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TEMPERED SUBGROUPS AND REPRESENTATIONS WITH MINIMAL DECAY OF MATRIX COEFFICIENTS

BY HEE OH (*)

ABSTRACT. — We present a function F for each simple real linear Lie group G with real rank at least 2 such that F bounds from above all the K -matrix coefficients of non-trivial irreducible spherical unitary representations of G where K is a maximal compact subgroup of G . This enables us to determine when a closed subgroup H is a (G, K) -tempered subgroup of G : for example, if the restriction $F|_H$ of F to H lies in $L^{1-\epsilon}(H)$. We also prove that this function F is the best possible for G a real-split group of type A_n or C_n and as a consequence, we obtain that if H is semisimple, then H is a (G, K) -tempered subgroup of G if and only if $F|_H$ lies in $L^1(H)$.

RÉSUMÉ. — SOUS-GROUPES TEMPÉRÉS ET REPRÉSENTATIONS. — Nous associons une fonction F à chaque groupe de Lie G , linéaire, réel simple de rang réel au moins 2, telle que F donne une borne supérieure pour tous les coefficients matriciels K -finis des représentations unitaires sphériques irréductibles de G , où K un sous-groupe compact maximal de G . Ceci nous permet de déterminer quand un sous-groupe fermé H de G est (G, K) -tempéré; c'est le cas par exemple si la restriction de F à H est dans $L^{1-\epsilon}(H)$. Nous prouvons aussi que cette fonction F est la meilleure possible pour un groupe réel déployé G de type A_n ou C_n , et comme conséquence, nous obtenons que si H est semi-simple, alors H est un sous-groupe (G, K) -tempéré de G si et seulement si $F|_H$ est dans $L^1(H)$.

1. Introduction

Let G be a connected semisimple linear Lie group and K a maximal compact subgroup of G . We say that a unitary representation of G is *spherical* if it has a K -invariant vector. For a unitary spherical represen-

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tation ρ , we will use the term “ K -matrix coefficients of ρ ” to refer to its matrix coefficients with respect to K -invariant unit vectors.

In this paper we are interested in the asymptotic behavior of the K -matrix coefficients of spherical unitary representations of G when restricted to a closed subgroup H of G . One motivation comes from the notion “ (G, K) -tempered subgroups” of G defined by Margulis [10]. That is, a closed subgroup H of G is called (G, K) -tempered if there exists a (positive) function $q \in L^1(H)$ such that for any non-trivial irreducible spherical unitary representation ρ of G ,

$$|\langle \rho(h)v, w \rangle| \leq q(h)\|v\| \cdot \|w\|$$

for all $h \in H$ and any K -fixed vectors v and w . Note that any compact subgroup of G is a (G, K) -tempered subgroup for a trivial reason. Margulis also showed in [10] that if a closed subgroup H is a (G, K) -tempered subgroup, then for any non-compact subgroup F of H , the quotient G/F does not allow a compact quotient by a discrete subgroup of G (see [6] for a survey on the general problem).

We denote by \hat{G} (resp. \hat{G}_K) the set of equivalence classes of non-trivial irreducible unitary (resp. spherical) representations.

In this paper we first present a “good upper bound function” for K -matrix coefficients for all representations in \hat{G}_K for a simple real linear Lie group G with real rank at least 2. Secondly we show that in simple real-split linear Lie group of type A_n or C_n this function is in fact the best possible by exhibiting a spherical representation of G in \hat{G}_K whose K -matrix coefficients are bounded below by this function. We now formulate the main results.

The notation $[x]$ denotes the largest integer which is not greater than x .

THEOREM A. — *Let G be a connected simple real linear Lie group with real rank $n \geq 2$, K a maximal compact subgroup, B a minimal parabolic subgroup, $A \subset B$ a maximal \mathbb{R} -split torus, $A^+ \subset A$ the positive Weyl chamber given by the choice of B . Denote by Φ' the set of all non-multipliable roots in the relative root system $\Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ where \mathfrak{a} and \mathfrak{g} are the Lie algebras of A and G respectively. Let $\alpha_1, \dots, \alpha_n$ be the basis of Φ' whose subscripts are determined by the highest weight given in section 2.1.*

Then for any $\epsilon > 0$, there exists a constant C (depending on ϵ) such that for any $\rho \in \hat{G}_K$ and f_0 a K -invariant unit vector of ρ ,

$$|\langle \rho(g)f_0, f_0 \rangle| \leq CF(g)^{1-\epsilon} \quad \text{for any } g \in G$$

where F is the K -bi-invariant function defined on A^+ as follows according to the type of Φ' :

Φ'	$-\log F$
$A_n, n \geq 2$	$\begin{cases} \sum_{i=1}^{\frac{1}{2}(n-1)} \frac{1}{2} i \alpha_i + \sum_{i=\frac{1}{2}(n+1)}^n \frac{1}{2} (n-i+1) \alpha_i & \text{for } n \text{ odd,} \\ \sum_{i=1}^{\frac{1}{2}n} \frac{1}{2} i \alpha_i + \frac{1}{4} n \alpha_{\frac{1}{2}n+1} + \sum_{i=\frac{1}{2}n+2}^n \frac{1}{2} (n-i+1) \alpha_i & \text{for } n \text{ even,} \end{cases}$
$B_n, n \geq 2$	$\sum_{i=1}^{[\frac{1}{2}n]} i \alpha_i + \sum_{i=[\frac{1}{2}n+1]}^n \frac{1}{2} n \alpha_i,$
$C_n, n \geq 2$	$\sum_{i=1}^{n-1} i \alpha_i + \frac{1}{2} n \alpha_n,$
$D_n, n \geq 4$	$\sum_{i=1}^{[\frac{1}{2}n]} i \alpha_i + \sum_{i=[\frac{1}{2}n+1]}^{n-2} \frac{1}{2} n \alpha_i + \frac{1}{4} n (\alpha_{n-1} + \alpha_n),$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$
E_7	$2\alpha_1 + \frac{7}{2} \alpha_2 + 4\alpha_3 + 6\alpha_4 + \frac{9}{2} \alpha_5 + 3\alpha_6 + \frac{3}{2} \alpha_7,$
E_8	$2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 8\alpha_4 + 7\alpha_5 + 5\alpha_6 + 3\alpha_7 + \alpha_8,$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,$
G_2	$2\alpha_1 + \alpha_2.$

COROLLARY B. — *With the same notation as in Theorem A, let H be a closed subgroup of G . If the restriction $F|_H$ of F to H is in $L^{1-\epsilon}(H)$ for some $\epsilon > 0$, then H is a (G, K) -tempered subgroup of G .*

REMARKS

1) Suppose further that H is a connected semisimple Lie subgroup of G such that $A \cap H$ is a maximal \mathbb{R} -split torus of H and $B \cap H$ is a minimal parabolic subgroup of H . Let δ_H denote the modular function of $B \cap H$, that is, the product of all positive roots including the multiplicity.

Let $\lambda_1, \dots, \lambda_m$ be the fundamental weights of the Lie algebra of H corresponding to $A \cap H$ and $B \cap H$.

For any two weights α and β of the Lie algebra of H , we write

$$\alpha < \beta \quad \text{if } (\alpha, \lambda_j) < (\beta, \lambda_j) \quad \text{for all } 1 \leq j \leq m.$$

Then the condition $F|_H \in L^{1-\epsilon}(H)$ is equivalent to

$$-\log F|_{A^+ \cap H} > \log \delta_H;$$

which is again equivalent to the condition $F|_H \in L^1(H)$.

2) If the restriction $F|_H$ is $L^{k-\epsilon}(H)$ -integrable for some $\epsilon > 0$ and some positive integer k , then the diagonal embedding $\delta_k(H)$ of H into the group $\prod_{i=1}^k G_i$ is a $(\prod_{i=1}^k G_i, \prod_{i=1}^k K_i)$ -tempered subgroup of $\prod_{i=1}^k G_i$ where $G_i = G$ and $K_i = K$ for all $1 \leq i \leq k$. To see this, it is enough to note that for any non-trivial irreducible spherical representation ρ of $\prod_{i=1}^k G_i$, the restrictions of the K -matrix coefficients of ρ to $\delta_k(H)$ are bounded by $(F|_H)^{k(1-\epsilon)}$.

