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NONLINEAR STRUCTURES DETERMINED BY MEASURES ON BANACH SPACES

By K. David ELWORTHY

0. INTRODUCTION.

A. A Gaussian measure γ on a separable Banach space E , together with the topological vector space structure of E , determines a continuous linear injection $i : H \rightarrow E$, of a Hilbert space H , such that γ is induced by the canonical cylinder set measure of H . Although the image of H has measure zero, nevertheless H plays a dominant role in both linear and nonlinear analysis involving γ , [8], [9], [10]. The most direct approach to obtaining measures on a Banach manifold M , related to its differential structure, requires a lot of extra structure on the manifold : for example a linear map $i_x : H \rightarrow T_x M$ for each x in M , and even a subset M_H of M which has the structure of a Hilbert manifold, [6], [7]. In the manifold case it has not been clear how much of this additional structure is really required ; or, slightly reformulated : do certain measures on an infinite dimensional manifold M , together with the differential structure of M , determine any such additional structures ? As a special case of this we can ask whether every diffeomorphism of E which preserves the Gaussian measure γ necessarily maps $i(H)$ to itself, or has derivatives which preserve $i(H)$.

Along similar lines, it is plausible that the Hilbert manifold $L_{x_0}^{2,1}(X)$ of $L^{2,1}$ paths starting at x_0 on a Riemannian manifold X may play a central role for the Wiener measure on the manifold $C_{x_0}(X)$ of continuous paths in X , [6], [7]. If so it should be possible to characterise $L_{x_0}^{2,1}(X)$ in terms of that measure and the differentiable structure of $C_{x_0}(X)$.

Although we do not answer these questions, we show here that any strictly positive Radon measure on a smooth manifold determines some structure : namely a partition of M into subsets invariant under measure preserving diffeomorphisms, and subspaces in the tangent spaces to M invariant under the derivatives of such diffeomorphisms. For infinite dimensional M these are shown to be non-trivial in a class of important cases : and the partition may well be non trivial in general, in infinite dimensions. A concrete consequence is obtained in Corollary 4 A : *the group of diffeomorphisms of an infinite dimensional separable Banach space E preserving a given Gaussian measure does not act transitively on E* . This is false for the group of measure class preserving diffeomorphisms : Theorem 1B. Another consequence, concerning group actions, is given in § 3C.

The precise definitions of the invariants may seem rather unnatural : they have been chosen from a wide range of similar definitions simply in order to make the theorems true and to show that non-trivial invariants exist, not because of any obvious intrinsic geometric meaning. A particularly interesting point is that the interpolation K-functors, as described by PEETRE in [13], play an important role in several different places : especially Corollary 2 C and Proposition 4 B. Full proofs and a more detailed discussion will be made available elsewhere.

B. We are concerned with measures on topological spaces : by which we mean positive Borel measures, usually locally finite (or even finite) and strictly positive ; so every point has a neighbourhood of finite measure, and each open set has non-zero measure. Moreover in order that our constructions are non-trivial we shall often have to assume that the measures μ are *tight* i.e. for each Borel set B

$$\mu(B) = \sup \{ \mu(K) : K \subset B, K \text{ compact} \}.$$

This follows automatically, when μ is finite, if the space is separable and admits a complete metric, see [12], [15]. Recall that a Borel measure is a *Radon* measure if it is locally finite and tight.

Two measures λ, μ on X are *equivalent*, $\lambda \approx \mu$, if they have the same sets of measure zero. If so the Radon-Nikodym derivatives $\frac{d\lambda}{d\mu}, \frac{d\mu}{d\lambda}$ are defined, almost everywhere, as measurable functions. This relation between measures seems to be too weak for our purposes (see Theorem 1B) : so for x in X we define λ and μ to be *pointwise equivalent at x*

$$\lambda \approx \mu \text{ pointwise at } x$$

if for all neighbourhood bases $\mathcal{U}$ at x , directed by inclusion, and all Borel sets B

$$\lim_{U \in \mathcal{U}} \frac{\mu(U \cap B)}{\lambda(U \cap B)} > 0 \quad \text{and} \quad \lim_{U \in \mathcal{U}} \frac{\lambda(U \cap B)}{\mu(U \cap B)} > 0$$

where, in the computation of the lower limits, we replace $\frac{0}{0}$ by 1 and $\frac{r}{0}$ by ∞ , if $r > 0$.

