A. C. M. VAN ROOIJ
W. H. SCHIKHOF

Group representations in non-archimedean Banach spaces

Mémoires de la S. M. F., tome 39-40 (1974), p. 329-340
<http://www.numdam.org/item?id=MSMF_1974__39-40__329_0>

Table Ronde Anal. non archim. (1972, Paris)
Bull. Soc. math. France,
Mémoire 39-40 1974, p. 329-340

GROUP REPRESENTATIONS IN NON-ARCHIMEDEAN BANACH SPACES

A.C.M. van ROOIJ and W.H. SCHIKHOF

INTRODUCTION.

This paper deals with continuous representations of locally compact groups G into non-archimedean Banach spaces E. In order that G has sufficiently many of such representations G must be totally disconnected, which we assume from now on. If G carries a K-valued Haar measure (where K is the (non-archimedean valued) scalar field) we have a 1-1 correspondence between the continuous representations of G and those of the group algebra L(G). If G is compact, then L(G) can be decomposed as a direct sum of full matrix algebras over skew fields (Theorem 2.5), which yields as a corollary that every irreductible continuous representation of G is equivalent to a minimal left ideal of L(G). Further, all continuous representations of G can be classified (Theorem 2.8). The theory for compact groups as it is given here is a generalizacion of the results of [2]. If G is locally compact and torsional (i.e., every compact set is contained in a compact subgroup) the results are satisfactory : G then has sufficiently many continuous irreductible representations ; every two-sided closed ideal in L(G) is the intersection of maximal left ideals (Theorem 3.1, and corollaries). About non-torsional G little is known.

1. The Banach algebra L(G).

K is a field with a (possibly trivial) non-Archimedean valuation $|\ \ |$ such that K is complete relative to the metric induced by $|\ \ |$. The residue class field of K is k. If $\lambda \in K, |\lambda| \leq 1$ then $\bar{\lambda}$ denotes the corresponding element of k. The characteristic of k is p (which may be 0).

G is a totally disconnected locally compact group, $\mathcal{N}$ the collection of all open compact subgroups of G, $\mathcal{B}$ the ring of sets generated by the left cosets of the elements of $\mathcal{N}$. It is known that $\mathcal{B}$ consists of the compact open subsets of G and is a base for the topology of G.

A totally disconnected compact group H is called p-free if no open subgroup of H has an index in H that is divisible by p. (Every H is 0-free). We assume that G has a p-free compact open subgroup G_o .

Then there exists a unique $m : \mathcal{B} \longrightarrow K$ with properties

(1) m is additive

(2) m is left invariant, i.e. $m(xA) = m(A)$ $(x \in G;\ A \in \mathcal{B})$

(3) $m(G_o) = 1$.

This m is a left Haar measure on G.

Let $C_\infty(G)$ be the K-Banach space of all continuous functions $G \longrightarrow K$ that vanish at infinity. (If G is compact we also call this space C(G)). More generally, for a Banach space E, $C_\infty(G \longrightarrow E)$ will denote the Banach space of all continuous functions $G \longrightarrow E$ that vanish at infinity. A left Haar measure m on G induces a unique E-valued continuous linear map $f \longmapsto \int f(x)dm(x)$ on $C_\infty(G \longrightarrow E)$ for which

$$\int 1_A(x)\ \xi\ dm(x) = m(A)\xi \qquad (A \in \mathcal{B}\ ;\ \xi \in E),$$

1_A denoting the K-valued characteristic function of A. In particular (E=K)

$$\int 1_A(x)\ dm(x) = m(A) \qquad (A \in \mathcal{B}).$$

For all $f \in C_\infty(G \longrightarrow E)$,

$$\left\| \int f(x)\ dm(x) \right\| \leq \| f \| .$$

This integration enables us to make $C_\infty(G)$ into a K-algebra by defining a multiplication $*$; for $f,g \in C_\infty(G)$, $y \in G$:

$$(f * g)(y) = \int f(x)g(x^{-1}y)dm(x) = \int f(yx^{-1})g(x)dm(x).$$

In fact, it turns out that $f * g \in C_\infty(G)$ and $\|f * g\| \leq \|f\|\,\|g\|$.

