

INVOLUTIVITY OF TRUNCATED MICROSUPPORTS

BY MASAKI KASHIWARA, TÉRÉSA MONTEIRO FERNANDES
& PIERRE SCHAPIRA

ABSTRACT. — Using a result of J.-M. Bony, we prove the weak involutivity of truncated microsupports. More precisely, given a sheaf F on a real manifold and $k \in \mathbb{Z}$, if two functions vanish on $SS_k(F)$, then so does their Poisson bracket.

RÉSUMÉ (*Involutivité des micro-soutports tronqués*). — Utilisant un résultat de J.-M. Bony, nous prouvons l'involutivité faible des micro-soutports tronqués. Plus précisément, étant donné un faisceau F sur une variété réelle et $k \in \mathbb{Z}$, si deux fonctions s'annulent sur $SS_k(F)$, il en est de même de leur crochet de Poisson.

Texte reçu le 21 mai 2002, accepté le 24 septembre 2002

MASAKI KASHIWARA, Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 (Japan) • *E-mail* : masaki@kurims.kyoto-u.ac.jp

TÉRÉSA MONTEIRO FERNANDES, Centro de Álgebra da Universidade de Lisboa, Complexo 2, 2 Avenida Prof. Gama Pinto, 1699 Lisboa codex (Portugal)

E-mail : tmf@ptmat.lmc.fc.ul.pt

PIERRE SCHAPIRA, Université Pierre et Marie Curie, case 82, Institut de Mathématiques, 4 place Jussieu, 75252 Paris cedex 05 (France) • *E-mail* : schapira@math.jussieu.fr

Url : <http://www.institut.math.jussieu.fr/~schapira/>

2000 Mathematics Subject Classification. — Primary: 35A27; Secondary: 32C38.

Key words and phrases. — Sheaves, microsupport, involutivity.

M.K.: Research partially supported by Grant-in-Aid for Scientific Research (B) 13440006, Japan Society for the Promotion of Science.

T.M.F.: Research supported by FCT and Programa Ciência, Tecnologia e Inovação do Quadro Comunitário de Apoio.

1. Introduction

The notion of microsupport of sheaves was introduced by two of the present authors (M.K. and P.S.) in the course of the study of the theory of linear partial differential equations. References are made to [3]. These authors also introduced a variant of this notion, that of “truncated microsupport” and developed its study with the third author in [2].

A crucial result in the microlocal theory of sheaves is the involutivity of the microsupport. This property does not hold true any more for the truncated microsupport, but a weak form of it does. More precisely, we prove here that the truncated microsupport is stable by Poisson bracket, that is, if two functions vanish on it, so does their Poisson bracket. The main technical tool is a result of J.-M. Bony which asserts the same property for the normal cone to a closed subset.

2. Notations and review

We will follow the notations of [3] and [2]. For the reader convenience, we recall some of them as well as the definition of the truncated microsupport. Let X be a real analytic manifold. We denote by $\tau: TX \rightarrow X$ the tangent bundle to X and by $\pi: T^*X \rightarrow X$ the cotangent bundle. For a smooth submanifold Y of X , $T_Y X$ denotes the normal bundle to Y and $T_Y^* X$ the conormal bundle. In particular, $T_X^* X$ is the zero section of T^*X . We set $\dot{T}^*X = T^*X \setminus X$, and denote by $\dot{\pi}: \dot{T}^*X \rightarrow X$ the restriction of π to \dot{T}^*X .

For a morphism $f: X \rightarrow Y$ of real manifolds, we denote the associated morphisms by

$$f_\pi: X \times_Y T^*Y \rightarrow T^*Y \quad \text{and} \quad f_d: X \times_Y T^*Y \rightarrow T^*X.$$

For a subset A of T^*X , we denote by A^a the image of A by the antipodal map $a: (x; \xi) \mapsto (x; -\xi)$. The closure of A is denoted by \bar{A} .

For a cone $\gamma \subset TX$, the polar cone γ° to γ is the convex cone in T^*X defined by

$$\gamma^\circ = \{(x; \xi) \in T^*X; x \in \pi(\gamma) \text{ and } \langle v, \xi \rangle \geq 0 \text{ for any } (x; v) \in \gamma\}.$$

A closed convex cone γ is called *proper* if $0 \in \gamma$ and $\text{Int}(\gamma^\circ) \neq \emptyset$.

