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ON SQUARE FUNCTIONS ASSOCIATED TO SECTORIAL OPERATORS

by Christian Le Merdy

Dedicated to Alan McIntosh on the occasion of his 60th birthday

Abstract. — We give new results on square functions

$$\|x\|_F = \Big\| \Big(\int_0^\infty \big|F(tA)x\big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_p$$

associated to a sectorial operator A on L^p for $1 < p < \infty$. Under the assumption that A is actually R-sectorial, we prove equivalences of the form $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ for suitable functions F, G. We also show that A has a bounded H^∞ functional calculus with respect to $\|\,.\,\|_F$. Then we apply our results to the study of conditions under which we have an estimate $\|(\int_0^\infty |C\mathrm{e}^{-tA}(x)|^2 \mathrm{d}t)^{1/2}\|_q \leq M\|x\|_p$, when $-A$ generates a bounded semigroup e^{-tA} on L^p and $C\colon D(A) \to L^q$ is a linear mapping.

Résumé (*Sur les fonctions carrées associées aux opérateurs sectoriels*)
Nous obtenons de nouveaux résultats sur les fonctions carrées

$$\|x\|_F = \Big\| \Big(\int_0^\infty \big|F(tA)x\big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_p$$

associées à un opérateur sectoriel A sur L^p pour $1 < p < \infty$. Quand A est en fait R-sectoriel, on montre des équivalences de la forme $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ pour des fonctions F, G appropriées. On démontre également que A possède un calcul fonctionnel H^∞ borné par rapport à $\|\,.\,\|_F$. Puis nous appliquons nos résultats à l'étude de conditions impliquant une inégalité du type $\|(\int_0^\infty |C\mathrm{e}^{-tA}(x)|^2 \mathrm{d}t)^{1/2}\|_q \leq M\|x\|_p$, où $-A$ engendre un semigroupe borné e^{-tA} sur L^p et $C\colon D(A) \to L^q$ est une application linéaire.

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1. Introduction

The main objects of this paper will be bounded analytic semigroups and sectorial operators on L^p-spaces, their H^∞ functional calculus, and their associated square functions. This beautiful and powerful subject grew out of McIntosh's seminal paper [18] and subsequent important works by McIntosh-Yagi [19] and Cowling-Doust-McIntosh-Yagi [6].

We first briefly recall a few classical notions which are the starting point of the whole theory. Given a Banach space X, we will denote by $B(X)$ the Banach algebra of all bounded operators on X. For any $\omega \in (0,\pi)$, we let

$$\Sigma_\omega = \bigl\{ z \in \mathbb{C}^* \,;\, |\mathrm{Arg}(z)| < \omega \bigr\}$$

be the open sector of angle 2ω around the half-line $(0,\infty)$. Let A be a possibly unbounded operator A on X and assume that A is closed and densely defined. For any z in the resolvent set of A we let $R(z,A) = (z-A)^{-1}$ denote the corresponding resolvent operator. Let $\sigma(A)$ denote the spectrum of A. Then by definition, A is *sectorial of type* ω if the following three conditions are fulfilled:

(S1) $\sigma(A) \subset \overline{\Sigma}_\omega$.

(S2) For any $\theta \in (\omega,\pi)$ there is a constant $K_\theta > 0$ such that

$$\bigl\| zR(z,A) \bigr\| \le K_\theta, \quad z \in \overline{\Sigma}_\theta^{c}.$$

(S3) A has a dense range.

Very often, (S3) is unnecessary and omitted in the definition of sectoriality. However we include it here to avoid tedious technical discussions. Note the well-known fact that A is one-to-one if it satisfies (S1), (S2) and (S3) above.

Given any $\theta \in (0,\pi)$, we let $H^\infty(\Sigma_\theta)$ be the algebra of all bounded analytic functions $f : \Sigma_\theta \to \mathbb{C}$ and we let $H_0^\infty(\Sigma_\theta)$ be the subalgebra of all $f \in H^\infty(\Sigma_\theta)$ for which there exist two positive numbers $s, c > 0$ such that

$$\bigl| f(z) \bigr| \le c \frac{|z|^s}{(1+|z|)^{2s}}, \quad z \in \Sigma_\theta. \tag{1.1}$$

Now given a sectorial operator A of type $\omega \in (0,\pi)$ on a Banach space X, a number $\theta \in (\omega,\pi)$, and a function $f \in H_0^\infty(\Sigma_\theta)$, one may define an operator $f(A) \in B(X)$ as follows. We let $\gamma \in (\omega,\theta)$ be an intermediate angle and consider the oriented contour Γ_γ defined by

$$\Gamma_\gamma(t) = \begin{cases} -t\mathrm{e}^{i\gamma} & t \in \mathbb{R}_-, \\ t\mathrm{e}^{-i\gamma} & t \in \mathbb{R}_+. \end{cases}$$

Then we let

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} f(z) R(z,A) \,\mathrm{d}z. \tag{1.2}$$

It follows from Cauchy's Theorem that the definition of $f(A)$ does not depend on the choice of γ and it can be shown that the mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$. The next step in H^∞ functional calculus consists in the definition of a possibly unbounded operator $f(A)$ associated to any $f \in H^\infty(\Sigma_\theta)$. Since we shall not use this construction here, we omit it and refer the reader to [18], [19] and [6] for details. We merely recall that by definition, A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if $f(A)$ is bounded for any $f \in H^\infty(\Sigma_\theta)$. In that case, the mapping $f \mapsto f(A)$ is a bounded homomorphism from $H^\infty(\Sigma_\theta)$ into $B(X)$, provided that $H^\infty(\Sigma_\theta)$ is equipped with the norm

$$\|f\|_{\infty,\theta} = \sup\bigl\{|f(z)|\ ;\ z \in \Sigma_\theta\bigr\}.$$

We shall be mainly concerned by square functions associated to sectorial operators in the case when X is an L^p-space. For any $\omega \in (0,\pi)$, we introduce

$$H_0^\infty(\Sigma_{\omega+}) = \bigcup_{\theta>\omega} H_0^\infty(\Sigma_\theta).$$

Assume first that $X = H$ is a Hilbert space. Given a sectorial operator A of type ω on H and $F \in H_0^\infty(\Sigma_{\omega+})$, we consider

$$\|x\|_F = \Bigl(\int_0^\infty \|F(tA)x\|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}, \quad x \in H,$$

which may be either finite or infinite. These square function norms were introduced in [18] where it is shown that for any $\theta > \omega$ and any non zero $F \in H_0^\infty(\Sigma_{\omega+})$, A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus if and only if $\|.\|_F$ is equivalent to the original norm of H. In [19, Theorem 5], McIntosh-Yagi established the following two remarkable properties. First these square function norms are pairwise equivalent, that is, for any two non zero functions F and G in $H_0^\infty(\Sigma_{\omega+})$ there exists a constant $K>0$ such that $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ for any $x \in H$. Second, A always has a bounded H^∞ functional calculus with respect to $\| \ \|_F$. More precisely, for any $\theta > \omega$ and for any $F \in H_0^\infty(\Sigma_\theta)$, there is a constant $K>0$ such that $\|f(A)x\|_F \leq K\|f\|_{\infty,\theta}\|x\|_F$ for any $f \in H^\infty(\Sigma_\theta)$ and any $x \in H$. Further properties and applications of square functions $\|.\|_F$ were investigated in [3], to which we refer the interested reader.

