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Mémoires de la S. M. F., tome 39-40 (1974), p. 329-340

<http://www.numdam.org/item?id=MSMF_1974__39-40__329_0>

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GROUP REPRESENTATIONS IN NON-ARCHIMEDEAN BANACH SPACES

A.C.M. van ROOIJ and W.H. SCHIKHOF

INTRODUCTION.

This paper deals with continuous representations of locally compact groups G into non-archimedean Banach spaces E . In order that G has sufficiently many of such representations G must be totally disconnected, which we assume from now on. If G carries a K -valued Haar measure (where K is the (non-archimedean valued) scalar field) we have a 1-1 correspondence between the continuous representations of G and those of the group algebra $L(G)$. If G is compact, then $L(G)$ can be decomposed as a direct sum of full matrix algebras over skew fields (Theorem 2.5), which yields as a corollary that every irreducible continuous representation of G is equivalent to a minimal left ideal of $L(G)$. Further, all continuous representations of G can be classified (Theorem 2.8). The theory for compact groups as it is given here is a generalization of the results of [2]. If G is locally compact and torsional (i.e., every compact set is contained in a compact subgroup) the results are satisfactory : G then has sufficiently many continuous irreducible representations ; every two-sided closed ideal in $L(G)$ is the intersection of maximal left ideals (Theorem 3.1, and corollaries). About non-torsional G little is known.

1. The Banach algebra $L(G)$.

K is a field with a (possibly trivial) non-Archimedean valuation $|\cdot|$ such that K is complete relative to the metric induced by $|\cdot|$. The residue class field of K is k . If $\lambda \in K, |\lambda| \leq 1$ then $\bar{\lambda}$ denotes the corresponding element of k . The characteristic of k is p (which may be 0).

G is a totally disconnected locally compact group, \mathcal{K} the collection of all open compact subgroups of G , \mathcal{B} the ring of sets generated by the left cosets of the elements of \mathcal{K} . It is known that \mathcal{B} consists of the compact open subsets of G .

and is a base for the topology of G .

A totally disconnected compact group H is called p-free if no open subgroup of H has an index in H that is divisible by p . (Every H is 0-free). We assume that G has a p-free compact open subgroup G_0 .

Then there exists a unique $m : \mathfrak{B} \rightarrow K$ with properties

- (1) m is additive
- (2) m is left invariant, i.e. $m(xA) = m(A)$ ($x \in G$; $A \in \mathfrak{B}$)
- (3) $m(G_0) = 1$.

This m is a left Haar measure on G .

Let $C_\infty(G)$ be the K -Banach space of all continuous functions $G \rightarrow K$ that vanish at infinity. (If G is compact we also call this space $C(G)$). More generally, for a Banach space E , $C_\infty(G \rightarrow E)$ will denote the Banach space of all continuous functions $G \rightarrow E$ that vanish at infinity. A left Haar measure m on G induces a unique E -valued continuous linear map. $f \mapsto \int f(x) dm(x)$ on $C_\infty(G \rightarrow E)$ for which

$$\int 1_A(x) \xi dm(x) = m(A)\xi \quad (A \in \mathfrak{B} ; \xi \in E),$$

1_A denoting the K -valued characteristic function of A . In particular ($E=K$)

$$\int 1_A(x) dm(x) = m(A) \quad (A \in \mathfrak{B}).$$

For all $f \in C_\infty(G \rightarrow E)$,

$$\left\| \int f(x) dm(x) \right\| \leq \|f\|.$$

This integration enables us to make $C_\infty(G)$ into a K -algebra by defining a multiplication $*$; for $f, g \in C_\infty(G)$, $y \in G$:

$$(f * g)(y) = \int f(x) g(x^{-1}y) dm(x) = \int f(yx^{-1}) g(x) dm(x).$$

In fact, it turns out that $f * g \in C_\infty(G)$ and $\|f * g\| \leq \|f\| \|g\|$.

