and with every element X the space X^\perp as well. Assume (4) or -if the form is tracevalued- (4_{tr}). Let $\tau : \mathcal{U} \rightarrow \mathcal{V}$ be a lattice isomorphism which respects indices and satisfies (0) and (1). In order that τ be induced by a metric automorphism of E it is sufficient that \mathcal{U} satisfies (3), (5) and the descending chain condition.

Remark 1. In applications of the main theorem condition (5) can often be bypassed by chopping off finite dimensional orthogonal summands. A good example is theorem 2 below.

Remark 2. There are many involutorial divisionrings k such that every form on an \aleph_0 -dimensional k -space will automatically satisfy condition (4). We shall list a few examples here. For the sake of illustration we shall stick to symmetric forms (see [6] for further examples) ; the (commutative) fields which we shall mention all share the following condition (mentioned in Theorem 3 of [10]).

(6) There exists $m \in \mathbb{N}$ depending on k solely such that every symmetric form in $n > m$ variables possesses a non banal zero.

Example 1. All fields k with "char $k \neq 2$ & k non formally real & k^*/k^{*2} (the multiplicative group of k modulo square factors) finite". Here m is the order of the group k^*/k^{*2} (necessarily a power of 2). Fields of power series provide examples for $|k^*/k^{*2}|$ any power of 2. Further examples are the Hilbertfields of [5].



The lattice $\mathcal{V}(V)$ of Chap. I.2, generated by a subspace V of the non degenerate sesquilinear space E .

Example 2. The functionfields k in r variables over a finite constant field. Here $m = 2^{r+1}$ (for $r = 1$ this is a classical result of Hasse theory, for arbitrary r it is a result of [12]).

1. 2. Witt's Theorem

As an application to Theorem 1 we let $\mathcal{V} = \mathcal{V}(V)$ be the lattice generated by one single subspace V of the sesquilinear space E under the operations $+$, \cap , \perp . $\mathcal{V}(V)$ contains 14 elements [10]; in fact, it is the union of two chains,

$$(7) \quad \mathcal{V}(V) = \{ \{0\} \subset V \cap V^\perp \subset (V \cap V^\perp)^\perp \subset (V^\perp \cap V^\perp) \subset V^\perp \subset V + V^\perp \subset V^\perp^\perp + \\ + V^\perp \subset (V + V^\perp)^\perp \subset (V \cap V^\perp)^\perp \subset E \} \cup \{ V \subset V + (V \cap V^\perp)^\perp \subset V + \\ + (V^\perp \cap V^\perp) \subset V^\perp \}$$

so that $\mathcal{V}(F)$ must be distributive. From the main theorem one can deduce the following result concerning Problem 1 of the introduction (an earlier more direct although less perspicuous proof of Theorem 2 is contained in [8]):

Theorem 2. Let E be a non degenerate sesquilinear space of dimension \aleph_0 , V and \bar{V} isometric subspaces of E satisfying

- 0) $V^\perp \cong \bar{V}^\perp$ (isometrically)
- 1) $\dim(V \cap V^\perp)^\perp / V \cap V^\perp = \dim(\bar{V} \cap \bar{V}^\perp)^\perp / \bar{V} \cap \bar{V}^\perp$
- 2) $\dim(V^\perp \cap V^\perp) / (V \cap V^\perp)^\perp = \dim(\bar{V}^\perp \cap \bar{V}^\perp) / (\bar{V} \cap \bar{V}^\perp)^\perp$
- 3) $\dim(V^\perp + V^\perp) / (V^\perp + V) = \dim(\bar{V}^\perp + \bar{V}^\perp) / (\bar{V}^\perp + \bar{V})$
- 4) $\dim(V + V^\perp)^\perp / (V^\perp + V^\perp) = \dim(\bar{V} + \bar{V}^\perp)^\perp / (\bar{V}^\perp + \bar{V}^\perp)$

In order that there exist a metric automorphism T of E with $TV = \bar{V}$ the following conditions are sufficient

- 5) if $\dim V / V \cap V^\perp = \aleph_0$ then condition (4) is satisfied with $D = V$ or -if the form is tracevalued- (4_{tr}) holds for $D = V$ or $V = \sum \{ Z \in \mathcal{V}(V) \mid Z \not\subset V \}$
- 6) if $\dim V^\perp / V^\perp \cap V^\perp = \aleph_0$ then condition (4) is satisfied with $D = V$ or -if the form is tracevalued- (4_{tr}) holds for $D = V$ or $V = \sum \{ Z \in \mathcal{V}(V) \mid Z \not\subset V \}$

Remark 3. Conditions 0) though 4) are obviously necessary for an automorphism of the required sort to exist. They are not, in general, sufficient. See § 3.7 in [8].