We now turn to L^p-spaces. Let $1 \leq p < \infty$ be a number, let Ω be an arbitrary measure space, and consider the Banach space $X = L^p(\Omega)$. Given a sectorial operator A of type ω on $L^p(\Omega)$ and $F \in H_0^\infty(\Sigma_{\omega+})$, we let

$$\|x\|_F = \Bigl\|\Bigl(\int_0^\infty \bigl|F(tA)x\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}\Bigr\|_{L^p(\Omega)}, \quad x \in L^p(\Omega).$$

Again $\|x\|_F$ may be either finite or infinite. These square function norms were introduced in [6] and play a key role in the study of bounded H^∞ functional calculus on L^p-spaces (see Corollary 2.3 below). The latter definition obviously extends the previous one that we recover when $p = 2$. However it is unknown whether the results from [19] reviewed above extend to the case when $p \neq 2$. In particular it is unknown whether square function norms are pairwise equivalent on L^p-spaces. In a recent work [2], Auscher-Duong-McIntosh succeded in proving such an equivalence in the case when $-A$ generates a bounded analytic semigroup acting on $L^2(\Omega)$ with suitable upper bounds on its heat kernels. We shall prove that the results from [19, Theorem 5] actually extend to all operators which are not only sectorial but R-sectorial. This notion which arose from some recent work of Weis [22] will be explained at the beginning of the next section.

THEOREM 1.1. — *Let A be an R-sectorial operator of R-type $\omega \in (0,\pi)$ on a space $L^p(\Omega)$, with $1 \leq p < \infty$. Let $\theta \in (\omega,\pi)$ and let F and G be two non zero functions belonging to $H_0^\infty(\Sigma_\theta)$.*

1) *There exists a constant $K > 0$ such that for any $f \in H^\infty(\Sigma_\theta)$ and any $x \in L^p(\Omega)$, we have*

$$\Big\|\Big(\int_0^\infty \big|f(A)F(tA)x\big|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_{L^p(\Omega)} \leq K\|f\|_{\infty,\theta}\Big\|\Big(\int_0^\infty \big|G(tA)x\big|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_{L^p(\Omega)}. \tag{1.3}$$

2) *There exists a constant $K > 0$ such that*

$$K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G, \quad x \in L^p(\Omega).$$

This result will be proved in Section 2 below, where we also include some relevant comments. Then Section 3 is devoted to an application of Theorem 1.1 to the study of R-admissibility. This new concept is a natural extension of the classical notion of admissibility considered *e.g.* in [24], [23], [25], [8] or [16]. Given a bounded analytic semigroup $T_t = \mathrm{e}^{-tA}$ on $L^p(\Omega)$ and a linear mapping C from the domain of A into some $L^q(\Sigma)$, we will study conditions under which we have an estimate of the form

$$\Big\|\Big(\int_0^\infty \big|CT_t(x)\big|^2 \mathrm{d}t\Big)^{1/2}\Big\|_{L^q(\Sigma)} \leq M\|x\|_{L^p(\Omega)}.$$

In particular we will show that such an estimate holds if A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{1}{2}\pi$ and the set $\big\{(-s)^{1/2}CR(s,A)\ ;\ s \in \mathbb{R},\ s < 0\big\}$ is R-bounded. This extends a result of ours ([16]) corresponding to the case when $p = 2$.

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2. Equivalence of square function norms

The main purpose of this section is the proof of Theorem 1.1. We first recall the key concepts of R-boundedness (see [4]) and R-sectoriality (see [22], [21], [14]). Consider a Rademacher sequence $(\varepsilon_k)_{k\geq 1}$ on a probability space $(\Omega_0, \mathbb{P})$. That is, the ε_k's are pairwise independent random variables on Ω_0 and $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}$ for any $k \geq 1$. Then for any finite family $x_1, \ldots, x_n$ in a Banach space X, we let

$$\Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_{\mathrm{Rad}(X)} = \int_{\Omega_0} \Big\|\sum_{k=1}^{n} \varepsilon_k(s) x_k\Big\|_X \, \mathrm{d}\mathbb{P}(s).$$

Let X, Y be two Banach spaces and let $B(X, Y)$ denote the space of all bounded operators from X into Y. By definition, a set $\mathcal{T} \subset B(X, Y)$ is R-bounded if there is a constant $C \geq 0$ such that for any finite families $T_1, \ldots, T_n$ in $\mathcal{T}$, and $x_1, \ldots, x_n$ in X, we have

$$\Big\|\sum_{k=1}^{n} \varepsilon_k T_k(x_k)\Big\|_{\mathrm{Rad}(Y)} \leq C \Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_{\mathrm{Rad}(X)}.$$

In that case, the smallest possible C is called the *R-boundedness constant* of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$. If A is a sectorial operator on X and $\omega \in (0, \pi)$ is a number, we say that A is *R-sectorial of R-type* ω if for any $\theta \in (\omega, \pi)$, the set $\{zR(z, A) \,;\, z \in \overline{\Sigma}_\theta^c\} \subset B(X)$ is R-bounded.

To describe the range of applications of our result, we first recall that if X is a Hilbert space, then any bounded subset of $B(X)$ is R-bounded, hence any sectorial operator of type ω on X is actually R-sectorial of R-type ω. Thus Theorem 1.1 comprises [19, Theorem 5] that we recover when $p = 2$. Note that our proof reduces to that of [19] in this case. If X is not isomorphic to a Hilbert space, then there exist bounded subsets of $B(X)$ which are not R-bounded (see *e.g.* [1, Proposition 1.13]). The notion of R-sectoriality on non Hilbertian Banach spaces is closely related to maximal L^p-regularity. Namely, it was proved in [13] and [22] that if A is a sectorial operator of type $< \frac{1}{2}\pi$ on a Banach space X with maximal L^p-regularity, then A is R-sectorial of R-type $< \frac{1}{2}\pi$. Thus the counterexamples to maximal L^p-regularity obtained by Kalton-Lancien [13] show that when $p \neq 2$, there exist sectorial operators on L^p-spaces which are not R-sectorial. Conversely, it was proved in [22] that if X is a UMD Banach space, and A is R-sectorial of R-type $< \frac{1}{2}\pi$ on X, then A has maximal L^p-regularity. Thus for $1 < p < \infty$ and $\omega < \frac{1}{2}\pi$, Theorem 1.1 exactly applies when the operator A has maximal L^p-regularity. In particular it applies to the operators considered in [2].

If $X = L^p(\Omega)$ for some $1 \leq p < \infty$, then there is a constant $C_0 > 0$ such that we both have

$$\Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_{\mathrm{Rad}(L^p(\Omega))} \leq C_0 \Big\|\Big(\sum_{k=1}^{n} |x_k|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \tag{2.1}$$

and

$$\Big\|\Big(\sum_{k=1}^{n} |x_k|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \leq C_0 \Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_{\mathrm{Rad}(L^p(\Omega))} \tag{2.2}$$

for any finite family $x_1, \dots, x_n$ in $L^p(\Omega)$. Thus $\mathcal{T} \subset B(L^p(\Omega))$ is R-bounded provided that

$$\Big\|\Big(\sum_{k=1}^{n} \big|T_k(x_k)\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \leq C \Big\|\Big(\sum_{k=1}^{n} |x_k|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \tag{2.3}$$

for some constant $C \geq 0$, and for any $T_1, \dots, T_n$ in $\mathcal{T}$ and $x_1, \dots, x_n$ in $L^p(\Omega)$. In the proof of Theorem 1.1, we shall need the following continuous version of (2.3) which was first observed by Weis [21, 4.a].