Thus, $C_\infty(G)$ actually is a Banach algebra over K . $*$ is called convolution. When we view $C_\infty(G)$ as a Banach algebra under convolution, we usually call it $L(G)$.

If $H \in \mathcal{H}$ is contained in G_0 , then $m(H) = [G_0 : H]^{-1}$, so $|m(H)| = 1$.

Set $u_H = m(H)^{-1} \cdot 1_H$. Then

$$\|u_H\| = 1, \quad \int u_H(x) dm(x) = 1$$

$$u_H * u_H = u_H.$$

Let E be a Banach space. A representation of G in E is a homomorphism $U : x \mapsto U_x$ of G into the group of all isometric linear bijections $E \rightarrow E$. Such a representation U is called continuous if $x \mapsto U_x \xi$ is continuous for each $\xi \in E$.

A linear subspace D of E is U -invariant if $U_x(D) \subset D$ for every $x \in G$. If $\{0\}$ and E are the only U -invariant linear subspaces of E , the representation U is called algebraically irreducible. If $\{0\}$ and E are the only closed U -invariant subspaces, U is irreducible.

For $f \in C_0(G)$ and $a \in G$, define the function $L_a f$ on G by

$$(L_a f)(x) = f(a^{-1}x) \quad (x \in G).$$

In this way we have constructed a continuous representation L of G in $C_0(G)$, the regular representation.

For all $f, g \in C_0(G)$ we have the identity

$$f * g = \int f(x) L_x g dm(x).$$

More generally, let U be a continuous representation of G in some Banach space E . For $f \in L(G)$ and $\xi \in E$ we define

$$(i) \quad f * \xi = \int f(x) U_x \xi dm(x).$$

Thus, E becomes a module over the ring $L(G)$ for which

$$(ii) \quad \|f * \xi\| \leq \|f\| \|\xi\| \quad (f \in L(G); \xi \in E)$$

and

$$(iii) \quad U_x(f * \xi) = (L_x f) * \xi \quad (f \in L(G); \xi \in E; x \in G).$$

If $\xi \in E$ and $\varepsilon > 0$, then $\{x \in G \mid \|U_x \xi - \xi\| < \varepsilon\}$ is an open subgroup of G .

If $H \in \mathcal{N}$ is contained in this subgroup, then $\|u_H * \xi - \xi\| < \varepsilon$. Ordering \mathcal{N} in the obvious way we obtain

$$(iv) \quad \lim_{H \in \mathcal{N}} u_H * \xi = \xi \quad (\xi \in E).$$

In particular, the u_H form a left approximate identity for $L(G)$. Without any trouble one proves that they actually form a two-sided approximate identity.

A closed linear subspace of E is U -invariant if and only if it is a submodule of E . A continuous linear map $E \rightarrow E$ commutes with every U_x if and only if it is a module homomorphism.

Conversely, a Banach $L(G)$ -module is a Banach space E provided with a bilinear map $*$: $L(G) \times E \rightarrow E$ such that $f * (g * \xi) = (f * g) * \xi$ ($f, g \in L(G)$; $\xi \in E$) and such that (ii) holds. Such a Banach $L(G)$ -module is continuous (or essential) if (iv) is also valid. If E is any Banach $L(G)$ -module, the closed linear hull of $\{f * \xi \mid f \in L(G); \xi \in E\}$ is the largest continuous submodule of E .

In any continuous Banach $L(G)$ -module E , formula (iii) defines a continuous representation U of G that fulfils (i): we have a one-to-one correspondence between continuous $L(G)$ -modules and continuous representations of G .

2 - The structure of $L(G)$ for compact G .

In this chapter we assume that G itself is compact and p -free. We work with the left Haar measure m for which $m(G) = 1$.

Let \mathcal{N}_0 denote the set of all normal open subgroups of G . It was proved by Pontryagin that every element of \mathcal{N} contains an element of \mathcal{N}_0 . It follows that the u_H ($H \in \mathcal{N}_0$) form a left approximate identity for $L(G)$.

