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**NOTE ON PULL-BACK AND  
LELONG NUMBER OF CURRENTS**

BY CHARLES FAVRE (\*)

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ABSTRACT. — We prove a uniform estimate of the Lelong number of the pull-back of a plurisubharmonic function by a holomorphic map in term of the original Lelong number of this function.

RÉSUMÉ. — NOTE SUR LE NOMBRE DE LELONG DES PULL-BACK DE COURANTS. Cet article est consacré à l'étude du nombre de Lelong  $\nu(f^*u, 0)$  du pull-back d'une fonction plurisousharmonique  $u$  par une application holomorphe  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  génériquement de rang maximal. Nous prouvons l'estimée  $\nu(f^*u, 0) \leq C_f \times \nu(u, 0)$  avec une constante  $C_f$  uniforme en  $u$ .

**1. Statement of the main result**

Fix  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  a holomorphic germ, and  $T$  a positive closed current of bidegree  $(1, 1)$  defined in a neighborhood of the origin in  $(\mathbb{C}^n, 0)$ . Let  $u \in \text{PSH}(\mathbb{C}^n, 0)$  be a plurisubharmonic (psh) potential for  $T$  such that  $T = dd^c u$ . One can set

$$f^*T := dd^c(u \circ f)$$

as soon as the psh function  $u \circ f$  is not identically  $-\infty$ .

DEFINITION 1 (Lelong number, see [LG86]). — Let  $u \in \text{PSH}(\mathbb{C}^n, 0)$ . The function  $r \mapsto \sup_{|z|=r} u(z)$  is an increasing convex function of  $\log r$ .

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- We can hence define the *Lelong number* of  $u$  at 0 by setting

$$\nu(u, 0) := \max\{c \geq 0; \text{ such that } u(z) \leq c \log |z| + O(1)\}$$

which is a finite non-negative real number.

- For a positive closed  $(1,1)$  current  $T$  in  $(\mathbb{C}^n, 0)$ , the *Lelong number* of  $T$  at 0 is

$$\nu(T, 0) := \nu(u, 0)$$

for any psh potential  $T = dd^c u$ .

For a given positive closed current  $T$  of bidegree  $(1,1)$  so that  $f^*T$  exists, we are interested in estimating the Lelong number of the pull-back  $\nu(f^*T, 0)$  in terms of  $\nu(T, 0)$ . Our theorem can be stated as follows.

**THEOREM 2.** — *Let  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  be a holomorphic map. Then the following conditions are equivalent:*

- 1) *the map  $f$  has generic (maximal) rank equal to  $n$ ;*
- 2) *for any positive closed current  $T$  of bidegree  $(1,1)$   $f^*T$  is well defined, and the operator  $f^*$  is continuous for the weak topology of currents;*
- 3) *the range of  $f$  is not pluripolar;*
- 4) *for any positive closed current  $T$  of bidegree  $(1,1)$   $f^*T$  is well defined, and there exists a constant  $C > 0$  (depending only on  $f$ ) such that one has the inequality*

$$\nu(T, 0) \leq \nu(f^*T, 0) \leq C \cdot \nu(T, 0)$$

*between Lelong numbers of  $T$  and  $f^*T$  at the origin.*

**REMARK 3.** — The proof gives an estimate for the constant  $C$  above. Assume  $n = m$  and 1) is satisfied. Then 4) holds with

$$C = 1 + 2(\mu(Jf, 0) + n - 1),$$

where  $\mu(Jf, 0)$  is the order of vanishing of the Jacobian determinant of  $f$  at 0.

Using this remark, we also have a semi local version of Theorem 2.

**COROLLARY 4.** — *Let  $X$  and  $Y$  be two connected complex manifolds, and  $f: X \rightarrow Y$  be a holomorphic map whose generic rank is maximal equal to  $\dim(Y)$ . Then for any compact set  $K \subset X$ , there exists a constant  $C_K > 0$  such that for all positive closed current  $T$  of bidegree  $(1,1)$  and all  $p \in K$ , one has the inequality*

$$\nu(T, p) \leq \nu(f^*T, p) \leq C_K \cdot \nu(T, f(p))$$

*between Lelong numbers.*

Before giving a proof of this theorem and of its corollary, we will make some remarks about the stated results.