Corollary 1. Let V be a subspace of the nondegenerate alternate space E , $\dim E = \aleph_0$. The finitely many cardinal numbers defined by the lattice $\mathcal{V}(V)$ (dimensions of quotients of neighbouring elements) are a complete set of orthogonal invariants for the subspace V .

This proves (cf. 8, p. 162) an old conjecture of Kaplansky ([10], p. 11). The corresponding statement is false when $\dim V > \aleph_0$, counter examples may be found in ([7], p. 132) (cf. question 3 in [10]).

Corollary 2 [10]. Let E be a non degenerate tracevalued sesquilinear space of dimension λ_0 and R a totally isotropic subspace. If $R = R^{\perp\perp}$ then there exists a totally isotropic subspace $R' \subset E$ such that $R + R'$ is an orthogonal summand of E ("Witt decomposition"); $R + R'$ is then a sum of hyperbolic planes with R, R' spanned by the two halves of a symplectic basis.

The two corollaries are typical for a host of applications which can be made of Theorem 2. We shall proceed with further applications of the Main Theorem.

1. 3. Orthogonal and Symplectic Separation

Notation. If in a direct sum $F_1 \oplus F_2$ of sesquilinear spaces both summands F_1, F_2 are totally isotropic we shall write " $F_1 \overset{\circ}{\oplus} F_2$ ".

Definition. Let F_1, F_2 be subspaces of the nondegenerate sesquilinear space E . The pair F_1, F_2 is said to be orthogonally [resp. symplectically] separated in E if and only if there exists a decomposition $E = E_1 \overset{\circ}{\oplus} E_2$ [resp. $E = E_1 \overset{\circ}{\oplus} E_2$] with $F_i \subset E_i$ ($i = 1, 2$).

Notice that F_1, F_2 are separated if and only if $F_1^{\perp\perp}, F_2^{\perp\perp}$ are separated (in either sense); we shall therefore assume without loss of generality that

$$F_1 = F_1^{\perp\perp}, \quad F_2 = F_2^{\perp\perp}.$$

In order that F_1, F_2 be separated in either sense it is evidently necessary that $F_1 \cap F_2 = (0)$ ("disjoint pair") and

