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L-FUNCTIONS FOR SYMPLECTIC GROUPS

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ABSTRACT. — We construct global integrals of Shimura type, which represent the standard (partial) L-function $L^S(\pi \otimes \sigma, s)$, for $\pi \otimes \sigma$, an irreducible, automorphic, cuspidal and generic representation of $\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathrm{GL}_k(\mathbb{A})$. We present two different constructions : one for the case $n > k$ and one for the case $n \leq k$. These constructions are, in a certain sense, dual to each other. We also study the (completely analogous) case where π is a representation of the metaplectic group $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Here we have to first fix a choice of a non-trivial additive character ψ, in order to define the L-function $L^S_\psi(\pi \otimes \sigma, s)$. The integrals depend on a cusp form of π, a theta series on $\widetilde{\mathrm{Sp}}_{2\ell}(\mathbb{A})$ ($\ell = \min(n,k)$) and an Eisenstein series on $\mathrm{Sp}_{2k}(\mathbb{A})$ (or $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$) induced from σ.

RÉSUMÉ. — FONCTIONS L POUR GROUPES SYMPLECTIQUES. — On construit des intégrales globales de type de Shimura, qui représentent la fonction L standard (partielle) $L^S(\pi \otimes \sigma, s)$ pour une représentation $\pi \otimes \sigma$, irréductible, automorphe, cuspidale et générique de $\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathrm{GL}_k(\mathbb{A})$. On présente deux constructions différentes : une pour le cas $n > k$, et une pour le cas $n \leq k$. Ces constructions sont, dans un certain sens, duales l'une de l'autre. On étudie de même le cas (tout à fait analogue) où π est une représentation du groupe métaplectique $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Ici, on doit d'abord fixer le choix d'un caractère additif et non trivial ψ, pour définir la fonction L, $L^S_\psi(\pi \otimes \sigma, s)$. Les intégrales dépendent d'une forme cuspidale de π, d'une série thêta sur $\widetilde{\mathrm{Sp}}_{2\ell}(\mathbb{A})$ ($\ell = \min(n,k)$) et d'une série d'Eisenstein sur $\mathrm{Sp}_{2k}(\mathbb{A})$ (ou $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$) induite par σ.

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Introduction

We present a global integral of Shimura type, which interpolates the standard L-function for generic cusp forms on $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ (resp. on $\widetilde{\mathrm{Sp}}_{2n} \times \mathrm{GL}_k$, where $\widetilde{\ }$ denotes the metaplectic cover.) The structure of the global integral has one of two forms, according to whether $n \geq k$ or $n < k$. However, these two forms of integrals are dual to each other, in the sense that the roles of cusp forms and Eisenstein series are interchanged. This construction for $\mathrm{Sp}_{2n} \times \mathrm{GL}_n$ was already done in [GPS]. Let us give some more details on these global integrals. Let π (resp. $\widetilde{\pi}$) be an irreducible automorphic, cuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ (resp. $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ ($\mathbb{A}$ – the adèles of a global field F). We assume that π (resp. $\widetilde{\pi}$) is generic, *i.e.* π (resp. $\widetilde{\pi}$) has a nontrivial Whittaker-Fourier coefficient. Let σ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_k(\mathbb{A})$. Let $E(g, f_{\sigma,s})$ be the Siegel-Eisenstein series on $\mathrm{Sp}_{2k}(\mathbb{A})$ corresponding to a holomorphic K-finite section $f_{\sigma,s}$ in the representation of $\mathrm{Sp}_{2k}(\mathbb{A})$ induced from the Siegel parabolic subgroup and σ. In a similar appropriate fashion, we may consider the representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ induced from σ (twisted by the Weil symbol), and construct, for a section $\tilde{f}_{\sigma,s}$, the corresponding Eisenstein series $\widetilde{E}(g, \tilde{f}_{\sigma,s})$ on $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. Let ψ be a nontrivial character of $F \backslash \mathbb{A}$. Denote by $\omega_\psi^{(\ell)}$ the Weil representation of $H_\ell(\mathbb{A}) \cdot \widetilde{\mathrm{Sp}}_{2\ell}(\mathbb{A})$, where H_ℓ is the corresponding Heisenberg group (of dimension $2\ell + 1$). Let $\theta_\psi^{(\ell)}$ be the corresponding realization by theta series. Assume that $n \geq k$ and let φ be a cusp form in the space of π. Then the global integral for $\pi \times \sigma$ has the form

$$I_1(s) = \int\limits_{\mathrm{Sp}_{2k}(F)\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int\limits_{V_F^{(n,k)}\backslash V_A^{(n,k)}} \varphi(vg)\theta_\psi^{(k)}(v'g)\psi'(v)\,\mathrm{d}v \cdot \widetilde{E}(g, \tilde{f}_{\sigma,s})\,\mathrm{d}g.$$

$V^{(n,k)}$ is the unipotent radical in Sp_{2n}, of the parabolic subgroup preserving a an $(n-k)$-dimensional isotropic flag. $V^{(n,k)}$ has projection $v \mapsto v'$ to the Heisenberg group H_k. ψ' is a certain character of $V_F^{(n,k)} \backslash V_{\mathbb{A}}^{(n,k)}$. Sp_{2k} is embedded in Sp_{2n} in the "middle" $2k \times 2k$ block. For $\widetilde{\pi}$ on $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$, we consider, for $\widetilde{\varphi}$ in the space of $\widetilde{\pi}$,

$$I_2(s) = \int\limits_{\mathrm{Sp}_{2k}(F)\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int\limits_{V_F^{(n,k)}\backslash V_{\mathbb{A}}^{(n,k)}} \widetilde{\varphi}(vg)\theta_\psi^{(k)}(v'g)\psi'(v)\,\mathrm{d}v \cdot E(g, f_{\sigma,s})\,\mathrm{d}g.$$

In case $n < k$, we switch the roles of k and n and of φ (resp. $\widetilde{\varphi}$) and $\widetilde{E}$

(resp. E) and consider

$$J_1(s) = \int\limits_{\mathrm{Sp}_{2n}(F)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \varphi(g) \int\limits_{V_F^{(k,n)}\backslash V_{\mathbb{A}}^{(k,n)}} \widetilde{E}(vg, \tilde{f}_{\sigma,s})\theta_\psi^{(n)}(v'g)\psi'(v)\,\mathrm{d}v\,\mathrm{d}g,$$

$$J_2(s) = \int\limits_{\mathrm{Sp}_{2n}(F)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \widetilde{\varphi}(g) \int\limits_{V_F^{(k,n)}\backslash V_{\mathbb{A}}^{(k,n)}} E(vg, f_{\sigma,s})\theta_\psi^{(n)}(v'g)\psi'(v)\,\mathrm{d}v\,\mathrm{d}g.$$

These integrals are of course meromorphic functions of s. We prove, for decomposable data, that the integrals are Eulerian. The corresponding local integrals satisfy the expected properties of the associated local theory (meromorphic continuation, nonvanishing and local functional equation). When all data is standard unramified at a place p the local integrals at p give the following quotient

$$\text{Case } \mathrm{Sp}_{2n} \times \mathrm{GL}_k : \quad \frac{L(\pi_p \otimes \sigma_p, s(k+1) - \frac{1}{2}k)}{L(\sigma_p, V^2, 2s(k+1) - k)},$$

$$\text{Case } \widetilde{\mathrm{Sp}}_{2n} \times \mathrm{GL}_k : \quad \frac{L_{\psi_p}(\widetilde{\pi}_p \otimes \sigma_p, s(k+1) - \frac{1}{2}k)}{L(\sigma_p, s(k+1) - \frac{1}{2}(k-1))L(\sigma_p, \Lambda^2, 2s(k+1) - k)}.$$

Here V^2 and Λ^2 are respectively the symmetric square and exterior square representations of $\mathrm{GL}_k(\mathbb{C})$. There is no canonical definition of the local L-factor for $\widetilde{\pi}_p \otimes \sigma_p$ (on $\widetilde{\mathrm{Sp}}_{2n}(F_p) \otimes \mathrm{GL}_k(F_p)$). We have to first make a choice of a nontrivial character of F_p. Here we choose ψ_p. The choice of ψ_p determines the unramified character $\eta = \eta_1 \otimes \cdots \otimes \eta_n$, of the diagonal subgroup of $\widetilde{\mathrm{Sp}}_{2n}(F)$, which corresponds to $\widetilde{\pi}_p$. See Section 3.1 for our precise definition. We then have

$$L_{\psi_p}(\widetilde{\pi}_p \otimes \sigma_p, s) = \prod_{i=1}^{n} L(\sigma_p \otimes \eta_i, s)L(\sigma_p \otimes \eta_i^{-1}, s).$$

We obtain these unramified computations, in cases $n \geq k$, using invariant theory in a direct fashion. We do not know how to do this in case $n < k$. In this case, we prove, first, a certain identity relating (up to slight modifications) the local integrals for $\big(\mathrm{Ind}_{\overline{P}_{2n,n}}^{\mathrm{Sp}_{2n}} \sigma'\big) \times \sigma$ and $\big(\mathrm{Ind}_{\widetilde{\overline{P}}_{2n,n}}^{\widetilde{\mathrm{Sp}}_{2n}} \sigma\big) \times \sigma'$ (resp. $\big(\mathrm{Ind}_{\widetilde{\overline{P}}_{2n,n}}^{\widetilde{\mathrm{Sp}}_{2n}} \sigma'\big) \times \sigma$ and $\big(\mathrm{Ind}_{\overline{P}_{2n,n}}^{\mathrm{Sp}_{2n}} \sigma\big) \times \sigma'$) and then we use the unramified calculation of the first case (see [S1], where similar identities are obtained for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$.)

The importance of achieving L-functions via explicit integrals of the above type is, apart from establishing their meromorphicity, the possibility of locating their poles and relating their existence to functorial liftings and nonvanishing of (generalized) periods. We already studied one such example in case $\mathrm{Sp}_{2n} \times \mathrm{GL}_1$ in [GRS1], where we showed that the only possible pole of the partial L-function $L^S(\pi, s)$ is at $s = 1$, and this pole occurs if and only if π has a nontrivial theta lift to a cuspidal generic representation of $\mathrm{SO}_{n,n}(\mathbb{A})$, and in this case a certain (generalized) period is nontrivial on π. In forthcoming works [GRS2], [GRS3] we study the existence of a pole at $s = 1$, when $k > 1$, and relate it to (explicit) functorial lifts between generic representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ and of $\mathrm{GL}_{2n}(\mathbb{A})$ and between generic representations of $\mathrm{Sp}_{2n}(\mathbb{A})$ and of $\mathrm{GL}_{2n+1}(\mathbb{A})$. We plan to study this explicit functorial lift for the nongeneric case as well, using a similar theory of L-functions of arbitrary (not necessarily generic) cuspidal representations of $\mathrm{Sp}_{2n}(\mathbb{A})$ (resp. $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$), due to the first two named authors. There "generic" is replaced by the existence of a certain Fourier-Jacobi model. The advantage of having all these different constructions is the possibility of relating generic and nongeneric representations, having the same functorial lift to the appropriate GL_n.

1. Notations

1. — Let J_r denote the $r \times r$ matrix $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. We define

$$\mathrm{Sp}_{2r} = \left\{ g \in \mathrm{GL}_{2r} : {}^t g \begin{pmatrix} & J_r \\ -J_r & \end{pmatrix} g = \begin{pmatrix} & J_r \\ -J_r & \end{pmatrix} \right\}.$$

2. — For our construction we will need the following subgroups of Sp_{2r}. For $0 \leq k \leq r$ let $P_{2r,k}$ denote the parabolic subgroup of Sp_{2r} which stabilizes a k-dimensional isotropic space. Thus

$$P_{2r,k} = \left(\mathrm{GL}_k \times \mathrm{Sp}_{2(r-k)}\right) U_{2r,k},$$

where $U_{2r,k}$ is the unipotent radical of $P_{2r,k}$. When $k = 0$ we understand that $P_{2r,0} = \mathrm{Sp}_{2r}$ and $\mathrm{GL}_0 = U_{2r,0} = \{1\}$. In terms of matrices, we consider the embedding of $P_{2r,k}$ in Sp_{2r} as follows :

$$(h, g) \longmapsto \begin{pmatrix} h & & \\ & \tilde{g} & \\ & & h^* \end{pmatrix}, \quad h \in \mathrm{GL}_k,\ g \in \mathrm{Sp}_{2(r-k)},$$

where h^* is such that the above matrix is in Sp_{2r}, and given $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathrm{Sp}_{2(r-k)}$, then $\tilde{g} = \begin{pmatrix} A & \frac{1}{2}B \\ 2C & D \end{pmatrix}$. The group $U_{2r,k}$ is embedded as all matrices in Sp_{2r} of the form

$$\begin{pmatrix} I_k & & * \\ & I_{2(r-k)} & \\ & & I_k \end{pmatrix},$$

where I_ℓ is the $\ell \times \ell$ identity matrix. Let $Q_{2r,k} \subset P_{2r,k}$ denote the parabolic subgroup of Sp_{2r} whose Levi part is $\mathrm{GL}_1^k \times \mathrm{Sp}_{2(r-k)}$. We shall denote its unipotent radical by $V_{2r,k}$.

3. — Let H_n denote the Heisenberg group with $2n+1$ variables. We shall identify an element $h \in H_n$ with a triple (x,y,z), where $x, y \in M_{1\times n}$ and $z \in M_{1\times 1}$. We define the following subgroups of H_n :

$$X_n = \{(x,0,0) \in H_n\},\ Y_n = \{(0,y,0) \in H_n\},\ Z_n = \{(0,0,z) \in H_n\}.$$

The product in H_n is given by

$$(x_1,y_1,z_1)(x_2,y_2,z_2) = \big(x_1+x_2, y_1+y_2, z_1+z_2+\tfrac{1}{2}(x_1J_ny_2^t - y_1J_nx_2^t)\big).$$

It is well-known that H_n is isomorphic to $U_{2n+2,1}$. We shall denote this isomorphism by τ. In coordinates we have

$$\tau(h) = \tau(x,y,z) = \begin{pmatrix} 1 & x & \frac{1}{2}y & z \\ & I_n & & y^* \\ & & I_n & x^* \\ & & & 1 \end{pmatrix}.$$

Here x^* and y^* are such that the above matrix is in Sp_{2n+2}. Thus,

$$y^* = \tfrac{1}{2}J_ny^t \quad \text{and} \quad x^* = -J_nx^t.$$

4. — Let F be a global field and $\mathbb{A}$ its ring of adèles. Given a linear algebraic group G, we shall denote by $G(\mathbb{A})$ the $\mathbb{A}$ points of G, *etc.* Fix n and k in $\mathbb{N}$. Let π (resp. σ) denote an irreducible automorphic cuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ (resp. $\mathrm{GL}_k(\mathbb{A})$). We shall denote by V_π (resp. V_σ) its realization in the space of cusp forms. We shall always assume that π is generic. We recall this notation. Let ψ denote a nontrivial additive character of $F \setminus \mathbb{A}$. Given $1 \le \ell \le n$ we define a character ψ_ℓ on $V_{2n,\ell}$ as follows. Let $v = (v_{ij}) \in V_{2n,\ell}$ be a matrix realization of $V_{2n,\ell}$ as defined in Section 2. If $1 \le \ell \le n$ we denote

$$\psi_\ell(v) = \psi\Big(\sum_{i=1}^{\ell} v_{i,i+1}\Big)$$

and we shall also need

$$\widetilde{\psi}_n(v) = \psi\Big(\sum_{i=1}^{n-1} v_{i,i+1} + \tfrac{1}{2} v_{n,n+1}\Big).$$

Let N_k denote the maximal unipotent subgroup of GL_k consisting of upper triangular matrices. For $\bar{n} = (\bar{n}_{ij}) \in N_k$ we define

$$\psi_{N_k}(\bar{n}) = \psi\Big(\sum_{i=1}^{k-1} \bar{n}_{i,i+1}\Big).$$

We say that π is generic with respect to ψ_n if the space of functions

$$W_{\varphi_\pi}(g) = \int_{V_{2n,n}(F)\backslash V_{2n,n}(\mathbb{A})} \varphi_\pi(vg)\psi_n(v)\,\mathrm{d}v$$

is not identically zero. Here $\varphi_\pi \in V_\pi$ and $g \in \mathrm{Sp}_{2n}(\mathbb{A})$. We shall denote this space of functions by $\mathcal{W}(\pi, \psi_n)$. A similar definition holds for $\mathcal{W}(\pi, \widetilde{\psi}_n)$. Similarly, for σ, we denote by $\mathcal{W}(\sigma, \psi_{N_k})$ the space of functions of the form

$$W_{\varphi_\sigma}(h) = \int_{N_k(F)\backslash N_k(\mathbb{A})} \varphi_\sigma(\bar{n}h)\psi_{N_k}(\bar{n})\,\mathrm{d}\bar{n}$$

for $\varphi_\sigma \in V_\sigma$ and $h \in \mathrm{GL}_k(\mathbb{A})$. By Shalika's well-known theorem, given σ as above $\mathcal{W}(\sigma, \psi_{N_k}) \not\equiv 0$.

5. — Let $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ denote the metaplectic double cover of $\mathrm{Sp}_{2k}(\mathbb{A})$. Given a subgroup G of Sp_{2k} we shall denote by $\widetilde{G}$ its full inverse image in $\widetilde{\mathrm{Sp}}_{2k}$. If a subgroup G of $\widetilde{\mathrm{Sp}}_{2k}$ splits under the cover, we shall view G as a subgroup of $\widetilde{\mathrm{Sp}}_{2k}$. Choosing the covering in a suitable way, it is well-known that the groups $\mathrm{Sp}_{2k}(F)$ and $V_{2k,k}(\mathbb{A})$ split.

