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CORRIGENDA: 'ON SYSTEMS OF LINEAR INEQUALITIES'

by Masami Fujimori

Abstract. — There are two mistakes in the referred paper. One is ridiculous and one is significant. But none is serious.

Résumé (*Corrigenda : « Sur certains systèmes d'inégalités linéaires »*)
Il y a deux erreurs dans l'article mentionné. L'une est ridicule, l'autre est significative. Mais aucune n'est sérieuse.

There are two mistakes in the paper [1]. One is ridiculous and one is significant. But none is serious.

The principal error: the author has overlooked the basic fact that a linear inequality with coefficients in an extension field is practically equivalent to a system of n linear inequalities with coefficients in the ground field, where n is the (local) extension degree of the fields. Hence the discussion of the second section of the paper requires 'local Galois stability' of systems of linear inequalities at each place of the field. Accordingly, when defining the category of systems of linear inequalities in the first section, we should demand 'local Galois invariance' of filtrations at all places of the field.

Specific corrections follow:

• Lemma 1.9 should be deleted since it is wrong; indeed its proof uses $M(W/\operatorname{Im} f) \geq 0$, which need not be. The only place where this lemma is

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used is during the proof of Lemma 1.18. Consequently, its proof should be replaced by the one below.

Proof. — What has to be proved is that the direct sum of objects in $\mathcal{C}_0^{\mathrm{ss}}$ is semi-stable.

Let V and W be any objects in $\mathcal{C}_0^{\mathrm{ss}}$ and let S be any subobject in $\mathcal{C}$ of $V \oplus W$. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & V \oplus W & \longrightarrow & (V \oplus W)/V & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle i} & & \\ 0 & \to & S \cap V & \longrightarrow & S & \longrightarrow & (S+V)/V & \longrightarrow & 0. \end{array}$$

We endow $(S+V)/V$ with the quotient filtration induced by that of S. The inclusion $i\colon (S+V)/V \to (V \oplus W)/V$ is filtered (though not necessarily strict). From Lemma 1.3, we see

$$M(S)\dim_k S = M(S \cap V)\dim_k(S \cap V) + M\big((S+V)/V\big)\dim_k\big((S+V)/V\big).$$

Since $S \cap V$ is a suboject of V which is assumed to be semi-stable, we get

$$M(S \cap V) \leq M(V).$$

On the other hand, we have $(V \oplus W)/V \simeq W$ as filtered vector spaces. Using Lemma 1.8 and the semi-stability of W, we obtain

$$M\big((S+V)/V\big) \leq M(\operatorname{Im} i) \leq M(W).$$

Due to the fact that $M(V) = M(W)(= 0)$, and thanks to the equality in Definition 1.10

$$\begin{aligned} M(S)\dim_k S &\leq M(V \oplus W)\dim_k(S \cap V) + M(V \oplus W)\dim_k((S+V)/V) \\ &= M(V \oplus W)\dim_k S, \end{aligned}$$

which means that $V \oplus W$ is semi-stable. Note that when the weights of V and W are all integral or rational, the weights of $V \oplus W$ are respectively also integral or rational by definition. □

Let $\mathfrak{M}(k)$ be the set of places of the base field k.

• Throughout the paper, we assume the field L is a (finite or infinite) Galois extension of k. Also from the beginning, we fix an extension of each $v \in \mathfrak{M}(k)$ to L once for all and denote it by the same letter v.

Write G_v for the decomposition subgroup at v of $\mathrm{Gal}(L/k)$.

• In Definiton 1.13, at every place $v \in \mathfrak{M}(k)$, the L-filtration $V_v^{\bullet}$ is asked to be (G_v-)*invariant* by the restriction to G_v of the natural action of $\mathrm{Gal}(L/k)$ on $L \otimes_k V$, the main modification in the first section.

The additional requirement does not cause a visible chage in Section 1.