For any Banach space F and for $n \in \mathbb{N}$ we consistently view F^n as a Banach space under the max-norm:

$$\|(\xi_1, \dots, \xi_n)\| = \max_i \|\xi_i\| \quad (\xi_1, \dots, \xi_n \in F).$$

If D is a closed linear subspace of a Banach space E , a projection of E onto D is a linear $P : E \rightarrow E$ for which

- (1) $\|P\| \leq 1.$
- (2) $P(E) \subset D.$
- (3) $Px = x$ for all $x \in D.$

The following lemma is well-known.

2.1. Lemma. Let D be a linear subspace of K^n . Then as a normed vector space, D is isomorphic to some K^m . There exists a projection of K^n onto D .

The same reasoning used in the classical theory for representations in Banach spaces now leads to

2.2. Lemma. Let U be a continuous representation of G in K^n . Let D be a U -invariant linear subspace of K^n . Then there exists a projection P of K^n onto D that commutes with every U_x .

Every $\xi \in K^n$ for which $\|\xi\| \leq 1$ determines in a natural way a $\bar{\xi} \in k^n$. Consequently, a K -linear $A : K^n \rightarrow K^n$ with $\|A\| \leq 1$ determines a k -linear $\bar{A} : k^n \rightarrow k^n$ by

$$\bar{A}(\bar{\xi}) = \overline{A\xi} \quad (\xi \in K^n, \|\xi\| \leq 1).$$

In particular, a representation U of G in K^n induces a representation $\bar{U} : x \mapsto \bar{U}_x$ in k^n . The following lemma can be proved as an application of lemma 2.2.

2.3. Lemma. Let U be a continuous representation of G in K^n . Then U is irreducible if and only if \bar{U} is irreducible.

A useful consequence :

2.4. Lemma. Let U, V be continuous representations of G in K^n and in a Banach space E , respectively. Suppose U to be irreducible. If $T : K^n \rightarrow E$ is a linear map such that $TU_x = VT_x$ ($x \in G$), then

$$\|T\xi\| = \|T\| \|\xi\| \quad (\xi \in K^n).$$

If U, V are representations of G in non-trivial Banach spaces E, F , respectively, we say that they are equivalent if there exists a surjective linear $T : E \rightarrow F$ with $TU_x = V_x T$ for all $x \in G$ and with

$$\|T\xi\| = \|T\| \|\xi\| \quad (\xi \in K^n).$$

Similarly, two non-zero Banach $L(G)$ -modules, E and F , are called equivalent if there exists a surjective module isomorphism $T : E \rightarrow F$ such that

$$\|T\xi\| = \|T\| \|\xi\| \quad (\xi \in K^n).$$

In either case, if T is an isometry we speak of isomorphism rather than equivalence.

For every $H \in \mathcal{H}_0$, $u_H * L(G)$ is a two-sided ideal in $L(G)$ consisting of all functions $G \rightarrow K$ that are constant on the cosets of H . Thus, $u_H * L(G)$ is finite-dimensional, and, as a normed vector space, is isomorphic to $K[G:H]$. We have already observed that the $u_H (H \in \mathcal{H}_0)$ form a left approximate identity in $L(G)$. Then

$$\sum \{u_H * L(G) \mid H \in \mathcal{H}_0\} \text{ is dense in } L(G).$$

In the set of all central idempotent elements of $L(G)$ we introduce an ordering \leq by

$$e_1 \leq e_2 \text{ if } e_1 * L(G) \subset e_2 * L(G).$$

Let \mathcal{E} be the set of all minimal non-zero central idempotents. The elements of \mathcal{E} are linearly independent and have norm 1. Then for every $H \in \mathcal{H}_0$ only finitely many elements of \mathcal{E} are $\leq u_H$. One proves easily that $u_H = \sum \{e \in \mathcal{E} : e \leq u_H\}$. For every $e \in \mathcal{E}$ there exists an $H \in \mathcal{H}_0$ with $\|u_H * e - e\| < 1$; then $e * u_H \neq 0$. By the minimality of e it follows that $e = e * u_H$, so

$$e * L(G) = e * u_H * L(G) = u_H * e * L(G) \subset u_H * L(G).$$

By lemma 2.1, $e * L(G)$ is isomorphic to some K^n .

