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ON p-ADIC DIFFERENTIAL EQUATIONS I
THE FROBENIUS STRUCTURE OF DIFFERENTIAL EQUATIONS

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Our object is to attach an operation of Frobenius to differential equations. We give no proofs but do give some examples and state some conjectures which convey the point of view which we wish to associate with the general results stated at the end of the article. Our object is also to broaden in certain respects the account of Katz [6] who does not admit a crystalline structure for such simple objects as the exponential function. The section entitled "General Results" was the subject of the actual lecture given at the table ronde sur l'analyse ultrametrique in september 1972. Full proofs will appear elsewhere [4,5] .

Let K be a field complete under a non-archimedean valuation and with algebraically closed residue class field of characteristic p. Let Ω be an algebraically closed, complete, extension field with a sufficiently large valuation group. Let σ be a lifting to Ω of the Frobenius automorphism of the residue class field and we shall assume that σ induces an automorphism of K.

Let G be an $n \times n$ matrix with coefficients in $K(X)$, the field of rational functions in one variable with coefficients in K. Let L denote the differential equation :

(L) $$\frac{d\underline{X}}{dX} = \underline{X}G$$

where $\underline{X}$ denotes an indeterminate n-row vector.

Recall that L is said to have rank $q = q(\beta, L)$ at singular point β of L if q is minimal such that $q + 1 \geqslant 0$ and $(X-\beta)^{q+1}$ G has no pole at β . For each $W \in GL(n, \Omega(X))$ let L_W be the differential equation obtained from L by the change in variable, $\underline{X} \to \underline{X}W$. We define the reduced rank of L at β by

$$q_0(\beta,L) = \text{Inf } q(\beta,L_W)$$

the infimum being over all $W \in GL(n,\Omega(X))$. The definition of $q(\infty,L)$ is obvious.

We define the total multiplicity of singularity of L to be

$$\text{tms}(L) = \sum_\beta (1+q_0(\beta,L))$$

the sum being over all $\beta \in \Omega$.

For $\alpha \in K$, α not a singular point of L, let $\underline{Y}_\alpha$ be a local solution matrix of L and let $\rho(\alpha)$ be the radius of convergence of the pair $(\underline{Y}_\alpha, \det Y_\alpha^{-1})$. If $\rho(\alpha)$ is finite let γ_α be an element of Ω such that

$$|\gamma_\alpha| = \rho(\alpha).$$

For $\alpha \in K$, $\beta \in \Omega$ let $T_{\alpha,\beta}$ denote the mapping

$$x \to \beta x + \alpha .$$

Let ϕ be the semilinear endomorphism of $\Omega[[X]]$ which applies σ to the coefficients and replace X by X^p. (Clearly this choice of $\phi(X)$ is arbitrary, but it may be adequate for our purposes).

If $\rho(\alpha)$ is infinite then the coefficients of $\underline{Y}_\alpha$ lie in $K[[X]]$ (as well as in $K[[X-\alpha]]$) and hence we associate a Frobenius structure with L by putting

$$(\underline{Y}_\alpha^\phi)^{-1}\, \underline{Y}_\alpha = A$$

a matrix with entire coefficients.

If $\rho(\alpha)$ is finite then let R_α be the complement in $D(\alpha,\rho(\alpha)^+)$ of the union of all disks $D(w,\rho(\alpha)^-)$ such that w is a singular point of L. We say that L has a weak Frobenius structure at α if there exists a differential equation $L^{(1)}$,

$$\frac{d\underline{X}}{dX} = \underline{X}G^{(1)}$$

where $G^{(1)}$ is again an n x n matrix with coefficients in K(X) such that the following two conditions are satisfied