Lemma 2.1. — *Let $I \subset \mathbb{R}$ be an interval and let $S\colon I \to B(L^p(\Omega))$ be a strongly continuous function, with $1 \leq p < \infty$. Then the set $\mathcal{T} = \{S(t)\ ;\ t \in I\}$ is R-bounded if and only if there is a constant $C \geq 0$ such that*

$$\Big\|\Big(\int_I \big|S(t)u(t)\big|^2 \mathrm{d}t\Big)^{1/2}\Big\|_{L^p(\Omega)} \leq C \Big\|\Big(\int_I \big|u(t)\big|^2 \mathrm{d}t\Big)^{1/2}\Big\|_{L^p(\Omega)}$$

for any $u \in L^p(\Omega; L^2(I))$. Moreover the smallest possible C is equivalent to $R(\mathcal{T})$.

We will also use the following well-known consequence of [4, Lemma 3.2].

Lemma 2.2. — *Let $I \subset \mathbb{R}$ be an interval and let $\mathcal{T} \subset B(L^p(\Omega))$ be an R-bounded set, with $1 \leq p < \infty$. Then the set*

$$\Big\{\int_I a(r)R(r)\,\mathrm{d}r\ ;\ R\colon I \to \mathcal{T} \ \text{ is continuous, } a \in L^1(I) \text{ and } \|a\|_1 \leq 1\Big\}$$

is R-bounded as well and its R-boundedness constant is $\leq 2R(\mathcal{T})$.

We finally recall some well-known facts concerning $H_0^\infty(\Sigma_\theta)$ that will be used without further reference. First of all, if $\varphi \in H_0^\infty(\Sigma_\theta)$ and A is a sectorial operator of type $\omega < \theta$ on X, then $t \mapsto \varphi(tA)$ is a continuous and bounded function from $(0, \infty)$ into $B(X)$. Second, if $\gamma < \theta$ then $\int_{\Gamma_\gamma} |\varphi(z)| \cdot |\mathrm{d}z/z| < \infty$ by (1.1). Third, changing z into tz shows that

$$\int_{\Gamma_\gamma} \big|\varphi(tz)\big| \cdot \Big|\frac{\mathrm{d}z}{z}\Big| = \int_{\Gamma_\gamma} \big|\varphi(z)\big| \cdot \Big|\frac{\mathrm{d}z}{z}\Big|$$

for any $t > 0$. Fourth, a simple change of variables also shows that

$$\sup_{z\in\Gamma_\gamma} \int_0^\infty \bigl|\varphi(tz)\bigr| \frac{\mathrm{d}t}{t} < \infty.$$

Proof of Theorem 1.1. — The proof is a generalization of the one of [19, Theorem 5]. By assumption, A is an R-sectorial operator of R-type $\omega \in (0,\pi)$ on $L^p(\Omega)$ and we consider $F, G \in H_0^\infty(\Sigma_\theta)\setminus\{0\}$ for some $\theta \in (\omega,\pi)$. Note that the second assertion follows from the first one in Theorem 1.1. Indeed applying 1) with the constant function $f(z) = 1$ yields an estimate $\|x\|_F \leq K\|x\|_G$. Then 2) follows by switching the roles of F and G. Also observe that to prove 1), we may assume that $f \in H_0^\infty(\Sigma_\theta)$. Indeed assume (1.3) for any element of $H_0^\infty(\Sigma_\theta)$, and let $f \in H^\infty(\Sigma_\theta)$ be an arbitrary function. Then according to the so-called Convergence Lemma (see [6, Lemma 2.1]), there exists a constant $C > 0$ not depending on f and a bounded sequence $(f_n)_{n\geq 1} \subset H_0^\infty(\Sigma_\theta)$ such that $\|f_n\|_{\infty,\theta} \leq C\|f\|_{\infty,\theta}$ for any $n \geq 1$ and $\lim_{n\to\infty} \|f_n(A)F(tA)x - f(A)F(tA)x\| = 0$ for any $x \in X$ and any $t > 0$. Applying Fatou's Lemma, we may therefore deduce that

$$\begin{aligned}
\Bigl\|\Bigl(\int_0^\infty \bigl|f(A)F(tA)x\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}\Bigr\|_p &\leq \liminf_{n\to\infty} \Bigl\|\Bigl(\int_0^\infty \bigl|f_n(A)F(tA)x\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}\Bigr\|_p \\
&\leq K \liminf_{n\to\infty} \|f_n\|_{\infty,\theta} \Bigl\|\Bigl(\int_0^\infty \bigl|G(tA)x\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}\Bigr\|_p \\
&\leq KC\|f\|_{\infty,\theta} \Bigl\|\Bigl(\int_0^\infty \bigl|G(tA)x\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}\Bigr\|_p.
\end{aligned}$$

Throughout the rest of this proof, x will be an element of $L^p(\Omega)$ such that $\|x\|_G < \infty$ and f will be an element of $H_0^\infty(\Sigma_\theta)$. We will denote by $C_1, C_2, C_3, \ldots$ various constants not depending either on f or on x. We fix an angle $\gamma \in (\omega,\theta)$ for which we will use the integral representation (1.2). We record for further use that by our R-sectoriality assumption, the set

$$\bigl\{zR(z,A)\,;\, z \in \Gamma_\gamma\bigr\} \quad \text{is } R\text{-bounded.} \tag{2.4}$$

Then we consider two auxiliary functions φ and ψ in $H_0^\infty(\Sigma_\theta)$ such that

$$\int_0^\infty \varphi(t)\psi(t)G(t)\frac{\mathrm{d}t}{t} = 1. \tag{2.5}$$

We will reach (1.3) after five steps, the identity (2.5) being used only in the last one.

First step. — By (1.2) we have for any $t > 0$

$$f(A)\psi(tA) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} f(z)\psi(tz)zR(z,A)\frac{\mathrm{d}z}{z}\cdot$$

Moreover, letting $C_1 = \int_{\Gamma_\gamma} |\psi(z)| \cdot |\mathrm{d}z/z|$, we have

$$\int_{\Gamma_\gamma} \bigl|f(z)\psi(tz)\bigr| \cdot \Bigl|\frac{\mathrm{d}z}{z}\Bigr| \leq \|f\|_{\infty,\theta} \int_{\Gamma_\gamma} \bigl|\psi(tz)\bigr| \cdot \Bigl|\frac{\mathrm{d}z}{z}\Bigr| = C_1 \|f\|_{\infty,\theta}.$$

By Lemma 2.2 and (2.4), we therefore deduce that the operators $f(A)\psi(tA)$ form an R-bounded set and that we have an estimate

$$R\bigl(\{f(A)\psi(tA)\ ;\ t>0\}\bigr) \leq C_2 \|f\|_{\infty,\theta}.$$

Hence applying Lemma 2.1 with $I = (0,\infty)$, $S(t) = f(A)\psi(tA)$, and $u(t) = G(tA)x/\sqrt{t}$, we obtain an estimate

$$\Bigl\|\Bigl(\int_0^\infty \bigl|f(A)\psi(tA)G(tA)x\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}\Bigr\|_p \leq C_3 \|f\|_{\infty,\theta} \cdot \|x\|_G. \tag{2.6}$$

Second step. — We define a continuous function $u\colon \Gamma_\gamma \to L^p(\Omega)$ by letting

$$u(z) = \int_0^\infty \varphi(tz) f(A)\psi(tA)G(tA)x \frac{\mathrm{d}t}{t}, \quad z \in \Gamma_\gamma. \tag{2.7}$$