We need one more definition before we can formulate the structure theorem for $L(G)$. Let $(A_i)_{i \in I}$ be a family of Banach spaces. We set

$$\bigoplus_{i \in I} A_i = \{x \in \prod_{i \in I} A_i \mid \text{if } \varepsilon > 0, \text{ then } \|x_i\| > \varepsilon \text{ for only finitely many } i\}.$$

In a natural way, $\bigoplus_{i \in I} A_i$ is a Banach space under the norm defined by $\|x\| = \sup_{i \in I} \|x_i\|$. If all the A_i are Banach algebras (or $L(G)$ -modules), $\bigoplus_{i \in I} A_i$ becomes a Banach algebra (an $L(G)$ -module).

It is now relatively easy to prove the following analog to a classical structure theorem for finite groups.

2.5. Theorem. For $e \in \mathcal{E}$ set $L(G)_e = e * L(G)$. As a Banach space, $L(G)_e$ is isomorphic to some K^n . Every $L(G)_e$ is a two-sided ideal in $L(G)$. If $f \in L(G)$, then

$$f = \sum_{e \in \mathcal{E}} e * f \text{ and } \|f\| = \sup_{e \in \mathcal{E}} \|e * f\|. \text{ The formula}$$

$$(Sf)_e = e * f \quad (e \in \mathcal{E}; f \in L(G))$$

yields an isomorphism of Banach algebras

$$S : L(G) \longrightarrow \bigoplus_{e \in \mathcal{E}} L(G)_e$$

For every $X \subset \mathcal{E}$, $\{f \in L(G) \mid e * f = 0 \text{ for every } e \in X\}$ is a closed two-sided ideal in $L(G)$; all closed two-sided ideals of $L(G)$ are of this form. The minimal non-zero two-sided ideals are just the $L(G)_e$.

In the following lines, instead of "minimal non-zero left ideal of $L(G)$ " we simply say "minimal ideal". $L(G)_e$, being a finite-dimensional left ideal of $L(G)$, contains minimal ideals. As in the purely algebraic representation theory of finite groups, each $L(G)_e$ is a sum of minimal ideals; every minimal ideal lies in some $L(G)_e$; and two minimal ideals are isomorphic (as $L(G)$ -modules) if and only if they are contained in the same $L(G)_e$.

Let $n(e)$ be the dimension (as a K -vector space) of a minimal ideal that is contained in $L(G)_e$. It follows from lemma 2.1 that for every $e \in \mathcal{E}$ we can choose an $L(G)$ -module structure on $K^{n(e)}$, so that the resulting module $I^{(e)}$ is isomorphic to the minimal ideals that lie in $L(G)_e$. The module structure of $I^{(e)}$ induces a continuous representation $W^{(e)}$ of G in $K^{n(e)}$.

The following generalization of 2.5 is not hard to prove.

2.6. Theorem. Let U be a continuous representation of G in a Banach space E ; let $*$ be the corresponding module operation $L(G) \times E \longrightarrow E$. For $e \in \mathcal{E}$ set $E_e = \{e * \xi \mid \xi \in E\}$. Each E_e is a closed submodule of E . The formula

$$(S\xi)_e = e * \xi \quad (\xi \in E)$$

yields an isomorphism of Banach $L(G)$ -modules

$$S : E \longrightarrow \bigoplus_{e \in \mathcal{E}} E_e.$$

The restriction of U to E_e is called the e -homogeneous part of U .

If $E_e = E$, then U itself is called e -homogeneous. (Observe that always $(E_e)_e = E_e$).