Let \mathbf{k} be a field. One denotes by $\text{Mod}(\mathbf{k}_X)$ the abelian category of sheaves of \mathbf{k} -vector spaces and by $D^b(\mathbf{k}_X)$ its bounded derived category. One denotes by $D_{\mathbb{R}\text{-c}}^b(\mathbf{k}_X)$ the full triangulated subcategory of $D^b(\mathbf{k}_X)$ consisting of objects with \mathbb{R} -constructible cohomology. If X is a complex manifold, one denotes by $D_{\mathbb{C}\text{-c}}^b(\mathbf{k}_X)$ the full triangulated subcategory of $D^b(\mathbf{k}_X)$ consisting of objects with \mathbb{C} -constructible cohomology.

If S is a locally closed subset of X , one denotes by \mathbf{k}_{XS} the sheaf on X which is the constant sheaf with stalk \mathbf{k} on S and 0 on $X \setminus S$. If there is no risk of confusion, we may write \mathbf{k}_S instead of \mathbf{k}_{XS} .

For $k \in \mathbb{Z}$, we denote as usual by $D^{\geq k}(\mathbf{k}_X)$ (resp. $D^{\leq k}(\mathbf{k}_X)$) the full additive subcategory of $D^b(\mathbf{k}_X)$ consisting of objects F satisfying $H^j(F) = 0$ for any $j < k$ (resp. $H^j(F) = 0$ for any $j > k$).

We denote by $\tau^{\leq k}: D(\mathbf{k}_X) \rightarrow D^{\leq k}(\mathbf{k}_X)$ the truncation functor.

Recall that for $F \in D(\mathbf{k}_X)$ the morphism $\tau^{\leq k}F \rightarrow F$ induces isomorphisms $H^j(\tau^{\leq k}F) \xrightarrow{\sim} H^j(F)$ for $j \leq k$ and $H^j(\tau^{\leq k}F) = 0$ for $j > k$.

If F is an object of $D^b(\mathbf{k}_X)$, $\text{SS}(F)$ denotes its microsupport, a closed \mathbb{R}^+ -conic subset of T^*X . For $p \in T^*X$, $D^b(\mathbf{k}_X; p)$ denotes the localization of $D^b(\mathbf{k}_X)$ by the full triangulated subcategory consisting of objects F such that $p \notin \text{SS}(F)$. A property holds “microlocally on a subset S of T^*X ” if it holds in the category $D^b(\mathbf{k}_X; p)$ for any $p \in S$.

We recall the definition of involutivity of [3]. This notion makes use of that of “normal cone” (see [3, Def. 4.1.1]). For a pair of subsets S_1 and S_2 of a manifold X , the normal cone $C_p(S_1, S_2)$ at $p \in X$ is defined as follows: it is a closed cone in the tangent space T_pX consisting of points v such that, for a local coordinate system, there exist a sequence $\{x_n\}_n$ in S_1 , $\{y_n\}_n$ in S_2 and a sequence $\{a_n\}_n$ in $\mathbb{R}_{\geq 0}$ such that x_n and y_n converge to p and $a_n(x_n - y_n)$ converges to v . For a subset S , $C_p(S, \{p\})$ is denoted by $C_p(S)$.

DEFINITION 2.1. — (see [3, Definition 6.5.1]) Let S be a locally closed subset of T^*X and let $p \in S$. One says that S is *involutive* at p if for any $\theta \in T_p^*(T^*X)$ such that the normal cone $C_p(S, S)$ is contained in the hyperplane $\{v \in T_p(T^*X); \langle v, \theta \rangle = 0\}$ one has $H(\theta) \in C_p(S)$. Here $H: T_p^*(T^*X) \xrightarrow{\sim} T_p(T^*X)$ is the Hamiltonian isomorphism.

If S is involutive at each $p \in S$, one says that S is *involutive*.

The involutivity theorem of [3, Theorem 6.5.4] asserts that the microsupport of sheaves is involutive.

The following definition was introduced by the authors of [3] and developed in [2].

DEFINITION 2.2. — Let X be a real analytic manifold and let $p \in T^*X$. Let $F \in D^b(\mathbf{k}_X)$ and $k \in \mathbb{Z}$. The closed conic subset $\text{SS}_k(F)$ of T^*X is defined by $p \notin \text{SS}_k(F)$ if and only if the following condition is satisfied.