The main result of Theorem 2 is contained in the implication 1)  $\Rightarrow$  4). All the others are either obvious, or were known before.

The second assertion is contained in [M96]. We also refer the reader to this article for more general problems concerning pull-back of positive closed currents by holomorphic mappings.

The upper estimate given in 4) was already known in several different cases (the other inequality is easy to prove).

PROPOSITION 5 (see [De93]). — *Let  $f$  be a finite holomorphic germ  $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$  of local degree  $d$  and  $T$  a positive closed current (of any bidegree). Then*

$$\nu(f^*T, 0) \leq d \times \nu(T, 0).$$

C. Kiselman also proved 1)  $\Rightarrow$  4) for monomial morphisms.

PROPOSITION 6 (see [K87]). — *Let  $M = [a_{ij}] \in M(n, \mathbb{N})$  be an  $n \times n$  matrix with non-negative integer coefficients. We assume that  $\det M \neq 0$ . If*

$$f(z) = \left( \prod_{j=1}^n z_j^{a_{1j}}, \dots, \prod_{j=1}^n z_j^{a_{nj}} \right)$$

then for any positive closed (1,1) current  $T$

$$\nu(f^*T, 0) \leq \max_i \left\{ \sum_j a_{ij} \right\} \cdot \nu(T, 0).$$

Diller in [D98] also proved the main estimate 4) for birational mappings of  $\mathbb{P}^2$ .

A warning concerning the implication 1)  $\Rightarrow$  3). When  $f$  does not have generic maximal rank, it is not true in general that the image of  $f$  is contained in a countable union of hypersurfaces. It is contained in a countable union of polydisks of dimension strictly less than  $n$ .

EXAMPLE 7 (see [H73, 4.2]). — *Define  $f: (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$  by*

$$f(z, w, t) = (z, ze^w, ze^{e^w}).$$

*Note that  $f$  is independent of the last variable  $t$ . Then the set  $f(\mathbb{C}^3, 0)$  is pluripolar, but it is not included in a countable union of hypersurfaces.*

*Proof.* — We give a short proof of these facts. We begin proving that  $f(\mathbb{C}^3, 0)$  is pluripolar. Decompose the mapping  $f = \pi \circ g \circ p$  with

$$\begin{aligned} p(z, w, t) &= (z, w), \\ g(x, y) &= (y, e^x, e^{e^x}), \\ \pi(z, w, t) &= (z, zw, zt). \end{aligned}$$

The range of  $g$  is included in the hypersurface  $g(\mathbb{C}^2, 0) \subset \{e^w = t\}$ , hence is pluripolar. The morphism  $\pi$  is an isomorphism outside  $\{z = 0\}$ . As countable union of pluripolar sets remains pluripolar, we see that the image

$$\begin{aligned} f(\mathbb{C}^3, 0) &= \pi \circ g \circ p(\mathbb{C}^3, 0) = \pi(g(\mathbb{C}^2, 0)) \\ &= \{0\} \bigcup_{k \geq 0} \pi(g(\mathbb{C}^2, 0) \cap \{|z| > 1/k\}) \end{aligned}$$

is also pluripolar.

For the second fact, we proceed as follows. Assume first that  $f(\mathbb{C}^3, 0)$  is included in an hypersurface defined by a non identically zero holomorphic map  $h$ . We thus have the identity

$$h(z, ze^w, ze^{e^w}) = 0$$

for every  $z, w$  in a neighborhood of  $0 \in \mathbb{C}$ . Expand  $h$  in power series  $h = \sum_{k \geq 0} h_k$  where  $h_k$  is a homogeneous polynomial of degree  $k$  in three variables. Take an index  $k_0 \in \mathbb{N}$  such that  $h_{k_0} \not\equiv 0$ . Then

$$h_{k_0}(z, ze^w, ze^{e^w}) = z^{k_0} h_{k_0}(1, e^w, e^{e^w}) \equiv 0.$$

This would contradict the fact that the three functions  $(1, e^w, e^{e^w})$  are algebraically independent.