Thus, $C_\infty(G)$ actually is a Banach algebra over K. $*$ is called convolution. When we view $C_\infty(G)$ as a Banach algebra under convolution, we usually call it L(G).

If $H \in \mathcal{N}$ is contained in G_o, then $m(H) = [G_o : H]^{-1}$, so $|m(H)| = 1$. Set $u_H = m(H)^{-1}.1_H$. Then

$$\|u_H\| = 1, \quad \int u_H(x)dm(x) = 1$$

$$u_H * u_H = u_H .$$

Let E be a Banach space. A <u>representation</u> of G in E is a homomorphism $U : x \longmapsto U_x$ of G into the group of all isometric linear bijections $E \longrightarrow E$. Such a representation U is called <u>continuous</u> if $x \longrightarrow U_x \xi$ is continuous for each $\xi \in E$.

A linear subspace D of E is U-<u>invariant</u> if $U_x(D) \subset D$ for every $x \in G$. If $\{0\}$ and E are the only U-invariant linear subspaces of E, the representation U is called <u>algebraically irreducible</u>. If $\{0\}$ and E are the only closed U-invariant subspaces, U is <u>irreducible</u>.

For $f \in C_\infty(G)$ and $a \in G$, define the function $L_a f$ on G by

$$(L_a f)(x) = f(a^{-1}x) \qquad (x \in G).$$

In this way we have constructed a continuous representation L of G in $C_\infty(G)$, the <u>regular representation</u>.

For all $f,g \in C_\infty(G)$ we have the identity

$$f * g = \int f(x)\, L_x\, g\, dm(x).$$

More generally, let U be a continuous representation of G in some Banach space E. For $f \in L(G)$ and $\xi \in E$ we define

(i) $$f * \xi = \int f(x) U_x \xi \, dm(x).$$

Thus, E becomes a module over the ring L(G) for which

(ii) $$\|f * \xi\| \leq \|f\| \; \|\xi\| \qquad (f \in L(G);\ \xi \in E)$$

and

(iii) $$U_x(f * \xi) = (L_x f) * \xi \qquad (f \in L(G); \xi \in E;\ x \in G).$$

If $\xi \in E$ and $\varepsilon > 0$, then $\{x \in G \mid \|U_x \xi - \xi\| < \varepsilon\}$ is an open subgroup of G. If $H \in \mathcal{N}$ is contained in this subgroup, then $\|u_H * \xi - \xi\| \leq \varepsilon$. Ordering $\mathcal{N}$ in the obvious way we obtain

(iv) $$\lim_{H \in \mathcal{N}} u_H * \xi = \xi \qquad (\xi \in E).$$

In particular, the u_H form a left approximate identity for L(G). Without any trouble one proves that they actually form a two-sided approximate identity.

<u>A closed linear subspace of</u> E <u>is</u> U-<u>invariant if and only if it is a submodule of</u> E. <u>A continuous linear map</u> $E \longrightarrow E$ <u>commutes with every</u> U_x <u>if and only if it is a module homomorphism</u>.

Conversely, a <u>Banach</u> L(G)-<u>module</u> is a Banach space E provided with a bilinear map $* : L(G) \times E \longrightarrow E$ such that $f * (g * \xi) = (f * g) * \xi$ $(f, g \in L(G); \xi \in E)$ and such that (ii) holds. Such a Banach L(G)-module is <u>continuous</u> (or <u>essential</u>) if (iv) is also valid. If E is any Banach L(G)-module, the closed linear hull of $\{f * \xi \mid f \in L(G); \xi \in E\}$ is the largest continuous submodule of E.

In any continuous Banach L(G)-module E, formula (iii) defines a continuous representation U of G that fulfils (i) : we have a <u>one-to-one correspondence between continuous</u> L(G)-<u>modules and continuous representations of</u> G.

2 - <u>The structure of</u> L(G) <u>for compact</u> G.

In this chapter we assume that G itself is compact and p-free. We work with the left Haar measure m for which m(G) = 1.