Letting $w(t) = f(A)\psi(tA)G(tA)x$ for $t > 0$, we see using the Cauchy-Schwarz inequality and Fubini's Theorem that u satisfies the following pointwise estimates:

$$\begin{aligned}
\int_{\Gamma_\gamma} \bigl|u(z)\bigr|^2 \cdot \Bigl|\frac{\mathrm{d}z}{z}\Bigr| &\leq \int_{\Gamma_\gamma} \Bigl(\int_0^\infty \bigl|\varphi(tz)\bigr| \cdot \bigl|w(t)\bigr| \frac{\mathrm{d}t}{t}\Bigr)^2 \Bigl|\frac{\mathrm{d}z}{z}\Bigr| \\
&\leq \int_{\Gamma_\gamma} \Bigl(\int_0^\infty \bigl|\varphi(tz)\bigr| \frac{\mathrm{d}t}{t}\Bigr)\Bigl(\int_0^\infty \bigl|\varphi(tz)\bigr| \cdot \bigl|w(t)\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)\Bigl|\frac{\mathrm{d}z}{z}\Bigr| \\
&\leq \Bigl(\sup_{z\in\Gamma_\gamma} \int_0^\infty \bigl|\varphi(tz)\bigr| \frac{\mathrm{d}t}{t}\Bigr) \int_0^\infty \int_{\Gamma_\gamma} \bigl|\varphi(tz)\bigr| \cdot \bigl|w(t)\bigr|^2 \cdot \Bigl|\frac{\mathrm{d}z}{z}\Bigr| \frac{\mathrm{d}t}{t} \\
&\leq \Bigl(\sup_{z\in\Gamma_\gamma} \int_0^\infty \bigl|\varphi(tz)\bigr| \frac{\mathrm{d}t}{t}\Bigr)\Bigl(\sup_{t>0} \int_{\Gamma_\gamma} \bigl|\varphi(tz)\bigr| \cdot \Bigl|\frac{\mathrm{d}z}{z}\Bigr|\Bigr) \int_0^\infty \bigl|w(t)\bigr|^2 \frac{\mathrm{d}t}{t}.
\end{aligned}$$

According to the discussion preceding this proof, the two suprema appearing here are finite hence applying (2.6) yields an estimate

$$\Bigl\|\Bigl(\int_{\Gamma_\gamma} \bigl|u(z)\bigr|^2 \cdot \Bigl|\frac{\mathrm{d}z}{z}\Bigr|\Bigr)^{1/2}\Bigr\|_p \leq C_4 \|f\|_{\infty,\theta} \cdot \|x\|_G. \tag{2.8}$$

Third step. — We now apply Lemma 2.1 with $I = \Gamma_\gamma$ and $S(z) = zR(z,A)$. By (2.4) and (2.8), we obtain a new estimate

$$\Bigl\|\Bigl(\int_{\Gamma_\gamma} \bigl|zR(z,A)u(z)\bigr|^2 \cdot \Bigl|\frac{\mathrm{d}z}{z}\Bigr|\Bigr)^{1/2}\Bigr\|_p \leq C_5 \|f\|_{\infty,\theta} \cdot \|x\|_G. \tag{2.9}$$

Fourth step. — This fourth step is similar to the second one. We define a continuous function $v\colon (0,\infty)\to L^p(\Omega)$ by letting

$$v(s)=\frac{1}{2\pi i}\int_{\Gamma_\gamma} F(sz)R(z,A)u(z)\,\mathrm{d}z,\quad s>0. \tag{2.10}$$

Then arguing as in the second step we find a constant $C_6\geq 0$ such that

$$\Big\|\Big(\int_0^\infty \big|v(s)\big|^2\frac{\mathrm{d}s}{s}\Big)^{1/2}\Big\|_p \leq C_6\Big\|\Big(\int_{\Gamma_\gamma}\big|zR(z,A)u(z)\big|^2\cdot\Big|\frac{\mathrm{d}z}{z}\Big|\Big)^{1/2}\Big\|_p.$$

Combining with (2.9), we obtain the final estimate

$$\Big\|\Big(\int_0^\infty \big|v(s)\big|^2\frac{\mathrm{d}s}{s}\Big)^{1/2}\Big\|_p \leq C_7\|f\|_{\infty,\theta}\cdot\|x\|_G.$$

Fifth step. — We conclude our proof by showing that for any $s>0$, $f(A)F(sA)x=v(s)$. By the Principle of Analytic Continuation, (2.5) implies that for any $z\in\Sigma_\theta$,

$$\int_0^\infty \varphi(tz)\psi(tz)G(tz)\frac{\mathrm{d}t}{t}=1.$$

Since $f\in H_0^\infty(\Sigma_\theta)$, we deduce by applying (1.2) and Fubini's Theorem that

$$f(A)=\int_0^\infty \varphi(tA)\psi(tA)G(tA)f(A)\frac{\mathrm{d}t}{t},$$

the latter integral being absolutely convergent. Therefore we have for any $s>0$,

$$\begin{aligned}
f(A)F(sA)x &= \int_0^\infty F(sA)\varphi(tA)\psi(tA)G(tA)f(A)x\frac{\mathrm{d}t}{t}\\
&= \int_0^\infty \Big(\frac{1}{2\pi i}\int_{\Gamma_\gamma}F(sz)\varphi(tz)R(z,A)\,\mathrm{d}z\Big)\psi(tA)G(tA)f(A)x\frac{\mathrm{d}t}{t}\\
&\qquad\qquad\text{by (1.2)},\\
&= \frac{1}{2\pi i}\int_{\Gamma_\gamma}F(sz)R(z,A)\Big(\int_0^\infty \varphi(tz)\psi(tA)G(tA)f(A)x\frac{\mathrm{d}t}{t}\Big)\mathrm{d}z\\
&\qquad\qquad\text{by Fubini's Theorem},\\
&= \frac{1}{2\pi i}\int_{\Gamma_\gamma}F(sz)R(z,A)u(z)\,\mathrm{d}z \qquad \text{by (2.7)},\\
&= v(s) \qquad \text{by (2.10)}.
\end{aligned}$$

□

Assume now that $1<p<\infty$ and let A be a sectorial operator on $L^p(\Omega)$ with a bounded $H^\infty(\Sigma_\theta)$ functional calculus. The following two results were proved by Cowling-Doust-McIntosh-Yagi [6, Section 6]. First, for any $F\in H_0^\infty(\Sigma_{\theta+})$, there is a constant $K>0$ such that $\|x\|_F\leq K\|x\|$ for any $x\in L^p(\Omega)$. Second, there exists $F\in H_0^\infty(\Sigma_{\theta+})$ as above such that for some suitable $K>0$, we have $K^{-1}\|x\|\leq\|x\|_F\leq K\|x\|$ for any $x\in L^p(\Omega)$. On the other hand, it follows from [14, Theorem 5.3] that A is R-sectorial of R-type θ provided that A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Combining with Theorem 1.1, we deduce the following strengthening of the above mentioned result.