(i) $\mathrm{tms}(L^{(1)}) = \mathrm{tms}(L)$

(ii) There exists a solution matrix $\underline{Y}_\alpha^{(1)}$ of $L^{(1)}$ at α such that the matrix function

$$A = ((\underline{Y}_\alpha \circ T_{\alpha,\gamma(\alpha)})^{\Phi} \circ T^{-1}_{\alpha,\gamma(\alpha)})^{-1}\, \underline{Y}_\alpha^{(1)}$$

and the scalar function det A^{-1} are analytic elements (in the sense of Krasner) on R_α. (Note that $\underline{Y}_\alpha$ being a function on $D(\alpha, \rho(\alpha)^-)$, $\underline{Y}_\alpha \circ T_{\alpha,\gamma(\alpha)}$ is a function on $D(0,1^-)$ and hence the coefficients lie in $\Omega[[X]]$. Thus Φ is well defined and so $(\underline{Y}_\alpha \circ T_{\alpha,\gamma(\alpha)})^{\Phi}$ is a function on $D(0,1^-)$. Thus finally

$$((\underline{Y}_\alpha \circ T_{\alpha,\gamma(\alpha)})^{\Phi} \circ T^{-1}_{\alpha,\gamma(\alpha)})(X) = Y^{\sigma}_\alpha(\alpha^\sigma + \gamma(\alpha)^\sigma \left\{\frac{X-\alpha}{\gamma(\alpha)}\right\}^p)).$$

a function on $D(\alpha, \rho(\alpha)^-)$.

The purpose of condition (i) is to avoid (in so far as possible) the trivial choice , A = 1, or the equally trivial choice

$$\underline{Y}_1 = (\underline{Y}_\alpha \circ T_{\alpha,\gamma(\alpha)})^{\Phi_1} \circ T^{-1}_{\alpha,\gamma(\alpha)}$$

where Φ_1 is the semilinear transformation of $\Omega[[X]]$ obtained by applying σ to coefficients and replacing X by $\Psi_1(X)$, an element of $X\Omega[[X]]$ which is analytic on the complement in $D(0,1^+)$ of a finite number of residue classes and which has the further property that $\Phi_1(X) - X^p$ is uniformly bounded on this complement by some real which is strictly less than unity.

It can be shown that the existence and choice of $\underline{Y}_\alpha^{(1)}$ does not depend upon the choice of $\gamma(\alpha)$ as representative of $\rho(\alpha)$ in the value group. Clearly $\underline{Y}_\alpha^{(1)}$ is at best unique as a coset of GL(n,K(X)). It can also be shown that if L has weak Frobenius structure at α then the same holds at each point of R_α.

We give some elementary examples of Frobenius structure for n = 1. For this purpose let π be chosen such that $\pi^{p-1} = -p$. It is clear that $\pi^{\sigma-1}$ is a p-1 root of unity.

1) $L = \frac{d}{dX} - 1$, $\alpha = 0$, $\underline{Y}(X) = \exp X$, $\underline{Y}^{(1)}(X) = \exp(\pi^{\sigma-1}X)$

2) $L = \frac{d}{dX} - \frac{a}{1+X}$, $\alpha = 0$, $\underline{Y}(X) = (1+X)^a$, $\underline{Y}^{(1)}(X) = (1+X)^{a'}$

where $a' = p\,a$ if $a \in \mathbb{Z}_p$ (here $A = 1$).

$= \pi^{\sigma-1}a$ ortherwise

(of course a' may be changed by addition of any element of $\mathbb{Z}$).

Conjecture 1.

L has weak Frobenius structure at each nonsingular point α of L.

This conjecture implies the existence of an infinite sequence of differential equations $\{L^{(i)}\}_{i=0}^{\infty}$, $L^{(0)} = L$, such that for each i, $L^{(i+1)}$ affirms the existence of a weak Frobenius structure of $L^{(i)}$. Let us call such a sequence a Frobenius sequence. We make no precise conjecture as to the uniqueness of such a sequence.

Definition. L will be said to have a strong Frobenius structure if L lies in a periodic Frobenius sequence.