For a unitary representation ρ of G , ρ is said to be *strongly* L^q if there is a dense subset V in the Hilbert space attached to ρ such that the matrix coefficients of ρ with respect to the vectors in V lie in $L^q(G)$. Let $p(G)$ be the smallest real number such that for any $\rho \in \widehat{G}$, ρ is strongly L^q for any $q > p(G)$ (cf. [7]). Similarly let $p_K(G)$ be the smallest real number such that for any $\rho \in \widehat{G}_K$, the K -matrix coefficients of ρ are $L^q(G)$ -integrable for any $q > p_K(G)$. The estimate of the Harish-Chandra function Ξ of G shows that $p_K(G)$ is at least 2 (cf. [3]) and hence G cannot be a (G, K) -tempered subgroup of itself. The method used in proving Theorem A yields upper bounds for both $p(G)$ and $p_K(G)$.

The following follows from Remark 1 after Corollary B.

COROLLARY C. — *With the same notation as in Theorem A, let δ_G be the modular function of B (cf. Table 3.7). Define*

$$r(G) = \max \left\{ \frac{\text{the coefficient of } \alpha_i \text{ in } \log \delta_G}{\text{the coefficient of } \alpha_i \text{ in } -\log F}; i = 1, \dots, n \right\}.$$

Then $p(G) \leq r(G)$ and $p_K(G) \leq r(G)$.

If G is split over \mathbb{R} , $r(G)$ is as follows :

$\Phi = \Phi' :$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$r(G) :$	$2n$	$2n$	$2n$	$2(n-1)$	16	18	58	11	6.

For $n \geq 3$, Vogan's classification of unitary duals for $GL_n(D)$ yields that

- for $G = SL_n(D)$, $D = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , $p(G)$ is $2(n-1)$, $2(n-1)$ and $2n-1$ respectively and
- for $Sp_{2n}(\mathbb{R})$, it follows from Howe's result in [3] that $p(G) = 2n$.

The number $p(G)$ in other classical group cases was calculated by Li [7] and was given an upper bound by Li and Zhu [8] in exceptional split group cases. We remark that the numbers $r(G)$ in Corollary C coincide with $p(G)$ calculated in [7] for all classical real split groups except B_n type. For a split group of type E_6 , by obtaining $r(G) = 16$, we improve the bound for $p(G)$ in [8].

We state a theorem which yields a necessary and sufficient condition for a closed semisimple subgroup to be a (G, K) -tempered subgroup in a simple real split linear Lie group G of type A_n or C_n . Let G be either $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. The group $Sp_{2n}(\mathbb{R})$ is defined by the bi-linear form $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where \bar{I}_n denotes the skew diagonal $n \times n$ -identity matrix. Set

$$K = SO_n(\mathbb{R}) \quad \text{and} \quad Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R})$$

respectively. Define the parabolic subgroup P of G as follows :

- for $G = SL_n(\mathbb{R})$, $P = \{(g_{ij}) \in G \mid g_{i1} = 0 \text{ if } i \neq 1\}$,
- for $G = Sp_{2n}(\mathbb{R})$, $P = \{(g_{ij}) \in G \mid g_{i1} = 0, g_{2nj} = 0 \text{ if } i \neq 1, j \neq 2n\}$.

Note that P is the maximal parabolic subgroup which stabilizes the line $\mathbb{R}e_1$. We fix an ordering in the root system of G so that the positive Weyl chamber A^+ is as follows :

$$\begin{aligned}
 SL_n(\mathbb{R}), \quad A^+ &= \left\{ \text{diag}(a_1, \dots, a_n) \mid \prod_{i=1}^n a_i = 1, \right. \\
 &\quad \left. a_i \geq a_{i+1} \text{ for all } 1 \leq i \leq n-1 \right\}, \\
 Sp_{2n}(\mathbb{R}), \quad A^+ &= \left\{ \text{diag}(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \right. \\
 &\quad \left. \text{for all } 1 \leq i \leq n-1 \right\}.
 \end{aligned}$$

EXAMPLE. — The function F defined in Theorem A is as follows :

- for $G = \mathrm{SL}_n(\mathbb{R})$,

$$F(a) = \begin{cases} \prod_{i=1}^{\frac{1}{2}n} a_i^{-1} & \text{for } n \text{ even,} \\ a_{\frac{1}{2}(n+1)}^{-\frac{1}{2}} \prod_{i=1}^{\frac{1}{2}(n-1)} a_i^{-1} & \text{for } n \text{ odd;} \end{cases}$$

- for $G = \mathrm{Sp}_{2n}(\mathbb{R})$,

$$F(a) = \prod_{i=1}^n a_i^{-1} \quad \text{where } a \in A^+.$$

THEOREM D. — Let G be $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{Sp}_{2n}(\mathbb{R})$ and P , K and A^+ be as above.

- (1) For any $\epsilon > 0$, there exist constants C_1 and C_2 such that

$$C_1 F(a) \leq |\langle \mathrm{Ind}_P^G(I)(a)f_0, f_0 \rangle| \leq C_2 F(a)^{1-\epsilon}$$

for any $a \in A^+$ and for any K -invariant unit vector f_0 in $\mathrm{Ind}_P^G(I)$.

- (2) If a closed subgroup H of G is (G, K) -tempered, $F|_H$ is in $L^1(H)$.

(3) A closed semisimple subgroup H of G is (G, K) -tempered if and only if $F|_H$ is in $L^1(H)$.

- (4) $p_K(G) = r(G) = p(G)$.

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2. A maximal system of strongly orthogonal roots in each irreducible root system

2.1. — Let Φ be an irreducible reduced root system with a fixed ordering. Denote by Φ^+ the set of positive roots and by $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots of Φ . The subscripts of α_i 's are determined by the

following choice of the highest root [2].

Φ	the highest root
A_n	$\alpha_1 + \alpha_2 + \cdots + \alpha_n,$
B_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n,$
C_n	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n,$
D_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n,$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7,$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8,$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,$
G_2	$3\alpha_1 + 2\alpha_2.$

We define the number $N(\Phi)$ as follows :

$$N(\Phi) = \begin{cases} \left\lceil \frac{1}{2}(n+1) \right\rceil & \text{for } \Phi = A_n, \\ 2\left\lceil \frac{1}{2}n \right\rceil & \text{for } \Phi = D_n, \\ 4 & \text{for } \Phi = E_6, \\ \text{rank}(\Phi) & \text{for } \Phi = B, C, F_4, G_2, E_7, E_8. \end{cases}$$

2.2. Construction of some strongly orthogonal roots.

Two roots α and β are called *strongly orthogonal* if neither one of $\alpha \pm \beta$ is a root. Consider the family $\mathcal{S}(\Phi)$ of all subsets of Φ^+ whose elements are pairwise strongly orthogonal. We call an element \mathcal{O} in $\mathcal{S}(\Phi)$ a *strongly orthogonal system*.

Let f be the function on $\mathcal{S}(\Phi)$ given by

$$f(\mathcal{O}) = \sum_{\alpha \in \mathcal{O}} \alpha.$$

The aim of this section is to construct an element

$$\mathcal{Q}(\Phi) = \{\gamma_1, \dots, \gamma_{N(\Phi)}\}$$

in $\mathcal{S}(\Phi)$ on which f attains its maximum. For simplicity, we set $N(\Phi) = N$.

We define $\mathcal{Q}(\Phi)$ as follows :

Φ	$\mathcal{Q}(\Phi)$
A_n	$\begin{cases} \gamma_i = \alpha_i + \cdots + \alpha_{n-i+1} & \text{for } i \leq N-1, \\ \gamma_N = \begin{cases} \alpha_N & \text{for } n \text{ odd,} \\ \alpha_N + \alpha_{N+1} & \text{for } n \text{ even;} \end{cases} \end{cases}$
B_n	$\begin{cases} \gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_n, \\ \gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} & \text{for } i \leq [\frac{1}{2}n], \\ \gamma_n = \alpha_{\frac{1}{2}(n+1)} + \cdots + \alpha_n & \text{for } n \text{ odd;} \end{cases}$
C_n	$\begin{cases} \gamma_i = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n & \text{for } i \leq N-1, \\ \gamma_N = \alpha_n; \end{cases}$
D_n	$\begin{cases} \gamma_1 = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n, \\ \gamma_2 = \alpha_1 + \cdots + \alpha_{n-1}, \\ \gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} \\ \quad + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \\ \gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} & \text{for } 3 \leq i \leq [\frac{1}{2}n]; \end{cases}$
E_6	$\begin{cases} \gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \\ \gamma_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\ \gamma_3 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \\ \gamma_4 = \alpha_2; \end{cases}$
E_7	$\begin{cases} \gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\ \gamma_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\ \gamma_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \\ \gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\ \gamma_5 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \\ \gamma_6 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\ \gamma_7 = \alpha_2; \end{cases}$
E_8	$\begin{cases} \gamma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\ \gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \\ \gamma_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \\ \gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \\ \gamma_5 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \\ \gamma_6 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\ \gamma_7 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \\ \gamma_8 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6; \end{cases}$
F_4	$\begin{cases} \gamma_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \\ \gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \\ \gamma_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3, \\ \gamma_4 = \alpha_1; \end{cases}$

$$G_2 \begin{cases} \gamma_1 = 3\alpha_1 + 2\alpha_2, \\ \gamma_2 = \alpha_1. \end{cases}$$

The following lemma can be easily checked.