COROLLARY 2.3. — *Let A be a sectorial operator with a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L^p(\Omega)$, with $1 < p < \infty$. Then for any $F \in H_0^\infty(\Sigma_{\theta+})$, there is a constant $K > 0$ such that for any $x \in L^p(\Omega)$,*

$$K^{-1}\|x\| \le \Big\| \Big(\int_0^\infty \big|F(tA)x\big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_p \le K\|x\|. \tag{2.11}$$

REMARK 2.4. — The above corollary clearly has a converse. Indeed assume that A is R-sectorial of R-type ω and satisfies the equivalence (2.11) for some $\theta > \omega$ and some $F \in H_0^\infty(\Sigma_\theta)$. Then applying the first part of Theorem 1.1 with $F = G$, we see that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. This leads to the question of computing square functions for R-sectorial operators without a bounded H^∞ functional calculus. We give a simple example below.

EXAMPLE 2.5. — Let $1 < p \ne 2 < \infty$ and let $\mathbb{T} = \{z \in \mathbb{C} \,;\, |z| = 1\}$ be the unimodular complex group equipped with its Haar measure. For any integer $n \in \mathbb{Z}$, we let $e_n(z) = z^n$ for $z \in \mathbb{T}$. As far as we know, the simplest example of a sectorial operator on an L^p-space without a bounded H^∞ functional calculus is obtained by defining A as the Fourier multiplier associated to the sequence $(2^n)_n$ on $L^p(\mathbb{T})$. Namely we let A be the closure of the operator defined on $\mathrm{Span}\{e_n \,;\, n \in \mathbb{Z}\}$ by first taking e_n to $2^n e_n$ for any n and then extending by linearity. This operator is essentialy the discrete version of the one given in [6, Example 5.2]. The arguments given in the latter paper extend to this discrete version and show that our operator A is sectorial of any positive type, has no bounded H^∞ functional calculus and admits bounded imaginary powers with $\|A^{is}\| = 1$ for any $s \in \mathbb{R}$. According to [5, Theorem 4] or [22], this implies that A is R-sectorial of R-type ω for any $\omega > 0$. Hence by Theorem 1.1, all non zero square function norms associated to A are equivalent. We claim that they are actually all equivalent to the norm of $L^2(\mathbb{T})$. Here is a brief proof using Theorem 1.1. We give ourselves some $\theta > 0$ and some $F \in H_0^\infty(\Sigma_\theta) \setminus \{0\}$. We let $(\alpha_n)_n$ be a finite sequence of complex numbers and consider $x = \sum_n \alpha_n e_n$. For any $z \in \mathbb{T}$ and any $t > 0$, we have

$$\big(F(tA)x\big)(z) = \sum_n F(t2^n)\alpha_n e_n(z).$$

Likewise for every $f \in H^\infty(\Sigma_\theta)$, we have

$$\big(f(A)F(tA)x\big)(z) = \sum_n f(2^n)F(t2^n)\alpha_n e_n(z).$$

Hence if we let $\Lambda = L^p\big(\mathbb{T}; L^2(0,\infty; \mathrm{d}t/t)\big)$ and apply (1.3) with $F = G$, we obtain that

$$\Big\|\sum_n f(2^n)F(t2^n)\alpha_n e_n(z)\Big\|_\Lambda \leq K\|f\|_{\infty,\theta}\cdot \Big\|\sum_n F(t2^n)\alpha_n e_n(z)\Big\|_\Lambda.$$

Now using the fact that $(2^n)_n$ is an interpolation sequence for the open set Σ_θ, we deduce that for an appropriate constant $K_1 > 0$, we have

$$\Big\|\sum_n \varepsilon_n F(t2^n)\alpha_n e_n(z)\Big\|_\Lambda \leq K_1\Big\|\sum_n F(t2^n)\alpha_n e_n(z)\Big\|_\Lambda$$

for any $\{-1,1\}$-valued sequence $(\varepsilon_n)_n$ (see *e.g.* [7, Chapter VII] for details). Taking the average over all such possible sequences we find that

$$\Big\|\sum_n F(t2^n)\alpha_n e_n(z)\Big\|_\Lambda \asymp \Big\|\sum_n \varepsilon_n F(t2^n)\alpha_n e_n(z)\Big\|_{\mathrm{Rad}(\Lambda)}.$$

Using the well-known fact that (2.1) and (2.2) hold with Λ in place of $L^p(\Omega)$ we finally obtain that

$$\Big\|\Big(\int_0^\infty \Big|F(tA)\Big(\sum_n \alpha_n e_n\Big)\Big|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_p \asymp \Big\|\Big(\sum_n \big|F(t2^n)\alpha_n e_n(z)\big|^2\Big)^{1/2}\Big\|_\Lambda.$$

Now observe that since $|e_n(z)| = 1$ for any $z \in \mathbb{T}$ and $\int_0^\infty |F(t2^n)|^2 \mathrm{d}t/t = \int_0^\infty |F(t)|^2 \mathrm{d}t/t$ for any $n \in \mathbb{Z}$, we have

$$\begin{aligned}\Big\|\Big(\sum_n \big|F(t2^n)\alpha_n e_n(z)\big|^2\Big)^{1/2}\Big\|_\Lambda^2 &= \Big\|\Big(\sum_n \big|F(t2^n)\alpha_n\big|^2\Big)^{1/2}\Big\|^2_{L^2(0,\infty;\mathrm{d}t/t)}\\ &= \Big(\int_0^\infty \big|F(t)\big|^2\frac{\mathrm{d}t}{t}\Big)\sum_n|\alpha_n|^2\\ &= \Big(\int_0^\infty \big|F(t)\big|^2\frac{\mathrm{d}t}{t}\Big)\Big\|\sum_n \alpha_n e_n\Big\|_2^2,\end{aligned}$$

which proves the announced result.

Remark 2.6. — It was observed in [15] that most of the results established in [6] extend to the case when $L^p(\Omega)$ is replaced by a B-convex Banach lattice. It is also easy to check that our Theorem 1.1 extends to this setting and as a by-product, we find that Corollary 2.3 also extends to this setting.

3. Application to *R*-admissibility.

Let X be a Banach space and let $(T_t)_{t\geq 0}$ be a bounded c_0-semigroup on X. We let $-A$ denote its infinitesimal generator and we let $D(A)$ be the domain of A. We consider a linear mapping $C\colon D(A) \to Y$ valued in another Banach space Y. We assume that C is continuous with respect to the graph norm of $D(A)$, so what $t \mapsto CT_t(x)$ is a well-defined continuous function from $(0,\infty)$

into Y for any $x \in D(A)$. By definition, C is admissible for A if there is a constant $M > 0$ such that

$$\int_0^\infty \big\|CT_t(x)\big\|^2 \mathrm{d}t \leq M^2\|x\|^2, \quad x \in D(A). \tag{3.1}$$

This definition arises from Control Theory and is usually given with X and Y being Hilbert spaces. We refer the reader to [24], [23], [25], [8], [20], [9] and the references therein for some background and applications of this notion.

If C is admissible for A, then there is a constant $K > 0$ such that

$$\big\|\big(-\mathrm{Re}(\lambda)\big)^{1/2} CR(\lambda, A)\big\| \leq K, \quad \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) < 0. \tag{3.2}$$

Indeed if $\mathrm{Re}(\lambda) < 0$, define

$$a_\lambda(t) = -\big(-\mathrm{Re}(\lambda)\big)^{1/2} \mathrm{e}^{\lambda t}, \quad t > 0.$$

Then

$$a_\lambda \in L^2(0, \infty; dt) \quad \text{with} \quad \|a_\lambda\|_2 = \frac{1}{\sqrt{2}}, \tag{3.3}$$

and according to the Laplace Formula, we have

$$\big(-\mathrm{Re}(\lambda)\big)^{1/2} CR(\lambda, A)x = \int_0^\infty a_\lambda(t) CT_t(x)\mathrm{d}t, \quad \mathrm{Re}(\lambda) < 0, \tag{3.4}$$

for any $x \in D(A)$. Thus (3.1) implies (3.2) with $K = M/\sqrt{2}$ by the Cauchy-Schwarz inequality.