Now assume  $f(\mathbb{C}^3, 0) \subset \bigcup_{n \in \mathbb{N}} H_n$  is included in a countable union of hypersurfaces. For each  $n \in \mathbb{N}$ , the complex space  $f^{-1}H_n$  is also an hypersurface by what precedes. But we have

$$(\mathbb{C}^3, 0) \subset f^{-1}f(\mathbb{C}^3, 0) \subset \bigcup_{n \in \mathbb{N}} f^{-1}H_n,$$

which can not contain any open subset of  $(\mathbb{C}^3, 0)$ .  $\square$

Finally, a word about the motivations of this article. The author came to the problem of estimating Lelong numbers of pull-back of positive closed  $(1, 1)$  current while working on dynamics of rational maps of the projective space  $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$  with maximal generic rank. Let us give a simple application of Theorem 2 in this context. We first recall some well-known facts which can be found for instance in [Si99].

We let  $\pi: \mathbb{C}^{k+1} - \{0\} \rightarrow \mathbb{P}^k$  be the natural projection onto  $\mathbb{P}^k$ , and take  $F = (F_0, \dots, F_k): \mathbb{C}^{k+1} \rightarrow \mathbb{C}^{k+1}$  a polynomial lift of  $f$  so that

$$F \circ \pi = \pi \circ f.$$

We assume that the  $k + 1$  polynomials  $\{F_i\}_{0 \leq i \leq k}$  do not contain any common factors. The indeterminacy set of  $f$  is equal to

$$I(f) := \pi \left( \bigcap_{i=0}^k F_i^{-1} \{0\} \right).$$

Given any positive closed current  $T$  of bidegree  $(1, 1)$  on  $\mathbb{P}^k$ , one can find a psh function  $G$  on  $\mathbb{C}^{k+1}$ , called its *potential*, such that

1) there exists a constant  $c > 0$  for which for all  $Z \in \mathbb{C}^{k+1}$  and for all  $\lambda \in \mathbb{C}$ ,

$$G(\lambda Z) = c \log |\lambda| + G(Z);$$

2)  $\pi^* T = dd^c G$ .

Conversely, given a psh function  $G$  on  $\mathbb{C}^{k+1}$  satisfying the homogeneity condition 1), one can find a unique positive closed current  $T$  of bidegree  $(1, 1)$  on  $\mathbb{P}^k$  such that 2) holds.

DEFINITION 8. — Let  $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$  be a rational map of maximal generic rank  $k$ , and  $T$  be a positive current of bidegree  $(1, 1)$  with potential  $G$ . We define  $f^* T$  to be the positive closed current of bidegree  $(1, 1)$  whose potential is  $G \circ F$ .

The study of the operator  $f^*$  turns out to give many interesting informations on  $f$  and on its dynamics (see [Si99]). When  $f$  is not holomorphic, for any positive closed current  $T$  of bidegree  $(1, 1)$ , the current  $f^* T$  admits singularity points even if  $T$  has a smooth potential. The computation of Lelong numbers of  $f^* T$  can be viewed as a quantitative measure of how bad the singularities of this current are. The estimate 4) allows us to extend a result of [D98].