$$(8) \quad (F_1 + F_2)^{\perp\perp} = F_1 + F_2$$

$$(9) \quad F_1^{\perp} + F_2^{\perp} = E (= (F_1 \cap F_2)^{\perp})$$

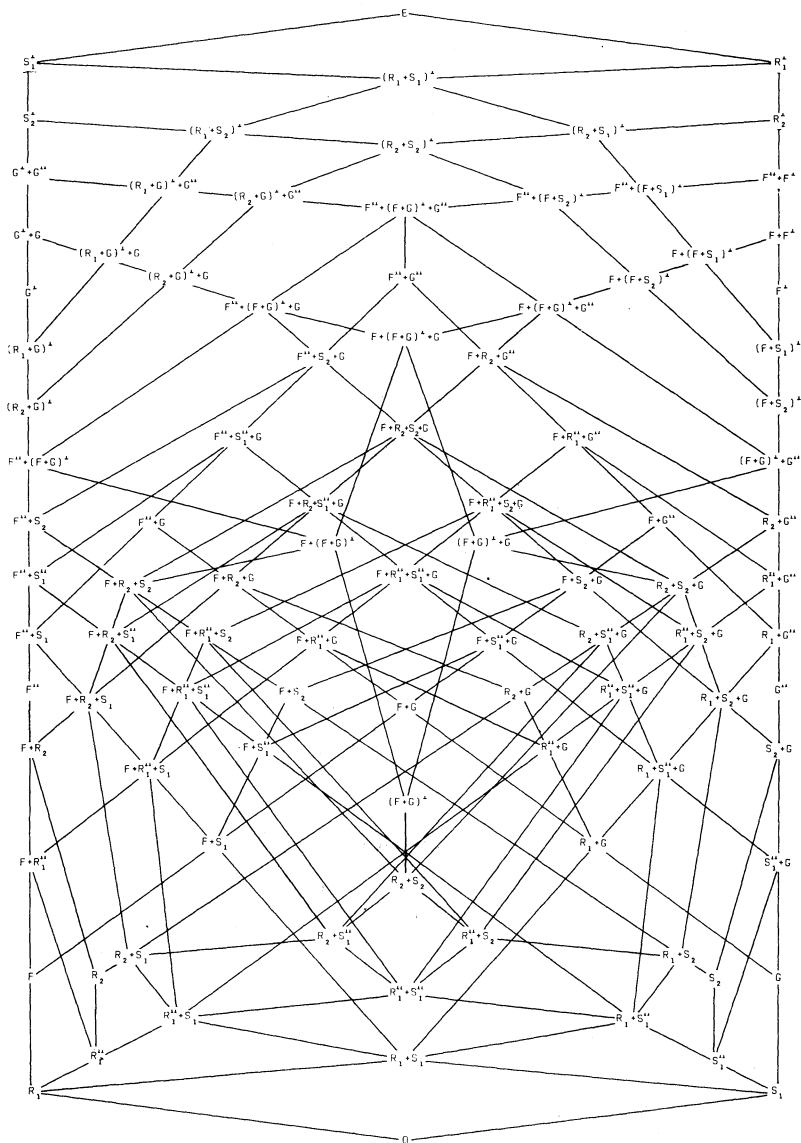
(8) and (9) may conveniently be interpreted in the lattice $L_{\perp\perp}(E)$ of all \perp -closed subspace X of E ($X^{\perp\perp} = X$). We first remark that $L_{\perp\perp}(E)$ happens to be a sublattice of the lattice $L(E)$ of all subspaces of E if and only if trivially so by the following.

Theorem 3. Let E be any nondegenerate sesquilinear space. $L_{\perp\perp}(E)$ is modular if and only if $\dim E$ is finite.

For $\text{char } k \neq 2$ and hermitean forms this was proved in [11]; using the same technique a proof can also be given in the general case. This theorem considerably generalizes a fact well known in the case of Hilbertspace (cf. Thm (32.17) in [13]). Now (8) and (9) say that F_1, F_2 is a disjoint modular and dual modular pair in the lattice $L_{\perp\perp}(E)$ (cf. Thm (33.4) in [13]).

Our result is that under certain general conditions such pairs F_1, F_2 must always be separated. In order to obtain this result via Theorem 1 we need

Theorem 4 [4]. Let E be a nondegenerate sesquilinear space of dimension λ_0 and F_1, F_2 an orthogonal [resp. totally isotropic] modular and dual modular pair



The lattice $\mathcal{U}(F, G)$ of Theorem 4. \mathcal{U} is generated by an orthogonal pair F, G of subspaces of the sesquilinear space E (E non degenerate and of countably infinite dimension) with $(F+G)^\perp = F^\perp + G^\perp$ and $F^\perp + G^\perp = E$.

(Abbrev. $R_1 = F^\perp \cap F$, $R_2 = F^\perp \cap F^\perp$, $S_1 = G^\perp \cap G$, $S_2 = G^\perp \cap G^\perp$)

in $L_{\perp\perp}(E)$. Let $\mathcal{V}(F_1, F_2)$ be the smallest sublattice of $L(E)$ (the lattice of all subspaces of E) which is stable under the operation \perp and which contains (0) , E, F_1, F_2 . $\mathcal{V}(F_1, F_2)$ is finite and distributive, in fact, it has 100 elements (in general) and is generated by two chains.

For F_1 and F_2 arbitrary subspaces of E the lattice $\mathcal{V}(F_1, F_2)$ will not, in general, be finite [6].