For $a \in A^*$ we let γ_a denote the Weil symbol. It depends on the choice of ψ. When we want to mark this dependence, we write $\gamma_{a,\psi}$. Let $\delta_{P_{2k,k}}$ denote the modular function of $P_{2k,k}$. As usual, we view it as a function of $\widetilde{P}_{2k,k}$ by composing it with the projection $\widetilde{P}_{2k,k} \to P_{2k,k}$. For $s \in \mathbb{C}$ and σ as in Section 4 we denote

$$\tilde{I}(\sigma, s) = \mathrm{Ind}_{\widetilde{P}_{2k,k}(\mathbb{A})}^{\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})} \big(\sigma \otimes \delta^s_{P_{2k,k}} \otimes \gamma^{-1}\big).$$

By definition, this is the space of all smooth functions $\tilde{f}_{\sigma,s} : \widetilde{\mathrm{Sp}}_{2k}(\mathbb{A}) \to V_\sigma$ satisfying

$$\tilde{f}_{\sigma,s}(pg) = \varepsilon\gamma^{-1}_{\det m}\delta^s_{P_{2k,k}}(m)\sigma(m)\tilde{f}_{\sigma,s}(g) \tag{1.1}$$

for all $g \in \widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$, $\varepsilon \in \{\pm 1\}$ and $p = \left\langle \begin{pmatrix} m & \\ & m^* \end{pmatrix} u, \varepsilon \right\rangle \in \widetilde{\mathrm{Sp}}_{2k,k}(\mathbb{A})$, where $m \in \mathrm{GL}_k(\mathbb{A})$ and $u \in U_{2k,k}(\mathbb{A})$. Here and henceforth we shall view elements of $\widetilde{\mathrm{Sp}}_{2k}$ as pairs $\langle g, \varepsilon \rangle$, where $g \in \mathrm{Sp}_{2k}$ and $\varepsilon \in \{\pm 1\}$ with multiplication as defined, for example, in [BFH]. Let us remark that (1.1) is well-defined since the subgroup GL_k of Sp_{2k} splits under the cover. We shall also denote

$$I(\sigma, s) = \mathrm{Ind}\,_{P_{2k,k}(\mathbb{A})}^{\mathrm{Sp}_{2k}(\mathbb{A})} \big(\sigma \otimes \delta^s_{P_{2k,k}} \big).$$

Here σ and s are as before. Thus $I(\sigma, s)$ is the space of all smooth functions $f_{\sigma,s} : \mathrm{Sp}_{2k}(\mathbb{A}) \to V_\sigma$, satisfying

$$f_{\sigma,s}(pg) = \delta^s_{P_{2k,k}}(m) \sigma(m) f_{\sigma,s}(g)$$

for all $g \in \mathrm{Sp}_{2k}(\mathbb{A})$ and $p = mu \in P_{2k,k}(\mathbb{A})$, where $m \in \mathrm{GL}_k(\mathbb{A})$ and $u \in U_{2k,k}(\mathbb{A})$.

In both cases, we view the function $f_{\sigma,s}(g)$ (resp. $\tilde{f}_{\sigma,s}(g)$) as complex valued functions on $\mathrm{Sp}_{2k}(\mathbb{A})$ (resp. $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$) which are left $U_{2k,k}(\mathbb{A})$ invariant, and for fixed $g \in \mathrm{Sp}_{2k}(\mathbb{A})$ (resp. $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$) the function $m \mapsto f_{\sigma,s}(mg)$ (resp. $m \mapsto \tilde{f}_{\sigma,s}(mg)$), with $m \in \mathrm{GL}_k(\mathbb{A})$, belongs to the space of cusp forms on $\mathrm{GL}_k(\mathbb{A})$ realizing the representation $\sigma \otimes \delta^s_{P_{2k,k}}$ (resp. $\sigma \otimes \delta^s_{P_{2k,k}} \otimes \gamma^{-1}$).

Next, we define the Eisenstein series we need. For $\tilde{f}_{\sigma,s} \in \tilde{I}(\sigma, s)$ define

$$\widetilde{E}(g, \tilde{f}_{\sigma,s}) = \sum_{\delta \in P_{2k,k}(F) \backslash \mathrm{Sp}_{2k}(F)} \tilde{f}_{\sigma,s}(\delta g),$$

where $g \in \widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. The above summation converges absolutely for $\mathrm{Re}(s)$ large and admits a meromorphic continuation to the whole complex plane with at most finitely many poles in $\mathrm{Re}(s) \geq \frac{1}{2}$. A similar definition holds for $E(g, f_{\sigma,s})$-the Siegel Eisenstein series on $\mathrm{Sp}_{2k}(\mathbb{A})$.

Let $\tilde{f}_{\sigma,s} \in \tilde{I}(\sigma, s)$. We denote

$$\tilde{f}_{W_{\sigma,s}}(g) = \int_{N_k(F) \backslash N_k(\mathbb{A})} \tilde{f}_{\sigma,s}(\bar{n} g) \psi_{N_k}(\bar{n}) \,\mathrm{d}\bar{n}$$

for $g \in \widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. Thus, for $g \in \mathrm{GL}_k(\mathbb{A})$, the function $\tilde{f}_{W_\sigma,s}(g)$ is in $\mathcal{W}(\sigma, \psi_{N_k})$. Similarly we define the function $f_{W_{\sigma,s}}(g)$.

6. — Let ω_ψ denote the Weil representation. It is a representation of the group $H_k(\mathbb{A})\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ and it is realized on the space $\mathcal{S}(\mathbb{A}^k)$ — the Schwartz space of $\mathbb{A}^k$. The following formulas are well-known (see [P])

$$\omega_\psi\big(\langle(0,y,z)(x,0,0),\varepsilon\rangle\big)\phi(\xi) = \varepsilon\psi(z+\xi J_k y^t)\phi(x+\xi), \tag{1.2}$$

$$\omega_\psi\Big(\Big\langle\begin{pmatrix} m & \\ & m^* \end{pmatrix},\varepsilon\Big\rangle\Big)\phi(\xi) = \varepsilon\gamma_{\det m}|\det m|^{\frac12}\phi(\xi m), \tag{1.3}$$

$$\omega_\psi\Big(\Big\langle\begin{pmatrix} I_k & T \\ & I_k \end{pmatrix},\varepsilon\Big\rangle\Big)\phi(\xi) = \varepsilon\psi\big(\tfrac12\xi T J_k\xi^t\big)\phi(\xi). \tag{1.4}$$

Here $\phi\in\mathcal{S}(\mathbb{A}^k)$, $(0,y,z)(x,0,0)\in H_k(\mathbb{A})$, $\varepsilon\in\{\pm1\}$, $m\in\mathrm{GL}_k(\mathbb{A})$ and $\begin{pmatrix} I_k & T \\ & I_k \end{pmatrix}\in U_{2k,k}(\mathbb{A})$. In the above formulas, we view $\xi\in\mathbb{A}^k$ as a row vector.

We now define the theta function,

$$\tilde\theta_\phi(hg) = \sum_{\xi\in F^k}\omega_\psi(hg)\phi(\xi)$$

for $\phi\in\mathcal{S}(\mathbb{A}^k)$, $h\in H_k(\mathbb{A})$ and $g\in\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. This function is an automorphic function of $H_k(\mathbb{A})\widetilde{\mathrm{Sp}}_{2k,k}(\mathbb{A})$ i.e. it is slowly increasing and invariant under the rational points $H_k(F)\,\mathrm{Sp}_{2k}(F)$.

2. The Global Integrals ($n > k$)

We keep the notations of Section 1. Assume that $n>k$. In this section we shall describe the global integral which will represent the tensor product L-function. Let

$$w_0 = \begin{pmatrix} 0 & I_k & & \\ I_{n-k} & 0 & & \\ & & 0 & I_{n-k} \\ & & I_k & 0 \end{pmatrix}$$

and define $j(g) = w_0 g w_0^{-1}$ for all $g\in\mathrm{Sp}_{2n}(\mathbb{A})$. The global integral we consider is

$$I_1(\varphi,\phi,\tilde f_{\sigma,s}) = \int\limits_{\mathrm{Sp}_{2k}(F)\backslash\mathrm{Sp}_{2k}(\mathbb{A})}\ \int\limits_{H_k(F)\backslash H_k(\mathbb{A})}\ \int\limits_{V_{2n,n-k-1}(F)\backslash V_{2n,n-k-1}(\mathbb{A})} \varphi\big(j(v\tau(h)g)\big)\tilde\theta_\phi(hg)\widetilde{E}(g,\tilde f_{\sigma,s})\psi_{n-k-1}(v)\,dv\,dh\,dg.$$

Here $\varphi \in V_\pi$, $\phi \in \mathcal{S}(\mathbb{A}^k)$ and $\tilde{f}_{\sigma,s} \in \tilde{I}(\sigma, s)$, and ψ_{n-k-1} is as defined in Chapter 1, Section 4. Also g embeds in Sp_{2n} as described in Section 1, part 2.

A similar construction is valid for irreducible cuspidal representations $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}_{2n}}(\mathbb{A})$. Indeed, given $\widetilde{\varphi} \in V_{\widetilde{\pi}}$, $\phi \in \mathcal{S}(\mathbb{A}^k)$ and $f_{\sigma,s} \in I_{\sigma,s}$ we define

$$I_2(\widetilde{\varphi}, \phi, f_{\sigma,s}) = \int_{\mathrm{Sp}_{2k}(F)\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int_{H_k(F)\backslash H_k(\mathbb{A})} \int_{V_{2n,n-k-1}(F)\backslash V_{2n,n-k-1}(\mathbb{A})} \widetilde{\varphi}\big(j(v\tau(h)g)\big)\tilde{\theta}_\phi(hg) E(g, f_{\sigma,s})\psi_{n-k-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g.$$

Let us remark that in both cases the integrals are well defined in the sense that the functions $\tilde{\theta}_\phi(hg)\widetilde{E}(g, \tilde{f}_{\sigma,s})$ and $\widetilde{\varphi}\Big(j(v\tau(h)g)\Big)\tilde{\theta}_\phi(hg)$ are non-genuine functions of $\widetilde{\mathrm{Sp}_{2k}}(\mathbb{A})$.

Let $R \subset V_{2n,n}$ be the unipotent subgroup consisting of all matrices of the form

$$R = \left\{ \begin{pmatrix} I_{n-k} & r & & \\ & I_k & & \\ & & I_k & r^* \\ & & & I_{n-k} \end{pmatrix} : \begin{array}{l} r \in M_{(n-k)\times n} \text{ where the} \\ \text{bottom row is zero} \end{array} \right\}$$

We are now ready to prove :

THEOREM 2.1. — *The integral $I_1(\varphi, \phi, \tilde{f}_{\sigma,s})$ converges absolutely for all s, except for those s for which the Eisenstein series has a pole. For* $\mathrm{Re}(s)$ *large :*

$$I_1(\varphi, \phi, \tilde{f}_{\sigma,s}) = \int_{V_{2k,k}(\mathbb{A})\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int_{R(\mathbb{A})} \int_{X_k(\mathbb{A})} W_\varphi\big(j(r\tau(x,0,0)g)\big) \omega_\psi(g)\phi(x)\tilde{f}_{W_\sigma,s}(g)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g.$$

Here $W_\varphi \in \mathcal{W}(\pi, \psi_n)$ and $\tilde{f}_{W_{\sigma,s}}$ is as defined in Chapter 1, *Section* 5.

Proof. — The proof of the absolute convergence follows as in [G] using the growth conditions of φ. We show the unfolding. Unfolding the Eisenstein series and the theta series, we obtain that $I_1(\varphi, \phi, \tilde{f}_{\sigma,s})$ equals

$$\int_{P_{2k,k}(F)\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int \varphi\big(j(v\tau(h)g)\big) \sum_{\xi\in F^k} \omega_\psi(hg)\phi(\xi)\tilde{f}_{\sigma,s}(g)\psi_{n-k-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g,$$

where h and v are integrated as before. From (1.2) it follows that

$$\omega_\psi(hg)\phi(\xi) = \omega_\psi\big((\xi,0,0)hg\big)\phi(0).$$

Write $h \in H_k(\mathbb{A})$ as $h = (0,y,z)(x,0,0)$ (see Chapter 1, Section 3). Changing variables using the left invariant properties of φ and collapsing the sum over F^k with the integration over $X_k(F)\setminus X_k(\mathbb{A})$, $I_1(\varphi,\phi,\tilde{f}_{\sigma,s})$ equals

$$\int \varphi\big(j(v\tau((0,y,z))\tau((x,0,0))g)\big)\,\omega_\psi\big((0,y,z)(x,0,0)g\big)\phi(0)$$
$$\tilde{f}_{\sigma,s}(g)\psi_{n-k-1}(v)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}g.$$

Here y is integrated over $Y_k(F)\setminus Y_k(\mathbb{A})$, z is integrated over $Z_k(F)\setminus Z_k(\mathbb{A})$, x over $X_k(\mathbb{A})$ and v and g are integrated as before. From (1.2) we obtain

$$\omega_\psi\big((0,y,z)(x,0,0)g\big)\phi(0) = \psi(z)\omega_\psi(g)\phi(x)$$

Thus $I_1(\varphi,\phi,\tilde{f}_{\sigma,s})$ equals

$$\int \varphi\big(j(v\tau((0,y,z))\tau((x,0,0))g)\big)\,\omega_\psi(g)\phi(x)$$
$$\tilde{f}_{\sigma,s}(g)\psi_{n-k-1}(v)\psi(z)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}g,$$

where all variables are integrated as before. We have the following Lemma :

LEMMA 2.2. — *For data as above*

$$\int \varphi\big(j(v\tau((0,y,z))ug)\big)\psi_{n-k-1}(v)\psi(z)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}u$$
$$= \int_{R(\mathbb{A})} \sum_{\delta\in N_k(F)\backslash \mathrm{GL}_k(F)} W_\varphi\left(\begin{pmatrix} \delta & & \\ & I_{2(n-k)} & \\ & & \delta^* \end{pmatrix} j(rg)\right)\mathrm{d}r.$$

Here u is integrated over $U_{2k,k}(F)\setminus U_{2k,k}(\mathbb{A})$, v over $V_{2n,n-k-1}(F)\setminus V_{2n,n-k-1}(\mathbb{A})$, y over $Y_k(F)\setminus Y_k(\mathbb{A})$ and z over $Z_k(F)\setminus Z_k(\mathbb{A})$.

The proof of Lemma 2.2 is as the proof of the Lemma in [G, p. 172]. We will give the details later.

Returning to the proof of the theorem, write $P_{2k,k} = \mathrm{GL}_k\, U_{2k,k}$ and

$$\int_{P_{2k,k}(F)\backslash \mathrm{Sp}_{2k}(\mathbb{A})} = \int_{\mathrm{GL}_k(F)U_{2k,k}(\mathbb{A})\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int_{U_{2k,k}(F)\backslash U_{2k,k}(\mathbb{A})}.$$

Using this and Lemma 2.2., $I_1(\varphi,\phi,\tilde{f}_{\sigma,s})$ equals

$$\int\limits_{\mathrm{GL}_k(F)U_{2k,k}(\mathbb{A})\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int\limits_{R(\mathbb{A})} \int\limits_{X_k(\mathbb{A})} \sum_{\delta\in N_k(F)\backslash \mathrm{GL}_k(F)} W_\varphi\big(j(\hat{\delta} r\tau(x,0,0)g)\big)\omega_\psi(g)\phi(x)\tilde{f}_{\sigma,s}(g)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g,$$

where $j(\hat{\delta}) = \begin{pmatrix} \delta & & \\ & I_{2(n-k)} & \\ & & \delta^* \end{pmatrix}$. Using the definition of j we see that $\hat{\delta}\in \mathrm{Sp}_{2k}(F)$. Hence, conjugating $\hat{\delta}$ across r and $\tau(x,0,0)$, and a change of variables in r and x, we may collapse the summation with the integration over $\mathrm{GL}_k(F)U_{2k,k}(\mathbb{A})\setminus \mathrm{Sp}_{2k}(\mathbb{A})$ to obtain

$$\int W_\varphi\big(j(r\tau(x,0,0)g)\big)\omega_\psi(g)\phi(x)\tilde{f}_{\sigma,s}(g)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g,$$

where g is integrated over $N_k(F)U_{2k,k}(\mathbb{A})\backslash \mathrm{Sp}_{2k}(\mathbb{A})$ and all other variables as before. Write

$$\int\limits_{N_k(F)U_{2k,k}(\mathbb{A})\backslash \mathrm{Sp}_{2k}(\mathbb{A})} = \int\limits_{N_k(\mathbb{A})U_{2k,k}(\mathbb{A})\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int\limits_{N_k(F)\backslash N_k(\mathbb{A})} .$$

We have $N_kU_{2k,k} = V_{2k,k}$. After a change of variables in r and x, $I_1(\varphi,\phi,\tilde{f}_{\sigma,s})$ equals

$$\int W_\varphi\big(j(r\tau(x,0,0)g)\big)\omega_\psi(g)\phi(x)\Big(\int\limits_{N_k(F)\backslash N_k(\mathbb{A})} \tilde{f}_{\sigma,s}(\bar{n}g)\psi_{N_k}(\bar{n})\,\mathrm{d}\bar{n}\Big)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g,$$

where g is integrated over $V_{2k,k}(\mathbb{A})\setminus \mathrm{Sp}_{2k}(\mathbb{A})$ and r and x as before. From this the Theorem follows. □