LEMMA. — *The set $\mathcal{Q}(\Phi)$ is a strongly orthogonal system.*

PROPOSITION 2.3. — *One has $f(\mathcal{Q}(\Phi)) = \max_{\mathcal{O} \in \mathcal{S}(\Phi)} f(\mathcal{O})$, that is, for any $\mathcal{O} \in \mathcal{S}(\Phi)$, the coefficient of α_i in $f(\mathcal{Q}(\Phi))$ is greater than or equal to the coefficient of α_i in $f(\mathcal{O})$ for each $1 \leq i \leq n$, where n is the rank of Φ .*

Proof. — Let \mathcal{O} be any element in $\mathcal{S}(\Phi)$. We prove this proposition by induction. We can easily check that the proposition is true for $n = 2$. Suppose that $n \geq 3$.

- For $\Phi = A_n$, take any element in \mathcal{O} , say $\alpha = \alpha_i + \cdots + \alpha_{j-1}$, $i < j - 1$. Then $\alpha \leq \gamma_1$ since γ_1 is the highest root. On the other hand $\mathcal{O} - \{\alpha\}$ is contained in $\{\alpha_m + \cdots + \alpha_{\ell-1} \mid m, \ell \notin \{i, j\}\}$, which is a root system of type A_{n-2} . Note that

$$\mathcal{Q}(A_n) \cap \{\alpha_m + \cdots + \alpha_{\ell-1} \mid m, \ell \notin \{i, j\}\} = \mathcal{Q}(A_{n-2}).$$

Therefore by the induction assumption, $f(\mathcal{O} - \{\alpha\}) \leq f(\mathcal{Q}(A_{n-2}))$. Hence we have $f(\mathcal{O}) \leq \gamma_1 + f(\mathcal{Q}(A_{n-2})) \leq f(\mathcal{Q}(A_n))$, proving the claim.

- For $\Phi = B_n$, note that for any $\alpha \in \Phi^+$, we have that the coefficient of α_1 in α is at most 1. Write \mathcal{O} as $\mathcal{O}_1 \cup \mathcal{O}_2$ so that $\beta \in \mathcal{O}_1$ if and only if the coefficient of α_1 in β is 1 and $\mathcal{O}_2 = \mathcal{O}_1^c$. It is not difficult to check that if three positive roots in B_n are mutually strongly orthogonal, then the coefficient of α_1 in at least one of them is 0. Therefore $|\mathcal{O}_1| \leq 2$. We can easily see that for any two strongly orthogonal roots $\beta_1, \beta_2 \in \Phi^+$ such that the coefficient of α_1 in β_i is 1 for both $i = 1, 2$, we have $\beta_1 + \beta_2 \leq \sum_{i=1}^n 2\alpha_i$;

hence $\sum_{\beta \in \mathcal{O}_1} \beta \leq \gamma_1 + \gamma_2$, because $|\mathcal{O}_1| \leq 2$ and $\gamma_1 + \gamma_2 = \sum_{i=1}^n 2\alpha_i$. For $\theta \subset \Delta$, the notation $[\theta]$ denotes the set of the roots in Φ which can be expressed as integral combinations of the roots in θ . Since $\mathcal{O}_2 \subset [\alpha_2, \cdots, \alpha_n]$, $\gamma_3 + \gamma_4 = \sum_{i=2}^n 2\alpha_i$ and $[\alpha_2, \cdots, \alpha_n]$ is a root system of type B_{n-1} , we can proceed by induction as before.

- The argument for D_n is exactly the same as the one for B_n ; so we omit it.

• If Φ is of type C_n , write $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ so that $\beta \in \mathcal{O}_1$ if and only if the coefficient of α_1 in β is positive and $\mathcal{O}_2 = \mathcal{O}_1^c$. It is easy to see that $|\mathcal{O}_1| \leq 1$. Therefore $\sum_{\alpha \in \mathcal{O}_1} \alpha \leq \gamma_1$, for γ_1 is the highest root in Φ .

Since $\mathcal{O}_2 \subset [\alpha_2, \dots, \alpha_n]$, it remains to use induction process.

• For exceptional root system cases, we can prove the proposition by checking each root system case by case. \square

As a corollary of the above proposition, we obtain that $\mathcal{Q}(\Phi)$ is a maximal element in $\mathcal{S}(\Phi)$ with respect to the inclusion ordering.

REMARK. — I learned from E. Vinberg that this construction of a strongly orthogonal system coincides with the so called Kostant's cascade construction (cf. [9]), if Φ is one of the types A_n, C_n or G_2 . But in all cases the cardinalities of the sets in Kostant's cascade construction coincide with the numbers $N(\Phi)$, which are the cardinalities of $\mathcal{Q}(\Phi)$ in our construction. We note that not all maximal strongly orthogonal systems in Φ have the same cardinality. For example, $\{\alpha_2, \alpha_4, 2\alpha_2 + 3\alpha_3 + 4\alpha_4\}$ is a maximal strongly orthogonal system in the root system of F_4 .

We remark that if Φ is none of A_n, C_n or G_2 , the function f attains its maximum in our construction but not in Kostant's cascade construction.

3. An upper bound function for matrix coefficients in simple non-compact linear Lie groups

3.1. — Let G be a connected semisimple non-compact linear Lie group, B a minimal parabolic subgroup, A a maximal \mathbb{R} -split torus contained in B , A^+ the positive Weyl chamber and K a maximal compact subgroup. Consider a Cartan decomposition of $G : G = KA^+K$. Since the K -matrix coefficients of a spherical unitary representation are K -bi-invariant, they are determined by their restrictions to the A^+ -part. Denote by \mathfrak{g} the Lie algebra of G and by \mathfrak{a} the Lie algebra of A . We denote by $\Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ the set of restricted roots of $(\mathfrak{g}, \mathfrak{a})$, which is endowed with the ordering given by B . If G is split over \mathbb{R} , then $\Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ will be simply denoted by $\Phi(\mathfrak{a}, \mathfrak{g})$. If we fix a Haar measure dg on G , then the modular function δ_G of B is given as

$$\delta_G = \prod_{\alpha \in \Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})} \exp \alpha.$$

It is well known (cf. [3]) that the induced representation $\text{Ind}_B^G(I)$ of the trivial representation of B is irreducible and has a unique (up to a

sign) K -invariant unit vector, say f_0 . The matrix coefficient of $\text{Ind}_B^G(I)$ defined by

$$g \mapsto \langle \text{Ind}_B^G(I)(g)f_0, f_0 \rangle$$

is called the *Harish-Chandra function* of G , which we will denote by Ξ_G . When there is no confusion, Ξ_G will simply be denoted by Ξ .

Harish-Chandra has shown the following :

PROPOSITION (cf. [3]). — *For any $\epsilon > 0$, there exist constants c_1 and c_2 such that, for all $a \in A^+$*

$$c_1 \delta_G^{-1/2}(a) \leq \Xi_G(a) \leq c_2 \delta_G^{-1/2+\epsilon}(a).$$

Moreover the value of Harish-Chandra function Ξ of $\text{SL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{R})$ at $\begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$ for $a_0 > 1$ is asymptotically $(\log a_0)/a_0$ up to some constant multiple.

3.2. — We can write the Haar measure dg of G in terms of the Cartan decomposition KA^+K as follows :

$$dg = \Delta(a) dk_1 da dk_2$$

where $\Delta(a)$ is a positive function on A^+ satisfying

$$d_1(t) \delta_G(a) \leq \Delta(a) \leq d_2 \delta_G(a)$$

for all $a \in \{a \in A^+ \mid |\alpha(a)| \geq t \text{ for all } \alpha \in \Phi_{\mathbb{R}}^+(\mathfrak{a}, \mathfrak{g})\}$ and for some constants $d_1(t)$ and d_2 if $t > 1$ (cf. [3]).