Let $\mathcal{N}_o$ denote the set of all normal open subgroups of G. It was proved by Pontryagin that every element of $\mathcal{N}$ contains an element of $\mathcal{N}_o$. It follows that the $u_H (H \in \mathcal{N}_o)$ form a left approximate identity for L(G).

For any Banach space F and for $n \in \mathbb{N}$ we consistently view F^n as a Banach space under the max-norm :

$$\|(\xi_1, \ldots, \xi_n)\| = \max_i \|\xi_i\| \quad (\xi_1, \ldots, \xi_n \in F).$$

If D is a closed linear subspace of a Banach space E, a <u>projection of</u> E <u>onto</u> D is a linear $P : E \longrightarrow E$ for which

(1) $\|P\| \leq 1$.

(2 $P(E) \subset D$.

(3) $Px = x$ for all $x \in D$.

The following lemma is well-known.

2.1. <u>Lemma</u>. <u>Let</u> D <u>be a linear subspace of</u> K^n . <u>Then as a normed vector space</u>, D <u>is isomorphic to some</u> K^m . <u>There exists a projection of</u> K^n <u>onto</u> D.

The same reasoning used in the classical theory for representations in Banach spaces now leads to

2.2. <u>Lemma</u>. <u>Let</u> U <u>be a continuous representation of</u> G <u>in</u> K^n . <u>Let</u> D <u>be a</u> U-<u>invariant linear subspace of</u> K^n . <u>Then there exists a projection</u> P <u>of</u> K^n <u>onto</u> D <u>that commutes with every</u> U_x .

Every $\xi \in K^n$ for which $\|\xi\| \leq 1$ determines in a natural way a $\bar{\xi} \in k^n$. Consequently, a K-linear $A : K^n \longrightarrow K^n$ with $\|A\| \leq 1$ determines a k-linear $\bar{A} : k^n \longrightarrow k^n$ by

$$\bar{A}(\bar{\xi}) = \overline{A\xi} \qquad (\xi \in K^n, \|\xi\| \leq 1).$$

In particular, a representation U of G in K^n induces a representation $\bar{U} : x \longmapsto \bar{U}_x$ in k^n . The following lemma can be proved as an application of lemma 2.2.

2.3. <u>Lemma</u>. <u>Let</u> U <u>be a continuous representation of</u> G <u>in</u> K^n . <u>Then</u> U <u>is irreductible if and only if</u> $\bar{U}$ <u>is irreducible</u>.

A useful consequence :

2.4. <u>Lemma</u>. <u>Let</u> U,V <u>be continuous representations of</u> G <u>in</u> K^n <u>and in a Banach space</u> E, <u>respectively</u>. <u>Suppose</u> U <u>to be irreducible.</u> <u>If</u> $T : K^n \longrightarrow E$ <u>is a linear map such that</u> $TU_x = V_x T (x \in G)$, <u>then</u>

$$\|T\xi\| = \|T\| \, \|\xi\| \qquad (\xi \in K^n).$$

If U,V are representations of G in non-trivial Banach spaces E,F, respectively, we say that they are equivalent if there exists a surjective linear $T : E \longrightarrow F$ with $TU_x = V_x T$ for all $x \in G$ and with

$$\|T\xi\| = \|T\| \, \|\xi\| \qquad (\xi \in K^n).$$

Similarly, two non-zero Banach L(G)-modules, E and F, are called equivalent if there exists a surjective module isomorphism $T : E \longrightarrow F$ such that

$$\|T\xi\| = \|T\| \, \|\xi\| \qquad (\xi \in K^n).$$

In either case, if T is an isometry we speak of isomorphism rather than equivalence.

For every $H \in \mathcal{N}_o$, $u_H * L(G)$ is a two-sided ideal in L(G) consisting of all functions $G \longrightarrow K$ that are constant on the cosets of H. Thus, $u_H * L(G)$ is finite-dimensional, and, as a normed vector space, is isomorphic to $K^{[G:H]}$. We have already observed that the $u_H (H \in \mathcal{N}_o)$ form a left approximate identity in L(G). Then $\sum \{u_H * L(G) \mid H \in \mathcal{N}_o\}$ is dense in L(G).