Example 1 and example 2 with $a \notin \mathbb{Z}_p$ are illustrations of differential equations with strong Frobenius structure. For example 2 with $a \in \mathbb{Z}_p$, L has strong Frobenius structure if and only if $a \in \mathbb{Q}$. We are thus led to a further conjecture.

Conjecture. If none of the irrational exponents of L lie in $\mathbb{Z}_p$ then L has a strong Frobenius structure.

Note : At regular singular points the exponents require no explanations, for irregular singular points we refer to the numbers $\rho_{i,h-1}$ of theorem II of Turrittin [7, p. 47], i.e. the exponents as given by the normal solutions of Thome and the subnormal solutions of Fabry. An account is also given in Chap. V of Wasow [8].

It seems reasonable to believe that it will be possible to check these conjectures for the case $n = 1$. Of course the phenomenon of strong Frobenius structure was first recognized in the case of differential equations arising from variation of de Rahm cohomology [1,2] and there one has the further property that A is analytic not only in R_α but also in a neighborhood of R_α , i.e. a set N, containing R_α for which there exists a real number $b \in (0,1)$ such that for each x in the complement of N the distance from x to N lies outside the interval $(b\rho(\alpha), b^{-1}\rho(\alpha))$.

To the best of our knowledge no example of these conjectures has been verified for any irreductible differential equation with $n > 1$ except in cases arising from algebraic geometry. However the methods of our earlier article [3] permit us to obtain some information in the case of hypergeometric-E functions. We consider a non-trivial example.

3) Bessel Equation

The differential equation

(1) $$X^2Y'' + XY' + X^2Y = 0$$

has two singular points, the origin which is a regular singularity (with zero exponents) and the point at infinity which is an iregular singular point with trivial local monodromy. The classical Bessel function

$$J_0(X) = \sum_{j=0}^{\infty} (-1)^j (X/2)^{2j}/j!^2$$

is the unique solution of (1) which is holomorphic at the origin. A second solution may be given in terms of log X. Since J_0 converges p-adically ($p \neq 2$) for $|X| < |\pi|$, (π as defined for examples 1,2 above) let

$$F(X) = J_0(\pi X)$$

so that F has unity as radius of (p-adic) disk of convergence. Thus F is a solution of

(1') $$X^2Y'' + XY' + \pi^2 X^2Y = 0.$$

A second solution of equation (1') is given by

$$v = F \,.\, \log X + \tilde{F} \tag{2}$$

where

$$\tilde{F}(X) = -2 \sum_{j=1}^{\infty} B_j \, S_j \, X^{2j} \ ,$$

B_j being the coefficient of X^{2j} in F and S_j being the sum,

$$S_j = \sum_{i=1}^{j} i^{-1} \ .$$

Thus $\tilde{F}$ is an analytic but unbounded function on $D(0,1^-)$. Let

$$f(X) = F(X)/F(X^p) \tag{3}$$

$$\eta(X) = F'/F \tag{4}$$

$$\tau(X) = \log X + \tilde{F}/F = v/F \tag{5}$$

Let $\underline{O}_1$ be the valuation ring of $\mathbb{Q}_p(\pi)$. Then by the methods of [3] we obtain the following results which we state without proof.

<u>Lemma</u> 1. <u>The functions</u> f <u>and</u> η <u>are analytic elements on</u> $D(0,1^+)$.

$$\tau(X^p) \equiv p\,\tau(X) \bmod p\, \underline{O}_1 [[X]] \ .$$

Thus certain functions defined in terms of local solutions of (1') at the singular point 0 have "global" properties. We extend this to disks $D(\alpha,1^-)$ containing no singular point. We replace (1') by

$$\frac{d}{dX}(Y_1 \ , \ Y_2) = (Y_1 \ , \ Y_2) \begin{bmatrix} 0 & -\pi^2 \\ 1 & -1/X \end{bmatrix} . \tag{6}$$

Let K be the maximal unramified extension of $\mathbb{Q}_p(\pi)$ and let σ be the relative Frobenius automorphism (thus $\pi^{\sigma-1} = 1$). For $\alpha \in K \cap C(0,1)$, let U_α be the two dimensional K space of formal solutions (in $K[[X-\alpha]]$) of (6). It can be shown that each element of U_α is analytic in $D(\alpha,1^-)$. We define Φ on $K[[X-\alpha]]$ by $\Phi(X-\alpha) = X^p - \alpha^\sigma$, Φ coincides with σ on K.