For a K -matrix coefficient $\phi(g) = \langle \rho(g)v, w \rangle$ of $\rho \in \widehat{G}_K$, it is well known that $\phi \in L^p(G)$ if and only if $\int_{A^+} |\phi(a)|^p \delta_G(a) da < \infty$.

PROPOSITION 3.3. — *Let H be $\text{SL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{R})$. Suppose that for some $k \geq 2$, H acts on \mathbb{R}^k by a rational representation so that the only H -invariant vector is the origin. Let $H \ltimes \mathbb{R}^k$ be the associated semidirect product. Let ρ be a unitary representation of $H \ltimes \mathbb{R}^k$ without any \mathbb{R}^k -invariant vectors. Then we have*

$$|\langle \rho|_H(h)v, w \rangle| \leq \Xi_H(h) (\dim \langle K \cdot v \rangle \dim \langle K \cdot w \rangle)^{1/2}$$

where $h \in H$, $K = \text{SO}_2(\mathbb{R})$ and v and w are K -finite unit vectors of ρ . Moreover if ρ is spherical, then the K -matrix coefficients of $\rho|_H$ are bounded by Ξ_H .

Proof. — By [12, Thm 7.3.9], the restriction $\rho|_H$ of ρ to H is weakly contained in the infinite sum of the regular representation of H . It is well known (cf. [4, Ch. V, Thm 3.2 1]) that the K -finite (or K -fixed) matrix coefficients of the regular representation of H satisfy the above inequality. \square

In the spirit of Howe's strategy (see [7]) we state the following proposition. The notation \mathfrak{u}_α for $\alpha \in \Phi_{\mathbb{R}}(\mathfrak{a}, \mathfrak{g})$ denotes the root space in \mathfrak{g} corresponding to α .

PROPOSITION 3.4. — *Let G be a connected simple real split linear Lie group. Let $\{\beta_1, \dots, \beta_m\} \subset \Phi^+(\mathfrak{a}, \mathfrak{g})$ be a strongly orthogonal system. Then for any $\epsilon > 0$, there exists a constant C such that for any $\rho \in \widehat{G}_K$ and K -fixed unit vectors v and w of ρ , and for any $a \in A^+$*

$$|\langle \rho(a)v, w \rangle| \leq C \prod_{i=1}^m \exp\left(\left(-\frac{1}{2} + \epsilon\right)\beta_i\right)(a).$$

Proof. — For each $1 \leq i \leq m$, let H_i be the connected closed subgroup of G whose Lie algebra is generated by $\mathfrak{u}_{\pm\beta_i}$. Note that for each $1 \leq i \leq m$,

- (1) H_i is isomorphic to $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$;
- (2) the subgroups $H_i \cap B$, $H_i \cap A$ and $H_i \cap K$ are a minimal parabolic subgroup, a maximal \mathbb{R} -split torus and a maximal compact subgroup of H_i respectively;
- (3) the positive Weyl chamber $A^+(H_i)$ of H_i is contained in A^+ .

It is not difficult to see that

$$A^+ \subset \prod_{i=1}^m A^+(H_i) C_G\left(\prod_{i=1}^m H_i\right)$$

where $C_G(\prod_{i=1}^m H_i)$ denotes the centralizer of $\prod_{i=1}^m H_i$. Since $\{\beta_1, \dots, \beta_m\}$ is a strongly orthogonal system, it follows that $x_i x_j = x_j x_i$ for all $i \neq j$, $x_i \in H_i$ and $x_j \in H_j$. By looking at the root system, it is not difficult to see that for each H_i , there exists an abelian unipotent subgroup U_i of G of dimension at least 2 such that H_i normalizes U_i and $C_G(H_i) \cap U_i$ is trivial. It follows that for each $1 \leq i \leq m$, $H_i \ltimes \mathbb{R}^{k_i}$ can be considered to be a subgroup of G where the semidirect product is as described in Proposition 3.3 and $k_i = \dim U_i$.

Let $\rho \in \widehat{G}_K$, and v and w be K -fixed unit vectors. The restriction $\rho|_{\prod_{i=1}^m H_i}$ can be written as a direct integral $\int_X \rho_\alpha d\mu(\alpha)$ where X is some Borel measure space with measure μ , $\alpha \in X$ and ρ_α is an irreducible

representation of $\prod_{i=1}^m H_i$ for all $\alpha \in X$ (cf. [3]). Without loss of generality, we may assume that all ρ_α 's are non-trivial spherical representations. Any element $a \in A^+$ can be uniquely written as $a = a_1 \cdots a_m c$ where $a_i \in A^+(H_i)$ and $c \in C_G(\prod_{i=1}^m H_i)$. Write $\rho(c)v$ and w as $\int v_\alpha d\mu(\alpha)$ and $\int w_\alpha d\mu(\alpha)$ respectively where v_α and w_α are vectors in ρ_α . Since c centralizes each H_i , $\rho(c)v$ is $K \cap H_i$ -fixed for all $1 \leq i \leq m$. Therefore there is no loss of generality in assuming that for all $\alpha \in X$, v_α and w_α are $K \cap H_i$ -fixed for all $1 \leq i \leq m$. Fix $\alpha \in X$. Then

$$\rho_\alpha|_{\prod_{i=1}^m H_i} = \bigotimes_{i=1}^m \rho_{\alpha i}, \quad v_\alpha = \bigotimes_{i=1}^m v_{\alpha i}, \quad w_\alpha = \bigotimes_{i=1}^m w_{\alpha i},$$

where $\rho_{\alpha i}$ is a spherical irreducible representation of H_i and $v_{\alpha i}$ and $w_{\alpha i}$ are $K \cap H_i$ -fixed vectors for each $1 \leq i \leq m$. By Moore's theorem (cf. [12, Thm 2.1.9]), for each $1 \leq i \leq m$, the representation $\rho_{\alpha i}$ is non-trivial and $\rho_{\alpha i}$ has no U_i -invariant vector.

By Proposition 3.3, we obtain that for each $1 \leq i \leq m$,

$$|\langle \rho_{\alpha i}(a_i)v_{\alpha i}, w_{\alpha i} \rangle| \leq \Xi_{H_i} \|v_{\alpha i}\| \cdot \|w_{\alpha i}\|.$$

Hence

$$\begin{aligned} |\langle \rho(a)v, w \rangle| &\leq \int_\alpha |\langle \rho_\alpha(\prod_{i=1}^m a_i)v_\alpha, w_\alpha \rangle| d\mu(\alpha) \\ &\leq \int_\alpha \prod_{i=1}^m |\langle \rho_{\alpha i}(a_i)v_{\alpha i}, w_{\alpha i} \rangle| d\mu(\alpha) \\ &\leq \int_\alpha \prod_{i=1}^m (\Xi_{H_i}(a_i) \|v_{\alpha i}\| \cdot \|w_{\alpha i}\|) d\mu(\alpha) \\ &= \prod_{i=1}^m \Xi_{H_i}(a_i) \|\rho(c)v\| \cdot \|w\| \\ &\leq \prod_{i=1}^m \Xi_{H_i}(a_i) \|v\| \cdot \|w\| = \prod_{i=1}^m \Xi_{H_i}(a_i). \end{aligned}$$

Note that the modular function δ_{H_i} of $H_i \cap B$ is equal to $\exp(\beta_i)$. Hence by Proposition 3.1, for each i , there exists a constant C_i (not depending on a) such that

$$\Xi_{H_i}(a_i) \leq C_i \exp(-\frac{1}{2} + \epsilon)\beta_i(a).$$

This proves the proposition. \square

3.5. Proof of Theorem A. — It is well known [1, Thm 7.2] that G contains a connected simple closed subgroup G_0 such that G_0 is split over \mathbb{R} , $\text{rank } G_0 = \mathbb{R}\text{-rank } G$ and Φ' is isomorphic to $\Phi(\mathfrak{g}_0, \mathfrak{g}_0 \cap \mathfrak{a})$ where Φ' is the set of all non-multipliable roots in $\Phi_{\mathbb{R}}(\mathfrak{g}, \mathfrak{a})$ and \mathfrak{g}_0 is the Lie algebra of G_0 . Recall the strongly orthogonal system $\mathcal{Q}(\Phi') = \{\gamma_1, \dots, \gamma_{N(\Phi')}\}$ of Φ' we constructed in section 2.2.