Theorem 5. Let E be a nondegenerate alternate space of dimension \aleph_0 and F_1, F_2 \perp -closed subspaces with $F_1 \perp F_2$ [resp. $F_1 \perp F_1, F_2 \perp F_2$]. If (8) and (9) are satisfied then F_1, F_2 is orthogonally [resp. symplectically] separated.

From Theorem 5 we obtain an answer to question 4 in [10], namely

Corollary 1. Let E be as in the theorem. If F_1, F_2 is a disjoint modular and dual modular pair and $F_1 \perp F_2$ [resp. $F_1 \perp F_1, F_2 \perp F_2$] then the finitely many cardinal numbers defined by the lattice $\mathcal{V}(F_1, F_2)$ are a complete set of orthogonal invariants for the pair F_1, F_2 .

One may also formulate theorem 5 for non alternate spaces ; one then has to put down some conditions in the vein of (4) or (4_{tr}). Direct proofs for these situations as well as for theorem 5 are given in [4].

Remark 4. Theorem 5 can be used to solve Problem 2 of the introduction for algebraic isometries $T_0 : F \rightarrow F \subset E$ whose polynomials split into different linear factors. See chap. II in [4].

I. 4. Extending Isometries

We give here some results concerning Problem 2 of the Introduction. We shall make use of the weak linear topology $\sigma(E)$ of a sesquilinear space E ; $\sigma(E)$ has $\{X^\perp \mid X \text{ linear subspace of } E \text{ \& dim } X < \infty\}$ as a 0-neighbourhoodbasis. A linear subspace Y of E is $\sigma(E)$ -closed if and only if it is \perp -closed ($Y^{\perp\perp} = Y$).

Theorem 6 ([1], p. 8). An isometry $T_0 : F \rightarrow \overline{F}$ between \perp -closed subspaces F, \overline{F} of the nondegenerate \aleph_0 -dimensional alternate space E can be extended to all of E if and only if the following two conditions hold.

(10) T_0 is homeomorphic with respect to $\sigma(E)$

(11) $\dim F^\perp / F^\perp \cap F = \dim \overline{F}^\perp / \overline{F}^\perp \cap \overline{F}$

Theorem 7 ([1], p. 16). Let E be as in Thm 6 and $T_0 : F \rightarrow \overline{F}$ an isometry between \perp -dense subspaces (i.e. $F^\perp = \overline{F}^\perp = (0)$). T_0 can be extended to an isometry of all of E if and only if

(12) U^\perp and $(T_0 U)^\perp$ are isometric for all $U \subset F$ with $\dim F/U \leq 2$.

Remark 5. Theorem 6 can be generalized to non alternate forms Φ . The following conditions, together with (10) and (11), prove sufficient for an extension of T_0 to exist

(13) If $\dim F^\perp/F^\perp \cap F < \infty$ then F^\perp and \overline{F}^\perp are isometric and Φ is tracevalued.

(14) If $\dim F^\perp/F^\perp \cap F = \infty$ then $\|E\|$ is an additive subgroup of k . If $\alpha \in \|E\|$, H a finite dimensional subspace of E and W a subspace of F^\perp with $\dim F^\perp/W + (F^\perp \cap F) < \infty$ there exists $x \in W \cap H^\perp$ with $\Phi(x, x) = \alpha$ (if Φ is tracevalued this condition is equivalent to the existence of an infinite dimensional totally isotropic subspace $G \subset F^\perp$ with $G \cap (F^\perp \cap F) = (0)$).

II. - Witt type theorems in the case of definite forms

II.1. Definite Forms

Let $(k, *)$ be an involutorial divisionring and (k_0, \langle) an ordered sub-divisionring. If (E, Φ) is a hermitean space over $(k, *)$ such that $\|E\| \subset k_0$ then we say that Φ is definite on the line $k(x_0) \subset E$ ($0 \neq x_0 \in E$) if and only if $\Phi(x_0, x_0) \cdot \Phi(\lambda x_0, \lambda x_0) > 0$ for all $0 \neq \lambda \in k$. We say that Φ is positive definite if $\Phi(x, x) > 0$ for all nonzero $x \in E$.