PROPOSITION 9. — *Let  $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$  be a rational map with maximal generic rank and  $T$  be a positive closed current of bidegree  $(1,1)$ . Then  $\nu(f^*T, p) > 0$  if and only if either  $p \in I(f)$  or  $\nu(T, f(p)) > 0$ .*

*Proof.* — Assume that  $p \notin I(f)$ . As  $f$  has generic maximal rank, we can apply Theorem 2. This yields a constant  $C_f > 0$  such that

$$\nu(T, f(p)) \leq \nu(f^*T, p) \leq C_f \cdot \nu(T, f(p)).$$

And it follows that  $\nu(f^*T, p) > 0$  if and only if  $\nu(T, f(p)) > 0$ . It remains to check that if  $p$  belongs to  $I(f)$ , then  $\nu(f^*T, p) > 0$ . Choose  $\sigma$  a local section of  $\pi$  around  $p$ , and  $G \in \text{PSH}(\mathbb{C}^{k+1})$  a potential for  $T$ . One can find a constant  $A > 0$  so that

$$|F(\sigma(z))| \leq A|z - p|$$

for points  $z$  near  $p$ . As the function  $G$  satisfies an homogeneity relation, one can bound it by

$$G(Z) \leq B \log |Z| + O(1),$$

with  $B > 0$ . We thus have

$$G(F(\sigma(z))) \leq B \log |z - p| + O(1) \quad \text{and} \quad \nu(f^*T, p) \geq B > 0,$$

which concludes the proof.  $\square$

NOTE. — The main theorem has been proved independently by C. Kiselman (see [K99]) with a different method. His proof relies on volume estimates of sublevel sets of psh functions.

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**2. Proof of the main theorem**

We shall first prove the equivalence between the first three assertions. We conclude by proving  $4) \Rightarrow 3)$ , and  $1) \Rightarrow 4)$ .

$1) \Rightarrow 2)$ . — We assume that  $f$  has generic maximal rank equal to  $n$ . If  $u \in \text{PSH}(\mathbb{C}^n, 0)$  is non degenerate, the psh function  $u \circ f$  can not be identically  $-\infty$  as the range of  $f$  contains some open ball. Hence  $f^*T$  is well-defined for any closed positive current  $T$  of bidegree  $(1, 1)$ . For a sequence of positive closed  $(1, 1)$  current  $T_j \rightarrow T$  converging weakly towards  $T$ , one can find a sequence  $u_j$  of psh potential of  $T_j$  converging in  $L^1_{\text{loc}}$  to  $u$  a psh potential for  $T$ . It remains to check that  $u_j \circ f \rightarrow u \circ f$  in  $L^1_{\text{loc}}$ .

As  $f$  has maximal generic rank,  $u_j \circ f \rightarrow u \circ f$  almost everywhere. Now one can extract a subsequence  $u_{j_k} \circ f$  converging in  $L^1_{\text{loc}}$  to a psh function (see [Ho83] p.94). As any such limit should be equal to  $u \circ f$ , we infer  $u_j \circ f \rightarrow u \circ f$  in  $L^1_{\text{loc}}$ , thus  $f^*T_j \rightarrow f^*T$  in the weak topology.

$2) \Rightarrow 3)$ . — If the range  $f(\mathbb{C}^n, 0)$  is pluripolar, one can find  $u \in \text{PSH}(\mathbb{C}^n, 0)$  non-degenerate such that  $u \circ f \equiv -\infty$ . In that case, if  $T := \text{dd}^c u$ ,  $f^*T$  is not defined.

We also give an example of a sequence of positive closed currents of bidegree  $(1, 1)$  so that  $T_n \rightarrow T$ ,  $f^*T_n$  and  $f^*T$  are all well-defined, but for which the sequence  $f^*T_n$  fails to converge to  $f^*T$ . For this, work in the unit ball, and take  $f(z, w) = (0, w)$ ,  $T_n = \text{dd}^c u_n$ , with

$$u_n(z, w) = \max\{n^{-1} \log |z|, -2 + |w|^2\}.$$

Then  $T_n \rightarrow 0$  but  $f^*T_n = \text{dd}^c |w|^2$ .