The notation \mathfrak{u}_{α} for $\alpha \in \Phi'$ denotes the one-dimensional root subalgebra of \mathfrak{g}_0 . Set $N = N(\Phi')$. For each $1 \leq i \leq N$, we define H_i to be the connected closed subgroup of G_0 whose Lie algebra is generated by $\mathfrak{u}_{\pm\gamma_i}$.

• Case $\Phi' \neq D_{n=2k+1}$. — We note that the restriction $F|_{A^+}$ of the function F to A^+ in Theorem A is equal to $\prod_{i=1}^N \delta_{H_i}^{-1/2}$ or equivalently,

$$F|_{A^+} = \prod_{i=1}^N \exp\left(-\frac{1}{2}\gamma_i\right).$$

Then Theorem A follows from Proposition 3.4.

• Case $\Phi' = D_{n=2k+1}$. — In this case, we define H'_N to be the connected closed subgroup of G_0 whose Lie algebra is generated by $\mathfrak{u}_{\pm(\alpha_{k+1}+2(\alpha_{k+2}+\dots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n)}$, $\mathfrak{u}_{\pm\alpha_k}$ and $\mathfrak{u}_{\pm\alpha_{k+1}}$. Note that the Lie algebra of H'_N is isomorphic to that of $\text{SO}(3, 3)$. We have that

$$\delta_{H'_N} = \exp(4\alpha_k) \left(\prod_{i=k+1}^{n-2} \exp(6\alpha_i) \right) \exp(3\alpha_{n-1}) \exp(3\alpha_n).$$

By [7, Lemma 4.1], the restriction $\rho|_{H'_N}$ is strongly $L^{4+\epsilon}$ for any $\rho \in \widehat{G}$. This implies [3, Cor. 7.2] that the restrictions to H'_N of the K -finite matrix coefficients (with respect to unit vectors) of ρ are bounded by $\Xi_{H'_N}^{1/2}$. It is not difficult to see from the proof of Proposition 3.4 that, when we replace H_N by H'_N , a statement similar to Proposition 3.4 holds, that is, the K -matrix coefficients of ρ for any $\rho \in \widehat{G}_K$ are bounded above by $(\prod_{i=1}^{N-1} \delta_{H_i}^{-1/2}) \delta_{H'_N}^{-1/4}$. Therefore it remains to observe that the function F in Theorem A is given by

$$\begin{aligned} F|_{A^+} = & \left(\prod_{i=1}^{n-2} \exp\left(-\frac{1}{2}\gamma_i\right) \right) \exp(\alpha_k) \left(\prod_{i=k+1}^{n-2} \exp\left(\frac{3}{2}\alpha_i\right) \right) \\ & \times \exp\left(\frac{3}{4}\alpha_{n-1}\right) \exp\left(\frac{3}{4}\alpha_n\right), \end{aligned}$$

which is equal to $\left(\prod_{i=1}^{n-2} \delta_{H_i}^{-1/2}\right) \delta_{H'_N}^{-1/4}$, to complete the proof.

REMARK. — The results in section 2.2 show that the function F is the best possible upper bound for K -matrix coefficients, which can be obtained using Proposition 3.4 when $\Phi' \neq D_{n=2k+1}$. Note that when $\Phi' = D_{n=2k+1}$, we improved F by replacing one $\mathrm{SL}_2(\mathbb{R})$ by $\mathrm{SO}(3, 3)$.

COROLLARY 3.6. — *With G , Φ' and $\alpha_1, \dots, \alpha_n$ as in section 3.5, suppose that $\mathcal{O} = \{\beta_1, \dots, \beta_t\}$ is a strongly orthogonal system of Φ' and that for some number r , the coefficient of α_j in $\sum_{i=1}^t r\beta_i$ is strictly bigger than the coefficient of α_j in $2\log(\delta_G)$ for each $1 \leq j \leq n$. Then*

$$p(G) \leq r \quad \text{and} \quad p_K(G) \leq r.$$

3.7. — In each simple real-split Lie group G , the modular function δ_G of B is given as below (cf. [2]), from which the remark after Corollary C follows.

Φ	$\log \delta$
A_n	$\sum_{i=1}^n i(n-i+1)\alpha_i,$
B_n	$\left(\sum_{i=1}^{n-1} (2ni - i^2)\alpha_i\right) + n^2\alpha_n,$
C_n	$\left(\sum_{i=1}^n (2ni - i^2 + i)\alpha_i\right) + \frac{1}{2}n(n+1)\alpha_n,$
D_n	$\left(\sum_{i=1}^{n-2} (2ni - i^2 - i)\alpha_i\right) + \frac{1}{2}n(n-1)(\alpha_{n-1} + \alpha_n),$
E_6	$16\alpha_1 + 22\alpha_2 + 30\alpha_3 + 42\alpha_4 + 30\alpha_5 + 16\alpha_6,$
E_7	$34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7,$
E_8	$92\alpha_1 + 136\alpha_2 + 182\alpha_3 + 270\alpha_4 + 220\alpha_5 + 168\alpha_6,$ $+ 114\alpha_7 + 58\alpha_8,$
F_4	$16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 22\alpha_4,$
G_2	$10\alpha_1 + 6\alpha_2.$

4. Spherical unitary representations with minimal decay in $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$

4.1. — In this section we will show that the upper bound function F we obtained in Theorem A is the best possible when $G = \mathrm{SL}_n(\mathbb{R})$ or $\mathrm{Sp}_{2n}(\mathbb{R})$. This will be proved by showing that there exists a spherical unitary representation of G whose K -matrix coefficients are bounded from below by a constant multiple of F . Those representations are the induced representations $\mathrm{Ind}_P^G(I)$ of the trivial representation where P is the maximal parabolic subgroup which stabilizes the line $\mathbb{R}e_1$.

4.2. — For the rest of section 4, let G be either $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{Sp}_{2n}(\mathbb{R})$. The group $\mathrm{Sp}_{2n}(\mathbb{R})$ is defined by the bi-linear form $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where \bar{I}_n denotes the skew diagonal $(n \times n)$ -identity matrix. Set

$$K = \mathrm{SO}_n(\mathbb{R}) \quad \text{and} \quad \mathrm{Sp}_{2n}(\mathbb{R}) \cap \mathrm{SO}_{2n}(\mathbb{R})$$

respectively. Define the maximal parabolic subgroup P of G as follows :

- for $G = \mathrm{SL}_n(\mathbb{R})$,

$$P = \{(g_{ij}) \in G \mid g_{i1} = 0 \text{ if } i \neq 1\};$$

- for $G = \mathrm{Sp}_{2n}(\mathbb{R})$,

$$P = \{(g_{ij}) \in G \mid g_{i1} = 0, g_{(2n)j} = 0 \text{ if } i \neq 1, j \neq 2n\}.$$

We fix an ordering in the root system of G so that the positive Weyl chamber A^+ is as follows :

$$\begin{aligned} \mathrm{SL}_n(\mathbb{R}), \quad A^+ = \{ \mathrm{diag}(a_1, \dots, a_n) \mid \prod_{i=1}^n a_i = 1, \\ a_i \geq a_{i+1} \text{ for all } 1 \leq i \leq n-1 \}; \end{aligned}$$

$$\begin{aligned} \mathrm{Sp}_{2n}(\mathbb{R}), \quad A^+ = \{ \mathrm{diag}(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \\ \text{for all } 1 \leq i \leq n-1 \}. \end{aligned}$$

4.3. — We recall the formula for the matrix coefficients of the induced representation $\mathrm{Ind}_P^G(I)$ (cf. [5]). Consider the Langlands decomposition of P : $P = MA_PN$. Denote by \bar{N} the unipotent radical of the opposite parabolic subgroup to P with the common Levi subgroup MA_P .

If g decomposes under the decomposition $G = KMA_PN$, we denote by $\exp H(g)$ the A_P -component of g . It is well known that the representation

space of $\text{Ind}_P^G(I)$ of the trivial representation I of P can be realized as $L^2(\bar{N}, dx)$. If g decomposes under $\bar{N}MA_PN$ as

$$g = \bar{n}(g)m(g)a(g)n(g),$$

then the action is given by

$$\text{Ind}_P^G(I)(g)f(x) = e^{-\delta_0(\log a(g^{-1}x))} f(\bar{n}(g^{-1}x))$$

for any $f \in L^2(\bar{N}, dx)$ and $x \in \bar{N}$, where δ_0 is the half sum of positive N -roots.