- (i) There exists an open conic neighborhood U of p such that for any $x \in \pi(U)$ and for any \mathbb{R} -valued C^1 -function φ defined on a neighborhood of x such that $\varphi(x) = 0$, $d\varphi(x) \in U$, one has

$$(2.1) \quad H_{\{\varphi \geq 0\}}^j(F)_x = 0 \quad \text{for any } j \leq k.$$

We refer to [2] for equivalent definitions. In particular, it is proved in *loc. cit.* that one can replace the condition that φ is of class C^1 by φ is of class C^α , where $\alpha \in \mathbb{Z}_{\geq 1} \cup \{\infty, \omega\}$.

This condition is also equivalent to:

(ii) there exists $F' \in D^{\geq k+1}(\mathbf{k}_X)$ and an isomorphism $F \simeq F'$ in $D^b(\mathbf{k}_X; p)$.

3. Normal cone

Let us recall the notion of the normal vector due to J.-M. Bony. Here we set for $r \geq 0$

$$B_r(x) = \{y \in \mathbb{R}^n; \|y - x\| < r\},$$

the open ball with center x and radius r .

PROPOSITION 3.1. — *Let X be a real manifold, S a closed subset of X , and $\alpha \in \mathbb{Z}_{\geq 1} \cup \{\infty, \omega\}$. Then the following subsets of T^*X are equal:*

- (i) $SS_0(\mathbf{k}_S)$;
- (ii) $_{\alpha}$ the closure of the set of points $p = (x; \xi) \in T^*X$ such that $x \in S$ and there is a C^{α} function φ defined on a neighborhood U of x such that $\varphi(x) = 0$, $d\varphi(x) = \xi$ and $S \cap U \subset \varphi^{-1}(\mathbb{R}_{\geq 0})$;
- (iii) the closure of the set of points $p = (x; \xi) \in T^*X$ such that $x \in S$ and $C_x(S) \subset \{v \in T_x X; \langle v, \xi \rangle \geq 0\}$.

If X is an open subset of \mathbb{R}^n , then the above sets are equal to the following set:

- (iv) the closure of the set of points $p = (x; \xi) \in T^*X$ such that $x \in S$ and the open ball $B_{t\|\xi\|}(x - t\xi)$ does not intersect S for some $t > 0$.

Proof. — We may assume that $X = \mathbb{R}^n$.

(i) = (ii) $_{\alpha}$ is an immediate consequence of Definition 2.2. In particular, the set (ii) $_{\alpha}$ does not depend on α .

(ii) $_{\alpha}$ = (iv) for $\alpha \geq 2$ is obvious.

(ii) $_{\alpha}$ \subset (iii) is clear.

(iii) \subset (iv). Let $p = (x_0; \xi_0) \in T^*X$ with $x_0 \in S$ and

$$C_{x_0}(S) \subset \{v \in T_{x_0} X; \langle v, \xi_0 \rangle \geq 0\}.$$

We have to prove that p belongs to the set (iv). The proof is very similar to that of Lemma 3.3 of [2], but for the sake of completeness, we shall repeat it.

If $\xi_0 = 0$, then it is trivially true. Hence we may assume that $\xi_0 \neq 0$.

We shall show that for an arbitrary open conic neighborhood U of p , there exists a point $(x_1; \xi_1) \in U$ such that the open ball $B_{\|\xi_1\|}(x_1 - \xi_1)$ satisfies

$$(3.1) \quad x_1 \in S, \quad x_1 \notin \overline{B_{\|\xi_1\|}(x_1 - \xi_1)} \cap \overline{S}.$$

This implies $B_{t\|\xi_1\|}(x_1 - t\xi_1) \cap S = \emptyset$ for $0 < t \ll 1$.

Let us take an open neighborhood V and a proper closed convex cone γ such that $\xi_0 \in \text{Int}(\gamma^{\circ})$, $\overline{V} \times (\gamma^{\circ} \setminus \{0\}) \subset U$ and $\text{Int}(\gamma) \neq \emptyset$. Since $C_{x_0}(S) \cap \gamma^{\circ} \subset \{0\}$, there is $\rho > 0$ such that

$$H_- := \{x \in \mathbb{R}^n; \langle x - x_0, \xi_0 \rangle > -\rho\}$$

satisfies $\overline{H}_- \cap (x_0 + \gamma^a) \subset V$ and $S \cap \overline{H}_- \cap (x_0 + \gamma^a) = \{x_0\}$. Set $S_0 := S \cap \overline{H}_-$.