$3) \Rightarrow 1)$ . — We only sketch the proof. We proceed by induction on  $m$ . Assume  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  is a holomorphic germ such that  $\text{rk } Df_z$  the rank of  $Df_z$  is smaller than  $n - 1$  for any  $z \in (\mathbb{C}^m, 0)$ . Set

$$N := \max\{\text{rk } Df_z\} \leq n - 1,$$

and define for each  $k \leq N$ ,

$$V_k := \{z \in (\mathbb{C}^m, 0); \text{rk } Df_z \leq k\}.$$

By assumption,  $V_N$  contains an open neighborhood of the origin. Define

$$W := V_N - V_{N-1}.$$

The set  $V_k$  is the set where all minors of  $Df_z$  of size  $k + 1$  have zero determinant, and hence defines a closed analytic subspace of  $(\mathbb{C}^m, 0)$ .



Hence  $W$  is a Zariski open set of  $V_N$ . Now, on  $W$  the rank of the differential of  $f$  is constant equal to  $N$ . We can thus apply locally the constant rank theorem. Take any countable covering  $\{U_i\}_{i \in I}$  of  $W$  by open subsets such that for each  $i \in I$ , the set  $f(U_i)$  is a (non-closed) analytic subset of  $(\mathbb{C}^n, 0)$  of dimension  $N < n$ . For any  $i \in I$   $f(U_i)$  is pluripolar. A countable union of pluripolar sets remains pluripolar, hence  $f(W) = \bigcup_{i \in I} f(U_i)$  is pluripolar.

As  $\dim(V_{N-1}) < m$ , we can apply the induction hypothesis to conclude that

$$f(\mathbb{C}^m, 0) = f(W) \cup f(V_{N-1})$$

is pluripolar too.

The implication 4)  $\Rightarrow$  3) follows from 2)  $\Rightarrow$  3).

In fact, we even have that when the range of  $f$  is pluripolar, the supremum of  $(\nu(T, 0))^{-1} \nu(f^*T, 0)$  over all positive closed current  $T$  of bidegree  $(1, 1)$  for which  $f^*T$  is well-defined, is not finite.

Take  $u \in \text{PSH}(\mathbb{C}^n, 0)$  non-degenerate such that  $u \circ f \equiv -\infty$ . For any  $\alpha > 0$ , define

$$v_\alpha(z) := \max\{\alpha \log |z|, u(z) + \log |z|\}.$$

Then

$$\nu(f^*v_\alpha, 0) = \alpha \cdot \nu(\log |f|, 0), \quad \nu(v_\alpha, 0) = \min\{\alpha, \nu(u, 0) + 1\}.$$

Hence for  $\alpha \geq \nu(u, 0) + 1$ ,

$$(\nu(\text{dd}^c v_\alpha, 0))^{-1} \nu(f^* \text{dd}^c v_\alpha, 0) = C\alpha$$

with  $C = (\nu(u, 0) + 1)^{-1} \nu(\log |f|, 0)$ .

1)  $\Rightarrow$  4). — Let us first prove the following general result.

LEMMA 10. — *If  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  is an arbitrary holomorphic germ, and  $T$  is a positive closed current of bidegree  $(1, 1)$  so that  $f^*T$  is well-defined, one has the inequality*

$$\nu(f^*T, 0) \geq \nu(T, 0)$$

between Lelong numbers.

*Proof.* — We fix  $u \in \text{PSH}(\mathbb{C}^n, 0)$  a local potential for  $T$ . We always have  $|f(Z)| \leq A|Z|$  for some constant, so that the estimate

$$u(Z) \leq \nu(T, 0) \log |Z| + O(1)$$

implies

$$u(f(Z)) \leq \nu(T, 0) \log |Z| + O(1)$$

which gives us the stated inequality.  $\square$

We now proceed with the proof of the upper bound for  $\nu(f^*T, 0)$  given in 4). As before,  $u$  will denote a local potential for  $T$ .

Let us show how to reduce the proof of this estimate to the equidimensional case *i.e.* when  $n = m$ .