In the set of all central idempotent elements of L(G) we introduce an ordering $\leq$ by

$$e_1 \leq e_2 \text{ if } e_1 * L(G) \subset e_2 * L(G).$$

Let $\mathcal{E}$ be the set of all minimal non-zero central idempotents. The elements of $\mathcal{E}$ are linearly independent and have norm 1. Then for every $H \in \mathcal{N}_o$ only finitely many elements of $\mathcal{E}$ are $\leq u_H$. One proves easily that $u_H = \sum \{e \in \mathcal{E} : e \leq u_H\}$. For every $e \in \mathcal{E}$ there exists an $H \in \mathcal{N}_o$ with $\|u_H * e - e\| < 1$; then $e * u_H \neq 0$. By the minimality of e it follows that $e = e * u_H$, so

$$e * L(G) = e * u_H * L(G) = u_H * e * L(G) \subset u_H * L(G).$$

By lemma 2.1, $e * L(G)$ is isomorphic to some K^n.

We need one more definition before we can formulate the structure theorem for L(G). Let $(A_i)_{i \in I}$ be a family of Banach spaces. We set

$$\bigoplus_{i\in I} A_i = \{x \in \prod_{i\in I} A_i \mid \text{if } \varepsilon > 0, \text{ then } \|x_i\| > \varepsilon \text{ for only finitely many } i\}.$$

In a natural way, $\bigoplus_{i\in I} A_i$ is a Banach space under the norm defined by $\|x\| = \sup_{i\in I} \|x_i\|$. If all the A_i are Banach algebras (or L(G)-modules), $\bigoplus_{i\in I} A_i$ becomes a Banach algebra (an L(G)-module).

It is now relatively easy to prove the following analog to a classical structure theorem for finite groups.

2.5. Theorem. *For* $e \in \mathcal{E}$ *set* $L(G)_e = e * L(G)$. *As a Banach space,* $L(G)_e$ *is isomorphic to some* K^n. *Every* $L(G)_e$ *is a two-sided ideal in* L(G). *If* $f \in L(G)$, *then* $f = \sum_{e\in\mathcal{E}} e * f$ *and* $\|f\| = \sup_{e\in\mathcal{E}} \|e * f\|$. *The formula*

$$(Sf)_e = e * f \qquad (e \in \mathcal{E}\ ;\ f \in L(G))$$

yields an isomorphism of Banach algebras

$$S : L(G) \longrightarrow \bigoplus_{e\in\mathcal{E}} L(G)_e$$

For every $X \subset \mathcal{E}$, $\{f \in L(G) \mid e * f = 0 \text{ for every } e \in X\}$ *is a closed two-sided ideal in* L(G) ; *all closed two-sided ideals of* L(G) *are of this form. The minimal non-zero two-sided ideals are just the* $L(G)_e$.

In the following lines, instead of "minimal non-zero left ideal of L(G)" we simply say "minimal ideal". $L(G)_e$, being a finite-dimensional left ideal of L(G), contains minimal ideals. As in the purely algebraic representation theory of finite groups, each $L(G)_e$ is a sum of minimal ideals ; every minimal ideal lies in some $L(G)_e$; and two minimal ideals are isomorphic (as L(G)-modules) if and only if they are contained in the same $L(G)_e$.

Let n(e) be the dimension (as a K-vector space) of a minimal ideal that is contained in $L(G)_e$. It follows from lemma 2.1 that for every $e \in \mathcal{E}$ we can choose an L(G)-module structure on $K^{n(e)}$, so that the resulting module $I^{(e)}$ is isomorphic to the minimal ideals that lie in $L(G)_e$. The module structure of $I^{(e)}$ induces a continuous representation $W^{(e)}$ of G in $K^{n(e)}$

The following generalization of 2.5 is not hard to prove.

2.6. Theorem. Let U be a continuous representation of G in a Banach space E ; let $*$ be the corresponding module operation $L(G) \times E \longrightarrow E$. For $e \in \mathcal{E}$ set $E_e = \{e * \xi \mid \xi \in E\}$. Each E_e is a closed submodule of E. The formula

$$(S\xi)_e = e * \xi \qquad (\xi \in E)$$

yields an isomorphism of Banach L(G)-modules

$$S : E \longrightarrow \bigoplus_{e \in \xi} E_e .$$

The restriction of U to E_e is called the e-homogeneous part of U. If $E_e = E$, then U itself is called e-homogeneous. (Observe that always $(E_e)_e = E_e$).