Proof of Lemma 2.2. — As mentioned before the proof of this lemma follows the same pattern as the proof of the lemma in [G, p. 173–175]. We give some details. Let $N_{n-k-1}\subset N_n$ be embedded in N_n in the lower right corner. We define the following two unipotent subgroups of $\mathrm{Sp}_{2n}(\mathbb{A})$. First

$$T = \left\{ \begin{pmatrix} I_k & \hat{t} & 0 & 0 \\ & I_{n-k} & 0 & 0 \\ & & I_{n-k} & \hat{t}^* \\ & & & I_k \end{pmatrix} : \begin{array}{l} \hat{t}\in M_{k\times(n-k)} \\ \text{such that the first} \\ \text{column is zero} \end{array} \right\}$$

and second

$$L=\left\{\begin{pmatrix} I_k &&& \\ \hat{\ell} & I_{n-k} && \\ && I_{n-k} & \\ && \hat{\ell}^* & I_k \end{pmatrix} : \begin{array}{l} \hat{\ell}\in M_{(n-k)\times k} \\ \text{such that the last} \\ \text{row is zero} \end{array}\right\}.$$

Notice that $L=j(R)$. We now prove the lemma. By definition

$$W_\varphi(g)=\int\limits_{V_{2n,n}(F)\backslash V_{2n,n}(\mathbb{A})}\varphi(vg)\psi_n(v)\,\mathrm{d}v.$$

As in [G], and using the references given there we have

$$\sum_{\delta\in N_k(F)\backslash \mathrm{GL}_k(F)} W_\varphi\left[\begin{pmatrix}\delta && \\ & I_{2(n-k)} & \\ && \delta^*\end{pmatrix} g\right]=$$

$$\int\limits_{N_{n-k-1}(F)\backslash N_{n-k-1}(\mathbb{A})}\ \int\limits_{T(F)\backslash T(\mathbb{A})}\ \int\limits_{U_{2n,n}(F)\backslash U_{2n,n}(\mathbb{A})}\varphi(ut\bar{n}g)\psi_n(\bar{n}u)\,\mathrm{d}u\,\mathrm{d}t\,\mathrm{d}\bar{n}.$$

Here we view $\bar{n}$ as an Sp_{2n} matrix via the embedding of GL_n in Sp_{2n}. Also $\psi_n(\bar{n}u)$ is defined by restriction. Let $T_1\subset T$ be defined by

$$T_1=\big\{t\in T : \text{all columns of } \hat{t} \text{ are zero except the second one}\big\}.$$

Thus $T_1\simeq M_{k\times 1}$. Also, let $L_1\subset L$ be defined by

$$L_1=\big\{\ell\in L : \text{all rows of } \hat{\ell} \text{ are zero except the first one}\big\}.$$

Thus $L_1\simeq M_{1\times k}$. Following (3.5)–(3.8) in [G] we obtain

$$\int\limits_{L_1(\mathbb{A})}\ \sum_{\delta\in N_k(F)\backslash \mathrm{GL}_k(F)} W_\varphi\left[\begin{pmatrix}\delta && \\ & I_{2(n-k)} & \\ && \delta^*\end{pmatrix}\ell_1 g\right]$$

$$=\int\limits_{N_{n-k-1}(F)\backslash N_{n-k-1}(\mathbb{A})}\ \int\limits_{L_1(F)\backslash L_1(\mathbb{A})}\ \int\limits_{T(F)T_1(\mathbb{A})\backslash T(\mathbb{A})}\ \int\limits_{U_{2n,n}(F)\backslash U_{2n,n}(\mathbb{A})}\varphi(ut\ell_1\bar{n}g)\psi_n(\bar{n}u)\,\mathrm{d}u\,\mathrm{d}t\,\mathrm{d}\ell_1\,\mathrm{d}\bar{n}.$$

Here $\bar{n}$ and u are integrated as before. Continuing this process, (as in [G]), we obtain

$$\int_{L(\mathbb{A})} \sum_{\delta\in N_k(F)\backslash \mathrm{GL}_k(F)} W_\varphi\left[\begin{pmatrix} \delta & & \\ & I_{2(n-k)} & \\ & & \delta^* \end{pmatrix}\ell g\right]$$

$$= \int_{N_{n-k-1}(F)\backslash N_{n-k-1}(\mathbb{A})} \int_{L(F)\backslash L(\mathbb{A})} \int_{U_{2n,n}(F)\backslash U_{2n,n}(\mathbb{A})} \varphi(u\ell\bar{n}g)\psi_n(\bar{n}u)\,\mathrm{d}u\,\mathrm{d}\ell\,\mathrm{d}\bar{n}.$$

The lemma follows from the fact that

$$LN_{n-k-1}U_{2n,n} = j\big(V_{2n,n-k-1}U_{2k,k}\tau(0, Y_k, Z_k)\big).$$

Here N_{n-k-1} is embedded in Sp_{2n} via the embedding of $N_{n-k-1} \hookrightarrow N_n \hookrightarrow \mathrm{Sp}_{2n}$, where the first embedding is as described in the beginning of the proof. $\square$

A similar statement holds for $I_2(\widetilde{\varphi}, \phi, f_{\sigma,s})$.

THEOREM 2.3. — *The integral $I_2(\widetilde{\varphi}, \phi, f_{\sigma,s})$ converges absolutely for all s except for those s for which the Eisenstein series has a pole. For* $\mathrm{Re}(s)$ *large* :

$$I_2(\widetilde{\varphi}, \phi, f_{\sigma,s}) = \int_{V_{2k,k}(\mathbb{A})\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int_{R(\mathbb{A})} \int_{X_k(\mathbb{A})} W_{\widetilde{\varphi}}\big(j(r\tau((x,0,0))g)\big)$$
$$\omega_\psi(g)\phi(x)f_{W_{\sigma,s}}(g)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g.$$

Here $W_{\widetilde{\psi}} \in \mathcal{W}(\widetilde{\pi}, \psi_n)$ and $f_{W_\sigma,s}$ is as defined in Chapter 1, Section 5. $\square$

Theorems 2.1 and 2.3 show that both global integrals are Eulerian. In the next chapter we shall study the local integral obtained from these global integrals.

3. The Local Theory

In this chapter we shall study the local integral in question. We shall compute the unramified integrals and prove some nonvanishing results.

Let F be a local field. Let π, $\widetilde{\pi}$, σ be irreducible representations of $\mathrm{Sp}_{2n}(F)$, $\widetilde{\mathrm{Sp}}_{2n}(F)$ and $\mathrm{GL}_k(F)$ resp. As before we shall assume that $n > k$. When convenient, we shall write Sp_{2n} for $\mathrm{Sp}_{2n}(F)$, etc. Let ψ be a nontrivial additive character of F. We shall denote by $\mathcal{W}(\pi, \psi_n)$, $\mathcal{W}(\widetilde{\pi}, \psi_n)$

and $\mathcal{W}(\sigma,\psi_{N_k})$ the Whittaker space associated with π, $\tilde{\pi}$ and σ resp. The local Weil representation ω_ψ of the group $H_k\widetilde{\mathrm{Sp}}_{2k}$ acts on $\mathcal{S}(F^k)$ — the Schwartz space of F^k. We have the local version of Chapter 1, Section 6 of (1.2)–(1.4), for the action of ω_ψ.

Let $(\,,)$ denote the local Hilbert symbol and γ_a $(a\in F^*)$ the local Weil constant. Let

$$\tilde{I}(\mathcal{W}(\sigma,\psi_{N_k}),s)=\mathrm{Ind}_{\widetilde{P}_{2k,k}}^{\widetilde{\mathrm{Sp}}_{2k}}\big(\mathcal{W}(\sigma,\psi_{N_k})\otimes\delta^s_{P_{2k,k}}\otimes\gamma^{-1}\big).$$

Thus a function $\tilde{f}_{W_{\sigma,s}}$, or $\tilde{f}_s$ in short, in $\tilde{I}(\mathcal{W}(\sigma,\psi_{N_k}),s)$ is a smooth function on the group $\widetilde{\mathrm{Sp}}_{2k}$ which takes values in $\mathcal{W}(\sigma,\psi_{N_k})$. More precisely, given $g\in\widetilde{\mathrm{Sp}}_{2k}$ there is a function $W^g_{\sigma,s}\in\mathcal{W}(\sigma,\psi_{N_k})$ such that

$$\tilde{f}_s\left(\begin{pmatrix} m & * \\ & m^*\end{pmatrix}g\right)=\delta^s_{P_{2k,k}}(m)\gamma^{-1}_{\det m}W^g_{\sigma,s}(m),$$

where $m\in\mathrm{GL}_k$ and $\begin{pmatrix} m & * \\ & m^*\end{pmatrix}\in P_{2k,k}$. Similarly we define

$$I\big(\mathcal{W}(\sigma,\psi_{N_k}),s\big)=\mathrm{Ind}_{P_{2k,k}}^{\mathrm{Sp}_{2k}}\big(\mathcal{W}(\sigma,\psi_{N_k})\otimes\delta^s_{P_{2k,k}}\big).$$

The local integrals we study are

$$I_1(W,\phi,\tilde{f}_s)=\int\limits_{V_{2k,k}\backslash\mathrm{Sp}_{2k}}\ \int\limits_{R}\int\limits_{X_k}W\big(j(r\tau((x,0,0))g)\big)\,\omega_\psi(g)\phi(x)\tilde{f}_s(g)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g$$

and

$$I_2(\widetilde{W},\phi,f_s)=\int\limits_{V_{2k,k}\backslash\mathrm{Sp}_{2k}}\ \int\limits_{R}\int\limits_{X_k}\widetilde{W}\big(j(r\tau((x,0,0))g\big)\,\omega_\psi(g)\phi(x)f_s(g)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g.$$

Here $W\in\mathcal{W}(\pi,\psi_n)$, $\widetilde{W}\in\mathcal{W}(\tilde{\pi},\psi_n)$, $\phi\in\mathcal{S}(F^k)$, $\tilde{f}_s\in\tilde{I}(\mathcal{W}(\sigma,\psi_{N_k}),s)$ and $f_s\in I(\mathcal{W}(\sigma,\psi_{N_k}),s)$.

If F is a finite place we let p denote a generator of the maximal ideal in the ring of integers of F. Also $|p|=q^{-1}$. For any local field, $K(G)$ will denote the standard maximal compact subgroup of G. It is well-known that $K(\mathrm{Sp}_{2k})$ splits in $\widetilde{\mathrm{Sp}}_{2k}$ and we shall identify $K(\mathrm{Sp}_{2k})$ with its image in $\widetilde{\mathrm{Sp}}_{2k}$.

3.1. The Unramified Computations.—Let F be a nonarchimedean field. Assume all data is unramified. More precisely, we assume that there are $W \in \mathcal{W}(\pi, \psi_n)$, $W_\sigma \in \mathcal{W}(\sigma, \psi_{N_k})$ and $\tilde{f}_s \in \tilde{I}(\mathcal{W}(\sigma, \psi_{N_k}), s)$ which are fixed under the corresponding maximal compact subgroup. In other words, $W(\bar{k}) = W(e) = 1$ for all $\bar{k} \in K(\mathrm{Sp}_{2n})$, and $W_\sigma(\bar{k}) = W_\sigma(e) = 1$ for all $\bar{k} \in K(\mathrm{GL}_k)$ and $\tilde{f}_s(\bar{k}) = \tilde{f}_s(e) = 1$ for all $\bar{k} \in K(\mathrm{Sp}_{2k})$. In these cases ψ is an unramified character of F. Let $\phi \in \mathcal{S}(F^k)$ satisfy $\phi(x) = 1$ if $x \in \mathcal{O}^k$ and zero otherwise. Here $\mathcal{O}$ is a ring of integers in F. Similarly, we assume the existence of $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi_n)$ and $f_s \in I(\mathcal{W}(\sigma, \psi_{N_k}), s)$ with similar properties.

Next we describe the L-functions we study. From general theory π is a quotient of $\mathrm{Ind}_{B_n}^{\mathrm{Sp}_{2n}}(\alpha\delta_{B_n}^{\frac{1}{2}})$, where B_n is the Borel of Sp_{2n} and α an unramified character of B_n. Thus if $t = \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$ denotes the maximal torus of Sp_{2n} then $\alpha(t) = \alpha_1(t_1) \cdots \alpha_n(t_n)$, where α_i are unramified characters of F^*. Let

$$A_p = \mathrm{diag}\left(\alpha_1(p), \ldots, \alpha_n(p), 1, \alpha_n^{-1}(p), \ldots, \alpha_1^{-1}(p)\right)$$

be the semisimple conjugacy class of $\mathrm{SO}_{2n+1}(\mathbb{C})$ attached to π. Similarly, we may associate with σ a semisimple conjugacy class in $\mathrm{GL}_k(\mathbb{C})$ denoted by

$$B_p = \mathrm{diag}\left(\beta_1(p), \ldots, \beta_k(p)\right).$$

The local tensor product L-function is defined

$$L(\pi \otimes \sigma, s) = \det\left(I_{(2n+1)k} - A_p \otimes B_p q^{-s}\right)^{-1}.$$

We also define the local symmetric square L-function of σ by

$$L(\sigma, V^2, s) = \prod_{i,j=1}^{k} \left(1 - \beta_i(p)\beta_j(p)q^{-s}\right)^{-1}.$$

We prove

THEOREM 3.1. — *For all unramified data and for* $\mathrm{Re}(s)$ *large*

$$I_1(W, \phi, \tilde{f}_s) = \frac{L(\pi \otimes \sigma, s(k+1) - \frac{1}{2}k)}{L(\sigma, V^2, 2s(k+1) - k)}.$$

Proof. — Let T denote the maximal torus of GL_k. We parametrize T as follows :

$$t = \mathrm{diag}(a_1, a_2, \ldots, a_k), \quad a_i \in F^*.$$

Let $\hat{t}$ denote the image of t in Sp_{2k} under the embedding described in Chapter 1, Section 2. Thus

$$\hat{t} = \mathrm{diag}(a_1, \ldots, a_k, a_k^{-1}, \ldots, a_1^{-1}).$$

It is easy to check that $j(\hat{t})$ embeds in Sp_{2n} as

$$j(\hat{t}) = \mathrm{diag}(a_1, \ldots, a_k, 1, \ldots, 1, a_k^{-1}, \ldots, a_1^{-1}).$$

Given a split algebraic matrix group G, we shall denote by $B(G)$ its Borel subgroup consisting of upper triangular matrices. We have

$$\begin{aligned}
\delta_{B(\mathrm{Sp}_{2k})}(\hat{t}) &= |a_1|^{2k}|a_2|^{2(k-1)}\cdots|a_k|^2,\\
\delta_{B(\mathrm{Sp}_{2n})}\big(j(\hat{t})\big) &= |a_1|^{2n}|a_2|^{2(n-1)}\cdots|a_k|^{2(n-k+1)},\\
\delta_{B(\mathrm{GL}_k)}(t) &= |a_1|^{k-1}|a_2|^{k-3}\cdots|a_k|^{-(k-1)},\\
\delta_{P_{2k,k}}(\hat{t}) &= |a_1\cdots a_k|^{k+1}.
\end{aligned}$$

To compute our integral, we apply the Iwasawa decomposition to obtain that $I_1(W,\phi,\tilde{f}_s)$ equals

$$\int_T\int_R\int_{X_k} W\big(j\big(r\tau((x,0,0))\hat{t}\big)\big)\omega_\psi(\hat{t})\phi(x)\tilde{f}_s(\hat{t})\delta^{-1}_{B(\mathrm{Sp}_{2k})}(\hat{t})\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}t.$$

Using the definition of $\tilde{f}_s$ this equals

$$\int W\big(j\big(r\tau((x,0,0))\hat{t}\big)\big)\omega_\psi(\hat{t})\phi(x)W_\sigma(t)\gamma^{-1}_{\det t}\delta^s_{P_{2k,k}}(\hat{t})\delta^{-1}_{B(\mathrm{Sp}_{2k})}(\hat{t})\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}t.$$

Next we use the local version of Chapter 1, Section 6, and a change of variables in r and x to obtain

$$\int W\big(j\big(\hat{t}r\tau((x,0,0))\big)\big)W_\sigma(t)\phi(x)|\det t|^{\frac{1}{2}-(n-k)}\,\delta^s_{P_{2k,k}}(\hat{t})\delta^{-1}_{B(\mathrm{Sp}_{2k})}(\hat{t})\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}t.$$

Since ϕ is supported on $\mathcal{O}^k$ we may ignore the x integration. Also, as in [G, p. 176] we may show that the support of the function $r\mapsto W\big(j(\hat{t}r)\big)$ is in $R(\mathcal{O})$. Hence we may ignore the r integration as well to obtain

$$\int_T W\big(j(\hat{t})\big)W_\sigma(t)|\det t|^{\frac{1}{2}-(n-k)}\delta^s_{P_{2k,k}}(\hat{t})\delta^{-1}_{B(\mathrm{Sp}_{2k})}(\hat{t})\,\mathrm{d}t.$$

Denote

$$K\big(j(\hat{t})\big) = \delta_{B(\mathrm{Sp}_{2n})}^{-\frac{1}{2}}\big(j(\hat{t})\big)W\big(j(\hat{t})\big), \quad K_\sigma(t) = \delta_{B(\mathrm{GL}_k)}^{-\frac{1}{2}}(t)W_\sigma(t).$$

Plugging this to the above integral and using the fact that

$$\delta_{B(\mathrm{Sp}_{2n})}^{\frac{1}{2}}\big(j(\hat{t})\big)\delta_{B(\mathrm{Sp}_{2k})}^{-1}(\hat{t})\delta_{B(\mathrm{GL}_k)}^{\frac{1}{2}k}(t)|\det t|^{\frac{1}{2}-(n-k)} = |\det t|^{-\frac{1}{2}k}$$

we obtain

$$\int_T K\big(j(\hat{t})\big)K_\sigma(t)|\det t|^{s(k+1)-\frac{1}{2}k}\,\mathrm{d}t.$$