Let us define the function $\psi(x)$ on \mathbb{R}^n by

$$\psi(x) = \text{dist}(x, \gamma^a) := \inf \{ \|y - x\| ; y \in \gamma^a \}.$$

It is well known that ψ is a continuous function on \mathbb{R}^n , and C^1 on $\mathbb{R}^n \setminus \gamma^a$. More precisely for any $x \in \mathbb{R}^n \setminus \gamma^a$, there exists a unique $y \in \gamma^a$ such that $\psi(x) = \|x - y\|$. Moreover $d\psi(x) = \|x - y\|^{-1}(x - y) \in \gamma^\circ \setminus \{0\}$. Furthermore $B_{\psi(x)}(y)$ is contained in $\{z \in \mathbb{R}^n ; \psi(z) < \psi(x)\}$.

For $\varepsilon > 0$, we set

$$\gamma_\varepsilon^a = \{x \in \mathbb{R}^n ; \psi(x) < \varepsilon\}.$$

Then γ_ε^a is an open convex set. Moreover $\gamma_\varepsilon^a + \gamma^a = \gamma_\varepsilon^a$.

Let us take $v \in \text{Int}(\gamma)$, and $\delta > 0$ such that $x_0 - \delta v \in H_-$. Then take $\varepsilon > 0$ such that $\gamma_\varepsilon^a - \delta v \subset \text{Int}(\gamma^a)$, $(x_0 + \overline{\gamma_\varepsilon^a}) \cap \overline{H}_- \subset V$ and $(x_0 + \overline{\gamma_\varepsilon^a}) \cap S_0 \subset H_-$.

Set $W_t = x_0 + \gamma_\varepsilon^a + tv$ for $t \in \mathbb{R}$. Then one has

$$(3.2) \quad W_t = \bigcup_{t' < t} W_{t'} \text{ and } \overline{W}_t = \bigcap_{t' > t} W_{t'} = \bigcap_{t' > t} \overline{W}_{t'},$$

$$(3.3) \quad x_0 \in W_t \cap S \text{ for } t \geq 0, \text{ and } W_t \cap \overline{H}_- = \emptyset \text{ for } t \ll 0,$$

$$(3.4) \quad \overline{W}_t \cap S_0 \subset H_- \cap V \text{ for } t \leq 0.$$

Hence, for any closed subset K of \overline{H}_- and $t \in \mathbb{R}$ such that $K \cap \overline{W}_t = \emptyset$, there exists $t' > t$ such that $K \cap W_{t'} = \emptyset$.

Let us set $c = \sup\{t ; W_t \cap S_0 = \emptyset\}$. Then $c \leq 0$ and $W_c \cap S_0 = \emptyset$. By the above remark, one has $\overline{W}_c \cap S_0 \neq \emptyset$. Take $x_1 \in S_0 \cap \partial W_c$. Here $\partial W_c := \overline{W}_c \setminus W_c$ is the boundary of W_c . The point x_1 belongs to $H_- \cap V$.

As seen before, there exists an open ball $B_\varepsilon(y)$ such that $B_\varepsilon(y) \subset W_c$, $\|x_1 - y\| = \varepsilon$ and $x_1 - y \in \gamma^\circ$. Hence $B_\varepsilon(y) \cap S \cap H_- = \emptyset$, while H_- is a neighborhood of x_1 . Hence $(x_1; x_1 - y)$ belongs to U and satisfies condition (3.1). \square

DEFINITION 3.2. — For a closed subset S of X , the closed subset given in Proposition 3.1 is called the 0-conormal cone of S and denoted by $N_0^*(S)$.

Note that $N_0^*(S)$ is a closed cone and it satisfies

$$\pi_X(N_0^*(S)) = \pi_X(N_0^*(S) \cap T_X^*X) = S.$$

Note also that one has $N_0^*(S) = \text{SS}_0(\mathbf{k}_S) \subset \text{SS}(\mathbf{k}_S)$.

EXAMPLE 3.3. — (i) If $S = X$, then $N_0^*(S) = T_X^*X$.

(ii) If S is a closed submanifold, then $N_0^*(S) = T_S^*X$.