Let U be an irreducible continuous representation of G in a Banach space E. Choose $\xi \in E$, $\xi \neq 0$. There must exist an $e \in \mathcal{E}$ with $e * \xi \neq 0$. As $L(G)_e$ is a sum of minimal ideals, there must exist a minimal ideal $D \subset L(G)_e$ with $D * \xi \neq (0)$. Applying lemma 2.4 (consider the map $f \longmapsto f * \xi$ $(f \in D)$) we get

2.7 Corollary. Every irreducible continuous representation of G is equivalent to one of the $W^{(e)}$. In particular, it is finite dimensional.

Now let F be any Banach space. Every $n \times n$-matrix induces in a natural way a map $F^n \longrightarrow F^n$. Thus, every $W^{(e)}$ induces a continuous e-homogeneous representation $W^{(e)} \otimes Id_F$ in $F^{n(e)}$. (To explain the notation we observe that F^n is linearly isometric to $K^n \otimes_K F^n$). Together with Theorem 2.6 the following gives a complete classification of all continuous representations of G.

2.8. Theorem. Every e-homogeneous continuous representation of G is isomorphic to $W^{(e)} \otimes Id_F$ for some Banach space F. The given representation determines F up to an isomorphism of Banach spaces.

For $e \in \mathcal{E}$ let $\mathfrak{A}_e$ be the set of all linear module homomorphisms $I^{(e)} \longrightarrow I^{(e)}$. Obviously, $\mathfrak{A}_e$ is a K-Banach algebra. But it follows from lemma 2.4 that $\mathfrak{A}_e$ even is a valued skew field containing K. It turns out that every commutative subfield of $\mathfrak{A}_e$ is obtainable by adjunction of roots of 1 to K. Hence, if K contains "enough" roots of 1, then $\mathfrak{A}_e = K$.

In a natural way, $I^{(e)}$ becomes a normed vector space over $\mathfrak{A}_e$. As in the algebraic theory, $L(G)_e$ (as an algebra or an L(G)-module) is isomorphic to the algebra of all $\mathfrak{A}_e$-linear maps $I^{(e)} \longrightarrow I^{(e)}$. But this time the isomorphism is also an isometry. It follows that, if G is abelian, then every $L(G)_e$ is a valued field, and L(G) is power-multiplicative. (A Banach algebra A is power-multiplicative if $\|a^n\| = \|a\|^n$ for all $a \in A$ and $n \in N$).

As a Banach space, $I^{(e)}$ is isomorphic to $(\mathfrak{A}_e)^{n(e)}$ for some $n(e) \in N$. It follows that $L(G)_e$ (as a Banach algebra or a Banach L(G)-module) is isomorphic to the algebra of all $n(e) \times n(e)$ matrices with entries form $\mathfrak{A}_e$. Here the norm of a matrix is the maximum of the norms of its entries.

3 - Representations of locally compact groups.

K,k,p,G are as in chapter 1. We assume every element of $\mathcal{N}$ to be p-free.

G is called torsional if every compact subset is contained in a compact subgroup. If G is torsional then so is every closed subgroup and every quotient of G by a closed normal subgroup.

The additive group of a non-trivial valued local field is torsional : for each $n \in \mathbb{N}$, $\{x \mid |x| \leq n\}$ is a compact open subgroup. The multiplicative group is not torsional : if $|x| > 1$, then $\lim |x^n| = \infty$. The general and special linear groups are not torsional. However, the following group G of triangular $m \times m$ matrices

$$G = \{(\alpha_{ij}) \mid \alpha_{ij} = 0 \text{ if } i < j;\ |\alpha_{ii}| = 1 \text{ for all } i\}$$

is torsional : for each $n \in \mathbb{N}$, $H_n = \{(\alpha_{ij}) \in G \mid |\alpha_{ij}| \leq n^{i-j} \text{ for all } i,j\}$ is a compact open subgroup.

