

MÉMOIRES DE LA S. M. F.

A. C. ZAAANEN

The linear modulus of an integral operator

Mémoires de la S. M. F., tome 31-32 (1972), p. 399-400

<http://www.numdam.org/item?id=MSMF_1972__31-32__399_0>

© Mémoires de la S. M. F., 1972, tous droits réservés.

L'accès aux archives de la revue « Mémoires de la S. M. F. » (<http://smf.emath.fr/Publications/Memoires/Presentation.html>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

THE LINEAR MODULUS OF AN INTEGRAL OPERATOR

by

A. C. ZAAZEN

Given the measure space (X, Λ, μ) , we denote by $M(X, \mu)$ the real vector space of all real (finite valued) μ -measurable functions on X (functions differing only on a set of μ -measure zero are identified). We assume, for simplicity, that the measure μ is (totally) σ -finite. The space $M(X, \mu)$ is partially ordered by defining that, for functions f and g in $M(X, \mu)$, we write $f \leq g$ whenever $f(x) \leq g(x)$ holds μ -almost everywhere on X . As well-known, $M(X, \mu)$ is a Dedekind complete Riesz space (espace de Riesz complètement réticulé).

Given the Riesz spaces L and M , the linear operator T from L into M is called positive whenever $Tf \geq 0$ for all $f \geq 0$, and T is called order bounded whenever $T = T_1 - T_2$ with T_1 and T_2 positive. The real vector space $\mathcal{L}_b(L, M)$ of all order bounded linear operators from L into M is partially ordered by defining that $T_1 \leq T_2$ means that $T_2 - T_1$ is positive. It is an important result due to F. Riesz and L.V. Kantorovitch that, for M Dedekind complete, the space $\mathcal{L}_b(L, M)$ is a Dedekind complete Riesz space with respect to the thus introduced partial ordering (cf. theorem VIII.2-1 in B.Z. Vulikh's book [3]). It follows that for any $T \in \mathcal{L}_b(L, M)$ the supremum $|T| = Tv(-T)$ is again an element of $\mathcal{L}_b(L, M)$. The operator $|T|$ is called the linear modulus of T .

Let (X, Λ, μ) and (Y, Γ, ν) be σ -finite measure spaces, and let $M(X, \mu)$ and $M(Y, \nu)$ be the corresponding Riesz spaces of real measurable functions (as defined above). Furthermore, let $T(x, y)$ be a real $(\mu \times \nu)$ -measurable function on $X \times Y$, and let $\text{dom } (T)$ be the set of all $f \in M(Y, \nu)$ for which

$$g(x) = \int_Y |T(x, y) f(y)| \, d\nu \in M(X, \mu)$$

holds. The set $\text{dom } (T)$ is an order ideal in the Riesz space $M(Y, \nu)$, and so $\text{dom } (T)$ is a Dedekind complete Riesz space in its own right. It follows easily that the integral operator T , defined by :

$$T : f \mapsto \int_Y T(x, y) f(y) \, d\nu(y),$$

is an order bounded linear operator from the Riesz space $\text{dom } (T)$ into the Riesz space $M(X, \mu)$. It is an obvious conjecture that the linear modulus $|T|$ of T is the integral operator with kernel $|T(x, y)|$. The conjecture is true ; the proof is not trivial. The proof presented in the book "Integral Operators in Spaces of Summable Functions" by M. A. Krasnoselskii and three other authors [2] in theorem

4.2 seems to be incorrect. Integral operators of the kind as introduced above were also considered in a paper by N. Aronszajn and P. Szeptycki [1], but these authors did not consider the linear modulus.

The result mentioned here, together with several other results about the linear modulus, will be published in a joint paper with W. A. J. Luxemburg in the Proceedings of the Dutch Academy of Science, Amsterdam [4].

BIBLIOGRAPHIE

- [1] ARONSZAJN (N.) and SZEPTYCKI (P.). - On general integral transformations, Math. Annalen 163, 127-154 (1966).
- [2] KRASNOSELSKII (M.A.), ZABREIKO (P.P.), PUSTYLNİK (E.I.) and SOBOLEVSKII (P.E.) Integral Operators in Spaces of Summable Functions, Moscow (1966).
- [3] VULIKH (B.Z.). - Introduction to the theory of partially ordered spaces, Moscow (1961), English translation, Groningen (1967).
- [4] LUXEMBURG (W.A.J.) and ZAAZEN (A.C.). - The linear modulus of an order bounded linear transformation, Proc. Acad. of Sci. Amsterdam, 74, 422-447 (1971).

Mathematics Institute
University of Leiden
LEIDEN (Pays-Bas)