Define the vector f_0 of $\text{Ind}_P^G(I)$ as follows :

$$f_0(x) = e^{-\delta_0(H(x))}.$$

It is not difficult to see that f_0 is K -fixed and the matrix coefficient of $\text{Ind}_P^G(I)$ with respect to f_0 is as follows :

$$\langle \text{Ind}_P^G(I)(g)f_0, f_0 \rangle = \int_{\bar{N}} e^{-\delta_0(\log a(g^{-1}x))} e^{-\delta_0(H(\bar{n}(g^{-1}x)))} e^{-\delta_0(H(x))} dx.$$

4.4. — Theorem D follows from the following proposition and Theorem A .

PROPOSITION. — *There exists a constant C such that*

$$CF(a) \leq |\langle \text{Ind}_P^G(I)(a)f_0, f_0 \rangle|$$

where $a \in A^+$ and F is as in Theorem A

Proof of Proposition 4.4.

- Case $G = \text{SL}_n(\mathbb{R})$, $n \geq 3$.

Denote by \tilde{a} the matrix $\text{diag}(a_1, \dots, a_n) \in \text{SL}_n(\mathbb{R})$ and by x the matrix in \bar{N} whose first column is $(1, x_2, \dots, x_n)$, that is, $x.e_1 = (1, x_2, \dots, x_n)$. To simplify notation, set $x_1 = 1$.

The decomposition of $\tilde{a}^{-1}x$ under $\bar{N}MA_PN$ is as follows :

$$a(\tilde{a}^{-1}x) = \text{diag}(a_1, a_1^{-1/(n-1)}, \dots, a_1^{-1/(n-1)}),$$

$$\bar{n}(\tilde{a}^{-1}x) \cdot e_1 = \left(1, \frac{a_1}{a_2}x_2, \dots, \frac{a_1}{a_n}x_n\right).$$

Then $\exp \delta_0(a(\tilde{a}^{-1}x)) = a_1^{-n/2}$ and $H(x) = \|x.e_1\| = \sqrt{\sum_{i=1}^n x_i^2}$.

Therefore

$$\begin{aligned} \langle \text{Ind}_P^G(I)(\tilde{a})f_0, f_0 \rangle &= \int_{\tilde{N}} a_1^{n/2} \|\tilde{n}(\tilde{a}^{-1}x).e_1\|^{-n/2} \|x.e_1\|^{-n/2} dx \\ &= \int_{\mathbb{R}^{n-1}} \left(\sum_{i=1}^n \left(\frac{1}{a_i} \right)^2 x_i^2 \right)^{-n/4} \left(\sum_{i=1}^n x_i^2 \right)^{-n/4} dm \end{aligned}$$

where dm is the standard measure in \mathbb{R}^{n-1} .

Set $k = [\frac{1}{2}(n+1)]$ and let T be the following set :

$$\left\{ (x_2, \dots, x_n) \mid 0 \leq x_i \leq 1 \text{ for } 2 \leq i \leq k-1, 1 \leq x_k \leq 2, \right. \\ \left. x_i \leq \frac{a_i}{a_k} x_k \text{ for } k+1 \leq i \leq n \right\}.$$

Note that if $(x_2, \dots, x_n) \in T$, then for each $1 \leq i \leq n$, we have

$$x_i \leq 2 \quad \text{and} \quad \frac{x_i}{a_i} \leq \frac{x_k}{a_k}.$$

Thus for $(x_2, \dots, x_n) \in T$, we have

$$\left(\sum_{i=1}^n \left(\frac{1}{a_i} \right)^2 x_i^2 \right)^{-n/4} \left(\sum_{i=1}^n x_i^2 \right)^{-n/4} \geq C a_k^{n/2}$$

for some constant $C > 0$. Therefore

$$|\langle \text{Ind}_P^G(I)(\tilde{a})f_0, f_0 \rangle| \geq C \int_T a_k^{n/2} dm \geq C a_k^{n/2} \prod_{i=k+1}^n \left(\frac{a_i}{a_k} \right) \geq CF(\tilde{a}).$$

• Case $G = \text{Sp}_{2n}(\mathbb{R})$, $n \geq 2$.

For $\tilde{a} = \text{diag}(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1}) \in \text{Sp}_{2n}(\mathbb{R})$, we have

$$\begin{aligned} \langle \text{Ind}_P^G(I)(\tilde{a})f_0, f_0 \rangle &= \int_{\mathbb{R}^{2n-1}} \left(\sum_{i=1}^n \left(\frac{x_i}{a_i} \right)^2 + \sum_{i=1}^n (a_i y_i)^2 \right)^{-n/2} \\ &\quad \times \left(\sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 \right)^{-n/2} dm \end{aligned}$$

where $x_1 = 1$. Let T be the following set :

$$\left\{ (x_1, \dots, x_n, y_n, \dots, y_1) \mid 1 \leq y_n \leq 2, y_i \leq \frac{a_n}{a_i} y_n, \right. \\ \left. 0 \leq x_i \leq 1, 0 \leq y_i \leq 2 \text{ for } 1 \leq i \leq n \right\}.$$

Note that if $(x_1, \dots, x_n, y_n, \dots, y_1) \in T$, then

$$x_i \leq a_i a_n y_n$$

since $a_i \geq 1$ for all $1 \leq i \leq n$. Therefore

$$|\langle \text{Ind}_P^G(I)(\tilde{a})f_0, f_0 \rangle| \geq C \int_T (a_n y_n)^{-n} dm \geq CF(\tilde{a})$$

where C is some positive constant, finishing the proof.

5. (G, K) -tempered subgroups and finite dimensional representations

5.1. — Let H be a linear connected non-compact semisimple Lie group. Let B_H be a minimal parabolic subgroup, A_H a maximal \mathbb{R} -split torus contained in B_H and K_H a maximal compact subgroup of H . Consider a Cartan decomposition of H : $H = K_H A_H^+ K_H$. Let A be the torus of $\text{SL}_n(\mathbb{R})$ consisting of all the diagonal elements and A^+ the positive Weyl chamber of $\text{SL}_n(\mathbb{R})$ given by

$$A^+ = \{\text{diag}(a_1, \dots, a_n) \mid a_i \geq a_{i+1} \text{ for all } 1 \leq i \leq n-1\}.$$

Let π be a representation of H to $\text{SL}_n(\mathbb{R})$ such that $\pi(A_H) \subset A$. For each $1 \leq i \leq n$, we define a weight β_i of $d\pi$ by

$$\beta_i(X) = (i, i)\text{-entry of the matrix } d\pi(X) \text{ for } X \in \log A_H,$$

where $d\pi$ denotes the differential of π . Denote by W the Weyl group of $\text{SL}_n(\mathbb{R})$. Using the well known isomorphism of W with the symmetric group on n letters, we can consider the action of W on $\{\beta_1, \dots, \beta_n\}$ by $w(\beta_i) = \beta_{w(i)}$ for each $1 \leq i \leq n$.

For each $w \in W$, set

$$\mathfrak{a}_w = \{X \in \log(A_H^+) \mid w(\beta_i)(X) \geq w(\beta_{i+1})(X) \text{ for all } 1 \leq i \leq n-1\}.$$

Note that since

$$\mathfrak{a}_w = \{X \in \log(A_H^+) \mid d\pi(X) \in w^{-1}(\log A^+)w\},$$

we have that $\log(A_H^+) = \bigcup_{w \in W} \mathfrak{a}_w$. It is not difficult to see that we can choose a subset $W_0 \subset W$ (not unique) so that $\log(A_H^+) = \bigcup_{w \in W_0} \mathfrak{a}_w$, the interior of \mathfrak{a}_w is non-empty for each $w \in W_0$, and the interiors of \mathfrak{a}_w 's, $w \in W_0$ are disjoint. For example, if $\pi(A_H^+) \subset A^+$, then we can choose W_0 to consist of only the identity element of W .

We keep the above notation, such as H , A^+ , π , β_1, \dots, β_n , W_0 , \mathfrak{a}_w , etc., for the rest of Chapter 5. Recall also that δ_H denotes the modular function of B_H .