The latter observation goes back to George Weiss [25] who investigated the converse implication, that is, whether the estimate (3.2) implies that C is admissible for A. He quickly proved that this converse does not hold on general Banach spaces but the question remained open for a long time under the name of "Weiss conjecture" in the case when X and Y are both Hilbert spaces. The Weiss conjecture has been disproved recently by Jacob-Partington-Pott [10]. Namely there exist Hilbert spaces X, Y, as well as $T_t = \mathrm{e}^{-tA}$ and C as above such that (3.2) holds for some K although C is not admissible for A. In fact it was proved by Jacob-Zwart [12] that such counterexamples exist with $Y = \mathbb{C}$. See also [11] for related work. The failure of the Weiss conjecture leads to the following question.

Which triples (X, A, Y) have the property that any continuous $C\colon D(A) \to Y$ is admissible for A provided that (3.2) holds?

In [9], it was shown that this property holds when X is a Hilbert space, $Y = \mathbb{C}$, and A is maximal accretive (equivalently, $(T_t)_{t\geq 0}$ is a contraction semigroup). In [16], we studied the case when $T_t = \mathrm{e}^{-tA}$ is a bounded analytic semigroup, that is, there exists $\alpha > 0$ such that $(T_t)_{t>0}$ extends to a bounded analytic family $(\mathrm{e}^{-zA})_{z\in\Sigma_\alpha} \subset B(X)$. We proved the following result (see [16, Theorem 4.1]).

THEOREM 3.1. — *Assume that* $T_t = \mathrm{e}^{-tA}$ *is a bounded analytic semigroup on a Banach space* X. *Then the following assertions are equivalent.*

(i) $A^{1/2}$ *is admissible for* A.

(ii) *For any Banach space* Y, *a continuous mapping* $C\colon D(A) \to Y$ *is admissible for* A *if and only if there is a constant* $K > 0$ *such that* $\big\|\big(-\mathrm{Re}(\lambda)\big)^{1/2} CR(\lambda, A)\big\| \leq K$ *for any* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) < 0$.

(iii) *For any Banach space* Y, *a continuous mapping* $C\colon D(A) \to Y$ *is admissible for* A *if and only if there is a constant* $K > 0$ *such that* $\big\|(-s)^{1/2} CR(s, A)\big\| \leq K$ *for any negative real number* $s < 0$.

Recall that $T_t = \mathrm{e}^{-tA}$ is a bounded analytic semigroup on X if and only if A satisfies the conditions (S1) and (S2) from Section 1 for some $\omega < \frac{1}{2}\pi$. Define

$$F_0(z) = z^{1/2}\mathrm{e}^{-z}, \quad z \in \mathbb{C}.$$

Then $F_0 \in H_0^\infty(\Sigma_\theta)$ for any $\theta \in (0, \frac{1}{2}\pi)$ and

$$A^{1/2}T_t(x) = \frac{F_0(tA)x}{\sqrt{t}}, \quad t > 0,\ x \in X. \tag{3.5}$$

Consequently, $A^{1/2}$ is admissible for A if and only if we have an estimate

$$\Big(\int_0^\infty \|F_0(tA)x\|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2} \leq M\|x\|, \quad x \in X.$$

This observation makes Theorem 3.1 especially interesting in the case when $X = H$ is a Hilbert space. Indeed in that case, an appeal to [18] shows that condition (i), hence conditions (ii) and (iii) in Theorem 3.1 are fulfilled provided that A admits a bounded H^∞ functional calculus. We refer the reader to [16, Section 5] for a more precise discussion of condition (i) of Theorem 3.1 in the case when $X = H$ is a Hilbert space.

When moving from Hilbert spaces to L^p-spaces, it is natural to introduce a variant of admissibility involving square function norms in the style of those considered so far in the previous two sections. We let $1 < p, q < \infty$ be two numbers, we let Ω and Σ be two measure spaces and we let $(T_t)_{t\geq 0}$ be a bounded c_0-semigroup on $L^p(\Omega)$ with generator denoted by $-A$. Then given a continuous linear mapping $C\colon D(A) \to L^q(\Sigma)$, we say that C is R-admissible for A if there is a constant $M > 0$ such that

$$\Big\|\Big(\int_0^\infty \big|CT_t(x)\big|^2 dt\Big)^{1/2}\Big\|_{L^q(\Sigma)} \leq M\|x\|_{L^p(\Omega)}, \quad x \in D(A).$$

Arguing as above, it is easy to check that this condition implies (3.2) with $K = M/\sqrt{2}$. It turns out that the following stronger property holds.

LEMMA 3.2. — *If* C *is* R*-admissible for* A, *then the following set is* R*-bounded:*

$$\big\{\big(-\mathrm{Re}(\lambda)\big)^{1/2} CR(\lambda, A)\ ;\ \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) < 0\big\}.$$

Indeed this follows from (3.3), (3.4) and the following statement of independent interest. Note the analogy with Lemma 2.2.

PROPOSITION 3.3. — *Let X be a Banach space, let $X_0 \subset X$ be a dense subspace, and let $t \mapsto \varphi_t$ be a strongly continuous function from an interval $I \subset \mathbb{R}$ into the space $L(X_0, L^q(\Sigma))$ of linear mappings from X_0 into $L^q(\Sigma)$. Assume that there is a constant $M > 0$ such that*

$$\Big\| \Big(\int_I \big|\varphi_t(x)\big|^2 \mathrm{d}t \Big)^{1/2} \Big\|_{L^q(\Sigma)} \leq M\|x\|, \quad x \in X_0.$$

For any $a \in L^2(I)$, let $\int_I a(t)\varphi_t \mathrm{d}t$ denote the element of $B(X, L^q(\Sigma))$ obtained by first taking $x \in X_0$ to $\int_I a(t)\varphi_t(x)\mathrm{d}t \in L^q(\Sigma)$ and then extending by continuity. Then the following set is R-bounded:

$$\Big\{ \int_I a(t)\varphi_t \mathrm{d}t \;;\; a \in L^2(I),\ \|a\|_2 \leq 1 \Big\}.$$

Proof. — We use the notation and definitions from the beginning of Section 2. For any $a \in L^2(I)$, we let

$$T_a = \int_I a(t)\varphi_t \mathrm{d}t$$

and we give ourselves a finite family $a_1, \dots, a_n$ of elements of $L^2(I)$ of norms less than or equal to one. Let $(e_1, \dots, e_m)$ be an orthonormal basis of $\mathrm{Span}\{a_1, \dots, a_n\} \subset L^2(I)$. Then we have $a_k = \sum_i \langle a_k, e_i \rangle e_i$ for any k, hence

$$T_{a_k} = \sum_{i=1}^m \langle a_k, e_i \rangle T_{e_i}, \quad 1 \leq k \leq n.$$