Lemma 2. There exists an element (u_α, u'_α) of U_α such that u_α is bounded on $D(\alpha,1^-)$. This solution is unique up to a factor in K. If properly normalized then

$$(7) \qquad u_\alpha/u_\alpha^\Phi = f$$

$$(8) \qquad u'_\alpha/u_\alpha = \eta \ .$$

Furthermore the second solution (v_α, v'_α) of U_α may be chosen such that $\tau_\alpha = v_a/u_\alpha$ has the property

$$(9) \qquad \tau_\alpha^\Phi = p\,\tau_\alpha \mod p\,\underline{O}_K[[X-\alpha]] \ .$$

These results support (but do not prove in this case) the conjecture that equation (6) has a strong Frobenius structure. We further conjecture in this case that the periodic Frobenius sequence has indeed period 1 and that the matrix A is analytic in $D(0,1^+)$ (i.e. there is no need to discard the disk $D(0,1^-)$). If this is the case then A must have the form given by

$$(10) \qquad \underline{Y}^\Phi A = \begin{bmatrix} 1 & 0 \\ b & p \end{bmatrix} \underline{Y}$$

(with $b \in Q_p(\pi)$),

where Y denotes the solution matrix

$$(11) \qquad Y = \begin{bmatrix} F & F' \\ v & v' \end{bmatrix}$$

at the origin, it being understood that $\Phi(\log X) = p \log X$. It follows from lemma 1 that b lies in $p\,\underline{O}_1$ (if it exists).

The conjecture in this case reduces to showing that b may be chosen such that

$$(12) \qquad b\, F^{\phi} \,.\, F^{\phi} + p\, \tilde{F}^{\phi}\, F^{\phi} - \tilde{F}^{\phi}\, F \in H(D(0,1^{+})),$$

the ring of analytic elements on $D(0,1^{+})$.

This conjecture is far stronger the congruence of Lemma 1 which is equivalent to

$$p\, \tilde{F}\, F^{\phi} - \tilde{F}^{\phi}\, F \in p\, F\, F^{\phi}\ \underline{O}_1[[X]] \ .$$

4) Legendre Normal form.

It may be instructive to compare example 3 with the well known case

$$X(1-X)Y'' + (1-2X)Y' - \frac{1}{4}\, Y = 0$$

which is the differential equation satisfied by the period of the first kind in the generic fiber of the family of elliptic curves given by the Legendre normal form. Here the unique solution analytic at the origin is given by

$$G(X) = F(\tfrac{1}{2},\tfrac{1}{2};\ 1,X) = \sum_{j=0}^{\infty} (\tfrac{1}{2})_j/j!)^2\, X^2 \ .$$

A second solution is given by

$$w = \tilde{G} + G \log X$$

where $\tilde{G}$ is analytic at the origin and normalized (non-naturally) by the condition $\tilde{G}(0) = 0$. Here G and $\tilde{G}$ are both analytic on $D(0,1^{-})$ but $\tilde{G}$ is unbounded on that disk. In this case Lemma 1 must be modified as G'/G and G/G^{ϕ} are analytic not on $D(0,1^{+})$ but rather only in the Hasse domain defined by the condition

$$\left| \sum_{j=0}^{(p-1)/2} (\tfrac{1}{2})_j/j!)^2 X^2 \right| \geqslant 1.$$

(This can be improved for G/G^{ϕ} if ϕ is chosen to be the Deligne-Tate mapping). In this case the existence of Frobenius structure for the differential equation is deduced by other methods but it is known [3 §.8] that if we set

$$\underline{Y} = \begin{bmatrix} G & G' \\ w & w' \end{bmatrix}$$

then equation (10) holds with A analytic everywhere except in a small neighborhood of 1 provided $b = \log 16^{p-1}$.