We assume the estimate has already been proved for  $n = m$ . By assumption, the rank of the Jacobian derivative of  $f$  is generically equal to  $n$ . We can therefore find a closed embedding

$$i: L = (\mathbb{C}^n, 0) \hookrightarrow (\mathbb{C}^m, 0)$$

of a piece of  $n$ -plane into  $(\mathbb{C}^m, 0)$  such that the rank of the Jacobian derivative of the restriction

$$\bar{f} := f \circ i$$

to  $(\mathbb{C}^n, 0)$  is also generically equal to  $n$ . We can now apply the estimate to  $\bar{f}$  and use Lemma 10. We get

$$\nu(f^*T, 0) \leq \nu(i^* \circ f^*T, 0) \leq \nu(\bar{f}^*T, 0) \leq C_{\bar{f}} \cdot \nu(T, 0).$$

Let us deal now with the equidimensional case. The assumption on  $f$  can be rewritten as its Jacobian derivative does not vanish identically on a neighborhood of the origin.

Take a line  $L$  passing through 0 intersecting  $\text{Crit}(f)$  the critical set of  $f$  only at 0, and not tangent to any irreducible component of  $\text{Crit}(f)$ . We can assume it is given in coordinates  $z = (z_1, \dots, z_n)$  by

$$L := \{z_2 = \dots = z_n = 0\}.$$

We can find an open cone around this line  $L$

$$\mathcal{C} := \{z \in U; \text{dist}(z, L) < \varepsilon|z|\}$$

such that  $\mathcal{C} \cap \text{Crit}(f) = \emptyset$ .

Instead of working in this cone, it is more convenient to work on an open set. We thus consider the blow-up  $\pi$  of the origin 0, and replace the germ  $f$  by the composition  $g := f \circ \pi$ . In coordinates,

$$\pi(z) = (z_1, z_1 z_2, \dots, z_1 z_n).$$

We look at  $g$  in the open set  $\overline{\pi^{-1}\{\mathcal{C}\}}$ . Define

$$E = \pi^{-1}\{0\} = \{z_1 = 0\}.$$

Let us point out some special properties of the map  $g$ .

- 1)  $\text{Crit}(g) = E$ .
- 2)  $g^{-1}\{0\} = E$ .

We can thus write the Jacobian determinant of  $g$  under the form

$$Jg(z) = z_1^N \psi(z)$$

for some integer  $N \in \mathbb{N}$  and some holomorphic function  $\psi$  which does not vanish at any point of  $E$ . In a sufficiently small neighborhood  $V$  of the origin, we can find a constant  $C > 0$  such that for all  $z \in V$

$$(1) \quad |Jg(z)| \geq C|z_1|^N.$$

For the proof of Remark 3 and Corollary 4, we will need the following estimation on the integer  $N$ . It gives precisely a control on the constant  $C$  of assertion 4) of the theorem.

LEMMA 11. — *The integer  $N$  introduced above can be chosen as*

$$N = \mu(Jf, 0) + n - 1,$$

where  $\mu(Jf, 0)$  is the order of vanishing of the holomorphic function  $Jf$  at the point 0.

*Proof.* — Set  $N_0 := \mu(Jf, 0)$ . We first check that for a (generic) suitable choice of line  $L$ , one has in a small cone  $\mathcal{C}$  around  $L$  as above

$$(2) \quad |Jf(z)| \geq C|z|^{N_0}.$$

Expand the holomorphic jacobian determinant  $Jf$  in power series

$$Jf = \sum_{k \geq N_0} h_k$$

where  $h_k$  is a homogeneous polynomial of degree  $k$  and  $h_{N_0}$  is not identically zero. Let  $\mathbb{P}^{n-1}$  be the set of complex lines in  $\mathbb{C}^n$  passing through the origin, and for a point  $z \in \mathbb{C}^n$  set  $L_z = \mathbb{C}z$ . By homogeneity of  $h_{N_0}$ , one can define the continuous function  $H: \mathbb{P}^{n-1} \rightarrow \mathbb{R}_+$  by

$$H(L_z) = |z|^{-N_0} |h_{N_0}(z)|.$$

Take a generic line  $L$  such that  $H(L) > 0$ . Then for all lines  $L'$  close to  $L$ , one has  $H(L') \geq \frac{1}{2}H(L)$ . Hence in a small cone  $\mathcal{C}$  around  $L$ , one has  $H(L_z) \geq \frac{1}{2}H(L)$ .