Let U be an irreducible continuous representation of G in a Banach space E . Choose $\xi \in E$, $\xi \neq 0$. There must exist an $e \in \mathcal{E}$ with $e * \xi \neq 0$. As $L(G)_e$ is a sum of minimal ideals, there must exist a minimal ideal $D \subset L(G)_e$ with $D * \xi \neq (0)$. Applying lemma 2.4 (consider the map $f \mapsto f * \xi$ ($f \in D$)) we get

2.7 Corollary. Every irreducible continuous representation of G is equivalent to one of the $W^{(e)}$. In particular, it is finite dimensional.

Now let F be any Banach space. Every $n \times n$ -matrix induces in a natural way a map $F^n \longrightarrow F^n$. Thus, every $W^{(e)}$ induces a continuous e -homogeneous representation $W^{(e)} \otimes \text{Id}_F$ in $F^{n(e)}$. (To explain the notation we observe that F^n is linearly isometric to $K^n \otimes_K F^n$). Together with Theorem 2.6 the following gives a complete classification of all continuous representations of G .

2.8. Theorem. Every e -homogeneous continuous representation of G is isomorphic to $W^{(e)} \otimes \text{Id}_F$ for some Banach space F . The given representation determines F up to an isomorphism of Banach spaces.

For $e \in \mathcal{E}$ let \mathcal{A}_e be the set of all linear module homomorphisms $I^{(e)} \rightarrow I^{(e)}$. Obviously, \mathcal{A}_e is a K -Banach algebra. But it follows from lemma 2.4 that \mathcal{A}_e even is a valued skew field containing K . It turns out that every commutative subfield

of \mathcal{A}_e is obtainable by adjunction of roots of 1 to K . Hence, if K contains "enough" roots of 1, then $\mathcal{A}_e = K$.

In a natural way, $I^{(e)}$ becomes a normed vector space over \mathcal{A}_e . As in the algebraic theory, $L(G)_e$ (as an algebra or an $L(G)$ -module) is isomorphic to the algebra of all \mathcal{A}_e -linear maps $I^{(e)} \rightarrow I^{(e)}$. But this time the isomorphism is also an isometry. It follows that, if G is abelian, then every $L(G)_e$ is a valued field, and $L(G)$ is power-multiplicative. (A Banach algebra A is power-multiplicative if $\|a^n\| = \|a\|^n$ for all $a \in A$ and $n \in \mathbb{N}$).

As a Banach space, $I^{(e)}$ is isomorphic to $(\mathcal{A}_e)^{n(e)}$ for some $n(e) \in \mathbb{N}$. It follows that $L(G)_e$ (as a Banach algebra or a Banach $L(G)$ -module) is isomorphic to the algebra of all $n(e) \times n(e)$ matrices with entries from \mathcal{A}_e . Here the norm of a matrix is the maximum of the norms of its entries.

3 - Representations of locally compact groups.

K, k, p, G are as in chapter 1. We assume every element of \mathcal{N} to be p -free.

G is called torsional if every compact subset is contained in a compact subgroup. If G is torsional then so is every closed subgroup and every quotient of G by a closed normal subgroup.

The additive group of a non-trivial valued local field is torsional : for each $n \in \mathbb{N}$, $\{x \mid |x| \leq n\}$ is a compact open subgroup. The multiplicative group is not torsional : if $|x| > 1$, then $\lim |x^n| = \infty$. The general and special linear groups are not torsional. However, the following group G of triangular $m \times m$ matrices

$$G = \{(\alpha_{ij}) \mid \alpha_{ij} = 0 \text{ if } i < j; |\alpha_{ii}| = 1 \text{ for all } i\}$$

is torsional : for each $n \in \mathbb{N}$, $H_n = \{(\alpha_{ij}) \in G \mid |\alpha_{ij}| \leq n^{i-j} \text{ for all } i, j\}$ is a compact open subgroup.