The following is an application of Theorem *D* when $G = \mathrm{SL}_n(\mathbb{R})$.

COROLLARY. — *The subgroup $\pi(H)$ is an $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup if and only if the following holds : for each $w \in W_0$ and all $X \in \mathfrak{a}_w$*

- *if n is even*

$$w(\beta_1)(X) + \cdots + w(\beta_{n/2})(X) > \log(\delta_H)(X);$$

- *if n is odd*

$$w(\beta_1)(X) + \cdots + w(\beta_{(n-1)/2})(X) + \frac{1}{2}w(\beta_{(n+1)/2})(X) > \log(\delta_H)(X).$$

Proof. — Note that

$$\int_{A_H^+} (F \circ \pi) \delta_H \, da = \sum_{w \in W_0} \int_{\exp \mathfrak{a}_w} (F \circ \pi) \delta_H \, da.$$

On the other hand, on each \mathfrak{a}_w , the restriction of $-\log F \circ \pi$ to $\log A_H^+$ is equal to the function in the left in the above inequality (see Example before Theorem *D*). This proves the claim by Theorem *D*. \square

EXAMPLE. — If H is simple and Ad is the adjoint representation of H , we can consider $\mathrm{Ad}(H)$ to be a subgroup of $\mathrm{SL}_n(\mathbb{R})$ where $n = \dim(\mathrm{Lie}(H))$. Since the restriction of $-\log F \circ \mathrm{Ad}$ to $\log A_H^+$ is equal to $\log \delta_H$, we have that $\mathrm{Ad}(H)$ is not an $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup by the above corollary.

5.2. — Let $\lambda_1, \dots, \lambda_k$ the fundamental weights of the Lie algebra of H corresponding to A_H^+ . For any weights γ_1 and γ_2 of the Lie algebra of H , we define a partial order $>$ so that $\gamma_1 > \gamma_2$ if and only if $(\gamma_1, \lambda_j) > (\gamma_2, \lambda_j)$ for all $1 \leq j \leq k$. This is equivalent to saying that the coefficient of each simple root in $\gamma_1 - \gamma_2$ is positive, or $\gamma_1(X) > \gamma_2(X)$ for all $X \in \log A_H^+$.

If λ is the highest weight of an irreducible representation, then the lowest weight, which we will denote by $\Lambda(\lambda)$, is given by

$$(\Lambda(\lambda), \lambda_j) = -(\lambda, i(\lambda_j)) \text{ for each } 1 \leq j \leq k,$$

where i is the opposition involution of the root system of $\mathrm{Lie}(H)$ (cf. [11]).

COROLLARY. — *Let H be a linear connected semisimple Lie group and π an irreducible representation with the highest weight λ . Suppose that*

$$\lambda - \Lambda(\lambda) > 2 \log \delta_H.$$

Then $\pi(H)$ is an $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup.

Proof. — Let $w \in W_0$. Since λ and $\Lambda(\lambda)$ are the highest weight and the lowest weight of π respectively, it follows from the definition of \mathfrak{a}_w that

$$w(\beta_1) = \lambda \quad \text{and} \quad w(\beta_n) = \Lambda(\lambda).$$

Let X be any element in \mathfrak{a}_w . Since $w(\beta_i)(X) \geq w(\beta_{i+1})(X)$ for each $1 \leq i \leq n-1$, we have that if n is even,

$$2 \sum_{i=1}^{n/2} w(\beta_i)(X) \geq 2w(\beta_1)(X) + \sum_{i=2}^{n-1} w(\beta_i)(X),$$

and if n is odd

$$\sum_{i=1}^{(n-1)/2} w(\beta_i)(X) + w(\beta_{(n+1)/2})(X) \geq 2w(\beta_1)(X) + \sum_{i=2}^{n-1} w(\beta_i)(X).$$

On the other hand, since $\sum_{i=1}^n \beta_i = 0$,

$$2w(\beta_1) + \sum_{i=2}^{n-1} w(\beta_i) = w(\beta_1) - w(\beta_n),$$

which is equal to $\lambda - \Lambda(\lambda)$. Therefore the assumption that $\lambda - \Lambda(\lambda) > 2 \log \delta_H$ implies the inequalities in Corollary 5.1, finishing the proof. \square

REMARK. — By the remark prior to Corollary 5.2 and the fact that

$$(\log \delta_H, \lambda_j) = (\log \delta_H, i(\lambda_j)) \quad \text{for each } 1 \leq j \leq k,$$

we have that if $\lambda > \log \delta_H$, then $\lambda - \Lambda(\lambda) > 2 \log \delta_H$; so the hypothesis of the above corollary is satisfied.

EXAMPLE. — If $H = \mathrm{SL}_{k+1}(\mathbb{R})$ in Corollary 5.2, then

$$\lambda - \Lambda(\lambda) > 2 \log \delta_H$$

is equivalent to the following :

$$c_j + c_{k+1-j} > 2j(k+1-j) \text{ for } 1 \leq j \leq k$$

where $c_j = (\lambda, \lambda_j)$.

EXAMPLE 5.3. — The following examples are applications of Corollary 5.1.

1) If π is an irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{SL}_n(\mathbb{R})$, then it is well known that $(\lambda, \lambda_1) = \frac{1}{2}(n-1)$; whereas $(\log \delta_H, \lambda_1) = 1$. Therefore $\pi(\mathrm{SL}_2(\mathbb{R}))$ is an $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup if and only if $n \geq 4$.

2) The embedding of $\mathrm{SL}_k(\mathbb{R})$ as the first k by k diagonal block matrix in $\mathrm{SL}_n(\mathbb{R})$ is not an $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup for any positive integers k and n .

3) For matrices A of order m and B of order k , the Kronecker product $A \otimes B$ of A and B is the matrix of order mk such that the (ij) -matrix block of $A \otimes B$ is $a_{ij}B$ where a_{ij} is the (ij) -entry of A .

The group $\mathrm{SL}_m(\mathbb{R}) \otimes I_k$ is an $(\mathrm{SL}_{mk}(\mathbb{R}), \mathrm{SO}_{mk}(\mathbb{R}))$ -tempered subgroup if and only if $k > 2(m-1)$.

5.4. — In this section we consider the case when π is symplectic or orthogonal. It is worthwhile to state the following fact, which enables us to tell when an irreducible representation π with the highest weight λ has such a property.

THEOREM (cf. [11, Ch. 3, Thm 2.15]). — *The representation π is self-dual if and only if $\lambda = -\Lambda(\lambda)$. In such cases, π is orthogonal (resp. symplectic) if $\sum_{j=1}^k (\log \delta_H, \lambda_j)(\lambda, \lambda_j)$ is even (resp. odd).*

We remark that all finite dimensional irreducible representations of H are self-dual unless H is of type A_n , D_{2k+1} or E_6 .

5.5. — We have the following corollary of Theorem D when $G = \mathrm{Sp}_n(\mathbb{R})$, which is analogous to Corollaries 5.1 and 5.2.

We use the same realization of $\mathrm{Sp}_n(\mathbb{R})$ as in section 4.2 so that a positive Weyl chamber of $\mathrm{Sp}_n(\mathbb{R})$ is the following :

$$\begin{aligned} \mathrm{Sp}_n(\mathbb{R}) \cap A^+ = \{ \mathrm{diag}(a_1, \dots, a_{n/2}, a_{n/2}^{-1}, \dots, a_1^{-1}) \mid \\ a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq \frac{1}{2}n-1 \}. \end{aligned}$$

COROLLARY. — *Let H be a linear connected semisimple Lie group and π a representation such that $\pi(H) \subset Sp_n(\mathbb{R})$.*

(1) *The subgroup $\pi(H)$ is an $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$ -tempered subgroup if and only if for each $w \in W_0$,*

$$w(\beta_1)(X) + \cdots + w(\beta_{n/2})(X) > \log \delta_H(X) \text{ for all } X \in \mathfrak{a}_w.$$

(2) *Furthermore assume that π is irreducible with the highest weight λ . Suppose that*

$$\lambda > \log \delta_H.$$

Then $\pi(H)$ is an $(Sp_n(\mathbb{R}), Sp_n(\mathbb{R}) \cap SO_n(\mathbb{R}))$ -tempered subgroup.