From the support properties of the Whittaker function on Sp_{2n} this equals

$$\sum_{n_1\geq n_2\geq\cdots\geq n_k\geq 0} K\big(j(p^{n_1},\ldots,p^{n_k})^\wedge\big)K_\sigma\big((p^{n_1},\ldots,p^{n_k})\big)x^{n_1+\cdots+n_k},$$

where $x = q^{-s(k+1)+\frac{1}{2}k}$ and

$$(p^{n_1},\ldots,p^{n_k}) = \mathrm{diag}(p^{n_1},\ldots,p^{n_k}),$$
$$j(p^{n_1},\ldots,p^{n_k})^\wedge = \mathrm{diag}(p^{n_1},\ldots,p^{n_k},1,\ldots,1,p^{-n_k},\ldots,p^{-n_1}).$$

For any k positive numbers $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ let

$$\mathrm{tr}(m_1,m_2,\ldots,m_k,0,\ldots 0 \mid m_1,m_2,\ldots,m_k)$$

denote the trace of the irreducible finite dimensional representation of $\mathrm{SO}_{2n+1}(\mathbb{C}) \times \mathrm{GL}_k(\mathbb{C})$ applied to a semisimple representative corresponding to π and σ, whose highest weight is $(m_1,\ldots,m_k,0,\ldots,0)$ in the SO_{2n+1} component and $(m_1,\ldots,m_k)$ in the GL_k component. Using the Casselman-Shalika formula [CS], $I_1(W,\phi,\tilde{f}_s)$ equals

$$\sum_{n_1\geq n_2\geq\cdots\geq n_k\geq 0} \mathrm{tr}(n_1,\ldots,n_k,0,\ldots 0 \mid n_1,\ldots,n_k)x^{n_1+n_2+\cdots+n_k}.$$

Next, we use the Poincaré identity

$$L\big(\pi\otimes\sigma, s(k+1)-\tfrac{1}{2}k\big) = \sum_{\ell=0}^{\infty}\mathrm{tr}\,\big(S^\ell(A_p\otimes B_p)\big)x^\ell.$$

Here S^ℓ denote the symmetric ℓ-th power operation. It follows from [T1] that the right-hand-side of the above identity equals

$$\sum_{m=0}^{\infty} \operatorname{tr}\bigl(S^m(S^2(B_p))\bigr)x^{2m}$$
$$\sum_{n_1\geq\cdots\geq n_k\geq 0} \operatorname{tr}\bigl(n_1,\dots,n_k,0,\dots,0 \mid n_1,\dots,n_k\bigr)x^{n_1+\cdots+n_k}$$

From this the theorem follows. $\square$

Next we compute $I_2(\widetilde{W},\phi,f_s)$ at unramified places. The definition of the local L-factor which corresponds to $\widetilde{\pi}\otimes\sigma$ is not canonical. We have to first make a choice of a nontrivial additive character ψ' of F. We use ψ' to write the unramified character of the torus of $\mathrm{Sp}_{2n}(F)$ which corresponds to $\widetilde{\pi}$. Thus, $\widetilde{\pi}$ is a quotient of the representation induced from $\widetilde{B}_n$ and the character

$$\bigl(\operatorname{diag}(t_1,\dots,t_n,t_n^{-1},\dots,t_1),\varepsilon\bigr) \longmapsto \varepsilon\eta_1(t_1)\cdots\eta_n(t_n)\gamma^{-1}_{t_1\cdots t_n,\psi'},$$

where $\eta_1,\dots,\eta_n$ are unramified characters of F^*. Let

$$C'_p = \operatorname{diag}\bigl(\eta_1(p),\dots,\eta_n(p),\eta_n^{-1}(p),\dots,\eta_1^{-1}(p)\bigr).$$

We define

$$L_{\psi'}(\widetilde{\pi}\otimes\sigma,s) = \det(I_{2nk} - q^{-s}C'_p\otimes B_p)^{-1}.$$

Note that

$$L_{\psi^a}(\widetilde{\pi}\otimes\sigma,s) = L_\psi\bigl(\widetilde{\pi}\otimes(\sigma\otimes\chi_a),s\bigr),$$

where $\chi_a(t)=(t,a)$, the quadratic character, which corresponds to a.

Note also that

$$L_{\psi'}(\widetilde{\pi}\otimes\sigma,s) = L\bigl(\theta_{\psi'}(\widetilde{\pi})\otimes\sigma,s\bigr),$$

where $\theta_{\psi'}(\widetilde{\pi})$ is the image of $\widetilde{\pi}$ under the local theta correspondence from $\widetilde{\mathrm{Sp}}_{2n}(F)$ to $\mathrm{SO}_{2n+1}(F)$, with respect to ψ'. This means that $\pi=\theta_{\psi'}(\widetilde{\pi})$ is such that

$$\operatorname{Hom}\bigl(\omega_{\psi'}^{(n(2n+1))}\otimes\widetilde{\pi},\pi\bigr)\neq 0,$$

where $\omega_{\psi'}^{(n(2n+1))}$ is the Weil representation of

$$\widetilde{\mathrm{Sp}}_{2n(2n+1)}(F) \quad\text{and}\quad \mathrm{Sp}_{2n}(F)\times\mathrm{SO}_{2n+1}(F)\hookrightarrow\mathrm{Sp}_{2n(2n+1)}(F)$$

as a dual pair.

We also define the exterior square L function of σ

$$L(\sigma,\Lambda^2,s)=\prod_{1\le i<j\le k}\left(1-\beta_i(p)\beta_j(p)q^{-s}\right)^{-1}$$

and the standard L function of σ

$$L(\sigma,s)=\det\left(I_k-B_pq^{-s}\right)^{-1}$$

We prove :

THEOREM 3.2. — *For all unramified data and for* $\mathrm{Re}(s)$ *large*

$$I_2(\widetilde{W},\phi,f_s)=\frac{L_\psi(\widetilde{\pi}\otimes\sigma,s(k+1)-\frac{1}{2}k)}{L(\sigma,s(k+1)-\frac{1}{2}(k-1))L(\sigma,\Lambda^2,2s(k+1)-k)}$$

Proof. — Using similar notations as in the proof of Theorem 3.1 we obtain that $I_2(\widetilde{W},\phi,f_s)$ equals

$$\int_T \widetilde{W}\big(j(\hat{t})\big)W_\sigma(t)|\det t|^{\frac{1}{2}-(n-k)}\delta^s_{P_{2k,k}}(\hat{t})\delta^{-1}_{B(\mathrm{Sp}_{2k})}(\hat{t})\gamma_{\det t}\,\mathrm{d}t,$$

where $\gamma_{\det t}$ is obtained from the formulas in Chapter 1, Section 6. Denote

$$\widetilde{K}\big(j(\hat{t})\big)=\delta^{-\frac{1}{2}}_{B(\mathrm{Sp}_{2n})}\big(j(\hat{t})\big)\gamma^{-1}_{\det t}\widetilde{W}\big(j(t)\big).$$

As in Theorem 3.1 we obtain that $I_2(\widetilde{W},\phi,f_s,s)$ equals

$$\sum_{n_1\ge n_2\ge\cdots\ge n_k\ge 0}\widetilde{K}\big(j(p^{n_1},\ldots,p^{n_k})^\wedge\big)K_\sigma\big((p^{n_1},\ldots,p^{n_k})\big)\\ \chi(p)^{n_1+\cdots+n_k}x^{n_1+\cdots+n_k}$$

Here we used the relation $\gamma_{\det t}\gamma_{\det t}=(\det t,\det t)$ and that $(\varepsilon,\varepsilon)=1$ if $|\varepsilon|=1$. We denote

$$\chi(x)=(x,x)=(-1,x).$$

To prove the Theorem we will follow [BFG, Section 5], using the formula for $\widetilde{K}$ as established in [BFH]. In order to use [BFH] we parametrize $\widetilde{\pi}$ as in (1.9) of [BFH], such that $\alpha_{n+1-i}=\mu_i(p)$. Thus, the unramified character, corresponding to $\widetilde{\pi}$ in (1.9) of [BFH] is

$$\left(\mathrm{diag}(t_1,\ldots,t_n,t_n^{-1},\ldots,t_1^{-1}),\varepsilon\right)\longmapsto\mu_1(t_1)\cdots\mu_n(t_n)\gamma_{t_1\cdots t_n,\psi}.$$

Since $\gamma_t = (t,t)\gamma_{t,\psi}^{-1} = \chi(t)\gamma_{t,\psi}^{-1}$, it follows that

$$L_\psi(\widetilde{\pi}\otimes\sigma, s) = \det\bigl(I_{2nk} - q^{-s}\chi(p)C_p\otimes B_p\bigr)^{-1},$$

where $C_p = \operatorname{diag}(\mu_1(p),\dots,\mu_n(p),\mu_n^{-1}(p),\dots,\mu_1^{-1}(p))$.

Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the alternator in the group algebra

$$\mathbb{C}\bigl[\mu_1(p)^{\pm1},\dots,\mu_n^{\pm1}(p)\bigr]$$

(resp. $\mathbb{C}[\beta_1(p),\dots,\beta_k(p)]$) corresponding to the Weyl group of $\mathrm{Sp}_{2n}(\mathbb{C})$ (resp. $\mathrm{GL}_k(\mathbb{C})$). As in [BFG, p. 53] we have

$$\chi_{\mathrm{Sp}(2n)}(\ell_1,\dots,\ell_k) = \Delta_{\mathrm{Sp}(2n)}^{-1}\mathcal{A}\bigl(\mu_1^{\ell_1+n}\mu_2^{\ell_2+n-1}\cdots\mu_n^{\ell_n}\bigr)$$

and

$$\chi_{\mathrm{GL}(k)}(m_1,\dots,m_k) = \Delta_{\mathrm{GL}(k)}^{-1}\mathcal{B}\bigl(\beta_1^{m_1+k-1}\beta_2^{m_2+k-2}\cdots\beta_k^{m_k}\bigr)$$

Here

$$\ell_1\ge\ell_2\ge\cdots\ge\ell_n\ge0,\quad m_1\ge m_2\ge\cdots\ge m_k\ge0,$$
$$\Delta_{\mathrm{Sp}(2n)} = \chi_{\mathrm{Sp}(2n)}(0,\dots,0)\quad\text{and}\quad\Delta_{\mathrm{GL}(k)} = \chi_{\mathrm{GL}(k)}(0,\dots,0).$$

To express our integral in terms of the alternator, we need to use the formula given in [BFH, Thm 1.2]. We have

$$\text{(3.1)}\quad \widetilde{K}\bigl(j(p^{n_1},\dots,p^{n_k})^\wedge\bigr) = \Delta_{\mathrm{Sp}(m)}^{-1}\mathcal{A}\bigl(\mu_1^n\mu_2^{n-1}\cdots\mu_n\prod_{i=1}^k\mu_i^{n_i}\prod_{j=1}^n(1-(p,p)q^{-\frac12}\mu_j^{-1})\bigr).$$

We recall that the difference between the above identity and the one stated in [BFH, Thm 1.2], is due to a different way of the parametrization of the semisimple conjugacy class associated with $\widetilde{\pi}$.

Next, using the Poincaré identity,

$$L\bigl(\widetilde{\pi}\otimes\sigma\otimes\chi, s(k+1)-\tfrac12k\bigr) = \sum_{\ell=0}^\infty \operatorname{tr}\bigl(S^\ell(B_p\otimes C_p)\bigr)\chi(p)^\ell x^\ell,$$

using Theorem 2.5 in [T2], the right-hand-side equals

$$\sum_{m=0}^\infty \operatorname{tr}\bigl(S^m(\Lambda^2(B_p))\bigr)x^{2m}C(\mu,\beta;x),$$

where

$$C(\mu,\beta;x)=\sum_{n_1\geq\cdots\geq n_k\geq 0}\chi_{\mathrm{GL}(k)}(n_1,\ldots,n_k)\chi_{\mathrm{Sp}(2n)}(n_1,\ldots,n_k,0,\ldots,0)\,\chi(p)^{n_1+\cdots+n_k}x^{n_1+\cdots+n_k}.$$

We have

$$(3.2)\quad C(\mu,\beta;x)=\prod_{i=1}^{k}(1-\beta_i q^{-\frac12}x)^{-1}\Delta_{\mathrm{Sp}(2n)}^{-1}\sum_{n_1\geq n_2\geq\cdots\geq n_k\geq 0}\chi_{\mathrm{GL}(k)}(n_1,\ldots,n_k)\,\mathcal{A}\Big(\mu_1^n\mu_2^{n-1}\cdots\mu_n\prod_{i=1}^{k}\mu_i^{n_i}\prod_{j=1}^{n}(1-(p,p)q^{-\frac12}\mu_j^{-1})\Big)\,\chi(p)^{n_1+\cdots+n_k}x^{n_1+\cdots+n_k}.$$

Indeed, this is the analog of our case to identity (5.4) in [BFG]. The proof of our formula is exactly the same. We omit the details. Using (3.1) we see that (3.2) becomes

$$C(\mu,\beta;x)=L\big(\sigma,s(k+1)-\tfrac12(k-1)\big)\sum_{n_1\geq n_2\geq\cdots\geq n_k\geq 0}\widetilde{K}\big(j(p^{n_1},\ldots,p^{n_k})^\wedge\big)K_\sigma\big((p^{n_1},\ldots,p^{n_k})\big)\,\chi(p)^{n_1+\cdots+n_k}x^{n_1+\cdots+n_k}.$$

Here we used the [CS] formula

$$K_\sigma\big((p^{n_1},\ldots,p^{n_k})\big)=\chi_{\mathrm{GL}(k)}(n_1,\ldots,n_k).$$

On the other hand

$$\sum_{m=0}^{\infty}\mathrm{tr}\big(S^m(\Lambda^2(B_p))\big)x^{2m}=L\big(\sigma,\Lambda^2,2s(k+1)-k\big)$$

Hence,

$$\begin{aligned}I_2(\widetilde{W},\phi,f_s)&=\sum_{n_1\geq\cdots\geq n_k\geq 0}\widetilde{K}\big(j(p^{n_1},\ldots,p^{n_k})^\wedge\big)\,K_\sigma\big((p^{n_1},\ldots,p^{n_k})\big)\chi(p)^{n_1+\cdots+n_k}x^{n_1+\cdots+n_k}\\&=\frac{C(\mu,\beta;x)}{L(\sigma,s(k+1)-\frac12(k-1))}\\&=\frac{L_\psi(\widetilde{\pi}\otimes\sigma,s(k+1)-\frac12 k)}{L(\sigma,s(k+1)-\frac12(k-1))L(\sigma,\Lambda^2,2s(k+1)-k)}.\qquad\square\end{aligned}$$

3.2. A Nonvanishing Result. — Let F be a local field. The main result in this section is to prove that given $s_0 \in \mathbb{C}$ there is a choice of data such that our local integrals are nonzero. As in [JS], [S1] and [S2] we have the following asymptotic expansion for the Whittaker spaces. Let

$$t = \operatorname{diag}\left(t_1 \cdots t_n, t_2 \cdots t_n, \ldots, t_n, t_n^{-1} \ldots\right)$$

be a parametrization of the maximal torus T_n of Sp_{2n}. We have

LEMMA 3.3. — *There is a finite set Δ of finite functions of $(F^*)^n$ such that for all W in $\mathcal{W}(\pi, \psi_n)$ and for $\alpha \in \Delta$ there is $\phi_\alpha \in \mathcal{S}(F^n \times K(\mathrm{Sp}_{2k}))$ such that*

$$W(tk) = \sum_{\alpha \in \Delta} \phi_\alpha(t_1, \ldots, t_n, m)\alpha(t_1, \ldots, t_k).$$

Here $t \in T$ parametrized as above and $m \in K(\mathrm{Sp}_{2n})$. $\square$

The asymptotic expansion for functions in $\mathcal{W}(\sigma, \psi_{N_k})$ is given in [JS]. We use this to prove :

LEMMA 3.4. — *For all W in $\mathcal{W}(\pi, \psi_n)$, ϕ in $\mathcal{S}(F^k)$ and $\tilde{f}_s$ in $I(\mathcal{W}(\sigma, \psi_{N_k}), s)$, the integral $I_1(W, \phi, \tilde{f}_s)$ converges absolutely for* $\mathrm{Re}(s)$ *large.*

Proof. — Let T denote the maximal torus of GL_k parameterized by

$$t = \left\{(t_1 \cdots t_k, t_2 \cdots t_k, \ldots, t_k)\right\}.$$

We let $\hat{t}$ denote the embedding of t in Sp_{2k}. Following the same steps as in the unramified computation (Theorem 3.1), $I_1(W, \phi, \tilde{f}_s)$ equals

$$\int_T \int_R \int_{X_k} \int_{K(\mathrm{Sp}_{2k})} W\big(j\big(\hat{t} r \tau((x,0,0)) m\big)\big) \omega_\psi(m) \phi(x) \; \tilde{f}_s(\hat{t} m) |\det t|^{\alpha_1 s + \alpha_2}\, dr\, dx\, dm\, dt.$$

Here $\alpha_1 \in \mathbb{N}$ and $\alpha_2 \in \mathbb{Z}$. Let $\widetilde{R}$ be the subgroup of GL_n defined by

$$\widetilde{R} = \left\{\begin{pmatrix} I_k & \\ \tilde{r} & I_{n-k} \end{pmatrix} : \tilde{r} \in M_{(n-k) \times k}\right\}.$$

We embed $\widetilde{R}$ in Sp_{2n} via the embedding of GL_n in Sp_{2n}. One can check that $j(R\tau(X_k)) = \widetilde{R}$. Ignoring the compact it is enough to prove the absolute convergence of

$$\int_T \int_{\widetilde{R}} W\big(j(\hat{t})\tilde{r}\big) \phi(\tilde{r}_{n-k}) W_\sigma(t) |\det t|^{\alpha_3 s + \alpha_4}\, d\tilde{r}\, dt.$$

Here $\alpha_3 \in \mathbb{N}$, $\alpha_4 \in \mathbb{Z}$ and $\tilde{r}_{n-k}$ denotes the last row of $\tilde{r}$. Notice that $T\widetilde{R} \subset \mathrm{GL}_n$. Now we can proceed as in Proposition 4.2 of [S1] or Proposition 7.1 in [JS]. $\square$