We now formulate the main

3.1. Theorem. Let G be torsional and let $I \subset L(G)$ be a proper closed two-sided ideal. For every $f \in L(G)$ there exists a maximal modular left ideal $N \supset I$ such that $\|f \bmod I\| = \|f \bmod N\|$.

Proof. First, assume that G is compact. Then $L(G) = \bigoplus_{e \in \mathcal{E}} L(G)_e$ where \mathcal{E} is the collection of minimal central idempotents of $L(G)$. (Theorem 2.5).

Then $I = \bigoplus_{e \in \mathcal{D}} L(G)_e$ for some $\mathcal{D} \subset \mathcal{E}$, $\mathcal{D} \neq \mathcal{E}$, and $f = \sum_{e \in \mathcal{E}} e * f$.

Clearly, $\|f \bmod I\| = \max_{e \notin \mathcal{D}} \|f * e\| = \|f * d\|$ for certain $d \notin \mathcal{D}$.

We identify $L(G)_d$ with the algebra of all $n(d) \times n(d)$ matrices over \mathcal{A}_d . (See the end of Chapter 2). There exists a $\xi \in (\mathcal{A}_d)^{n(d)}$ with

$\|(d * f)(\xi)\| = \|d * f\| \|\xi\|$. Let $N_d = \{g \in L(G)_d \mid g(\xi) = 0\}$; then

$\|d * f \bmod N_d\| = \|d * f\|$. For $e \in \mathcal{E}$, $e \neq d$ set $N_e = L(G)_e$, and let

$N \subset L(G)$ be the closure of $\sum_{e \in \mathcal{E}} N_e$. Then N is a maximal modular left ideal containing I , and $\|f \bmod N\| = \|d * f \bmod N_d\| = \|d * f\| = \|f \bmod I\|$.

Observe that one can make a non-zero $n(d) \times n(d)$ matrix s over \mathcal{A}_d such that $N_d * s = \{0\}$ and $s * s = s$. (The columns of s are suitable multiples of ξ). We need this remark in the second part of this proof.

For the general case we may assume that f has compact support, so $f = 0$ outside a compact open subgroup H . We have the obvious embedding $L(H) \hookrightarrow L(G)$.

By the foregoing there exists a maximal modular left ideal M of $L(H)$, with identity e_0 , for which $M \supset I \cap L(H)$ and $\|f \bmod I \cap L(H)\| = \|f \bmod M\|$, and there exists an idempotent $s \in L(H)$ with $M * s = \{0\}$. By maximality, $M = \{g \in L(H) \mid g * s = 0\}$. Set $J = \overline{L(G) * M + I}$. J is a closed left ideal of $L(G)$, containing I . For all $g \in L(G)$

$$g * e_0 - g = \lim_{v \in \mathcal{U}} (g * u_v * e_0 - g * u_v) \in \overline{L(G) * M} \subset J,$$

so J is modular. We next prove $J \neq L(G)$.

Let $j \in J \cap L(H)$. Then $(j - j * s) * s = 0$, so $j - j * s \in M$. Also, $j * s \in \overline{L(G) * M + I} * s \subset I * s \subset I$ and $j * s \in L(H)$, so $j * s \in M$. Therefore, $J \cap L(H) \subset M$, so that $J \neq L(G)$. Trivially, $J \cap L(H) \supset M$, so $J \cap L(H) = M$.

Being a proper modular left ideal, J extends to a maximal modular left ideal N of $L(G)$. By the maximality of M we still have $N \cap L(H) = M$.