For strictly positive measures λ, μ , we see that if λ, μ are orthogonal they are not pointwise equivalent at any point, while, for X first countable, if λ and μ are equivalent they are pointwise equivalent at x iff both Radon-Nikodym derivatives are essentially bounded in some neighbourhood of x .

1. GAUSSIAN MEASURES.

A. Since Gaussian measures furnish our main test bed we quickly give the defini-

tion and most relevant properties. For simplicity we consider only strictly positive, mean zero measures.

Let E be a separable real Banach space. A measure γ on E is *Gaussian* if for all continuous linear surjections with finite dimensional range :

$$T : E \rightarrow F_T$$

the induced measure $T(\gamma)$ on F_T

$$T(\gamma)(B) = \gamma T^{-1}(B) \quad B \in \text{Borel}(F_T)$$

is given by

$$T(\gamma)(B) = (2\pi)^{-n/2} \int_B \exp\left(-\frac{|x|^2}{2}\right) dx$$

where $n = \dim F_T$, and the Lebesgue measure and norm come from some inner product $\langle \cdot, \cdot \rangle_T$ on F_T .

For such a measure γ :

G1 : [4], there is a compact linear injective map $i : H \rightarrow E$, of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ into E such that the inner product $\langle \cdot, \cdot \rangle_T$ in the definition is the quotient inner product under the map $T \circ i : H \rightarrow F_T$.

G2 : the image of H , $i(H)$, has γ -measure 0.

G3 : translation, $T_z : E \rightarrow E$, by an element z of E preserves sets of measure zero, i.e. $T_z(\gamma) \approx \gamma$, iff z lies in the image of H .

G4 : if $j : E^* \rightarrow H$ denotes the adjoint of i then $T_z(\gamma) \approx \gamma$ pointwise at some point iff z lies in the image of E^* by $j \circ i$, in which case $T_z(\gamma) \approx \gamma$ pointwise at every point of E .

G5 : the image of H in E is the intersection of all measurable linear subspaces of E with non-zero measure. Such subspaces have measure 1 (see [2] for a short proof).

B. Given a Gaussian measure γ on E let $\varphi : U \rightarrow V$ be a C^1 diffeomorphism of open subsets of E , having the form $\varphi(x) = x + i \circ j \circ \alpha(x)$ where $\alpha : U \rightarrow E^*$ is C^1 . Then H-H. KUO [10], proved that φ preserves sets of measure zero and its "jacobian"

$$\frac{d\varphi^{-1}(\gamma)}{d\gamma} = U \rightarrow \mathbb{R}$$

is given by $x \mapsto |\det D\varphi(x)| \exp \left\{ \frac{1}{2} [-2\alpha(x)(x) - |j \circ \alpha(x)|^2] \right\}$ (the determinant

refers to $D\varphi(x)|_H : H \rightarrow H$, and is proved to exist).

It follows that

$$\varphi(\gamma|U) \approx \gamma|V \text{ pointwise on } V.$$

Ramer [14] has a stronger version of Kuo's theorem.

When E , and hence H , is infinite dimensional we can follow the construction used by BESSAGA [1], or [3], to show that $H - \{0\}$ is diffeomorphic to H , and for any r in E we can obtain a C^∞ diffeomorphism $\varphi : E \rightarrow E$ with $\varphi(0) = v$ and such that

$$\varphi|_{E - \{0\}} : E - \{0\} \rightarrow E - \{v\}$$

satisfies the conditions of Kuo's theorem. It follows that still $\varphi(\gamma) \approx \gamma$: although now the pointwise equivalence at v will be lost in general. This proves :

THEOREM 1B. - *Let γ be a strictly positive Gaussian measure on a separable Banach space E . Then the group of C^∞ diffeomorphisms preserving γ up to equivalence, acts transitively on E .*

We show below, Corollary 4A, that the theorem is false for infinite dimensional E when equivalence is replaced by pointwise equivalence at all points of E . In any case the theorem does not necessarily imply that measure class preserving diffeomorphisms can behave in a completely abandoned way : for example

Problem. - With the notation of the theorem : does there exist a diffeomorphism $\varphi : E \rightarrow E$ with

$$\begin{aligned} \varphi(\gamma) &\approx \gamma \\ \text{and } \varphi(i(H)) \cap i(H) &= \emptyset ? \end{aligned}$$

2. TANGENT CONES AND INTERPOLATION FUNCTORS.

A.† Let A be a subset of the real Banach space E .