We infer for all  $z \in \mathcal{C}$ ,

$$\begin{aligned} |f(z)| &\geq \left| h_{N_0}(z) - \sum_{k \geq N_0+1} h_k(z) \right| \\ &\geq |h_{N_0}(z)| - \left| \sum_{k \geq N_0+1} h_k(z) \right| \\ &\geq 2^{-1}H(L)|z|^{N_0} - C'|z|^{N_0+1} \geq C|z|^{N_0}, \end{aligned}$$

for some constants  $C, C' > 0$ .

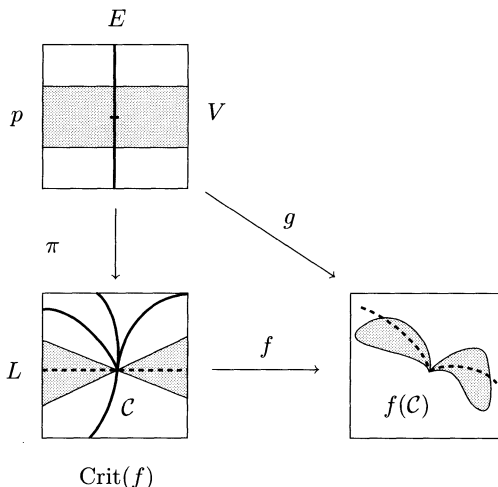
Now a direct computation yields

$$\det(D\pi_z) = z_1^{n-1}.$$

Therefore, if we have chosen a line  $L$  so that equation (2) applies, we get for all  $z \in \mathcal{C}$ ,

$$\begin{aligned} |\det(Dg_z)| &= |\det(D\pi_z) \cdot \det(Df_{\pi(z)})| \\ &\geq |z_1|^{n-1} \cdot C|z_1|^{\mu(Jf,0)}, \end{aligned}$$

which concludes the proof of Lemma 11.  $\square$



In the sequel, we will assume that  $V$  is a small ball in  $\mathbb{C}^n$  endowed with the usual euclidean metric. If  $r > 0$  and  $K$  is a compact set, we set

$$B(K, r) := \{z; \text{dist}(z, K) < r\}.$$

The key lemma is:

LEMMA 12. — *There exists two integers  $N_0, N_1 \in \mathbb{N}^*$ , and two positive constants  $C_0, C_1 > 0$  such that for all  $z \in V$ ,*

$$g(B(z, C_0|z_1|^{N_0})) \supset B(g(z), C_1|z_1|^{N_1}).$$

Moreover, we can choose  $N_0 = N + 1$ , and  $N_1 = 2N + 1$  (with the above notations).

*Proof.* — The idea is to approximate the range of  $g(B(z, r))$  by  $Dg_z(B(z, r))$  and estimate the size of the latter.

We have  $|Jg(z)| \geq C|z_1|^N$  for all  $z \in V$ . In  $V$ , all eigenvalues of  $Dg_z$  are uniformly bounded by some constant  $D > 0$ . Therefore for all  $z \in V - E$ ,

$$|Dg_z^{-1}|^{-1} \geq \inf\{|\lambda|; \lambda \in \text{Spec}(Dg_z)\} \geq \frac{C}{D^{n-1}}|z_1|^N.$$

And for all  $z \in V$ , for all  $r > 0$ ,

$$Dg_z(B(z, r)) \supset B(g(z), C'|z_1|^N r),$$

for some constant  $C' > 0$ . Now by Taylor's formula, there exists another constant  $C'' > 0$  such that for all  $z, w \in V$ ,

$$|g(w) - g(z) - Dg_z \cdot (w - z)| \leq C''|w - z|^2.$$

If we choose  $M > N$  and take  $r = |z_1|^M$ , we infer for  $z$  sufficiently small

$$g(B(z, |z_1|^M)) \supset B(g(z), C'|z_1|^{N+M} - C''|z_1|^{2M}),$$

which gives the desired result with  $N_1 = N + M$ .  $\square$

To conclude, we follow Diller [D98]. Define

$$\Delta_r := L \cap \{|z| \leq r\}.$$

We first apply Lemma 12 to each point of the set  $\partial\Delta_r$ . We obtain

$$(3) \quad g(B(\partial\Delta_r, C_0r^{N_0})) \supset B(\partial g(\Delta_r), C_1r^{N_1}).$$