We now formulate the main

3.1. Theorem. *Let* G *be torsional and let* $I \subset L(G)$ *be a proper closed two-sided ideal. For every* $f \in L(G)$ *there exists a maximal modular left ideal* $N \supset I$ *such that* $\|f \bmod I\| = \|f \bmod N\|$.

Proof. First, assume that G is compact. Then $L(G) = \bigoplus_{e \in \mathcal{E}} L(G)_e$ where $\mathcal{E}$ is the collection of minimal central idempotents of $L(G)$. (Theorem 2.5).

Then $I = \bigoplus_{e \in \mathcal{D}} L(G)_e$ for some $\mathcal{D} \subset \mathcal{E}$, $\mathcal{D} \neq \mathcal{E}$, and $f = \sum_{e \in \mathcal{E}} e * f$.

Clearly, $\|f \bmod I\| = \max_{e \notin \mathcal{D}} \|f * e\| = \|f * d\|$ for certain $d \notin \mathcal{D}$.

We identify $L(G)_d$ with the algebra of all $n(d) \times n(d)$ matrices over $\mathfrak{A}_d$. (See the end of Chapter 2). There exists a $\xi \in (\mathfrak{A}_d)^{n(d)}$ with $\|(d * f)(\xi)\| = \|d * f\| \, \|\xi\|$. Let $N_d = \{g \in L(G)_d \mid g(\xi) = 0\}$; then $\|d * f \bmod N_d\| = \|d * f\|$. For $e \in \mathcal{E}$, $e \neq \mathcal{D}$ set $N_e = L(G)_e$, and let $N \subset L(G)$ be the closure of $\sum_{e \in \mathcal{E}} N_e$. Then N is a maximal modular left ideal containing I, and $\|f \bmod N\| = \|d * f \bmod N_d\| = \|d * f\| = \|f \bmod I\|$.

Observe that one can make a non-zero $n(d) \times n(d)$ matrix s over $\mathfrak{A}_d$ such that $N_d * s = \{0\}$ and $s * s = s$. (The columns of s are suitable multiples of ξ). We need this remark in the second part of this proof.

For the general case we may assume that f has compact support, so $f = 0$ outside a compact open subgroup H. We have the obvious embedding $L(H) \hookrightarrow L(G)$.

By the foregoing there exists a maximal modular left ideal M of $L(H)$, with identity e_o, for which $M \supset I \cap L(H)$ and $\|f \bmod I \cap L(H)\| = \|f \bmod M\|$, and there exists an idempotent $s \in L(H)$ with $M * s = \{0\}$. By maximality, $M = \{g \in L(H) \mid g * s = 0\}$. Set $J = \overline{L(G) * M + I}$. J is a closed left ideal of $L(G)$, containing I. For all $g \in L(G)$

$$g * e_o - g = \lim_{V \in \mathcal{V}} (g * u_V * e_o - g * u_V) \in \overline{L(G) * M} \subset J,$$

so J is modular. We next prove $J \neq L(G)$.

Let $j \in J \cap L(H)$. Then $(j - j * s) * s = 0$, so $j - j * s \in M$. Also, $j * s \in (\overline{L(G) * M + I}) * s \subset \overline{I * s} \subset I$ and $j * s \in L(H)$, so $j * s \in M$. Therefore, $J \cap L(H) \subset M$, so that $J \neq L(G)$. Trivially, $J \cap L(H) \supset M$, so $J \cap L(H) = M$.

Being a proper modular left ideal, J extends to a maximal modular left ideal N of L(G). By the maximality of M we still have $N \cap L(H) = M$.

By lemma 2.4, the canonical map

$$\rho: L(H)/M \longrightarrow L(G)/N$$

satisfies $\|\rho(\eta)\| = \|\rho\| \ \|\eta\|$ $(\eta \in L(H)/M)$. Using the fact that

$\lim_{V\in\mathcal{N}} \| u_V \bmod M\| = \lim_{V\in\mathcal{N}} \| u_V \bmod N\| = 1$ we see that $\|\rho\| = 1$, so ρ is an isometry. Hence, $\| f \bmod N \| = \|f \bmod M\| = \|f \bmod I \cap L(H)\| \geq$

$\|f \bmod I\| \geq \|f \bmod N\|$.