By lemma 2.4, the canonical map

$$\rho: L(H)/M \longrightarrow L(G)/N$$

satisfies $\|\rho(\eta)\| = \|\rho\| \|\eta\|$ ($\eta \in L(H)/M$). Using the fact that

$$\lim_{v \in \mathcal{V}} \|u_v \text{ mod } M\| = \lim_{v \in \mathcal{V}} \|u_v \text{ mod } N\| = 1 \text{ we see that } \|\rho\| = 1, \text{ so } \rho \text{ is an}$$

isometry. Hence, $\|f \text{ mod } N\| = \|f \text{ mod } M\| = \|f \text{ mod } I \cap L(H)\| >$

$$\|f \text{ mod } I\| > \|f \text{ mod } N\|.$$

3.2. Corollary. Let $H \in \mathcal{H}$ and let I be a closed two-sided ideal in $L(G)$.

Then the canonical map $L(H)/I \cap L(H) \longrightarrow L(G)/I$ is an isometry.

3.3. Corollary. If G is abelian and if I is a maximal modular ideal of $L(G)$, then $L(G)/I$ is a valued field which is the completion of an algebraic extension of K .

Proof. For every $H \in \mathcal{H}$, $I \cap L(H)$ is a maximal ideal of $L(H)$ of finite codimension, and $L(H)/I \cap L(H)$ is a valued field.

The corollary now follows from the observation that the union of the canonical images of the $L(H)/I \cap L(H)$ ($H \in \mathcal{H}$) is dense in $L(G)/I$.

3.4. Corollary. For each two-sided closed ideal $I \subset L(G)$ the Banach algebra $L(G)/I$ is reduced ("Spectral synthesis"). In particular, for each $f \in L(G)$ there exists an (algebraically) irreducible continuous representation T of $L(G)$ in some Banach space such that $\|T_f\| = \|f\|$ ($f \in L(G)$). ("The Fourier transformation is an isometry"). For each $x \in G$, $x \neq e$ there exists a continuous irreducible representation U of G in some Banach space such that $U_x \neq I$. ("Gelfand-Raikov Theorem").

The representation space of an irreducible representation of an abelian group may have dimension greater than 1. If K is "big enough" this cannot happen :

3.5. Theorem. Let G be an abelian torsional group and suppose that the equation $\zeta^n = 1$ has n distinct roots in K for every $n \in \{[H_2 : H_1] : H_1, H_2 \in \mathcal{N}; H_2/H_1 \text{ cyclic}\}$. Let \hat{G} be the group of all continuous homomorphisms of G into $\{\alpha \in K : |\alpha| = 1\}$, topologized with the compact open topology. Then every maximal modular ideal M of $L(G)$ has codimension 1 and there is an $\alpha_M \in \hat{G}$ such that the homomorphism $L(G) \rightarrow L(G)/M$ has the form

$$f \mapsto f(\alpha_M) = \int f(x) \alpha_M(x^{-1}) dx \quad (f \in L(G))$$

The map $M \mapsto \alpha_M$ is a homeomorphism of the collection of maximal modular ideals, with the Gelfand topology, onto \hat{G} . The dual group \hat{G} is also torsional and the Fourier transformation $f \mapsto \hat{f}$, given by

$$\hat{f}(\alpha) = \int f(x) \alpha(x^{-1}) dx \quad (f \in L(G))$$

is an isometrical isomorphism of $L(G)$ onto $C_\infty(\hat{G})$. Finally, the canonical map $G \rightarrow \hat{\hat{G}}$ is an isomorphism of topological groups.

Proof. See Corollary 3.4 and [1], 4.3.16 and 5.2.11.

We mention (without proof) a result for not-necessarily torsional groups. Define $B(G) = \{x \in G : U_x = I \text{ for every continuous irreducible representation } U \text{ of } G\}$. It is clear from the definition that $B(G)$ is a closed normal subgroup.

3.6. Theorem. $B(G)$ is a discrete torsion-free subgroup of G , and is contained in every open normal subgroup of G . If G is either abelian or discrete or torsional then $B(G) = \{e\}$. $B(G)$ has a trivial intersection with the center of G .

We end with a

Conjecture : Let G be a locally compact totally disconnected group, such that all elements of \mathcal{N} are p -free, where p is the characteristic of the residue class field k of K . Then $B(G) = \{e\}$, i.e. G has sufficiently many continuous irreducible representations.

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