For a point a in the closure $\bar{A}$ of A we shall define the *tangent cone* $TC_a(A)$ to A at a by

$$TC_a(A) = \{v \in E \text{ s.t. } d(a + sv, A) = O(s^2) \text{ as } s \rightarrow 0\}.$$

Note that the more natural definition would have $O(s)$ instead of our $O(s^2)$. This would have the advantage of being invariant under C^1 diffeomorphisms, but Corollary 2C below would not hold with that definition. Our construction is easily seen to

have the following properties.

TC(i) : If $v \in TC_a(A)$ and $\lambda > 0$ then $\lambda v \in TC_a(A)$.

TC(ii) : If A is convex then so is $TC_a(A)$.

TC(iii) : If $\varphi : U \rightarrow V$ is a C^2 map of open sets of Banach spaces, and $\bar{A} \subset U$, then $D\varphi(a)(TC_a(A)) \subset TC_{\varphi(a)}(\varphi(A))$.

From TC(iii) it follows that tangent cones are defined for subsets A of C^2 Banach manifolds M . They then lie in the tangent spaces :

$$TC_a(A) \subset T_a M.$$

B. Let $\vec{E}$ denote a pair of Banach spaces (E_1, E) with a given continuous linear injection $i : E_1 \rightarrow E$. As in PEETRE [13], for $0 < t < \infty$ and $v \in E$ define

$$K(t, v) = \inf \{ \|v - v_1\| + t \|v_1\|_1 : v_1 \in E_1 \},$$

where $\| \cdot \|$ and $\| \cdot \|_1$ denote the norms of E and E_1 respectively, and elements of E_1 are identified with their image in E .

For $0 < \theta < 1$ define

$$\vec{E}_{\theta, \infty} = \{v \in E : K(t, v) = O(t^\theta)\}$$

and set $\|v\|_{\theta, \infty} = \sup_{t > 0} \frac{K(t, v)}{t^\theta}$ if $v \in \vec{E}_{\theta, \infty}$.

This is a special case of the more general construction of K -functors described in [13]. The properties we need are

K1 : $\vec{E}_{\theta, \infty}, \| \cdot \|_{\theta, \infty}$ is a Banach space.

K2 : The map i factorizes by continuous linear maps

$$E_1 \xrightarrow{\alpha} \vec{E}_{\theta, \infty} \xrightarrow{\beta} E.$$

K3 : If i was compact then so are both α and β .

C. For $\vec{E}$ as above, let $B_1(x; r)$ denote the closed ball

$$\{y : \|x - y\|_1 \leq r\}$$

about x , radius r , in E_1 .

Define the *contact space* $\tau(E_1, E)$ of $\vec{E}$ by

$$\tau(E_1, E) = TC_0(i[B_1(0; 1)]).$$

PROPOSITION 2C. - As subsets of E

$$\tau(E_1, E) = E_1^{\rightarrow}, \infty.$$

From property K3 and the fact that any compact subset of E lies in the image of the unit ball of Banach space mapped into E by a compact linear map, the proposition yields :

COROLLARY 2C. - If K is a compact subset of the infinite dimensional Banach space E then for all $a \in K$

$$\text{Linear span } TC_a(K) \neq E.$$

[However $TC_a(K)$ can certainly be dense in E .]

3. INFINITESIMAL PROPERTIES OF MEASURES.

A. Let μ be a strictly positive, locally finite, measure on a metric space (M, d) . We say that the Borel subset A of M *infinitesimally supports* μ at the point a of M , $A \in \text{Supp}(\mu ; a)$, if for all $r > 0$ $\frac{\mu(B(a ; t) - A)}{\mu(B(a ; rt^2))} \rightarrow 0$ as $t \rightarrow 0$ where $B(a ; t)$

denotes the closed ball about a , radius t . It is easy to see that this definition depends only on the local Lipschitz class of the metric d , and on the pointwise equivalence class at a of μ .

PROPOSITION 3A. - If μ is a Radon measure, (e.g. if (M, d) is complete and if $A \in \text{Supp}(\mu ; a)$) then there is a compact set K with

$$\begin{aligned} K &\subset A \cup \{a\} \\ \text{and} \quad K &\in \text{Supp} \{ \mu ; a \}. \end{aligned}$$

Now suppose that M is a separable C^2 Banach manifold and that the metric d is in the local Lipschitz class determined by the differentiable structure. For a in M define the *tangent cone*, $\tau c_a(\mu) = \cap \{TC_a(A) : A \in \text{Supp}(\mu ; a)\}$.

$$\text{Thus } \tau c_a(\mu) \subset T_a M.$$

THEOREM 3A :

(i) For every strictly positive Radon measure μ on the infinite dimensional metrizable C^2 Banach manifold M the tangent cone to μ at a general point a satisfies.

$$\text{Linear span } \tau c_a(\mu) \neq T_a M.$$

(ii) Let $\varphi : M \rightarrow M$ be a C^2 diffeomorphism such that

$\varphi(\mu) \approx \mu$ pointwise at $\varphi(a)$. Then

$$T_a \varphi [\tau c_a(\mu)] = \tau c_{\varphi(a)}(\mu).$$

Part (i) follows from Corollary 2C and Proposition 3A, and (ii) is straightforward.

