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THE LINEAR MODULUS OF AN INTEGRAL OPERATOR

by

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Given the measure space (X, Λ, μ), we denote by $M(X, \mu)$ the real vector space of all real (finite valued) μ-measurable functions on X (functions differing only on a set of μ-measure zero are identified). We assume, for simplicity, that the measure μ is (totally) σ-finite. The space $M(X, \mu)$ is partially ordered by defining that, for functions f and g in $M(X, \mu)$, we write $f \leqslant g$ whenever $f(x) \leqslant g(x)$ holds μ-almost everywhere on X. As well-known, $M(X, \mu)$ is a Dedekind complete Riesz space (espace de Riesz complètement réticulé).

Given the Riesz spaces L and M, the linear operator T from L into M is called positive whenever $Tf \geqslant 0$ for all $f \geqslant 0$, and T is called order bounded whenever $T = T_1 - T_2$ with T_1 and T_2 positive. The real vector space $\mathcal{L}_b(L, M)$ of all order bounded linear operators from L into M is partially ordered by defining that $T_1 \leqslant T_2$ means that $T_2 - T_1$ is positive. It is an important result due to F. Riesz and L.V. Kantorovitch that, for M Dedekind complete, the space $\mathcal{L}_b(L, M)$ is a Dedekind complete Riesz space with respect to the thus introduced partial ordering (cf. theorem VIII.2-1 in B.Z. Vulikh's book [3]). It follows that for any $T \in \mathcal{L}_b(L, M)$ the supremum $|T| = T \vee (-T)$ is again an element of $\mathcal{L}_b(L, M)$. The operator $|T|$ is called the linear modulus of T.

Let (X, Λ, μ) and (Y, Γ, ν) be σ-finite measure spaces, and let $M(X, \mu)$ and $M(Y, \nu)$ be the corresponding Riesz spaces of real measurable functions (as defined above). Furthermore, let $T(x, y)$ be a real $(\mu \times \nu)$-measurable function on $X \times Y$, and let dom (T) be the set of **all** $f \in M(Y, \nu)$ for which

$$g(x) = \int_Y |T(x,y)\, f(y)|\, d\nu \in M(X, \mu)$$

holds. The set dom (T) is an order ideal in the Riesz space $M(Y, \nu)$, and so dom (T) is a Dedekind complete Riesz space in its own right. It follows easily that the integral operator T, defined by :

$$T : f \to \int_Y T(x,y)\, f(y)\, d\nu(y) ,$$

is an order bounded linear operator from the Riesz space dom (T) into the Riesz space $M(X, \mu)$. It is an obvious conjecture that the linear modulus $|T|$ of T is the integral operator with kernel $|T(x,y)|$. The conjecture is true ; the proof is not trivial. The proof presented in the book "Integral Operators in Spaces of Summable Functions" by M. A. Krasnoselskii and three other authors [2] in theorem

4.2 seems to be incorrect. Integral operators of the kind as introduced above were also considered in a paper by N. Aronszajn and P. Szeptycki [1], but these authors did not consider the linear modulus.

The result mentioned here, together with several other results about the linear modulus, will be published in a joint paper with W. A. J. Luxemburg in the Proceedings of the Dutch Academy of Science, Amsterdam [4].

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