There exist non commutative involutorial divisionrings $(k, *)$, $* \neq 1$, which are ordered and which have the property " $\sum \lambda_i \lambda_i^* = 0 \Rightarrow \lambda_i = 0$ " ([6]). Clearly, such fields admit anisotropic hermitean forms, however, they do not admit definite forms by the following

Theorem 8 ([6]). Let $(k, *)$ and k_0 be as above and (E, ϕ) a non degenerate hermitean space over k with $\|E\| \subset k_0$. If $* \neq 1$ then the following are equivalent.

(i) ϕ is definite on all lines of E . (ii) ϕ is anisotropic and definite on at least one line of E . (iii) ϕ is definite and either $(k, *)$ is a quaternion algebra $(\frac{\alpha, \beta}{k_0})$ with $\alpha, \beta < 0$ and $*$ the usual "conjugation" or else $(k, *)$ is commutative and a quadratic extension of k_0 , $k = k_0(\sqrt{\gamma})$ for some $\gamma < 0$ and $(\alpha + \beta \sqrt{\gamma})^* = \alpha - \beta \sqrt{\gamma}$ for all $\alpha, \beta \in k_0$. If on the other hand $* = 1$ then k is commutative and $k = (k_0, \langle)$ is ordered.

In [6] we have treated Problem 1 of the introduction for arbitrary subspaces of definite spaces as defined here. As we can give but an illustration in this short survey we shall make the following simplifications here :

1. k_0 is archimedean ordered, hence $k_0 \subseteq \mathbb{R}$ without loss of generality ;
2. $k_0 = k$ and the form is symmetric.

Finally we put down the following condition (cf. Theorem 4 in [10]).

(15) There exists $m \in \mathbb{N}$ depending on k solely such that every positive symmetric form in $n \geq m$ variables represents 1 or -1 (or both).

Examples. Algebraic numberfields ($m = 4$).

A consequence of (14) is that every \aleph_0 -dimensional positive definite space admits an orthonormal basis.

II. 2. Invariants for \perp -dense subspaces

The general case is treated in [6]; here we shall merely consider subspaces V, \bar{V} of a positive definite space (E, Φ) which satisfy $V^\perp = \bar{V}^\perp = (0)$. By a standardbasis for the embedding $V \subset E$ we mean a basis $\mathcal{B} = (v_i)_{i \in \mathbb{N}} \cup (f_\lambda)_{\lambda \in J \setminus \mathbb{N}}$ such that $(v_i)_{i \in \mathbb{N}}$ is an orthonormal basis for V and $(f_\lambda)_{\lambda \in J}$ an orthonormal basis for some supplement of V in E . With respect to a fixed basis \mathcal{B} we set

$$(16) \quad \alpha_{\lambda i} = \Phi(f_\lambda, v_i) \in k \quad (\lambda \in J; i \in \mathbb{N})$$

$$(17) \quad A_{\lambda \chi}^n = \sum_{i=1}^n \alpha_{\lambda i} \alpha_{\chi i} \in k \quad (\lambda, \chi \in J; n \in \mathbb{N})$$

$$(18) \quad A_{\lambda \chi} = \lim_{n \rightarrow \infty} A_{\lambda \chi}^n \in \mathbb{R} \quad (\lambda, \chi \in J)$$

One proves that the real matrix $A = (A_{\lambda \chi})_{J \times J}$ is positive definite and $A - \mathbb{1}$ is negative semidefinite. We call A the matrix associated with \mathcal{B} . A may be interpreted as a point with coordinates $A_{\lambda \chi}$ in a real space of dimension $\frac{1}{2} n(n+1)$ where $n = \text{card } J = \dim E/V \leq \aleph_0$. Thus to every standardbasis there corresponds a point of the convex region \mathcal{R} which is the intersection of the two cones

$$(19) \quad \begin{aligned} C_1 &: (A_{\lambda \chi}) \text{ positive definite} \\ C_2 &: (A_{\lambda \chi}) - \mathbb{1} \text{ negative semidefinite.} \end{aligned}$$

Conversely one proves the following

Theorem 9 ([14], [6]). Let A be any positive $J \times J$ matrix over \mathbb{R} , $\text{card } J \leq \aleph_0$ such that $A - \mathbb{1}$ is negative semidefinite. There exists a positive definite symmetric space (E, ϕ) over k which contains a standardbasis $\mathcal{B} = (v_i)_{i \in \mathbb{N}} \cup (f_\lambda)_{\lambda \in J \setminus \mathbb{N}}$ with \perp -dense span of the v_i and with A the associated matrix.