(iii) If $X = \mathbb{R}^2$ and $S = \{(x, y) \in X ; x \geq 0 \text{ or } y \geq 0\}$. Then

$$\begin{aligned} N_0^*(S) = & \{(x, y; 0, 0) \in T_X^*X ; (x, y) \in S\} \\ & \cup \{(x, y; \xi, \eta) \in T^*X ; y = 0, x \leq 0, \eta \geq 0 \text{ and } \xi = 0\} \\ & \cup \{(x, y; \xi, \eta) \in T^*X ; x = 0, y \leq 0, \xi \geq 0 \text{ and } \eta = 0\}. \end{aligned}$$

One has the microlocal isomorphisms

$$\mathbf{k}_{XS} \simeq \begin{cases} \mathbf{k}_X & \text{on } \{(x, y; 0, 0); x > 0 \text{ or } y > 0\}, \\ \mathbf{k}_{\{y=0\}} & \text{on } \{(x, 0; 0, \eta); x < 0 \text{ and } \eta > 0\}, \\ \mathbf{k}_{\{x=0\}} & \text{on } \{(0, y; \xi, 0); y < 0 \text{ and } \xi > 0\}, \\ \mathbf{k}_{\{(0,0)\}}[-1] & \text{on } \{(0, 0; \xi, \eta); \xi > 0 \text{ and } \eta > 0\}. \end{cases}$$

Note that (see Theorem 4.5 below)

$$\text{SS}(\mathbf{k}_{XS}) = \text{SS}_1(\mathbf{k}_{XS}) = N_0^*(S) \cup \{(0, 0; \xi, \eta) \in T^*X; \xi, \eta \geq 0\}$$

and this set is different from $N_0^*(S)$.

PROPOSITION 3.4. — (i) *Let $f: X \rightarrow Y$ be a morphism of manifolds and let S be a closed subset of X . Assume that $f(S)$ is a closed subset of Y . Then*

$$N_0^*(f(S)) \subset f_\pi f_d^{-1}(N_0^*(S)).$$

If f is moreover a closed embedding, then one has $N_0^(f(S)) = f_\pi f_d^{-1}(N_0^*(S))$.*

(ii) *For a closed subset S_1 of X_1 and a closed subset S_2 of X_2 , one has*

$$N_0^*(S_1 \times S_2) = N_0^*(S_1) \times N_0^*(S_2).$$

(iii) *If a closed subset S of X satisfies $N_0^*(S) \subset T_X^*X$, then S is an open subset.*

(iv) *Let f be a C^1 -function on X , let S a closed subset, and let $c \in \mathbb{R}$. Assume that $f|_S: S \rightarrow \mathbb{R}$ is proper, $f(S) \subset [a, \infty)$ for $a \ll 0$. If $df(x) \notin N_0^*(S)$ for x such that $f(x) < c$, then $f(S) \subset [c, \infty)$.*

(v) *Let γ be a proper closed convex cone of \mathbb{R}^n , Ω an open subset of \mathbb{R}^n such that $\Omega + \gamma^\alpha = \Omega$, and S a closed subset of Ω such that S is relatively compact in \mathbb{R}^n . If $N_0^*(S) \cap (S \times \text{Int}(\gamma^\circ)) = \emptyset$, then S is an empty set.*

Proof. — (i), (ii), (iii) are easy exercises.

(iv) If $f|_S$ takes its minimal value at $x \in S$, then $df(x)$ belongs to $N_0^*(S)$.

(v) is nothing but Lemma 3.3 in [2]. □

The following property of involutivity is due to J.-M. Bony [1].

THEOREM 3.5. — *Let S be a closed subset of X and f, g two C^1 -functions on T^*X such that $N_0^*(S)$ is contained in the zeroes's set of f and that of g . Then $N_0^*(S)$ is contained in the zeroes's set of the Poisson bracket $\{f, g\}$.*

4. Involutivity of truncated microsupports

THEOREM 4.1 (Weak involutivity of truncated microsupports)

*Let $F \in D^b(\mathbf{k}_X)$, $k \in \mathbb{Z}$ and let f, g be two C^1 -functions on T^*X such that $\text{SS}_k(F)$ is contained in the zeroes's set of f and that of g . Then $\text{SS}_k(F)$ is contained in the zeroes's set of the Poisson bracket $\{f, g\}$.*

Proof. — Assume that $\overline{\text{SS}_k(F)} \subset \{f = 0\} \cap \{g = 0\}$ and let $p = (x_0; \xi_0)$ such that $\{f, g\}(p) \neq 0$. We have to show that for any function φ with $\varphi(x_0) = 0$ and $d\varphi(x_0) = \xi_0$, the local cohomology $H^k_{\{\varphi \geq 0\}}(F)_{x_0}$ vanishes.