• In Section 2, systems of linear inequalities need to be 'G_v-stable' in the next sense:

DEFINITION 1. — Let $f_1, \dots, f_n \in L \otimes_k V$ be an L-basis and $c(1), \dots, c(n)$ real numbers. The system $(f_1, \dots, f_n; c(1), \dots, c(n))$ is said to be G_v-*stable*, if for any $\tau \in G_v$, when we write

$$\tau(f_i) = \sum_{j=1}^{n} \tau_{ij} f_j \quad (\tau_{ij} \in L),$$

the coefficient τ_{ij} is non-zero only for the indices i, j such that $c(i) \leq c(j)$; *i.e.*,

$$\tau_{ij} = 0 \quad \text{for} \quad c(i) > c(j).$$

REMARK 2. — The system $(f_1, \dots, f_n; c(1), \dots, c(n))$ is G_v-stable if and only if the associated L-filtration on V is G_v-invariant.

- The essence of the second section is a naïve comparison of volumes. For recovery of validity of the whole argument, the following proposition suffices:

PROPOSITION 3. — *Let V^* be the dual space to V over k, K a positive constant, Q a positive parameter, and* $\mathrm{vol}_v = \mathrm{vol}_v(Q)$ *the volume of the bounded open set in $k_v \otimes_k V^*$ defined as*

$$\left|\langle f_i, t\rangle\right|_v < K \cdot Q^{-c(i)} \quad (t \in k_v \otimes_k V^*;\ i = 1, \dots, n).$$

If $(f_1, \dots, f_n; c(1), \dots, c(n))$ is G_v-stable, then there exist positive constants C' and C'' depending only on K, n, v, and $f_1, \dots, f_n$ such that

$$C' \cdot Q^{-\sum_{i=1}^{n} c(i)} < \mathrm{vol}_v < C'' \cdot Q^{-\sum_{i=1}^{n} c(i)} \quad (Q > 1).$$

Proof. — Straightforward using the subsequent lemma. □

Recall that $G_v \simeq \mathrm{Gal}(L_v/k_v)$ is acting on $L_v \otimes_k V$.

LEMMA 4. — *Let $f_1, \dots, f_r$ be L_v-linearly independent forms in $L_v \otimes_k V$,*

$$\widetilde{W} := L_v f_1 \oplus \cdots \oplus L_v f_r \subset L_v \otimes_k V,$$

and f the k_v-linear map given by

$$f := f_1 \oplus \cdots \oplus f_r \colon k_v \otimes_k V^* \to L_v^{\oplus r}.$$

If $\widetilde{W}$ is G_v-stable in the usual sense, then

$$\dim_{k_v} f(k_v \otimes_k V^*) = r.$$

Proof. — By virtue of the Hilbert Satz 90, the assumption implies the existence of a subspace W over k_v of $k_v \otimes_k V$ satisfying

$$\widetilde{W} = L_v \otimes_{k_v} W.$$

Choose any k_v-basis $w_1, \dots, w_r$ of W and define a k_v-linear map w via

$$w := w_1 \oplus \cdots \oplus w_r \colon k_v \otimes_k V^* \longrightarrow k_v^{\oplus r}.$$

Since $w_1, \dots, w_r$ are (k_v-)linearly independent, the homomorphism w is surjective. In particular

$$\dim_{k_v} w(k_v \otimes_k V^*) = r.$$

On the other hand, we have an invertible matrix $\ell \in \mathrm{GL}_r(L_v)$ such that

$$\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = \ell \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix}.$$

Regarding ℓ as a (k_v-)linear automorphism of $L_v^{\oplus r}$ and w as a k_v-linear homomorphism of $k_v \otimes_k V^*$ to $L_v^{\oplus r}$, we observe that

$$f = \ell \circ w.$$

Finally $\dim_{k_v} f(k_v \otimes_k V^*) = \dim_{k_v} w(k_v \otimes_k V^*) = r.$ □

At last, we list the precise spots where we resort to Proposition 3:

- p. 52, l. 9: … is big for …
- p. 52, ll. 12–13: … is small for …
- p. 53, l. 2.
- p. 53, l. 12 (the inequality involving the constant C).
- The (initial) equality in the proof of Lemma 2.7 must be replaced by the following type of inequalities:

$$CQ^{-\sum_{v\in\mathfrak{M}(k)}\sum_{i=1}^{n} c(i;v)} < \mathrm{vol}(\Pi) < C'''Q^{-\sum_{v\in\mathfrak{M}(k)}\sum_{i=1}^{n} c(i;v)}.$$

The resulting accompanied modification of the proof of Lemma 2.7 is very easy, which we omit here.

BIBLIOGRAPHY

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