B. We have yet to show that $\tau c_a(\mu)$ can ever be larger than $\{0\}$ when E is infinite dimensional. This can be done using a more geometric differential invariant of measures : For a point a of a C^2 manifold M and a strictly positive measure μ on M define

$$Q_a(\mu) \subset T_a M$$

to consist of those tangent vectors v for which there exists a C^2 vector field ξ on M with $\xi(a) = v$ such that there is a neighbourhood V of a and positive constants ϵ, α satisfying

(i) the flow $\sigma : V \times (-\epsilon, \epsilon) \rightarrow M$ of ξ is defined.

(ii) there is a base $\mathcal{B}$ for the neighbourhood system of a in V with

$$\mu(\sigma_t(B)) \geq \alpha \mu(B) \text{ for all } 0 < t < \epsilon, B \in \mathcal{B}.$$

THEOREM 3B. - $Q_a(\mu) \subset \tau c_a(\mu)$.

COROLLARY 3B1. - $Q_a(\mu)$ does not span $T_a M$, if M is infinite dimensional and separable.

COROLLARY 3B2. - For a Gaussian measure γ on a separable Banach space E ,

$$\tau c_a(\gamma) \neq \{0\} \quad \text{all } a \in E.$$

In 3B2 we have $i.o.j(E^*) \subset Q_a(\gamma) \subset \tau c_a(\gamma)$, each $a \in E$.

C. As an application of the above : if $G \times M \rightarrow M$ is a C^2 action of a Banach Lie group G on a metrizable Banach manifold M which preserves, up to pointwise equivalence, some strictly positive, Radon measure on M , then for each a in M the derivative map at the identity

$$T_e G \rightarrow T_a M$$

obtained from $g \mapsto g.a$ is compact.

However it seems likely that the above is true for group actions which only preserve the measure up to equivalence. For the linear case with G a group of translations see [16].

4. ORDERING INDUCED BY A MEASURE.

A. Let μ be a strictly positive measure on a metric space (M, d) . For x and y in M write

$$x < y, \text{ if } \lim_{s \rightarrow 0} \frac{\mu(B(y; s))}{\mu(B(x, rs))} = 0, \text{ for all } r > 0.$$

If neither $x < y$ nor $y < x$ write $x \sim y$. It is easy to see that this defines an equivalence relation on M .

PROPOSITION 4A. - Let $f : M \rightarrow M$ be a homeomorphism which is locally bi-Lipschitz and satisfies $f(\mu) \approx \mu$ pointwise on M . Then $x \sim f(x)$ all $x \in M$.

From the proposition, in order to show that the group of such homeomorphisms f of M does not act transitively on M it suffices to show that the equivalence relation $\sim$ is non-trivial. Possibly this is true for a general class of measures on M when M is infinite dimensional. The proof of the following theorem depends on the fact that Gaussian measures are convex in the sense of BORELL [2] : in fact the theorem is true for arbitrary convex measures.

THEOREM 4A. - For a Gaussian measure γ on the Banach space E ; if $\bar{0}$ denotes the equivalence class of 0 , we have

$$\bar{0} \subset \tau_{C_0}(\gamma).$$

This combines with Theorem 3A to give the required non-triviality, whence :

COROLLARY 4A. - Let E be a separable infinite dimensional Banach space and γ a Gaussian measure on E . Then the group of locally bi-Lipschitz homeomorphisms of E with $f(\gamma) \approx \gamma$ pointwise on E does not act transitively on E .

B. The problem remains of characterizing the orbits of 0 under the group of diffeomorphisms of Corollary 4A, (or under the group of measure preserving diffeomorphisms) or perhaps a simpler problem is to characterize the equivalence class $\bar{0}$ of Theorem 4A. The following is suggestive, at least of the type of characterizations which may be true.