By induction on k we may assume that $p \notin \text{SS}_{k-1}(F)$. Hence we may assume that $F \in D^{\geq k}(\mathbf{k}_X)$. Then $H^k_{\{\varphi \geq 0\}}(F)_{x_0} \simeq \Gamma_{\{\varphi \geq 0\}}(H^k(F))_{x_0}$. Assume that $s \in \Gamma_{\{\varphi \geq 0\}}(H^k(F))_{x_0}$ does not vanish. There exists an open neighborhood V of x_0 such that s extends to $\tilde{s} \in \Gamma_{\{\varphi \geq 0\}}(V; H^k(F))$. Then $S := \text{supp}(\tilde{s})$ satisfies $N_0^*(S) \subset \text{SS}_k(F)$. Hence $p \notin N_0^*(S)$ by Theorem 3.5, which is a contradiction. \square

REMARK 4.2. — The truncated microsupport is not involutive in the sense of Definition 2.1. Indeed, for $X = \mathbb{R}$ and $Z = \{x \in X; x > 0\}$, one has

$$\text{SS}_0(\mathbf{k}_Z) = \{(x; \xi); \xi = 0, x \geq 0\}.$$

Hence one has $C_p(\text{SS}_0(F), \text{SS}_0(F)) \subset \{-d\xi = 0\}$ with $p = (0; 0)$, but $-\partial/\partial x = H(-d\xi) \notin C_p(\text{SS}_0(F))$.

COROLLARY 4.3. — *Let S be a locally closed submanifold of T^*X such that $T_p S$ is not involutive for any $p \in S$. Then $S \cap \text{SS}_k(F) \subset \overline{\text{SS}_k(F)} \setminus S$.*

THEOREM 4.4. — *Let $F \in D^b(\mathbf{k}_X)$, $k \in \mathbb{Z}$ and let S be a subanalytic subset of T^*X of dimension smaller than $\dim X$. Then $\text{SS}_k(F) = \overline{\text{SS}_k(F)} \setminus S$.*

Proof. — We shall argue by the induction on $\dim S$. There is a closed subanalytic subset S_1 of S such that $\dim S_1 < \dim S$ and $S_0 := S \setminus S_1$ is non singular. Since $T_p S_0$ is not involutive for any $p \in S_0$, $\text{SS}_k(F) \subset S_1 \cup \overline{\text{SS}_k(F)} \setminus S$. Hence one has $\overline{\text{SS}_k(F)} \setminus S = \overline{\text{SS}_k(F)} \setminus S_1$, and the induction proceeds. \square

THEOREM 4.5. — *Let $F \in D^b_{\mathbb{R}-\mathbb{C}}(\mathbf{k}_X)$. Let $\{Y_\alpha\}_{\alpha \in A}$ be a locally finite family of real analytic submanifolds subanalytic in X , and let Λ_α be an open subset of $T^*_{Y_\alpha} X$ subanalytic in T^*X , such that $\text{SS}(F) \subset \bigcup_{\alpha \in A} \overline{\Lambda}_\alpha$. Let $K_\alpha \in D^b(\mathbf{k})$ and assume that F is microlocally isomorphic to $\mathbf{k}_{Y_\alpha} \otimes K_\alpha$ at every point of Λ_α . Set*

$$A_k := \{\alpha \in A; K_\alpha \notin D^{>k}(\mathbf{k})\}.$$

Then $\text{SS}_k(F) = \bigcup_{\alpha \in A_k} \overline{\Lambda}_\alpha$ for any $k \in \mathbb{Z}$.