We consider now translated of  $g(\Delta_r)$  by vectors  $z$  of norm  $|z| < C_1r^{N_1}$ . The estimate (3) tells us that  $\partial(z + g(\Delta_r))$  is still included in the range of  $g$ . We have more precisely for all  $|z| \leq C_1r^{N_1}$ ,

- 1)  $z \in z + g(\Delta_r)$ ,
- 2)  $\partial(z + g(\Delta_r)) \subset g(B(\partial\Delta_r, C_0r^{N_0}))$ .

We are now in position to prove the desired inequality. We start with

$$u(g(z)) \leq \nu(g^*u, 0) \log |z| + D$$

for some constant  $D \in \mathbb{R}$ . We want to prove an analog estimate for  $u$ . Fix  $z \in V$  and  $r > 0$  such that  $|z| < C_1 r^{N_1}$ . Then the maximum principle applied to  $u$  on the analytic disk  $z + g(\Delta_r)$  yields

$$\begin{aligned} u(z) &\leq \max_{z+g(\Delta_r)} u \leq \max_{\partial(z+g(\Delta_r))} u \\ &\leq \max_{g(B(\partial\Delta_r, C_0 r^{N_0}))} u \\ &\leq \max_{w \in B(\partial\Delta_r, C_0 r^{N_0})} u(g(w)) \\ &\leq \max_{w \in B(\partial\Delta_r, C_0 r^{N_0})} \nu(g^*u, 0) \log |w| + D \\ &\leq \nu(g^*u, 0) \log r + D' \end{aligned}$$

for  $D' := D + \nu(g^*u, 0) \log(\frac{3}{2})$  (by possibly reducing  $C_0$  we can assume that  $C_0 r^{N_0-1} \leq \frac{1}{2}$ ). As this is true for any  $r$  satisfying  $|z| \leq C_1 r^{N_1}$ , we obtain

$$u(z) \leq \frac{1}{N_1} \nu(g^*u, 0) \log |z| + D''.$$

Thus  $\nu(u, 0) \geq N_1^{-1} \nu(g^*u, 0)$ . To conclude the proof we use the general inequality in Lemma 10

$$\nu(u, 0) \geq \frac{1}{N_1} \nu(g^*u, 0) \geq \frac{1}{N_1} \nu((f \circ \pi)^*u, 0) \geq \frac{1}{N_1} \nu(f^*u, 0).$$

The proof combined with Lemmas 11 and 12 gives more precisely (see Remark 3):

LEMMA 13. — *If  $f: (\mathbb{C}^n, z) \rightarrow (\mathbb{C}^n, f(z))$  is a germ of holomorphic map of maximal generic rank, then for any positive closed current  $T$  of bidegree  $(1,1)$ , one has the inequality*

$$\nu(f^*T, z) \leq (2(n - 1 + \mu(Jf, 0)) + 1) \cdot \nu(T, f(z))$$

between Lelong numbers.

*Proof of Corollary 4.* — We localize first the problem and assume that  $X = B^m(0, 1)$ ,  $Y = B^n(0, 1)$  are unit balls respectively in  $\mathbb{C}^m$  and  $\mathbb{C}^n$ . As before, it is sufficient to prove it in the equidimensional case *i.e.*  $X = Y = B^n(0, 1)$ .

As  $f$  has maximal generic rank, we can apply Lemma 13 at each point  $z \in K$ . Now on the compact set  $K$ , the function  $z \mapsto \mu(Jf, z)$  is upper semi continuous, hence bounded above by a constant  $C_K$ . This yields Corollary 4.  $\square$

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