Let $x_1, \dots, x_n$ be arbitrary elements of X_0. (Strictly speaking, we should take elements of X but the density of X_0 clearly allows this reduction.) Then for some numerical constant $C_0 \geq 0$, we have

$$\begin{aligned} \Big\| \sum_{k=1}^n \varepsilon_k T_{a_k}(x_k) \Big\|_{\mathrm{Rad}(L^q)} &\leq C_0 \Big\| \Big(\sum_{k=1}^n \big|T_{a_k}(x_k)\big|^2 \Big)^{1/2} \Big\|_{L^q} \\ &= C_0 \Big\| \Big(\sum_{k=1}^n \Big| \sum_{i=1}^m \langle a_k, e_i \rangle T_{e_i}(x_k) \Big|^2 \Big)^{1/2} \Big\|_{L^q}. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain the following pointwise estimates on $L^q(\Sigma)$.

$$\begin{aligned} \sum_{k=1}^n \Big| \sum_{i=1}^m \langle a_k, e_i \rangle T_{e_i}(x_k) \Big|^2 &\leq \sum_{k=1}^n \Big(\sum_{i=1}^m \big|\langle a_k, e_i \rangle\big|^2 \Big) \Big(\sum_{i=1}^m \big|T_{e_i}(x_k)\big|^2 \Big) \\ &= \sum_{k=1}^n \|a_k\|_2^2 \Big(\sum_{i=1}^m \big|T_{e_i}(x_k)\big|^2 \Big) \leq \sum_{i,k} \big|T_{e_i}(x_k)\big|^2. \end{aligned}$$

Combining with the preceding estimate, this yields

$$\Big\|\sum_{k=1}^{n}\varepsilon_k T_{a_k}(x_k)\Big\|_{\mathrm{Rad}(L^q)} \leq C_0\Big\|\Big(\sum_{i,k}\big|T_{e_i}(x_k)\big|^2\Big)^{1/2}\Big\|_{L^q}. \tag{3.6}$$

Now observe that since $(e_1,\ldots,e_m)$ is an orthonormal family of $L^2(I)$, we have

$$\sum_i\Big|\int_I e_i(t)\alpha(t)\,\mathrm{d}t\Big|^2 \leq \int_I|\alpha(t)|^2\,\mathrm{d}t$$

for any $\alpha\in L^2(I)$, hence we have a pointwise inequality

$$\sum_i\big|T_{e_i}(x)\big|^2 \leq \int_I|\varphi_t(x)|^2\,\mathrm{d}t$$

for any $x\in X_0$. Applying this to each x_k, we deduce that

$$\sum_{i,k}\big|T_{e_i}(x_k)\big|^2 \leq \int_I\sum_k|\varphi_t(x_k)|^2\,\mathrm{d}t.$$

Since $(\varepsilon_1,\ldots,\varepsilon_n)$ is an orthonormal family of $L^2(\Omega_0)$, the right handside of the latter inequality can be written as

$$\int_I\sum_k|\varphi_t(x_k)|^2\,\mathrm{d}t = \int_I\int_{\Omega_0}\Big|\sum_k\varepsilon_k(s)\varphi_t(x_k)\Big|^2\mathrm{d}\mathbb{P}(s)\,\mathrm{d}t.$$

Owing to the Khintchine-Kahane inequality (see *e.g.* [17, p. 74]), there is a numerical constant $C_1\geq 0$ such that

$$\begin{aligned}\Big(\int_{\Omega_0}\int_I\Big|\sum_k\varepsilon_k(s)\varphi_t(x_k)\Big|^2\mathrm{d}t\,\mathrm{d}\mathbb{P}(s)\Big)^{1/2}&\\ \leq C_1\int_{\Omega_0}\Big(\int_I\Big|\sum_k\varepsilon_k(s)\varphi_t(x_k)\Big|^2\mathrm{d}t\Big)^{1/2}\mathrm{d}\mathbb{P}(s).&\end{aligned}$$

We therefore obtain that

$$\begin{aligned}\Big(\sum_{i,k}\big|T_{e_i}(x_k)\big|^2\Big)^{1/2} &\leq C_1\int_{\Omega_0}\Big(\int_I\Big|\sum_k\varepsilon_k(s)\varphi_t(x_k)\Big|^2\mathrm{d}t\Big)^{1/2}\mathrm{d}\mathbb{P}(s)\\ &= C_1\int_{\Omega_0}\Big(\int_I\Big|\varphi_t\Big(\sum_k\varepsilon_k(s)x_k\Big)\Big|^2\mathrm{d}t\Big)^{1/2}\mathrm{d}\mathbb{P}(s).\end{aligned}$$

Hence by (3.6), we deduce that

$$\begin{aligned}\Big\|\sum_{k=1}^{n}\varepsilon_k T_{a_k}(x_k)\Big\|_{\mathrm{Rad}(L^q)} &\leq C_0C_1\Big\|\int_{\Omega_0}\Big(\int_I\Big|\varphi_t\Big(\sum_k\varepsilon_k(s)x_k\Big)\Big|^2\mathrm{d}t\Big)^{1/2}\mathrm{d}\mathbb{P}(s)\Big\|_{L^q}\\ &\leq C_0C_1\int_{\Omega_0}\Big\|\Big(\int_I\Big|\varphi_t\Big(\sum_k\varepsilon_k(s)x_k\Big)\Big|^2\mathrm{d}t\Big)^{1/2}\Big\|_{L^q}\mathrm{d}\mathbb{P}(s).\end{aligned}$$

It now remains to apply our assumption with $x = \sum_k \varepsilon_k(s)x_k$ for each $s \in \Omega_0$ to deduce that

$$\Big\|\sum_{k=1}^n \varepsilon_k T_{a_k}(x_k)\Big\|_{\mathrm{Rad}(L^q)} \leq C_0C_1M\Big\|\sum_k \varepsilon_k x_k\Big\|_{\mathrm{Rad}(X)},$$

which proves our R-boundedness property. □

We record here the simple consequence of Lemma 3.2.

LEMMA 3.4. — *If C is R-admissible for A, then the set*

$$\big\{(-s)^{1/2}CR(s,A)\,;\, s\in\mathbb{R},\, s<0\big\}$$

is R-bounded. The latter condition is equivalent to the existence of a constant $K>0$ such that

$$\Big\|\Big(\int_0^\infty \big|C(t+A)^{-1}u(t)\big|^2\,\mathrm{d}t\Big)^{1/2}\Big\|_{L^q(\Sigma)} \leq K\Big\|\Big(\int_0^\infty \big|u(t)\big|^2\frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_{L^p(\Omega)}$$

for any $u\in L^p\big(\Omega; L^2(0,\infty;\mathrm{d}t/t)\big)$.

Proof. — The first part follows from Lemma 3.2 whereas the second part follows by simply adapting the proof of Lemma 2.1 to the case of a function valued in $B(L^p(\Omega), L^q(\Sigma))$. We skip the details. □

We now come to the main result of this section, which is an analogue of Theorem 3.1 for R-admissibility. We will say that a bounded analytic semigroup $T_t = \mathrm{e}^{-tA}$ on X is an R-bounded one if there exists $\alpha>0$ such that the set $\big\{\mathrm{e}^{-zA}\,;\, z\in\Sigma_\alpha\big\}\subset B(X)$ is R-bounded. According to [22], this is equivalent to the existence of $\theta<\frac{1}{2}\pi$ such that $\big\{zR(z,A)\,;\, z\in\overline{\Sigma}_\theta^c\big\}$ is R-bounded, hence (modulo (S3)) to the property that A is R-sectorial of R-type $<\frac{1}{2}\pi$.