Proof. — Set $S = \bigcup_{\alpha \in A} (\overline{\Lambda}_\alpha \setminus \Lambda_\alpha)$. Then $\dim S < \dim X$. If $\alpha \in A_k$, then $\overline{\Lambda}_\alpha \subset \text{SS}_k(F)$ and if $\alpha \notin A_k$, then $\text{SS}_k(F) \cap \Lambda_\alpha = \emptyset$. Hence $\text{SS}_k(F) \setminus S = (\bigcup_{\alpha \in A_k} \overline{\Lambda}_\alpha) \setminus S$. Hence Theorem 4.4 implies the desired result. \square

The following corollary is proved in [2] when $\mathbf{k} = \mathbb{C}$ by a different method. Let X be a complex manifold. Recall that $F \in D^b_{\mathbb{C}-\mathbb{C}}(\mathbf{k}_X)$ is perverse if

$$\text{codim Supp}(H^k(F)) \geq k \text{ and } \text{codim Supp}(H^k(R\mathcal{H}om(F, \mathbf{k}_X))) \geq k$$

for any $k \in \mathbb{Z}$.

COROLLARY 4.6. — *Let X be a complex manifold. Let $F \in D_{\mathbb{C}-c}^b(\mathbf{k}_X)$ and let $\{X_\alpha\}_{\alpha \in A}$ be a family of complex submanifolds such that \overline{X}_α and $\overline{X}_\alpha \setminus X_\alpha$ are closed complex analytic subsets and $\text{SS}(F) = \bigcup_{\alpha \in A} \overline{T_{X_\alpha}^* X}$.*

(i) *If F is a perverse sheaf, then one has*

$$(4.1) \quad \text{SS}_k(F) = \bigcup_{\text{codim } X_\alpha \leq k} \overline{T_{X_\alpha}^* X} \quad \text{for any } k.$$

(ii) *Conversely if $F \in D^b(\mathbf{k}_X)$ satisfies*

$$(4.2) \quad \text{SS}_k(F) \cup \text{SS}_k(\text{RHom}(F, \mathbf{k}_X)) \subset \bigcup_{\text{codim } X_\alpha \leq k} \overline{T_{X_\alpha}^* X} \quad \text{for any } k,$$

then F is a perverse sheaf.

Proof. — Recall (see [3, Theorem 10.3.12]) that F is perverse if and only if F is microlocally isomorphic to a finite direct sum of copies of $\mathbf{k}_{X_\alpha}[-\text{codim } X_\alpha]$ at a generic point of $T_{X_\alpha}^* X$ for any α .

(i) Assume F is perverse. Using the notations of Theorem 4.5, we get

$$A_k = \{\alpha; \text{codim } X_\alpha \leq k\}$$

and the result follows from this theorem.

(ii) The proof goes as [2, Corollary 6.10]. For the sake of completeness, we repeat it. Recall that $\mu_Y(\cdot)$ denotes the Sato’s microlocalization functor. By [3], F is isomorphic to $\mathbf{k}_{X_\alpha}[-\text{codim } X_\alpha] \otimes K$ at a generic point of $T_{X_\alpha}^* X$ for some $K \in D^b(\mathbf{k})$. Since $\mu_{X_\alpha}(F)$ must be in $D^{\geq \text{codim } X_\alpha}(\mathbf{k}_{T_{X_\alpha}^* X})$ and

$$\mu_{X_\alpha}(F) \simeq \mathbf{k}_{T_{X_\alpha}^* X}[-\text{codim } X_\alpha] \otimes K,$$

one has $K \in D^{\geq 0}(\mathbf{k})$. Similarly,

$$\mu_{X_\alpha}(\text{RHom}(F, \mathbf{k}_X)) \simeq \mathbf{k}_{T_{X_\alpha}^* X}[-\text{codim } X_\alpha] \otimes \text{RHom}(K, \mathbf{k})$$

implies $K \in D^{\leq 0}(\mathbf{k})$. □

BIBLIOGRAPHY

- [1] BONY (J.-M.) – *Quasi-analyticité et unicité du problème de Cauchy pour les solutions d’équations aux dérivées partielles*, in *Équations aux dérivées partielles et analyse fonctionnelle*, Séminaire Goulaouic-Schwartz, École Polytechnique Paris, 1971, Exp. 10.
- [2] KASHIWARA (M.), MONTEIRO FERNANDES (T.) & SCHAPIRA (P.) – *Truncated microsupport and holomorphic solutions of D -modules*, ArXiv math.AG/0203091, Annales Sci. École Normale Sup., to appear.
- [3] KASHIWARA (M.) & SCHAPIRA (P.) – *Sheaves on manifolds*, Grundlehren der Math. Wiss., vol. 292, Springer Verlag, 1990.