Thus, conversely, to every point of the convex region $\mathcal{R} = C_1 \cap C_2$ there corresponds a dense embedding $V \subset E$ (E spanned by an orthonormal basis). The orthogonal invariants which we set up for the \perp -dense $V \subset E$ will enable us to replace the study of orbits in the set of \perp -dense $V \subset E$ under the orthogonal group of E by the study of orbits of points in the region \mathcal{R} under some more accessible group.

"Quantities" appropriate for the description of a \perp -dense embedding $V \subset E$ are not the matrices A associated with standard bases but rather the

matrices $A - \mathbb{1}$. Here is how they transform :

Let $\beta = (v_i)_{i \in \mathbb{N}} \cup (f_\lambda)_{\lambda \in J}$, $\bar{\beta} = (\bar{v}_i)_{i \in \mathbb{N}} \cup (\bar{f}_\lambda)_{\lambda \in J}$ be two standard bases for the embedding $V \subset E$, $V^\perp = (0)$ (over the field $k \subseteq \mathbb{R}$). We have

$$(20) \quad \bar{f}_\lambda = \sum_{\chi=1}^{\chi(\lambda)} \gamma_{\lambda\chi} f_\chi + \sum_{i=1}^{i(\lambda)} \xi_{\lambda i} v_i$$

for certain row finite matrices $(\gamma_{\lambda\chi}), (\xi_{\lambda i})$ over k ; $(\gamma_{\lambda\chi})$ is invertible. Let furthermore $(A_{\lambda\chi}), (\bar{A}_{\lambda\chi})$ be the matrices over \mathbb{R} associated with β and $\bar{\beta}$ respectively. Then

$$(21) \quad \bar{A}_{\lambda\chi} - \delta_{\lambda\chi} = \sum_{v,\mu=1}^{v(\lambda),\mu(\chi)} \gamma_{\lambda v} (A_{v\mu} - \delta_{v\mu}) \gamma_{\chi\mu} \quad (\gamma_{\lambda v} \in k)$$

where $(\delta_{\lambda\chi})$ is the unit matrix.

Our principal result in the present case is

Theorem 10 ([14], [6]). Let $V, \bar{V} \subset E$ be \perp -dense subspaces. There is a metric automorphism T of E with $TV = \bar{V}$ if and only if there are standard bases for $V \subset E$, $\bar{V} \subset E$ such that (21) holds, i.e. if and only if the real matrices $\bar{A} - \mathbb{1}$, $A - \mathbb{1}$ are equivalent over the subfield k .

Corollary 1. Let E be the usual inner product space over \mathbb{R} of dimension \aleph_0 . For every $n \leq \aleph_0$ there are precisely $n+1$ orbits (under the orthogonal group of E) of \perp -dense subspaces $V \subset E$ with $\dim E/V = n$. The nullity of the semidefinite $n \times n$ matrix $A - \mathbb{1}$ and n are the only invariants.

Corollary 2. Let E be as in Corollary 1. Every \perp -dense embedding $V \subset E$ splits, i.e. E is an orthogonal sum of $\dim E/V$ copies of E with each copy containing a \perp -dense hyperplane V_i such that $V = \sum V_i$.

Corollary 3. If $k \subset \mathbb{R}$ then the embedding $V \subset E$ splits if and only if the real matrix A associated with any standard basis for V in E can be diagonalized over the subfield k .

Corollary 4. If $k \subset \mathbb{R}$ (e.g. k the real closure of \mathbb{Q}) then there are 2^{\aleph_0} orbits of \perp -dense subspaces $V \subset E$ with $\dim E/V = n$; among them there are 2^{\aleph_0} orbits whose representatives do not split.