Note that according to the comments following Theorem 3.1, if $T_t=\mathrm{e}^{-tA}$ is a bounded analytic semigroup on $L^p(\Omega)$, then $A^{1/2}$ is R-admissible for A if and only if there is a constant $M>0$ such that

$$\|x\|_{F_0} = \Big\|\Big(\int_0^\infty \big|F_0(tA)x\big|^2\frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_{L^p(\Omega)} \leq M\|x\|_{L^p(\Omega)},\quad x\in L^p(\Omega). \tag{3.7}$$

Here F_0 is defined by $F_0(z)=z^{1/2}\mathrm{e}^{-z}$.

THEOREM 3.5. — *Let $T_t=\mathrm{e}^{-tA}$ be an R-bounded analytic semigroup on $L^p(\Omega)$, with $1<p<\infty$. Then the following assertions are equivalent.*

(i) *$A^{1/2}$ is R-admissible for A.*

(ii) *For any $1<q<\infty$ and any measure space Σ, a continuous mapping $C\colon D(A)\to L^q(\Sigma)$ is R-admissible for A if and only if the set $\big\{\big(-\mathrm{Re}(\lambda)\big)^{1/2}CR(\lambda,A)\,;\,\lambda\in\mathbb{C},\mathrm{Re}(\lambda)<0\big\}$ is R-bounded.*

(iii) *For any* $1 < q < \infty$ *and any measure space* Σ*, a continuous mapping* $C\colon D(A) \to L^q(\Sigma)$ *is* R*-admissible for* A *if and only if the set* $\big\{(-s)^{1/2}CR(s,A)\ ;\ s \in \mathbb{R}, s < 0\big\}$ *is* R*-bounded if and only if there is a constant* $K > 0$ *such that*

$$\Big\|\Big(\int_0^\infty \big|C(t+A)^{-1}u(t)\big|^2 \mathrm{d}t\Big)^{1/2}\Big\|_{L^q(\Sigma)} \leq K \Big\|\Big(\int_0^\infty \big|u(t)\big|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_{L^p(\Omega)}$$

for any $u \in L^p\big(\Omega; L^2(0,\infty; \mathrm{d}t/t)\big)$.

Proof. — Owing to the results proved before, the proof is now a simple adaptation of that of Theorem 3.1 (stated as Theorem 4.1 in [16]). We shall therefore only sketch it. It is well-known that since $X = L^p(\Omega)$ is reflexive, it is the direct sum of the kernel of A and of the closure of the range of A hence we may clearly assume that A has a dense range. We let $\omega < \frac{1}{2}\pi$ be such that A is R-sectorial of R-type ω.

It is obvious that (iii) implies (ii). To prove that (ii) implies (i), it suffices to show that the set

$$\big\{|\lambda|^{1/2}A^{1/2}R(\lambda,A)\ ;\ \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) < 0\big\}$$

is R-bounded. For we fix some angle $\gamma \in (\omega, \frac{1}{2}\pi)$ and we write

$$(-\lambda)^{1/2}A^{1/2}R(\lambda,A) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} \frac{(-\lambda)^{1/2}z^{1/2}}{\lambda - z}R(z,A)\mathrm{d}z, \quad \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) < 0.$$

Since the set $\{zR(z,A)\ ;\ z \in \Gamma_\gamma\}$ is R-bounded, Lemma 2.2 ensures that it suffices to prove that for a certain constant $K > 0$, we have

$$\int_{\Gamma_\gamma} \frac{|\lambda z|^{1/2}}{|\lambda - z|}\cdot\Big|\frac{\mathrm{d}z}{z}\Big| \leq K, \quad \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) < 0.$$

This estimate holds true and is established in the course of the proof of [16, Theorem 4.1].

We now assume that $A^{1/2}$ is R-admissible for A and will prove (iii). We consider a continuous mapping $C\colon D(A) \to L^q(\Sigma)$. In view of Lemma 3.4, we only need to prove that if the set

$$\big\{(-s)^{1/2}CR(s,A)\ ;\ s \in \mathbb{R}, s < 0\big\} \tag{3.8}$$

is R-bounded, then C is R-admissible for A. Arguing as in the proof of [16, Lemma 2.3], we obtain that the R-boundedness of (3.8) implies the existence of an angle $\nu \in (\omega, \pi)$ such that

$$\big\{|z|^{1/2}CR(z,A)\ ;\ |\mathrm{Arg}(z)| \geq \nu\big\} \tag{3.9}$$

is R-bounded as well. Then arguing as in the proof of [16, Theorem 4.1], we find $\theta \in (\nu, \pi)$ and functions $F_1, F_2 \in H_0^\infty(\Sigma_{\omega+})$ and $G_1, G_2 \in H_0^\infty(\Sigma_\theta)$ such that $F_0 = G_1F_1 + G_2F_2$. According to (3.5), this yields

$$CT_t(x) = \big[CA^{-1/2}G_1(tA)\big]\frac{F_1(tA)x}{\sqrt{t}} + \big[CA^{-1/2}G_2(tA)\big]\frac{F_2(tA)x}{\sqrt{t}} \tag{3.10}$$

for any $t > 0$ and every $x \in D(A)$. By our assumption (i), the estimate (3.7) holds for some $M > 0$. We therefore deduce from Theorem 1.1 that for some constants $M_1, M_2 > 0$, we also have estimates

$$\|x\|_{F_1} \leq M_1\|x\| \quad \text{and} \quad \|x\|_{F_2} \leq M_2\|x\|. \tag{3.11}$$

As we already said, Lemma 2.1 extends to the case of functions valued in $B(L^p(\Omega), L^q(\Sigma))$. Hence to deduce the R-admissibility of C from (3.10) and (3.11), it now suffices to check that for $j = 1, 2$, the set

$$\big\{CA^{-1/2}G_j(tA)\ ;\ t > 0\big\} \tag{3.12}$$

is R-bounded. According to the proof of [16, Theorem 4.1], each of the operators of the latter set has the following integral representation:

$$CA^{-1/2}G_j(tA) = \frac{1}{2\pi i}\int_{\Gamma_\nu} z^{-1/2}G_j(tz)CR(z, A)\,\mathrm{d}z.$$

Since the set (3.9) is R-bounded and $\int_{\Gamma_\nu}|G_j(tz)|\cdot|\mathrm{d}z/z| = \int_{\Gamma_\nu}|G_j(z)|\cdot|\mathrm{d}z/z| < \infty$ for any $t > 0$, we deduce from Lemma 2.2 that the set (3.12) is indeed R-bounded, which concludes our proof. □

Remark 3.6. — If A is a sectorial operator on $L^p(\Omega)$ with a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{1}{2}\pi$, then it satisfies (3.7) by [6] hence $A^{1/2}$ is admissible for A. Furthermore it is R-sectorial of R-type $< \frac{1}{2}\pi$ by [14, Theorem 5.3], hence $T_t = \mathrm{e}^{-tA}$ is an R-bounded analytic semigroup. Consequently, A satisfies the assertion (iii) of Theorem 3.5.

Remark 3.7. — In [16, Section 5], we exhibited a sectorial operator A_0 on ℓ^2 such that $A_0^{1/2}$ is admissible for A_0 although A_0 has no bounded H^∞ functional calculus. Using the fact that $L^p(\mathbb{R})$, say, contains a complemented subspace isomorphic to ℓ^2 when $1 < p < \infty$, it is easy to transfer A_0 to an operator A on $L^p(\mathbb{R})$ satisfying the assertions of Theorem 3.5 but having no bounded H^∞ functional calculus.

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Note added on proofs. — We learned that some of the results in Section 3 were obtained independently by Bernhard Haak (Karlsruhe). His work should appear soon in his Ph.D. thesis.