Proof. — The proof of the first claim is similar to that of Corollary 5.1; so we will omit it. Since λ is the highest weight, $w(\beta_1) = \lambda$ for each $w \in W_0$. Since $w(\beta_i)(X) \geq 0$ for any $X \in \mathfrak{a}_w$ and each $1 \leq i \leq \frac{1}{2}n$, we have $\sum_{i=1}^{n/2} w(\beta_i)(X) \geq \lambda(X)$. Now the second claim follows from the first one. \square

5.6. — We consider a realization of $SO(m, n-m)$, $m = [\frac{1}{2}n]$ so that a positive Weyl chamber of $SO(m, n-m)$ is given by $SO(m, n-m) \cap A^+$, that is, if n is even,

$$\{\text{diag}(a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq m-1\}$$

and if n is odd,

$$\{\text{diag}(a_1, \dots, a_m, 1, a_m^{-1}, \dots, a_1^{-1}) \mid a_i \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq m-1\}.$$

COROLLARY. — *Let H be a linear connected semisimple Lie group and π an n -dimensional irreducible representation with the highest weight λ such that $\pi(H) \subset SO(m, n-m)$ where $m = [\frac{1}{2}n]$. Suppose that*

$$\lambda > \log \delta_H.$$

Then $\pi(H)$ is an $(SO(m, n-m), SO(m, n-m) \cap SO_n(\mathbb{R}))$ -tempered subgroup.

Proof. — Consider the case when n is even. Let $p = [\frac{1}{4}n]$. Then for any $w \in W_0$ and any $X \in \mathfrak{a}_w$, the function F in Theorem A is such that

$$-\log F \circ \pi(X) = w(\beta_1)(X) + \cdots + w(\beta_p)(X).$$

Therefore by the same argument as in the previous corollary, it is enough to show that

$$w(\beta_1)(X) + \cdots + w(\beta_p)(X) > \log \delta_H(X).$$

This is true since $w(\beta_i)(X) \geq 0$ for all $1 \leq i \leq p$ and $w(\beta_1) = \lambda$. The proof in the case when n is odd is similar. \square

EXAMPLE. — If $H = \mathrm{SL}_{k+1}(\mathbb{R})$ and $c_j = (\lambda, \lambda_j)$ for $1 \leq j \leq k$, then π is self-dual if and only if $c_j = c_{k+1-j}$ for $1 \leq j \leq k$, and the condition $\lambda > \log \delta_H$ is equivalent to the condition $c_j > 2j(k+1-j)$ for each $j = 1, \dots, k$.

Therefore with these two conditions satisfied, if $\sum_{i=1}^k i(k+1-i)c_i$ is even, then $\pi(\mathrm{SL}_{k+1}(\mathbb{R}))$ is an $(\mathrm{SO}(m, n-m), \mathrm{SO}(m, n-m) \cap \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup where $m = \lfloor \frac{1}{2}n \rfloor$, and if $\sum_{i=1}^k i(k+1-i)c_i$ is odd, then $\pi(H)$ is an $(\mathrm{Sp}_n(\mathbb{R}), \mathrm{Sp}_n(\mathbb{R}) \cap \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup.

Moreover in the case when $H = \mathrm{SL}_2(\mathbb{R})$ and π is an n -dimensional irreducible representation with $n \geq 4$ (cf. Example 5.3), the subgroup $\pi(\mathrm{SL}_2(\mathbb{R}))$ is $(\mathrm{Sp}_n(\mathbb{R}), \mathrm{Sp}_n(\mathbb{R}) \cap \mathrm{SO}_n(\mathbb{R}))$ -tempered if n is even; otherwise it is $(\mathrm{SO}(m, n-m), \mathrm{SO}(m, n-m) \cap \mathrm{SO}_n(\mathbb{R}))$ -tempered.

5.7. Unipotent tempered subgroups. — Lastly we give examples of some unipotent tempered subgroups of $G = \mathrm{SL}_n(\mathbb{R})$. In order to apply Theorem D when H is not semisimple, we need to know how each element of H decomposes under the Cartan decomposition of G .

Consider the decomposition of the element $v_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ as $k_1 a k_2$ under the Cartan decomposition of $\mathrm{SL}_2(\mathbb{R})$ with $K = \mathrm{SO}_2(\mathbb{R})$ and A the torus consisting of all the diagonal elements. Since $v_s(v_s)^t = k_1 a^2 k_1^{-1}$, the eigenvalues of a^2 coincide with those of $v_s(v_s)^t$. If $a = \mathrm{diag}(b, b^{-1})$, then

$$b = \sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2 + 4})}.$$

Consider the one parameter unipotent subgroup U_{ij} of $\mathrm{SL}_n(\mathbb{R})$ consisting of the elements $u_{ij}(s) = I + sE_{ij}$, $s \in \mathbb{R}$, where $i \neq j$ and E_{ij} is the elementary matrix whose non-zero entry is 1 only at (i, j) . We keep the same notation as in section 5.1. Then the A^+ -component of $u_{ij}(s)$ under the Cartan decomposition of $\mathrm{SL}_n(\mathbb{R})$ is $\mathrm{diag}(b, 1, \dots, 1, b^{-1})$ where

$$b = \sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2 + 4})}$$

by the previous argument.

Therefore $F(u_{ij}(s))$ is equal to $\left(\sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2 + 4})}\right)^{-1}$.

PROPOSITION. — *Let $n \geq 2$ and $i \neq j$.*

(1) *For any $\epsilon > 0$, the restriction $F|_{U_{ij}}$ is $L^{1+\epsilon}(U_{ij})$ -integrable; hence U_{ij} is not an $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup.*

(2) *The diagonal embedding*

$$\delta(U_{ij}) = \{(g, g) \in \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) \mid g \in U_{ij}\}$$

is an $(\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R}) \times \mathrm{SO}_n(\mathbb{R}))$ -tempered subgroup.

Proof. — The part (1) is clear. For the second claim, see the remark following Corollary B. \square

Now consider the unipotent one-parameter subgroup U of $\mathrm{SL}_4(\mathbb{R})$ consisting of the elements

$$U(s) = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s \in \mathbb{R}.$$

It is easy to see that the following proposition holds.

PROPOSITION 5.8. — *The subgroup U is an $(\mathrm{SL}_4(\mathbb{R}), \mathrm{SO}_4(\mathbb{R}))$ -tempered subgroup.*

BIBLIOGRAPHIE

- [1] BOREL (A.), TITS (J.). — *Groupes réductifs*, Publ. Math. IHES, **27**, 1965, p. 55–151.
- [2] BOURBAKI (N.). — *Groupes et Algèbres de Lie*, Chap. IV–VI. — Hermann, Paris, 1968.
- [3] HOWE (R.E.). — *On a notion of rank for unitary representations of the classical groups*, Harmonic analysis and group representations II Circolo, Palazzone della Scuola normale superiore, CIME, 1980, p. 223–331.
- [4] HOWE (R.E.), TAN (E.C.). — *Non-Abelian Harmonic Analysis*. — Springer-Verlag, 1992.

- [5] KNAPP (A.W.). — *Representation theory of Semisimple groups – An overview based on examples*. — Princeton University Press, 1986.
- [6] KOBAYASHI (T.). — *Discontinuous groups and Clifford-Klein forms of pseudo-Riemannian homogeneous manifolds*, in *Lecture Notes of the European School on Group theory, Perspectives in Math., Algebraic and Analytic methods in Representation theory*, ed. B. Ørsted and H. Schlichtkrull, t. **17**, 1996, p. 99–165.
- [7] LI (J-S.). — *The minimal decay of matrix coefficients for classical groups*, *Math. Appl. (Harmonic Analysis in China)*, t. **327**, 1996, p. 146–169.
- [8] LI (J-S.), ZHU (C-B.). — *On the decay of matrix coefficients for exceptional groups*, *Math. Ann.*, t. **305**, 1996, p. 249–270.
- [9] LIPSMAN (R.L.), WOLF (J.A.). — *Canonical semi-invariants and the Plancherel formula for parabolic groups*, *Trans. AMS*, t. **269**, 1982, p. 111–131.
- [10] MARGULIS (G.A.). — *Existence of compact quotients of homogeneous spaces, measurably proper actions and decay of matrix coefficients*, *Bull. Soc. Math. France*, t. **125**, 1997, p. 447–456.
- [11] ONISHCHIK (A.), VINBERG (E.). — *Lie groups and Lie algebras III*. — Springer-Verlag, 1993.
- [12] ZIMMER (R.). — *Ergodic theory and semisimple groups*. — Birkhäuser, Boston, 1985.