$\mathbb{R} \otimes_k E$ is a normed vector space under the norm $\sqrt{\phi(x,x)}$ for $x \in \mathbb{R} \otimes_k E$. We endow E with the induced topology and let \hat{V} be the closure of the subspaces $V \subset E$. $\dim \hat{V}/V$ is an obvious orthogonal invariant of the subspace V . If $V^\perp = (0)$ and $m \leq \aleph_0$ is the nullity of a matrix A associated with the embedding $V \subset E$ then one proves that $m = \dim \hat{V}/V$. In particular we have

Corollary 5. Let E be the usual inner product space over \mathbb{R} of dimension \aleph_0 . If V is a \perp -dense subspace of E then the pair $\{\dim E/\hat{V}, \dim \hat{V}/V\}$ is a complete set of orthogonal invariants for V (here \hat{V} is the closure of V in the "natural" topology of E).

Remark 6. If E is the symmetric space spanned by an orthonormal basis over $k = \mathbb{C}$, then in contrast to corollary 1 there is only 1 orbit of \perp -dense subspaces $V \subset E$ for each $n = \dim E/V \leq \aleph_0$. (This is an immediate consequence of Theorem 2.)

II. 3. Extending Isometries

Let the field $k \subseteq \mathbb{R}$ be as in II. 1. and, as usual, $\dim E = \aleph_0$.

Theorem 11. Let $V, \bar{V} \subset E$ be dense subspaces with respect to the natural topology in E . A given isometry $T_0 : V \rightarrow \bar{V}$ admits an (isometric) extension to all of E if and only if T_0 is homeomorphic with respect to the weak linear topologies $\sigma(E)|_V, \sigma(E)|_{\bar{V}}$.

Even when $k = \mathbb{R}$ there does not seem to exist an "obvious" proof for Theorem 11 (suggested, say, by Hilbert space arguments). A proof for Theorem 11 is contained in [14], [6]; it proceeds by a recursive construction of the required extension. This also explains why a "topological" theorem of this sort carries along with it arithmetical assumptions such as (15).

III. - Bibliography

- [1] W. ALLENSPACH, Erweiterung von Isometrien in alternierenden Räumen, Zürich University Thesis 1973.
- [2] G. BIRKHOFF, Lattice Theory, AMS Providence, Rhode Island, 1973.
- [3] N. BOURBAKI, Eléments de Mathématiques, Algèbre Chap. IX, Formes sesquilineaires et formes quadratiques, Hermann, Paris, 1959.
- [4] L. BRAND, Erweiterung von algebraischen Isometrien in sesquilinearen Räumen. Zürich University Thesis 1974.
- [5] A. FRÖHLICH, Quadratic forms à la local theory. Proc. Camb. Phil. Soc. (1967), 63, 579-586.
- [6] H. GROSS, Sesquilinear forms and quadratic forms in infinite dimensional spaces, Vol. 1 : Spaces of countably infinite dimension. To appear.
- [7] H. GROSS, Der Euklidische Defekt bei quadratischen Räumen, Math. Ann. 180, 95-137 (1969).
- [8] H. GROSS, On Witt's Theorem in the denumerably infinite case, Math. Ann. 170, 145-165 (1967).
- [9] H. GROSS and H.R. FISCHER, Quadratic Forms and linear topologies I, Math. Ann. 157, 296-325 (1964).
- [10] I. KAPLANSKY, Forms in infinite-dimensional spaces, Ann. Acad. Bras. Ci. 22, 1-17 (1950).

- [11] H.A. KELLER, On the lattice of all closed subspaces of a Hermitean Space, Rev. Soc. Mat. Chile, Vol. 2 (1976).
- [12] S. LANG, On quasi algebraic closure, Ann. of Math. (1952) 55, 373-390. Important improvements can be found in M. Nagata : Note on a paper of Lang concerning quasi algebraic closure, Mem. Univ. Kyoto Ser. A 30 (1957) 237-241. For further developments and references see G. Terjanian : Dimension arithmétique d'un corps, Journ. of Algebra 22 (1972) 517-545 ; further G. Maxwell : A note on Artin's Diophantine Conjecture. Canad. Math. Bull. (1970) 13, 119-120.
- [13] F. MAEDA and S. MAEDA, Theory of Symmetric Lattices, Springer NY 1970.
- [14] U. SCHNEIDER, Beiträge zur Theorie der sesquilinearen Räume unendlicher Dimension, Zürich University Thesis 1975.

Mathematisches Institut
Universität Zürich
Freiestrasse 36

8032 ZÜRICH
SUISSE