LEMMA 3.5. — *With notations as in Lemma 3.4 the integrals $I_1(W, \phi, \tilde{f}_s)$ admits a meromorphic continuation to the whole complex plane. Moreover this continuation is continuous in W, ϕ and $\tilde{f}_s$.*

Proof. — Using the notations of Lemma 3.4, we consider first the integral

$$\int_T \int_{\widetilde{R}} W(j(\hat{t})\tilde{r})\phi(\tilde{r}_{n-k})W_\sigma(t)|\det t|^{\alpha_1 s+\alpha_2}\,\mathrm{d}\tilde{r}\,\mathrm{d}t. \tag{3.3}$$

Let $\widetilde{R}_{n-k-1}$ be the subgroup of $\widetilde{R}$ defined by

$$\widetilde{R}_{n-k-1} = \{\tilde{r} \in \widetilde{R} :\ \tilde{r}_{n-k} = 0\}.$$

Thus the above integral equals

$$\int_T \int_{\widetilde{R}_{n-k-1}} W_1\big(j(t)\tilde{r}\big)W_\sigma(t)|\det t|^{\alpha_1 s+\alpha_2}\,\mathrm{d}\tilde{r}\,\mathrm{d}t,$$

where

$$W_1(m) = \int_{F^k} W(m\tilde{r}_{n-k})\phi(\tilde{r}_{n-k})\,\mathrm{d}\tilde{r}_{n-k}.$$

Here $m \in \mathrm{Sp}_{2n}$ and we identified the last row of $\widetilde{R}$ with F^k. Thus to prove the meromorphic continuation of (3.3) it is enough to prove it for

$$\int_T \int_{\widetilde{R}_{n-k-1}} W\big(j(t)\tilde{r}\big)W_\sigma(t)|\det t|^{\alpha_1 s+\alpha_2}\,\mathrm{d}\tilde{r}\,\mathrm{d}t. \tag{3.4}$$

Denote

$$L_n = \Big\{I_n + \sum_{i=1}^{k} \ell_i e_{in} :\ \ell_i \in F\Big\} \subset \mathrm{GL}_n\,.$$

The function W is a linear combination of functions of the form

$$W_2 = \int_{L_n} \phi(\ell)\pi(\ell)W_1\,\mathrm{d}\ell,$$

where $\phi \in \mathcal{S}(L_n)$ and $W_1 \in \mathcal{W}(\pi, \psi)$. This is clear if F is nonarchimedean, and follows from [DM] if F is archimedean. Here, as before we view L_n as a subgroup of Sp_{2n} via the embedding of GL_n in Sp_{2n}. Thus to prove the meromorphic continuation of (3.4) we may consider

$$\int_T \int_{\widetilde{R}_{n-k-1}} \int_{L_n} W\big(j(t)\tilde{r}\ell\big)\phi(\ell)W_\sigma(t)|\det t|^{\alpha_1 s+\alpha_2}\,\mathrm{d}\ell\,\mathrm{d}\tilde{r}\,\mathrm{d}t.$$

Conjugating ℓ to the left this equals

$$\int_T \int_{\widetilde{R}_{n-k-1}} W\big(j(t)\tilde{r}\big)\widehat{\phi}(\tilde{r}_{n-k-1})W_\sigma(t)|\det t|^{\alpha_1 s+\alpha_2}\, \mathrm{d}\tilde{r}\, \mathrm{d}t,$$

where $\tilde{r}_{n-k-1}$ denotes the $n-k-1$ row of $\tilde{r}$ and $\widehat{\phi}$ the Fourier transform of ϕ. Let

$$\widetilde{R}_{n-k-2} = \big\{\tilde{r} \in \widetilde{R}_{n-k-1} :\ \tilde{r}_{n-k-1} = 0\big\}.$$

Hence to prove the meromorphic continuation of the above integral it is enough to consider

$$\int_T \int_{\widetilde{R}_{n-k-2}} W\big(j(t)\tilde{r}\big)W_\sigma(t)|\det t|^{\alpha_1 s+\alpha_2}\, \mathrm{d}\tilde{r}\, \mathrm{d}t.$$

Continuing this process we get rid of the unipotent integration and reduce to

$$\int_T W\big(j(t)\big)W_\sigma(t)|\det t|^{\alpha_1 s+\alpha_2}\, \mathrm{d}t.$$

The meromorphic continuation of this integral is obtained by Lemma 3.3. This establishes the meromorphic continuation of (3.3). Moreover, it follows from Theorem 2, Section 4, and Lemma 2, Section 5, in [S2] that (3.3) admits a meromorphic continuation to all $s \in \mathbb{C}$ which is continuous in W, W_σ and ϕ. Hence, as in Lemma 1 of [S2] we obtain the meromorphic continuation of

$$\int_T \int_R \int_{X_k} \int_{K(\mathrm{Sp}_{2k})} W\big(j\big(\hat{t}r\tau((x,0,0))m\big)\omega_\psi(m)\phi(x)\tilde{f}_s(\hat{t}m)\, \mathrm{d}r\, \mathrm{d}x\, \mathrm{d}m\, \mathrm{d}t.$$

From this the Lemma follows. $\square$

Finally we prove :

PROPOSITION 3.6. — *Given $s_0 \in \mathbb{C}$ there is $W \in \mathcal{W}(\pi, \psi_n)$, $\phi \in \mathcal{S}(F^k)$ and a $K(\mathrm{Sp}_{2k})$-finite function $\tilde{f}_s \in \tilde{I}(\mathcal{W}(\sigma, \psi_{N_k}), s)$ such that $I_1(W, \phi, \tilde{f}_s)$ is nonzero at s_0.*

Proof. — We construct the following family of sections in $\tilde{I}(\sigma, s)$. Let Φ be an arbitrary smooth function on $(P_{2k,k} \cap K(\mathrm{Sp}_{2k})) \setminus K(\mathrm{Sp}_{2k})$, and $\tilde{f} \in \mathrm{Ind}_{\widetilde{\mathrm{P}}_{2k,k}}^{\widetilde{\mathrm{Sp}}_{2k}} \big(\mathcal{W}(\sigma, \psi_{N_k}) \otimes \gamma^{-1}\big)$. We denote

$$\tilde{f}_s^\Phi(pm) = \delta_{P_{2k,k}}^s(p)\tilde{f}(pm)\Phi(m),$$

where $p \in \widetilde{P}_{2k,k}$ and $m \in K(\mathrm{Sp}_{2k})$. Thus $\tilde{f}^{\Phi}_{\sigma,s} \in \tilde{I}(\sigma, s)$. Denote

$$I_1^1(W,\phi,s;m) = \int\limits_{N_k\backslash \mathrm{GL}_k} \int\limits_{R} \int\limits_{X_k} W\big(j\big(r\tau((x,0,0))gm\big)\big) \omega_\psi(gm)\phi(x)\delta^{-1}_{P_{2k,k}}(g)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g.$$

Then, as can be checked, we have for $\mathrm{Re}(s)$ large

$$I_1(W,\phi,\tilde{f}^{\Phi}_s,s) = \int\limits_{K_1\backslash K(\mathrm{Sp}_{2k})} I_1^1(W,\phi,s;m)\Phi(m)\,\mathrm{d}m. \tag{3.5}$$

As in Lemma 3.5 we can show that $I_1^1(W,\phi,s;m)$ which converges for $\mathrm{Re}(s)$ large, admits a meromorphic continuation to the whole complex plane. Moreover, $I_1^1(W,\phi,s;m)$ is a continuous function in m. Suppose that $I_1(W,\phi,f_s)$ is zero at $s = s_0$ for all choice of data. This implies that the right-hand-side of (3.5) is zero at s_0. Since Φ is arbitrary we may deduce that the meromorphic continuation of $I_1^1(W,\phi,s;m)$ is zero at $s = s_0$ for all choice of data. Plugging $m = 1$ and applying the definition of $\tilde{f}$ we may deduce that the meromorphic continuation of

$$\int\limits_{N_k\backslash \mathrm{GL}_k} \int\limits_{R} \int\limits_{X_k} W\big(j\big(r\tau((x,0,0))g\big)\big)\omega_\psi(g)\phi(x) W_\sigma(g)|\det g|^{(s-1)(k+1)}\gamma^{-1}_{\det g}\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g$$

is zero at $s = s_0$, for all choice of data. Conjugating g to the left, and changing variables we may assume that the meromorphic continuation of

$$\int\limits_{N_k\backslash \mathrm{GL}_k} \int\limits_{R} \int\limits_{X_k} W\big(j\big(gr\tau((x,0,0))\big)\big)\phi(x) W_\sigma(g)|\det g|^{\alpha s+\beta}\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g \tag{3.6}$$

is zero at $s = s_0$ for all choice of data. Here $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{Z}$. Recall that $\phi \in \mathcal{S}(F^k)$. Consider, for $x \in F^k$

$$I_1^2(W,W_\sigma,s;x) = \int\limits_{N_k\backslash \mathrm{GL}_k} \int\limits_{R} W\big(j\big(gr\tau((x,0,0))\big)\big)W_\sigma(g)|\det g|^{\alpha s+\beta}\,\mathrm{d}r\,\mathrm{d}g.$$

As before, we can prove that $I_1^2(W,W_\sigma,s;x)$ converges absolutely for $\mathrm{Re}(s)$ large and admits a meromorphic continuation to the whole complex plane. Thus (3.6) equals

$$\int_{X_k} I_1^2(W,W_\sigma,s;x)\phi(x)\,\mathrm{d}x.$$

Hence we may deduce that the meromorphic continuation of the above integral is zero at $s = s_0$ for all choice of $\phi \in \mathcal{S}(F^k)$. Thus $I_1^2(W, W_\sigma, s; x)$ is zero at $s = s_0$ for all choice of data. Let $x = 0$, and hence we may assume that the meromorphic continuation of

$$\int_{N_k\backslash \mathrm{GL}_k}\int_R W\big(j(gr)\big)W_\sigma(g)|\det g|^{\alpha s+\beta}\,\mathrm{d}r\,\mathrm{d}g$$

is zero at $s = s_0$ for all W and W_σ. At this point we will use a similar process as in the proof of Lemma 3.5.

Using the notations there we may rewrite the above integral as

$$\int_{N_k\backslash \mathrm{GL}_k}\int_{\widetilde{R}_{n-k-1}} W\big(j(g)\tilde{r}\big)W_\sigma(g)|\det g|^{\alpha s+\beta}\,\mathrm{d}\tilde{r}\,\mathrm{d}g.$$

Let $\phi \in \mathcal{S}(L_n)$. Replace W by

$$\int_{L_n}\phi(\ell)\pi(\ell)W\,\mathrm{d}\ell.$$

Thus the above integral equals

$$\int_{N_k\backslash \mathrm{GL}_k}\int_{\widetilde{R}_{n-k-1}}\int_{L_n} W(j(g)\tilde{r}\ell)\phi(\ell)W_\sigma(g)|\det g|^{\alpha s+\beta}\,\mathrm{d}\ell\,\mathrm{d}\tilde{r}\,\mathrm{d}g$$
$$= \int_{N_k\backslash \mathrm{GL}_k}\int_{\widetilde{R}_{n-k-1}} W\big(j(g)\tilde{r}\big)\widehat{\phi}(\tilde{r}_{n-k-1})W_\sigma(g)|\det g|^{\alpha s+\beta}\,\mathrm{d}\tilde{r}\,\mathrm{d}g,$$

where $\widehat{\phi}$ is the Fourier transform of ϕ and $\tilde{r}_{n-k-1}$ denotes the $n-k-1$ row of $\widetilde{R}_{n-k-1}$. Arguing as before, we may deduce that the meromorphic continuation of

$$\int_{N_k\backslash \mathrm{GL}_k}\int_{\widetilde{R}_{n-k-2}} W\big(j(g)\tilde{r}\big)W_\sigma(g)|\det g|^{\alpha s+\beta}\,\mathrm{d}\tilde{r}\,\mathrm{d}g$$

is zero at $s = s_0$ for all choice of data. Continuing this process, we may assume that the meromorphic continuation of

$$\int_{Nk\backslash \mathrm{GL}_k} W\big(j(g)\big)W_\sigma(g)|\det g|^{\alpha s+\beta}\,\mathrm{d}g$$

is zero at $s = s_0$ for all choice of data.

Arguing as in [JS] this implies that $W(e)W_\sigma(e)$ is zero for all W and W_σ which is a contradiction. $\square$

We have a similar result for $I_2(\widetilde{W}, \phi, f_s)$. As in the case of $I_1(W, \phi, \tilde{f}_s)$ we may prove that $I_2(\widetilde{W}, \phi, f_s)$ converges absolutely for $\mathrm{Re}(s)$ large and admits a meromorphic continuation to the whole complex plane, which is also continuous in $\widetilde{W}, \phi$ and f_s. From this as in Proposition 3.6 we have :

PROPOSITION 3.7. — *Given $s_0 \in \mathbb{C}$ there is $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi_n)$, $\phi \in \mathcal{S}(F^k)$ and a $K(\mathrm{Sp}_{2k})$-finite function $f_s \in I(\mathcal{W}(\sigma, \psi_{N_k}), s)$ such that $I_2(\widetilde{W}, \phi, f_s)$ is nonzero at $s = s_0$.*

4. A Double Coset Decomposition

In this Chapter we shall prove a Lemma and fix some notations needed later. We fix n and k with $k > n$. Denote $Q^0_{2k,k-n} = \mathrm{Sp}_{2n} V_{2k,k-n}$. Clearly $Q^0_{2k,k-n}$ is a subgroup of $Q_{2k,k-n}$. We embed GL_{k-n} in Sp_{2k} as

$$g \longmapsto \begin{pmatrix} g & & \\ & I_{2n} & \\ & & g^* \end{pmatrix}, \quad g \in \mathrm{GL}_{k-n}$$

and all subgroups of GL_{k-n} will be embedded in Sp_{2k} via this embedding. Here g^* is such that the above matrix is in Sp_{2k}.

For given $0 \le i \le k-n$ denote by M_i the maximal parabolic of GL_{k-n} defined by

$$M_i = \big(\mathrm{GL}_i \times \mathrm{GL}_{k-n-i}\big) L_i.$$

We let $M_0 = M_{k-n} = \mathrm{GL}_{k-n}$. Here L_i is the unipotent radical of M_i and we choose it to consist of lower unipotent, that is,

$$L_i = \Big\{ \begin{pmatrix} I_i & \\ A & I_{k-n-i} \end{pmatrix} : \ A \in M_{(k-n-i)\times i} \Big\}.$$

For $0 \le i \le k-n$ let W_i denote the Weyl group of GL_i. We shall identify W_i with all $i \times i$ permutation matrices. Finally for $0 \le i \le k-n$ we set

$$\gamma_i = \begin{pmatrix} & & -I_i \\ & I_{2(k-i)} & \\ I_i & & \end{pmatrix}.$$

Thus γ_i is an element of the Weyl group of Sp_{2k}.

LEMMA 4.1. — *A set of representatives for $P_{2k,k} \setminus \mathrm{Sp}_{2k} / Q^0_{2k,k-n}$ is contained in the set of all matrices of the form $\gamma_i w$, where $0 \le i \le k-n$ and $w \in (W_i \times W_{k-n-i}) \setminus W_{k-n}$.*

Proof. — Clearly $P_{2k,k-n} \supset Q^0_{2k,k-n}$. It is not hard to check that γ_i, $0 \le i \le k-n$ is a set of representatives for $P_{2k,k} \setminus \mathrm{Sp}_{2k} / P_{2k,k-n}$. Indeed, the space $P_{2k,k} \setminus \mathrm{Sp}_{2k}$ can be identified with the set of all k dimensional isotropic subspaces. Hence, we can parametrize the above double cosets with all possible intersections of $k-n$ dimensional isotropic subspaces with all k dimensional isotropic subspaces and for these we can choose as representatives γ_i. Hence every representative of $P_{2k,k} \setminus \mathrm{Sp}_{2k} / Q^0_{2k,k-n}$ can be chosen in the set of all elements $\gamma_i w$, where

$$w \in (\gamma_i^{-1} P_{2k,k} \gamma_i \cap P_{2k,k-n}) \setminus P_{2k,k-n} / Q^0_{2k,k-n}.$$

Since $\gamma_i^{-1} P_{2k,k} \gamma_i \cap P_{2k,k-n} \supset M_i$ then we may choose a set of representatives for the set

$$(\gamma_i^{-1} P_{2k,k} \gamma_i \cap P_{2k,k-n}) \setminus P_{2k,k-n} / Q^0_{2k,k-n}$$

to be contained in the set $M_i \setminus \mathrm{GL}_{k-n} / N_{k-n}$. (The group N_{k-n} is defined in Chapter 1, Section 4. Clearly, a set of representatives for $M_i \setminus \mathrm{GL}_{k-n} / N_{k-n}$ can be chosen to be contained in $(W_i \times W_{k-n-i}) \setminus W_{k-n}$.) $\square$

Next we prove :

LEMMA 4.2. — *For $0 \le i \le n-k$, each representative of $(W_i \times W_{k-n-i}) \setminus W_{k-n}$ can be chosen to be a symmetric matrix.*

Proof. — Let $w_1, \dots, w_{k-n-1}$ denote the simple reflections of W_{k-n} so that $w_1, \dots, w_i$ are in W_i and $w_{i+2}, \dots, w_{k-n-1}$ are in W_{k-n-i}. Given $w \in W_{k-n}$ we shall denote by $w(r,j)$ its (r,j) entry. Let

$$w = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in W_{k-n},$$

where

$$A \in M_{i \times i},\ B \in M_{i \times (k-n-i)},\ C \in M_{(k-n-i) \times i},\ D \in M_{(k-n-i) \times (k-n-i)}.$$

By multiplying from the left by elements of $W_i \times W_{k-n-i}$ we need to bring w to a symmetric matrix. Recall that left multiplication is just changing the rows of w. We claim that by multiplication on the left by W_i we can bring w to the following form. First if $w(r,j) = 1$, where $r \le i$ and $j \le i$ then $r = j$. In other words, all nonzero entries of A are on the main diagonal. Secondly, suppose that $w(r_1, j_1) = w(r_2, j_2) = 1$, where $r_1 < r_2 \le i$ and $i+1 \le j_1, j_2 \le k-n$ then $j_1 < j_2$. In other words B can be put in row echelon form. Indeed, if $w(1,j) = 1$, where $j \le i$, we can use $w_1, \dots, w_{i-1}$ to assume that $j = 1$ and then proceed by induction. If $w(1,j) = 0$ for all $j \le i$, then clearly A contains at most $i-1$ ones and hence B is nonzero. Let

$$p_1 = \min_j \{ w(r,j) = 1 \text{ where } 1 \le r \le i,\ j \ge i+1 \}.$$

Using $w_1, \ldots, w_{i-1}$ we may assume that $w(1, p_1) = 1$. Now we proceed by induction *i.e.* using $w_2, \ldots, w_{i-1}$ we may arrange all rows in w between two and i in the desired way. Hence we may assume that A and B in w has the above pattern.