General Results.

It is known that if a differential equation, L, has strong Frobenius structure at a point α then

I. If $\beta \in D(\alpha, \rho(\alpha)^{+})$ but $D(\beta, \rho(\alpha)^{-})$ contains no singularity of L then the local solution matrix of L at β has $\rho(\alpha)$ as radius of convergence.

II. The solution matrix $\underline{Y}_{\alpha}$ has logarithmic growth in $D(\alpha, \rho(\alpha)^{-})$, i.e. if u is a coefficient of $\underline{Y}_{\alpha}$ then as $r \to \rho(\alpha)$,

$$|u|_{\alpha}(r) = 0(1/|\log(r/\rho(\alpha))|^{n-1}).$$

It is now known that these statements are valid without any hypothesis concerning existence of Frobenius structure. Full details will appear elsewhere. We explain the idea of the proof.

With no loss in generality one may take $\alpha = 0$, $\rho(\alpha) = 1$ and one may replace the system, by an nth order linear differential equation in one unknown, Y, with coefficients $E_0 = K(X)$. We define a sequence $\left\{B_m^{(\nu)}\right\}_{m \in \mathbb{Z}, \nu \in (0,n)}$ in E_0 by the relation

$$D^m/m! = \sum_{\nu=0}^{n-1} B_m^{(\nu)} D^{\nu} \mod L.$$

The general solution at x is then given by

$$y(x+z) = \sum_{m=0}^{\infty} z^m \sum_{\nu=0}^{n-1} B_m^{(\nu)}(x)\, y^{(\nu)}(X)$$

$(Y^{(\nu)}$ means ν^{th} derivative, but $B_m^{(\nu)}$ need not be ν^{th} derivative of $B_m^{(0)}$).

For the proof of I, we use the hypothesis to show that for each $\nu \in [0,n)$, and each $r < 1$,

$$|B_m^{(\nu)}|_0(1)\ r^m \to 0$$

as $m \to \infty$. Assertion I then follows from the maximum modulus theorem.

For the proof of II we note that the assertion is equivalent to the asymptotic estimate.

$$|B_m^{(\nu)}|_0(1) = O(m^{n-1})$$

for $\nu \in [0,n)$ as $m \to \infty$. Thus the question for solutions in $D(0,1^-)$ may be replaced by the corresponding question for solutions in $D(\alpha,1^-)$ where α may be chosen arbitrarily in $D(0,1^+)$ provided L has no singularities in $D(\alpha,1^-)$. The main idea is to let $\alpha = t$, where t mod $D(0,1^-)$ is transcendantal over the residue class. field of K. At the same time let E be the completion of E_0 under the Gauss norm $f \to |f|_0(1)$. The elements of E may be viewed as functions on $D(t,1^-)$. By the method outlined above we reduce II to the case of $\alpha = t$, $\rho(\alpha) = 1$, L defined over E. The proof is then completed by an induction argument, the case n = 1 being trivial and the reduction argument being given by the following irreducibility lemma : $(D = \frac{d}{dX})$.

An irreducible element $\tilde{L}$ of $E[D]$ can have no solution which is unbounded in $D(t,1^-)$.

In [4] we prove a stronger result of the same type :

Let $\ker_L$ denote the solutions at t of an irreducible element L of $E[D]$. The elements of $\ker_L$ are either all analytic and bounded on $D(t,1^-)$, or all have the same radius of convergence $\rho < 1$ and are bounded on $D(t,\rho^-)$. For further results of this type see [5, Corollary 3]. In that article we also show that the dimension of space of solutions analytic and bounded in $D(t,1^-)$ is not less than the dimension of the space of solutions bounded and analytic on $D(0,1^-)$.

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