PROPOSITION 4B. - For a Gaussian measure γ on E , with corresponding maps

$$E^* \xrightarrow{j} H \xrightarrow{i} E \text{ we have}$$

$$\tau(E^*, H) \subset \bar{0}$$

In fact for all $z \in H$ and $x \in \tau(E^*, H)$ we have $z \sim z+x$.

[We identify points of E^* and H with their images in E].

Proof. - Let $||$ denote the norm of H , and $\|$ that of E .

Since $x \in \tau(E^*, H)$ there is a function $e : (0, 1) \rightarrow E^*$ and a constant k with

$$|x - e(s)| < \frac{1}{2} s$$

and
$$\|e(s)\|_{E^*} \leq k/s.$$

By the change of variable formula, § 1B, for $e = e(s)$

$$\begin{aligned} \gamma(B(z; s)) &= \int_{B(z+e; s)} \exp(e(y) - \frac{1}{2} |e|^2) d\gamma(y) \\ &= \exp(e(z) + \frac{1}{2} |e|^2) \int_{B(z+e; s)} \exp(e(y-e-z)) d\gamma(y) \\ &\geq \exp(e(z) + \frac{1}{2} |e|^2) \exp(-s \|e\|_{E^*}) \gamma(B(z+e; s)) \\ &\geq \exp(\langle x, z \rangle - \langle x-e, z \rangle + \frac{1}{2} |e|^2 - k) \gamma(B(z+e; s)) \end{aligned}$$

Now $B(z+x; \frac{1}{2} s) \subset B(z+e; s)$, so we have

$$\lim_{s \rightarrow 0} \frac{\gamma(B(z; s))}{\gamma(B(z+x; \frac{1}{2} s))} \geq \exp(\langle x, z \rangle - k) > 0$$

$$\text{whence } z \leq z + x.$$

substitution shows that also

$$z + x \leq (z+x) - x$$

giving

$$z \sim z+x \text{ as required.}$$

REFERENCES

- [1] C. BESSAGA, *Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere*, Bull. Acad. Polon. Sc. XIV, 1 (1966), 27-31.
- [2] C. BORELL, *Convex measures on locally convex spaces*, Arkiv for Matematik (12) (1974), 239-252.
- [3] D. BURGHELEA and N.H. KUIPER, *Hilbert manifolds*, Ann. of Math. 90 (1969), 379-417.
- [4] R.M. DUDLEY, J. FELDMAN, and L. LECAM, *On seminorms and probabilities, and abstract Wiener spaces*, Annals of Math. 93 (1971), 390-408.
- [5] J. EELLS, *Integration on Banach manifolds*, Proc. 13th Biennial Seminar of the Canadian Mathematical Congress, Halifax, (1971), 41-49.

- [6] K.D. ELWORTHY, *Gaussian measures on Banach spaces and manifolds*, Proc. 1972, Summer Institute on Global Analysis, Trieste : *Global Analysis and its Applications*, Vol. II, 151-166, International Atomic Energy, Vienna, (1974).
- [7] K.D. ELWORTHY, *Measures on infinite dimensional manifolds*, *Functional Integration and its applications*, (A.M. Arthurs, Ed.) Oxford University Press, (1975).
- [8] L. GROSS, *Measurable functions on Hilbert space*, Trans. A.M.S. 105 (1962), 372-390.
- [9] L. GROSS, *Potential theory on Hilbert space*, J. of Functional Anal. 1, (1967) 123-181.
- [10] H.H. KUO, *Integration theory on infinite dimensional manifolds*, Trans. A.M.S. 159 (1971), 57-78.
- [11] H.H. KUO, *Gaussian measures in Banach spaces*, Lecture Notes in Math. 463, Springer-Verlag.
- [12] K.R. PARTHASARATHY, *Probability measures on metric spaces*, Academic Press, 1967.
- [13] J. PEETRE, *Interpolation functors and Banach couples*, Actes, Congres intern. Math., Nice, 1970, Tome 2, 373-378.
- [14] R. RAMER, *On nonlinear transformations of Gaussian measures*, J. of Functional Anal. 15, (1974), 166-187.
- [15] L. SCHWARTZ, *Radon measures on arbitrary topological spaces and cylindrical measures*, Tata Institute of Fundamental Research Studies in Mathematics 6, Oxford University Press., 1973.
- [16] A.M. VERSIK, *Duality in the theory of measure in linear spaces*, Soviet Math. Dokl. Vol. 7, (1966), n° 5, 1210-1214 = Dokl. Nauk SSSR, Tom 170 (1966), n° 3.

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