Next, using W_{k-n-i} we apply similar arguments to C and D. More precisely, suppose that $w(r_1, p_1) = 1$, where $1 \leq r_1 \leq i$, $i+1 \leq p_1$ and if $w(r_2, p_2) = 1$ with $1 \leq r_2 \leq i$ and $i+1 \leq p_2$ then $p_1 < p_2$. This means that $w(r, j) = 0$ for all $1 \leq r \leq i$ and $i+1 \leq j < p_1$. Hence, using $w_{i+2}, \ldots, w_{k-n-i}$ we may assume that $w(r, r) = 1$ for all $i+1 \leq r < p_1$. Since $w(r_1, p_1) = 1$ then $w(r_1, r_1) = 0$ and hence, from the fact that all nonzero entries of A are on the diagonal we may assume that if $w(r, r_1) = 1$ then $r > i$. But since $w(j, j) = 1$ for all $i+1 \leq j < p_1$ then if $w(r, r_1) = 1$ then $r \geq p_1$. Hence using $w_{p_1}, \ldots, w_{k-n-i}$ we may assume that $w(p_1, r_1) = 1$. Continuing this process we may assume that if $w(r, p) = 1$ then $w(p, r) = 1$ and all nonzero entries of A and D are on the main diagonal. In particular w is symmetric. $\square$

Let $\alpha_1, \ldots, \alpha_{k-n-1}$ denote the simple roots of GL_{k-n} corresponding to N_{k-n}. Let $x_\alpha(t)$ denote the one parameter unipotent subgroup of N_{k-n} corresponding to the root α. Thus,

$$x_{\alpha_j}(t) = I_{k-n} + te_{j,j+1}.$$

Here $e_{r,j}$ denotes the $(k-n) \times (k-n)$ matrix whose only nonzero entry is one in the (r, j) position.

We shall agree that $w = e$ is the representative of the coset $W_i \times W_{k-n-i}$ in W_{k-n}.

LEMMA 4.3. — *Let $w \in (W_i \times W_{k-n-i}) \setminus W_{k-n}$ such that $w \neq e$. Then there exists a simple root α_j such that $wx_{\alpha_j}(t)w^{-1} \in L_i$.*

Proof. — Let $e \neq w \in (W_i \times W_{k-n-i}) \setminus W_{k-n}$. We assume that w is as described in the proof of Lemma 4.2. In other words, if $w = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then w is symmetric and all nonzero entries of A and D are on the main diagonal. Since $w \neq e$, $C \neq 0$. Let r_1 be the smallest integer such that $w(r_1, j_1) = 1$, where $r_1 > i$ and $1 \leq j_1 \leq i$. If $r_1 > i+1$ then $w(r_1 - 1, r_1 - 1) = 1$. Hence

$$wx_{\alpha_{r_1-1}}(t)w^{-1} = I + twe_{r_1-1, r_1} = I + te_{r_1-1, j_1} \in L_i.$$

Hence we may assume that $r_1 = i+1$. Thus $w(i+1, j_1) = 1$. Let $r_2 > r_1$ be the smallest integer such that $w(r_2, j_2) = 1$, where $1 \leq j_2 \leq i$. Of course such an r_2 may not exist. However if it does, then arguing as with r_1, we may assume that $r_2 = i+2$. We claim that $j_2 = j_1 + 1$. Indeed, from the shape of w we may deduce that $j_2 > j_1$. If $j_2 \neq j_1 + 1$, then $w(j_1+1, j_1+1) = 1$ and hence

$$wx_{\alpha_{j_1}}(t)w^{-1} = I + twe_{j_1,j_1+1}w^{-1} = I + te_{i+1,j_1+1} \in L_i.$$

Thus we may conclude that w has the shape

$$w = \begin{pmatrix} I_p & & & & \\ & 0 & & I_s & \\ & & I_q & & \\ & I_s & & 0 & \\ & & & & I'_k \end{pmatrix},$$

where $p+q+s = i$ and $k' = k-p-q-2s$. p, q or k' can be zero but not s, since $w \neq e$. If $q \neq 0$, then $w(p+s+1, p+s+1) = w(i+s, p+s) = 1$ and then

$$wx_{\alpha_{p+s}}(t)w^{-1} = I + twe_{p+s,p+s+1}w^{-1} = I + te_{i+s,p+s+1} \in L_i.$$

If $q = 0$ then $wx_{\alpha_i}(t)w^{-1} \in L_i$. We are done. $\square$

5. The Global Integrals ($k > n$)

In this chapter, we introduce the global integrals which will represent the tensor product L-function in the case when $k > n$. These integrals are "dual" to the ones introduced in Chapter 2 in the sense that the cusp form and the Eisenstein series are interchanged. We define

$$J_1(\varphi, \phi, \tilde{f}_{\sigma,s}) = \int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{H_n(F)\backslash H_n(\mathbb{A})} \int_{V_{2k,k-n-1}(F)\backslash V_{2k,k-n-1}(\mathbb{A})} \varphi(g)\tilde{\theta}_\phi(hg)\widetilde{E}(v\tau(h)g)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g.$$

Here $\varphi \in V_\pi$, $\phi \in \mathcal{S}(\mathbb{A}^n)$ and $\tilde{f}_{\sigma,s} \in \tilde{I}(\sigma, s)$. We have the covering version,

$$J_2(\widetilde{\varphi}, \phi, f_{\sigma,s}) = \int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{H_n(F)\backslash H_n(\mathbb{A})} \int_{V_{2k,k-n-1}(F)\backslash V_{2k,k-n-1}(\mathbb{A})} \widetilde{\varphi}(g)\tilde{\theta}_\phi(hg)E(v\tau(h)g)\psi_{n-k-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g.$$

Here $\widetilde{\varphi} \in V_{\widetilde{\pi}}$ and $f_{\sigma,s} \in I(\sigma, s)$. In the next Theorem we shall show that these integrals are Eulerian.

First, let

$$\gamma = \begin{pmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & -I_{k-n} \\ I_{k-n} & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \end{pmatrix}.$$

We shall also need to consider the following unipotent subgroup of $V_{2k,k-n}$. Let

$$T_{k-n} = \left\{ \begin{pmatrix} I_{k-n} & 0 & t & 0 \\ & I_n & 0 & t^* \\ & & I_n & 0 \\ & & & I_{k-n} \end{pmatrix} : \begin{array}{l} t \in M_{(k-n)\times n} \text{ such that the} \\ \text{bottom row of } t \text{ is zero} \end{array} \right\}$$

and let $\xi_0 = (0, \ldots, 1) \in F^n$. Due to the embedding of H_n in Sp_{2n} as described in Chapter 1 Section (3), we need to change the character ψ_{N_k}. More precisely, let $\widetilde{\psi}_{N_k}$ be the character of N_k defined by

$$\widetilde{\psi}_{N_k}(\bar{n}) = \psi\Big(\sum_{i=1}^{n-1} \bar{n}_{i,i+1} + 2\bar{n}_{n,n+1} + \sum_{i=n+1}^{k-1} \bar{n}_{i,i+1} \Big).$$

Here $\bar{n} = (\bar{n}_{i_j}) \in N_k$. Since σ is a cuspidal representation of GL_k,

$$\mathcal{W}(\sigma, \psi_{N_k}) = \mathcal{W}(\sigma, \widetilde{\psi}_{N_k}).$$

As in Chapter 1, Section 5 we denote for $\tilde{f}_{\sigma,s} \in \tilde{I}(\sigma, s)$,

$$\tilde{f}_{W_{\sigma,s}}(g) = \int_{N_k(F)\backslash N_k(\mathbb{A})} \tilde{f}_{\sigma,s}(\bar{n}g)\widetilde{\psi}_{N_k}(\bar{n})\,\mathrm{d}\bar{n}.$$

A similar definition applies to $f_{\sigma,s} \in I(\sigma, s)$.

We have :

THEOREM 5.1. — *The integral $J_1(\varphi, \phi, \tilde{f}_{\sigma,s})$ converges absolutely for all s except for those s for which the Eisenstein series has a pole. For* $\mathrm{Re}(s)$ *large :*

$$J_1(\varphi, \phi, \tilde{f}_{\sigma,s}) = \int_{V_{2n,n}(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{Y_n(\mathbb{A})\backslash H_n(\mathbb{A})} \int_{T_{k-n}(\mathbb{A})N_{k-n}(\mathbb{A})\backslash V_{2k,k-n-1}(\mathbb{A})} W_\varphi(g)\omega_\psi(hg)\phi(\xi_0)\tilde{f}_{W_\sigma,s}(\gamma v\tau(h)g)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g.$$

Here $W_\varphi \in \mathcal{W}(\pi, \widetilde{\psi}_n)$ and as always we view N_{k-n} as a subgroup of Sp_{2k} *via the embedding of* GL_{k-n} *in* Sp_{2k}.

Proof. — We prove that $J_1(\varphi,\phi,\tilde{f}_{\sigma,s})$ is Eulerian. Unfolding the Eisenstein series and interchanging summation with integration we obtain that $J_1(\varphi,\phi,\tilde{f}_{\sigma,s})$ equals

$$(5.1)\quad \sum_{\delta} \int_{\mathrm{Sp}_{2n}^{\delta}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{H_n^{\delta}(F)\backslash H_n(\mathbb{A})} \int_{V_{2k,k-n-1}^{\delta}(F)\backslash V_{2k,k-n-1}(\mathbb{A})} \varphi(g)\tilde{\theta}_{\phi}(hg)\tilde{f}_{\sigma,s}(\delta v\tau(h)g)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g,$$

where $\delta \in P_{2k,k}(F)\setminus \mathrm{Sp}_{2k}(F)/Q^0_{2k,k-n}$ and where, for a given group $G\subset \mathrm{Sp}_{2k}$,

$$G^{\delta} = \delta^{-1}P_{2k,k}\delta\cap G \quad\text{and}\quad H_n^{\delta} = \tau^{-1}\big(\delta^{-1}P_{2k,k}\delta\cap\tau(H_n)\big).$$

Recall that $Q^0_{2k,k-n} = \mathrm{Sp}_{2n}\cdot\tau(H_n)V_{2k,k-n-1}$. From Lemma 4.1 we may choose $\delta=\gamma_i w$, where $0\le i\le k-n$ and $w\in (W_i\times W_{k-n-i})\setminus W_{k-n}$. Let $\delta=\gamma_i w$ with $0\le i\le k-n-1$ and $w\ne e$ as above. It follows from Lemma 4.3 that there exists a simple root α_j of GL_{k-n} such that $wx_{\alpha_j}(t)w^{-1}\in L_i$. (See Chapter 4 for notations.) It is easy to check that $\gamma_i L_i\gamma_i^{-1}\subset U_{2k,k}$. However $\tilde{f}_{\sigma,s}$ is left invariant under elements in $U_{2k,k}(\mathbb{A})$. Thus $\gamma_i w x_{\alpha_j}(t)(\gamma_i w)^{-1}\in U_{2k,k}(\mathbb{A})$. Since ψ_{k-n-1} restricted to $x_{\alpha_j}(t)$ is nonzero we end up with $\int_{F\backslash\mathbb{A}}\psi(t)\,\mathrm{d}t$ as an inner integral. Hence the contribution of such δ to $J_1(\varphi,\phi,\tilde{f}_{\sigma,s})$ is zero. Assume $\delta=\gamma_i$ with $0\le i\le k-n-1$. It is not hard to check that $\gamma_i\tau(Z_n)\gamma_i^{-1}=\tau(Z_n)\subset U_{2k,k}$. However $\tilde{\theta}_{\phi}((0,0,z)hg)=\psi(z)\tilde{\theta}_{\phi}(hg)$ for all $z\in Z_n(\mathbb{A})$. Hence, once again we get $\int_{F\backslash\mathbb{A}}\psi(z)\,\mathrm{d}z$ as an inner integral. Hence in (5.1) we are left with $\delta=\gamma_{k-n}$. Since we may change γ_{k-n} on the left with any Weyl element of GL_k, we may replace γ_{k-n} by γ. Simple matrix multiplication shows that

$$\mathrm{Sp}^{\gamma}_{2n} = P_{2n,n};\quad H_n^{\gamma}=Y_n;\quad V^{\gamma}_{2k,k-n-1} = N_{k-n}T_{k-n}.$$

Indeed, recall that if $g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}_{2n}$, its embedding in Sp_{2k} is

$$g\longmapsto \begin{pmatrix} I_k & & \\ & g & \\ & & I_k\end{pmatrix},$$

and after conjugating with γ, the image of the element above is

$$\begin{pmatrix} A & & & B\\ & I_k & & \\ & & I_k & \\ C & & & D\end{pmatrix}$$

such an element lies in $P_{2k,k}$ if and only if $C=0$, and hence $\mathrm{Sp}_{2n}^{\gamma}=P_{2n,n}$. The other identities follow similarly. Hence $J_1(\varphi,\phi,\tilde{f}_{\sigma,s})$ equals

$$\int \varphi(g)\tilde{\theta}_\phi(hg)\tilde{f}_{\sigma,s}\big(\gamma v\tau(h)g\big)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g.$$

Here g is integrated over $P_{2n,n}(F)\setminus \mathrm{Sp}_{2n}(\mathbb{A})$, h is integrated over $Y_n(F)\setminus H_n(\mathbb{A})$ and v over $T_{k-n}(F)N_{k-n}(F)\setminus V_{2k,k-n-1}(\mathbb{A})$.

Let $M_{n-1}^0\subset \mathrm{GL}_n$ denote the mirabolic subgroup of GL_n. Thus

$$M_{n-1}^0=\left\{\begin{pmatrix} * & * \\ 0 & 1\end{pmatrix}\in \mathrm{GL}_n\right\}.$$

Using Lemma 4.3.2 in [GPS] we have

$$\tilde{\theta}_\phi(hg)=\omega_\psi(hg)\phi(0)\;+\sum_{\delta\in M_{n-1}^0(F)\backslash \mathrm{GL}_n(F)}\omega_\psi(\delta hg)\phi(\xi_0),$$

where we recall that $\xi_0=(0,\dots,0,1)\in F^n$. We plug this expansion in the above integral. We consider the contribution of each summand. The first term contributes

$$\int \varphi(g)\omega_\psi(hg)\phi(0)\tilde{f}_{\sigma,s}\big(\gamma v\tau(h)g\big)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g. \tag{5.2}$$

We claim that this integral is zero. Indeed, let $P_{2n,n}=\mathrm{GL}_n\,U_{2n,n}$. Write

$$\int_{P_{2n,n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})}=\int_{\mathrm{GL}_n(F)U_{2n,n}(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})}\int_{U_{2n,n}(F)\backslash U_{2n,n}(\mathbb{A})}.$$

Thus we obtain as an inner integral to (5.2)

$$\int \varphi(ug)\omega_\psi(hug)\phi(0)\tilde{f}_{\sigma,s}\big(\gamma v\tau(h)ug\big)\psi_{k-n-1}(v)\,\mathrm{d}u\,\mathrm{d}h\,\mathrm{d}v,$$

where u is integrated over $U_{2n,n}(F)\setminus U_{2n,n}(\mathbb{A})$ and v and h as before. Conjugating u to the left, using the left invariant properties of ω_ψ and $\tilde{f}_{\sigma,s}$ and changing variables, we obtain

$$\int_{U_{2n,n}(F)\backslash U_{2n,n}(\mathbb{A})}\varphi(ug)\,\mathrm{d}u$$

as an inner integral, which is zero by cuspidality. Hence $J_1(\varphi,\phi,\tilde f_{\sigma,s})$ equals

$$\int \varphi(g) \sum_{\delta\in M^0_{n-1}(F)\backslash \mathrm{GL}_n(F)} \omega_\psi(\delta hg)\phi(\xi_0)\tilde f_{\sigma,s}\big(\gamma v\tau(h)g\big)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g.$$

Collapsing the summation with the integration this equals

$$\int \varphi(g)\omega_\psi(hg)\phi(\xi_0)\tilde f_{\sigma,s}\big(\gamma v\tau(h)g\big)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g,$$

where g is integrated over $M^0_{n-1}(F)U_{2n,n}(F)\setminus \mathrm{Sp}_{2n}(\mathbb{A})$. Write

$$\int_{M^0_{n-1}(F)U_{2n,n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} = \int_{M^0_{n-1}(F)U_{2n,n}(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{U_{2n,n}(F)\backslash U_{2n,n}(\mathbb{A})}.$$