3.2. Corollary. Let $H \in \mathcal{N}$ and let I be a closed two-sided ideal in L(G). Then the canonical map $L(H)/I \cap L(H) \longrightarrow L(G)/I$ is an isometry.

3.3. Corollary. If G is abelian and if I is a maximal modular ideal of L(G), then L(G)/I is a valued field which is the completion of an algebraic extension of K.

Proof. For every $H \in \mathcal{N}$, $I \cap L(H)$ is a maximal ideal of L(H) of finite codimension, and $L(H)/I \cap L(H)$ is a valued field.

The corollary now follows from the observation that the union of the canonical images of the $L(H)/I \cap L(H)$ $(H \in \mathcal{N})$ is dense in L(G)/I.

3.4. Corollary. For each two-sided closed ideal $I \subset L(G)$ the Banach algebra L(G)/I is reduced ("Spectral synthesis"). In particular, for each $f \in L(G)$ there exists an (algebraically) irreducible continuous representation T of L(G) in some Banach space such that $\|T_f\| = \|f\|$ $(f \in L(G))$. ("The Fourier transformation is an isometry"). For each $x \in G$, $x \neq e$ there exists a continuous irreducible representation U of G in some Banach space such that $U_x \neq I$. ("Gelfand-Raikov Theorem").

The representation space of an irreducible representation of an abelian group may have dimension greater than 1. If K is "big enough" this cannot happen :

3.5. Theorem. Let G be an abelian torsional group and suppose that the equation $\xi^n = 1$ has n distinct roots in K for every $n \in \{[H_2 : H_1] : H_1, H_2 \in \mathcal{N}; H_2/H_1$ cyclic$\}$. Let $G^\wedge$ be the group of all continuous homomorphisms of G into $\{\alpha \in K : |\alpha| = 1\}$, topologized with the compact open topology. Then every maximal modular ideal M of L(G) has codimension 1 and there is an $\alpha_M \in G^\wedge$ such that the homomorphism $L(G) \longrightarrow L(G)/M$ has the form

$$f \longmapsto f(\alpha_M) = \int f(x)\alpha_M(x^{-1})dx \qquad (f \in L(G))$$

The map $M \longrightarrow \alpha_M$ is a homeomorphism of the collection of maximal modular ideals, with the Gelfrand topology, onto $G^\wedge$. The dual group $G^\wedge$ is also torsional and the Fourier transformation $f \longmapsto f^\wedge$, given by

$$f^\wedge(\alpha) = \int f(x)\ \alpha(x^{-1})dx \qquad (f \in L(G))$$

is an isometrical isomorphism of L(G) onto $C_\infty(G^\wedge)$. Finally, the canonical map $G \longrightarrow G^{\wedge\wedge}$ is an isomorphism of topological groups.

Proof. See Corollary 3.4 and [1], 4.3.16 and 5.2.11.

We mention (without proof) a result for not-necessarily torsional groups. Define $B(G) = \{x \in G : U_x = I$ for every continuous irreductible representation U of $G\}$. It is clear from the definition that B(G) is a closed normal subgroup.

3.6. Theorem. B(G) is a discrete torsion-free subgroup of G, and is contained in every open normal subgroup of G. If G is either abelian or discrete or torisonal then $B(G) = \{e\}$. B(G) has a trivial intersection with the center of G.

We end with a

Conjecture : Let G be a locally compact totally disconnected group, such that all elements of $\mathcal{N}$ are p-free, where p is the characteristic of the residue classe field k of K. Then $B(G) = \{e\}$, i.e. G has sufficiently many continuous irreductible representations.

BIBLIOGRAPHY

[1] SCHIKHOF, W.H. Non-archimedean harmonic analysis. (Thesis). Nijmegen, 1967.

[2] SCHIKHOF, W.H. Non-archimedean representations of compact groups. Comp. Math. 23 (1971), 215-232

A.C.M. van ROOIJ and W.H. SCHIKHO
Université de Göttinger