Thus $J_1(\varphi,\phi,\tilde f_{\sigma,s})$ equals

$$\int \varphi(ug)\omega_\psi(hug)\phi(\xi_0)\tilde f_{\sigma,s}\big(\gamma v\tau(h)ug\big)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}u\,\mathrm{d}g.$$

Here g is integrated over $M^0_{n-1}(F)U_{2n,n}(\mathbb{A})\setminus \mathrm{Sp}_{2n}(\mathbb{A})$, u over $U_{2n,n}(F)\setminus U_{2n,n}(\mathbb{A})$ and h and v as before. We have, using Chapter 1, Section 6, formula (1.4),

$$\omega_\psi(uhg)\phi(\xi_0) = \widetilde{\psi}_n(u)\omega_\psi(hg)\phi(\xi_0), \quad u\in U_{2n,n}(\mathbb{A}).$$

Here $\widetilde{\psi}_n$ is as defined in Chapter 1, Section 4. Conjugating u to the left, changing variables in v and h the above integral equals

$$\int\Big(\int \varphi(ug)\widetilde{\psi}_n(u)\,\mathrm{d}u\Big)\omega_\psi(hg)\phi(\xi_0)\tilde f_{\sigma,s}\big(\gamma v\tau(h)g\big)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g.$$

Let $\mathrm{GL}_{n-1}\subset \mathrm{GL}_n$ be embedded as

$$g\longmapsto \begin{pmatrix} g & \\ & 1\end{pmatrix}, \quad g\in \mathrm{GL}_{n-1}.$$

As in [PS] we have

$$\int_{U_{2n,n}(F)\backslash U_{2n,n}(\mathbb{A})} \varphi(ug)\widetilde{\psi}_n(u)\,\mathrm{d}u = \sum_{\delta\in N_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\varphi(\delta g).$$

From the definition of M_{n-1}^0 we may collapse the summation with integration to obtain that $J_1(\varphi, \phi, \tilde{f}_{\sigma,s})$ equals

$$\int W_\varphi(g)\omega_\psi(hg)\phi(\xi_0)\tilde{f}_{\sigma,s}\big(\gamma v\tau(h)g\big)\psi_{k-n-1}(v)\,\mathrm{d}v\,\mathrm{d}h\,\mathrm{d}g,$$

where g is integrated over $N_n(F)U_{2n,n}(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})$ and v and h as before. Write

$$\int_{Y_n(F)\backslash H_n(\mathbb{A})} = \int_{Y_n(\mathbb{A})\backslash H_n(\mathbb{A})} \int_{Y_n(F)\backslash Y_n(\mathbb{A})}.$$

Let $y = (y_1, \ldots, y_n) \in Y_n$. It follows from Chapter 1, Section 6, formula (1.2) that

$$\omega_\psi\big((0,y,0)hg\big)\phi(\xi_0) = \psi(y_1)\omega_\psi(hg)\phi(\xi_0).$$

Write (recall that $V_{2n,n} = N_n U_{2n,n}$)

$$\int_{N_n(F)U_{2n,n}(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} = \int_{V_{2n,n}(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N_n(F)\backslash N_n(\mathbb{A})}.$$

We have $W_\varphi(\bar{n}g) = \psi_{N_n}^{-1}(\bar{n})W_\varphi(g)$ for all $\bar{n} \in N_n$. Also, write

$$\int_{T_{k-n}(F)N_{k-n}(F)\backslash V_{2k,k-n-1}(\mathbb{A})} = \int_{T_{k-n}(\mathbb{A})N_{k-n}(\mathbb{A})\backslash V_{2k,k-n-1}(\mathbb{A})} \int_{T_{k-n}(F)N_{k-n}(F)\backslash T_{k-n}(\mathbb{A})N_{k-n}(\mathbb{A})}.$$

Combining all this, we obtain as an inner integration

$$\int \tilde{f}_{\sigma,s}\big(\gamma v'\tau\big((0,y,0)\big)\bar{n}v\tau(h)g\big)\psi_{k-n-1}(v')\psi_{N_n}^{-1}(\bar{n})\psi(y_1)\,\mathrm{d}v'\,\mathrm{d}y\,\mathrm{d}\bar{n}.$$

Here v' is integrated over $T_{k-n}(F)N_{k-n}(F) \setminus T_{k-n}(\mathbb{A})N_{k-n}(\mathbb{A})$, y over $Y_n(F) \setminus Y_n(\mathbb{A})$ and $\bar{n}$ over $N_n(F) \setminus N_n(\mathbb{A})$. It follows from matrix multiplication that

$$\gamma T_{k-n}N_{k-n}\tau(Y_n)N_n\gamma^{-1} = N_k$$

and that if $\gamma v'\tau(0,y,0)\bar{n}\gamma^{-1} = n_k \in N_k$ then

$$\psi_{k-n-1}(v')\psi_{N_n}(\bar{n})\psi(y_1) = \widetilde{\psi}_{N_k}(n_k).$$

Hence the above integral equals to $\tilde{f}_{W_\sigma,s}(\gamma v\tau(h)g)$. From this the Theorem follows. □

We also have the same Theorem for the integral $J_2(\widetilde{\varphi}, \phi, f_{\sigma,s})$.

THEOREM 5.2. — *The $J_2(\widetilde{\varphi}, \phi, f_{\sigma,s})$ converges absolutely for all s except for those s for which the Eisenstein series has a pole. For* $\mathrm{Re}(s)$ *large,*

$$J_2(\widetilde{\varphi}, \phi, f_{\sigma,s}) = \int\limits_{V_{2n,n}(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int\limits_{Y_n(\mathbb{A})\backslash H_n(\mathbb{A})} \int\limits_{T_{k-n}(\mathbb{A})N_{k-n}(\mathbb{A})\backslash V_{2k,k-n-1}(\mathbb{A})} W_{\widetilde{\varphi}}(g)\omega_\psi(hg)\phi(\xi_0) f_{W_\sigma,s}\big(\gamma v \tau(h) g\big)\, \mathrm{d}v\, \mathrm{d}h\, \mathrm{d}g,$$

where $W_{\widetilde{\varphi}} \in \mathcal{W}(\widetilde{\pi}, \widetilde{\psi}_n)$.

6. The Local Theory ($k > n$)

In this section we shall study the local integrals in the case where $k > n$. The approach here will be based on the method developed in [S1] and [JPSS]. We shall keep the notations introduced in Section 3. The local integrals to be studied are the ones which come from the factorization of the global integrals introduced in Section 5. More precisely, define

$$J_1(W, \phi, \tilde{f}_s) = \int\limits_{V_{2n,n}\backslash \mathrm{Sp}_{2n}} \int\limits_{Y_n\backslash H_n} \int\limits_{T_{k-n}N_{k-n}\backslash V_{2k,k-n-1}} W(g)\omega_\psi(hg)\phi(\xi_0)\tilde{f}_{W_{\sigma,s}}\big(\gamma v \tau(h) g\big)\, \mathrm{d}v\, \mathrm{d}h\, \mathrm{d}g.$$

Here $W \in \mathcal{W}(\pi, \widetilde{\psi})$, $\phi \in S(F^n)$ and $\tilde{f}_{W_{\sigma,s}} \in \tilde{I}(\mathcal{W}(\sigma, \psi^{-1}), s)$. We have a similar integral for the covering case. That is, let

$$J_2(\widetilde{W}, \phi, f_s) = \int\limits_{V_{2n,n}\backslash \mathrm{Sp}_{2n}} \int\limits_{Y_n\backslash H_n} \int\limits_{T_{k-n}N_{k-n}\backslash V_{2k,k-n-1}} \widetilde{W}(g)\omega_\psi(hg)\phi(\xi_0) f_{W_\sigma,s}(\gamma v \tau(hg)\, \mathrm{d}v\, \mathrm{d}h\, \mathrm{d}g,$$

where now $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \widetilde{\psi})$ and $f_{W_{\sigma,s}} \in I(\mathcal{W}(\sigma, \psi^{-1}), s)$. As before we shall write $\tilde{f}_{\sigma,s}$ for $\tilde{f}_{W_{\sigma,s}}$, *etc.* Section 6.1, which follows, is presented in the formal level (*i.e.* ignoring convergence issues.) All justifications and basic properties of the local integrals are deferred to Section 6.3.

6.1. — We start our local analysis by proving a formal identity. This identity is analogous to the one proved in [S1, Section 11.4] and is based on similar ideas. We start with a few assumptions and notations.

Let $\overline{P}_{2n,n}$ denote the opposite parabolic to $P_{2n,n}$. We shall assume that

$$\pi = \operatorname{Ind}_{\overline{P}_{2n,n}}^{\mathrm{Sp}_{2n}} \big(\sigma' \otimes \delta_{\overline{P}_{2n,n}}^{\frac{1}{2}+\zeta}\big),$$

where σ' is an admissible generic representation of GL_n. We shall denote its central character by $\omega_{\sigma'}$. Given $\varphi_{\sigma',\zeta} \in \operatorname{Ind}_{\overline{P}_{2n,n}}^{\mathrm{Sp}_{2n}} \big(\mathcal{W}(\sigma',\psi) \otimes \delta_{\overline{P}_{2n,n}}^{\frac{1}{2}+\zeta}\big)$ we set

$$W(g) = \int_{U_{2n,n}} \varphi_{\sigma',\zeta}(ug)\widetilde{\psi}_n(u)\,\mathrm{d}u. \tag{6.1}$$

Here $g \in \mathrm{Sp}_{2n}$. The integral converges for $\mathrm{Re}(\zeta)$ large and as in [S1] it has an analytic continuation to the whole ζ plane and hence $W \in \mathcal{W}(\pi,\widetilde{\psi})$. Also, given $W \in \mathcal{W}(\pi,\widetilde{\psi})$ there is a function $\varphi_{\sigma',0}$ such that (6.1) holds in the sense of analytic continuation. Recall that

$$\tilde{I}(\mathcal{W}(\sigma,\psi^{-1}),s) = \operatorname{Ind}_{\mathrm{P}_{2k,k}}^{\widetilde{\mathrm{Sp}}_{2k}} \big(\mathcal{W}(\sigma,\psi^{-1}) \otimes \delta_{\mathrm{P}_{2k,k}}^{s} \otimes \gamma^{-1}\big).$$

Another induced space we need is

$$\tilde{I}(\mathcal{W}(\sigma\otimes\chi,\psi^{-1}),s) = \operatorname{Ind}_{\mathrm{P}_{2k,k}}^{\widetilde{\mathrm{Sp}}_{2k}} \big(\mathcal{W}(\sigma\otimes\chi,\psi^{-1}) \otimes \delta_{\mathrm{P}_{2k,k}}^{s} \otimes \gamma^{-1}\big),$$

where $\chi\begin{pmatrix} m & \\ & m^* \end{pmatrix} = (\det m, \det m)$ and $(\ ,\ )$ denotes the local Hilbert symbol. Due to the relation $\gamma_{\det m}^{-1} \cdot (\det m, \det m) = \gamma_{\det m}$,

$$\tilde{I}\big(\mathcal{W}(\sigma\otimes\chi,\psi^{-1}),s\big) = \operatorname{Ind}_{\mathrm{P}_{2k,k}}^{\widetilde{\mathrm{Sp}}_{2k}} \big(\mathcal{W}(\sigma,\psi^{-1}) \otimes \delta_{\mathrm{P}_{2k,k}}^{s} \otimes \gamma\big).$$

Given a function $\tilde{f}_{W_\sigma,s} \in \tilde{I}(\mathcal{W}(\sigma,\psi^{-1}),s)$ we associate to it a function $\tilde{f}_{W_\sigma,s,\chi} \in \tilde{I}(\mathcal{W}(\sigma\otimes\chi,\psi^{-1}),s)$ satisfying

$$\tilde{f}_{W_\sigma,s,\chi}\Big[\begin{pmatrix} m & * \\ & m^* \end{pmatrix} g\Big] = \chi(m)\tilde{f}_{W_\sigma,s}(g).$$

We shall write $\tilde{f}_{\sigma,s,\chi}$ for $\tilde{f}_{W_{\sigma,s,\chi}}$. We shall denote

$$w_n = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix} \quad \text{and let} \quad \bar{w}_0 = \begin{pmatrix} 0 & -2I_n & & \\ I_{k-n} & 0 & & \\ & & 0 & I_{k-n} \\ & & -\frac{1}{2}I_n & 0 \end{pmatrix}.$$

Also, given $g \in \mathrm{Sp}_{2n}$ we set $j'(g) = \bar{w}_0 g \bar{w}_0^{-1}$. Finally we shall denote by $\gamma(\sigma\times\sigma',s,\psi^{-1})$ the gamma factor attached to σ and σ' as defined in [JPSS]. We shall prove :

PROPOSITION 6.1. — *With the above notations let $W(g)$ be given by* (6.1). *We have*

$$\omega_{\sigma'}(-1)^{k-2}\gamma(\sigma\times\sigma',(k+1)s-(n+1)\zeta-\tfrac{1}{2}k,\psi^{-1})J_1(W,\phi,\tilde f_{\sigma,s})$$
$$=\int\varphi_{\sigma',\zeta}(w_ng)\omega_\psi(g)\phi(2x)$$
$$\tilde f_{\sigma,s,\chi}\big(w_kuj'\big(r\tau((x,0,0))g\big)\bar w_0\big)\psi_k(u)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g.$$

Here g is integrated over $V_{2n,n}\setminus\mathrm{Sp}_{2n}$, x over F^n, u over $U_{2k,k}$ and r over R (see Chapter 2 for the definition of R). Each integral converges in some (s,ζ) domain and admit a meromorphic continuation to all values of (s,ζ). The above equality is understood in the analytic continuation sense.

Proof. — At this point we shall ignore convergence issues and prove formally the above identity. The convergence properties will be dealt with in Section 6.3. As can be seen, we may identify the product of the groups $T_{k-n}N_{k-n}\setminus V_{2k,k-n-1}$ and $\tau(Y_n\setminus H_n)$ with the group for all matrices in Sp_{2k} of the form

$$(A,0,B):=\begin{pmatrix}I_{k-n}&A&0&B\\&I_n&0&0\\&&I_n&A^*\\&&&I_{k-n}\end{pmatrix},$$

where $A\in M_{(k-n)\times n}$, and $B\in M_{(k-n)\times k-n)}$ satisfying $B^tJ_{k-n}=J_{k-n}B$. Also, $A^*=-J_nA^tJ_{k-n}$. Under the above identification $(x,0,0)$ in H_n is identified with the last row of A and $(0,0,z)$ is identified with $b_{k-n,k-n}$. Hence we can write

$$J_1(W,\phi,\tilde f_{\sigma,s})=\int W(g)\omega_\psi\big((x,0,z)g\big)\phi(\xi_0)\tilde f_{\sigma,s}\big(\gamma(A,0,B)g\big)\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}g.$$

Here g is integrated over $V_{2n,n}\backslash\mathrm{Sp}_{2n}$ and A and B as above. Plugging (6.1) in the above integral and collapsing integrations, $J_1(W,\phi,\tilde f_{\sigma,s})$ equals

$$\int\varphi_{\sigma',\zeta}(g)\omega_\psi\big((x,0,z)g\big)\phi(\xi_0)\tilde f_{\sigma,s}\big(\gamma(A,0,B)g\big)\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}g,$$

where now g is integrated over $N_n\setminus\mathrm{Sp}_{2n}$. Proposition 6.8 says that the last integral converges absolutely in a domain of the form

$$\begin{cases}(n+1)\,\mathrm{Re}(\zeta+\tfrac{1}{2})+C<(k+1)\,\mathrm{Re}(s),\\(k+1)\,\mathrm{Re}(s)<(1+\varepsilon_0)(n+1)\,\mathrm{Re}(\zeta+\tfrac{1}{2})+Q,\\R<\mathrm{Re}(\zeta),\end{cases}$$

where Q, C, R, ε_0 are constants which depend on (σ, σ', k, n) and $\varepsilon_0 > 0$. We continue the calculation in this domain. See, also, the remark prior to Proposition 6.8.

Factor

$$\int_{N_n \backslash \mathrm{Sp}_{2n}} = \int_{\mathrm{GL}_n \backslash \mathrm{Sp}_{2n}} \int_{N_n \backslash \mathrm{GL}_n} .$$

Hence $J_1(W, \phi, \tilde{f}_{\sigma,s})$ equals

$$\int \varphi_{\sigma',\zeta}\Big(\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big) \omega_\psi\Big((x,0,z)\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big)\phi(\xi_0)$$
$$\tilde{f}_{\sigma,s}\left(\gamma(A,0,B)\begin{pmatrix} I_{k-n} & & & \\ & m & & \\ & & m^* & \\ & & & I_{k-n} \end{pmatrix} g\right) \mathrm{d}A\,\mathrm{d}B\,\mathrm{d}m\,\mathrm{d}g,$$

where m is integrated over $N_n \backslash \mathrm{GL}_n$ and g over $\mathrm{GL}_n \backslash \mathrm{Sp}_{2n}$. We have

$$\delta_{\overline{P}_{2n,n}}\begin{pmatrix} m & \\ & m^* \end{pmatrix} = |\det m|^{-(n+1)}.$$

From the definition of $\varphi_{\sigma',\zeta}$ we can write

$$\varphi_{\sigma',\zeta}\Big(\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big) = |\det m|^{-(n+1)(\frac{1}{2}+\zeta)} W_{\sigma',g,\zeta}(m),$$

where $W_{\sigma',g,\zeta} \in \mathcal{W}(\sigma', \psi)$. We also have

$$\omega_\psi\Big((x,0,z)\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big)\phi(\xi_0) = |\det m|^{\frac{1}{2}} \gamma_{\det m}\omega_\psi\big((xm,0,z)g\big)\phi(\xi_0 m).$$

Finally conjugating m to the left in $\tilde{f}_{\sigma,s}$ and changing variables in A, $J_1(W, \phi, \tilde{f}_{\sigma,s})$ equals

$$\int W_{\sigma',g,\zeta}(m)\omega_\psi\big((x,0,z))g\big)\phi(\xi_0 m)$$
$$\tilde{f}_{\sigma,s}\Big(\begin{pmatrix} m & & \\ & I_{2(k-n)} & \\ & & m^* \end{pmatrix}\gamma(A,0,B)g\Big)$$
$$\gamma_{\det m}|\det m|^{-(n+1)\zeta + \frac{1}{2}n - k}\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}m\,\mathrm{d}g,$$

where all the integration remains as before. ¿From the definition of $\tilde{f}_{\sigma,s}$ there is a function $F \in \mathcal{W}(\sigma, \psi)$ such that

$$\tilde{f}_{\sigma,s}\Big(\begin{pmatrix} m & * \\ 0 & m^* \end{pmatrix}, \varepsilon\Big) = |\det m|^{(k+1)s}\gamma_{\det m}^{-1} F_{\sigma,h}(m),$$

where $m \in \mathrm{GL}_k$ and $h \in \widetilde{\mathrm{Sp}_{2k}}$. To simplify notations, we shall write $\tilde{f}_{\sigma,s}(h;m)$ for $F_{\sigma,h}(m)$. Thus, $J_1(W,\phi,\tilde{f}_{\sigma,s})$ equals

$$\tag{6.2} \int W_{\sigma',g,\zeta}(m)\omega_\psi\big((x,0,z)g\big)\phi(\xi_0 m) \\ \tilde{f}_{\sigma,s}\Big(\gamma(A,0,B)g;\begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}\Big) \\ |\det m|^{\frac{1}{2}n-k+(k+1)s-(n+1)\zeta}\, \mathrm{d}m\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}g,$$

where the integrations are as before. To proceed let $W_\sigma(h) \in \mathcal{W}(\sigma,\psi)$, and given $\phi_1 \in S(F^n)$ define

$$\big(\sigma(\phi_1)W_\sigma\big)(h) = \int_{F^n} W_\sigma\left(h\begin{pmatrix} I_n & \mathrm{e}^t & 0 \\ & 1 & 0 \\ & & I_{k-n-1}\end{pmatrix}\right)\phi_1(e)\,\mathrm{d}e.$$

We also denote $\widehat{\phi}_1(\xi) = \displaystyle\int_{F^n} \phi_1(e)\psi(2\xi\,\mathrm{e}^t)\,\mathrm{d}e$ and hence $\widehat{\widehat{\phi}}_1(\xi) = \phi_1(-\xi)$. We have

$$\tag{6.3} \int_{N_n\backslash \mathrm{GL}_n} W_{\sigma'}(m)\big(\sigma(\widehat{\phi}_1)W_\sigma\big)\begin{pmatrix} m & \\ & I_{k-1n}\end{pmatrix}|\det m|^\alpha\,\mathrm{d}m$$

$$= \int_{N_n\backslash \mathrm{GL}_n} W_{\sigma'}(m)\int_{F^n}\widehat{\phi}_1(e)W_\sigma\left[\begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}\begin{pmatrix} I_n & e & 0 \\ & 1 & 0 \\ & & I_{k-n-1}\end{pmatrix}\right] \\ |\det m|^\alpha\,\mathrm{d}e\,\mathrm{d}m$$

$$= \int_{N_n\backslash \mathrm{GL}_n} W_{\sigma'}(m)W_\sigma\begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}|\det m|^\alpha\int_{F^n}\widehat{\phi}_1(e)\psi(2\xi_0 m\,\mathrm{e}^t)\,\mathrm{d}e\,\mathrm{d}m$$

$$= \int_{N_n\backslash \mathrm{GL}_n} W_{\sigma'}(m)W_\sigma\begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}\phi_1(-\xi_0 m)|\det m|^\alpha\,\mathrm{d}m.$$

In the above equalities $W_{\sigma'} \in \mathcal{W}(\sigma,'\psi)$, $W_\sigma \in \mathcal{W}(\sigma,\psi)$ and $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha)$ large. Recall that

$$W_\sigma\left[\begin{pmatrix} I_n & y \\ & I_{k-n}\end{pmatrix}h\right] = \psi(2y_{n,k-n})W_\sigma(h)$$

which explains the reason for the presence of the number 2. In (6.2) changing variables $m \mapsto -m$, we obtain as an inner integration

$$\omega_{\sigma'}(-1)\int_{N_n\backslash \mathrm{GL}_n} W_{\sigma',g,\zeta}(m)\phi_1(-\xi_0 m)\tilde{f}_{\sigma,s}\Big(h;\begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}\Big)|\det m|^\alpha\,\mathrm{d}m,$$

where $\phi_1 = \omega_\psi((x,0,z)g)\phi$ and $h = \gamma_0(A,0,B)g$. Here

$$\gamma_0 = \begin{pmatrix} 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & -I_{k-n} \\ I_{k-n} & 0 & 0 & 0 \\ 0 & 0 & -I_n & 0 \end{pmatrix}$$

and $\alpha = \frac{1}{2}n - k + (k+1)s - (n+1)\zeta$. Using the equalities in (6.3) the above integral equals

$$\omega_{\sigma'}(-1)\int_{N_n\backslash \mathrm{GL}_n} W_{\sigma',g,\zeta}(m)\big(\sigma(\widehat{\phi}_1)\tilde{f}_{\sigma,s}\big)\Big(h;\begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}\Big)|\det m|^{\alpha}\,\mathrm{d}m.$$

Next we shall use the local functional equation for $\mathrm{GL}_k\times \mathrm{GL}_n$. We shall apply it as in [S1, p. 70]. We have

$$\begin{aligned}&\omega_{\sigma'}(-1)^{k-1}\gamma(\sigma\times\sigma',\beta,\psi^{-1})\\ &\int_{N_n\backslash \mathrm{GL}_n} W_{\sigma',g,\zeta}(m)\big(\sigma(\widehat{\phi}_1)\tilde{f}_{\sigma,s}\big)\Big(h;\begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}\Big)\\ &\qquad\qquad\qquad\qquad|\det m|^{\beta-\frac{1}{2}(k-n)}\,\mathrm{d}m\\ &=\int_{N_n\backslash \mathrm{GL}_n}\int_{M_{n\times(k-n-1)}} W_{\sigma',g,\zeta}(m)\big(\sigma(\widehat{\phi}_1)\tilde{f}_{\sigma,s}\big)\left(h;\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ m & 0 & y\end{pmatrix}\right)\\ &\qquad\qquad\qquad\qquad|\det m|^{\beta-\frac{1}{2}(k+n)+n}\,\mathrm{d}y\,\mathrm{d}m.\end{aligned}$$

Here $\beta = (k+1)s - (n+1)\zeta - \frac{1}{2}k$. Multiplying (6.2) by

$$\omega_{\sigma'}(-1)^{k-2}\gamma\big(\sigma\times\sigma',(k+1)s-(n+1)\zeta-\tfrac{1}{2}k,\psi^{-1}\big),$$

using the definition of $(\sigma(\widehat{\phi}_1)\tilde{f}_{\sigma,s})$, we obtain first formally that

$$\omega_{\sigma'}(-1)^{k-2}\gamma\big(\sigma\times\sigma',(k+1)s-(n+1)\zeta-\tfrac{1}{2}k,\psi^{-1}\big)J_1(W,\phi,\tilde{f}_{\sigma,s})$$

equals

$$\begin{aligned}&\int W_{\sigma',g,\zeta}(m)\widehat{\omega_\psi\big((x,0,z)g\big)\phi}(e)\\ &\qquad\tilde{f}_{\sigma,s}\left(\gamma_0(A,0,B)g;\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ m & 0 & y\end{pmatrix}\begin{pmatrix} I_n & e^t & \\ & 1 & \\ & & I_{k-n-1}\end{pmatrix}\right)\\ &\qquad\qquad|\det m|^{(k+1)s-(n+1)\zeta-k+\frac{1}{2}n}\,\mathrm{d}y\,\mathrm{d}e\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}m\,\mathrm{d}g.\end{aligned}$$

Here e is integrated over F^n, y over $M_{n\times(k-n-1)}$ and the other variables as before. Recall that

$$\varphi_{\sigma,\zeta}\Big(\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big) = |\det m|^{-(n+1)(\frac{1}{2}+\zeta)} W_{\sigma',g,\zeta}(m).$$

The interpretation of this passage is as in [S1, 11.8]. We first prove, in Proposition 6.9, that the last integral converges absolutely in a certain domain (see (6.39)). Next we note that in this domain, the last integral must be proportional to J_1, and actually is a rational function in q^{-s}. The reason is that they both satisfy certain equivariance conditions, which hold uniquely up to scalars for almost all values of q^{-s}. In Proposition 6.10, we use a special substitution to calculate the proportionality factor which indeed is

$$\omega_{\sigma'}(-1)^{k-2}\gamma\big(\sigma\times\sigma', (k+1)s-(n+1)\zeta-\tfrac{1}{2}k, \psi^{-1}\big).$$

Write

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ m & 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ m & 0 & 0 \end{pmatrix} \begin{pmatrix} I_n & 0 & m^{-1}y \\ & 1 & 0 \\ & & I_{k-n-1} \end{pmatrix}.$$

Changing variables $y \mapsto my$, the above integral equals

$$\int \varphi_{\sigma',\zeta}\Big(\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big) \widehat{\omega_\psi((x,0,z)g)\phi}(e)$$
$$\tilde f_{\sigma,s}\left(\gamma_0(A,0,B)g; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ m & 0 & 0 \end{pmatrix} \begin{pmatrix} I_n & e^t & y \\ & 1 & 0 \\ & & I_{k-n-1} \end{pmatrix}\right)$$
$$|\det m|^{(k+1)s-\frac{1}{2}}\, \mathrm{d}y\, \mathrm{d}e\, \mathrm{d}A\, \mathrm{d}B\, \mathrm{d}m\, \mathrm{d}g.$$

It will be convenient to write $C = (\mathrm{e}^t, y)$. Thus C is an $n\times(k-n)$ matrix whose first column is e^t. Given an $n\times(k-n)$ matrix D we shall denote

$$(0,D,0) := \begin{pmatrix} I_{k-n} & 0 & -D^* & 0 \\ & I_n & 0 & D \\ & & I_n & 0 \\ & & & I_{k-n} \end{pmatrix},$$

where $D^* = -J_{k-n}D^tJ_n$. Thus the above integral equals

$$\int \varphi_{\sigma',\zeta}\Big(\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big) \widehat{\omega_\psi((x,0,z)g)\phi}(e)$$
$$\tilde f_{\sigma,s}\Big(\gamma_0(0,C,0)(A,0,B)g; \begin{pmatrix} 0 & I_{k-n} \\ I_n & 0 \end{pmatrix} \begin{pmatrix} m & \\ & I_{k-n} \end{pmatrix}\Big)$$
$$|\det m|^{(k+1)s-\frac{1}{2}}\, \mathrm{d}A\, \mathrm{d}B\, \mathrm{d}c\, \mathrm{d}m\, \mathrm{d}g.$$

Next we bring the m across to obtain

$$\int \varphi_{\sigma',\zeta}\Big(\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big) \widehat{\omega_\psi((xm,0,z)g)\phi(e\cdot {}^t m^{-1})}$$
$$\tilde f_{\sigma,s}\Big(\gamma_0(0,C,0)(A,0,B)\begin{pmatrix} m & \\ & m^* \end{pmatrix} g; \begin{pmatrix} & I_{k-n} \\ I_n & \end{pmatrix}\Big)$$
$$|\det m|^{-\frac12}\gamma_{\det m}\, \mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C\,\mathrm{d}m\,\mathrm{d}g.$$

Here we also performed a change of variables in A and C which explains the presence of xm and $e\cdot {}^t m^{-1}$ in $\omega_\psi(xm,0,z)g)\phi(e\cdot {}^t m^{-1})$. Also, the factor $\gamma_{\det m}$ appears from the definition of $\tilde f_{\sigma,s}$. Let

$$\gamma_1 = \begin{pmatrix} & & & -I_{k-n} \\ & -I_n & 0 & \\ & 0 & -I_n & \\ I_{k-n} & & & \end{pmatrix}.$$

We also have

$$\widehat{\omega_\psi((xm,0,z)g)\phi(e\cdot {}^t m^{-1})} = \omega_\psi\Big[\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}(xm,0,z)g\Big]\phi(2e\cdot {}^t m^{-1})$$
$$= \omega_\psi\Big[\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}\begin{pmatrix} m^{-1} & \\ & J_n m^t J_n \end{pmatrix}(x,0,z)\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big]\phi(2e\cdot {}^t m^{-1})$$
$$= |\det m|^{\frac12}\gamma_{\det m}\omega_\psi\Big[\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}(x,0,z)\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big]\phi(2e).$$

Plugging this in the above integral, we obtain

$$\int \varphi_{\sigma',\zeta}\Big(\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big)\omega_\psi\Big[\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}(x,0,z)\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big]\phi(2e)$$
$$\tilde f_{\sigma,s}\Big(\gamma_1(0,C,0)(A,0,B)\begin{pmatrix} m & \\ & m^* \end{pmatrix} g\Big)(\det m,\det m)\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C\,\mathrm{d}m\,\mathrm{d}g,$$

where we used the identity $\gamma_{\det m}\gamma_{\det m} = (\det m,\det m)$. Recall that g is integrated over $\mathrm{GL}_n\setminus\mathrm{Sp}_{2n}$ and m over $N_n\setminus\mathrm{GL}_n$. Hence we may collapse the two integrations to obtain

$$\int \varphi_{\sigma',\zeta}(g)\omega_\psi\Big[\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}(x,0,z)g\Big]\phi(2e)$$
$$\tilde f_{\sigma,s,\chi}(\gamma_1(0,C,0)(A,0,B)g)\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C\,\mathrm{d}g,$$

where now g is integrated over $N_n \setminus \mathrm{Sp}_{2n}$. We have

$$\begin{aligned}\omega_\psi\left[\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}(x,0,z)g\right]\phi(2e) &= \omega_\psi\left[(0,-xJ_n,z)\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}g\right]\phi(2e) \\ &= \psi(z-2ex^t)\omega_\psi\left[\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}g\right]\phi(2e).\end{aligned}$$

Plug this to the above integral and also conjugate $(0,C,0)$ across $(A,0,B)$ we obtain

$$\int \varphi_{\sigma',\zeta}(g)\omega_\psi\left[\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}g\right]\phi(2e) \\ \tilde{f}_{\sigma,s,\chi}(\gamma_1(A,0,B)(0,C,0)g)\psi(b_{k-n,k-n})\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C\,\mathrm{d}g.$$

Here we changed variables in B and we remind the reader that by our notations $b_{k-n,k-n}$ was z. Write $\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} = \begin{pmatrix} J_n & \\ & J_n \end{pmatrix} w_n^{-1}$. Changing variables $g \to w_n g$ we obtain

$$\int \varphi_{\sigma',\zeta}(w_n g)\omega_\psi(g)\phi(2eJ_n) \\ \tilde{f}_{\sigma,s,\chi}\left(\gamma_1(A,0,B)(0,C,0)\begin{pmatrix} I_{k-n} & & & \\ & 0 & -\frac{1}{2}I_n & \\ & 2I_n & 0 & \\ & & & I_{k-n}\end{pmatrix}g\right) \\ \psi(b_{k-n,k-n})\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C\,\mathrm{d}g.$$

Denote

$$\gamma_2 = \begin{pmatrix} & & & -I_{k-n} \\ & & \frac{1}{2}I_n & \\ & -2I_n & & \\ I_{k-n} & & & \end{pmatrix}.$$

Then, conjugating the Weyl element to the left the above integral equals

$$\int \varphi_{\sigma',\zeta}(w_n g)\omega_\psi(g)\phi(2eJ_n) \\ \tilde{f}_{\sigma,s,\chi}(\gamma_2(0,-\tfrac{1}{2}A^*,B)(-2C^*,0,0)g)\psi(b_{k-n,k-n})\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C\,\mathrm{d}g.$$

Recall that g is integrated over $N_n \setminus \mathrm{Sp}_{2n}$. Write

$$\int_{N_n\setminus \mathrm{Sp}_{2n}} = \int_{V_{2n,n}\setminus \mathrm{Sp}_{2n}} \int_{U_{2n,n}}.$$

We have

$$\omega_\psi\left[\begin{pmatrix} I_n & T \\ & I_n \end{pmatrix} g\right]\phi(2eJ_n) = \psi(2eJ_nT\,\mathrm{e}^t)\omega_\psi(g)\phi(2eJ_n).$$

Also, one can check that conjugating the matrix

$$\begin{pmatrix} I_{k-n} & & & \\ & I_n & \frac{1}{2}T & \\ & & I_n & \\ & & & I_{k-n} \end{pmatrix}$$

across $(-2C^*,0,0)$, $b_{k-n,k-n}$ is changed to $b_{k-n,k-n} + 2eJ_nT\,\mathrm{e}^t$. Hence, after a change of variables, we obtain

$$\int \varphi_{\sigma',\zeta}(w_n g)\omega_\psi(g)\phi(2eJ_n)$$
$$\tilde{f}_{\sigma,s,\chi}\left(\gamma_2(0,-\tfrac{1}{2}A^*,B)\begin{pmatrix} I_{k-n} & & & \\ & I_n & \frac{1}{2}T & \\ & & I_n & \\ & & & I_{k-n} \end{pmatrix}(-2C^*,0,0)g\right)$$
$$\psi(b_{k-n,k-n})\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C\,\mathrm{d}T\,\mathrm{d}g,$$

where now g is integrated over $V_{2n,n}\setminus \mathrm{Sp}_{2n}$ and T over all $n\times n$ matrices satisfying $T^tJ_n = J_nT$. Now $\gamma_2 = w_k\cdot\bar{w}_0$. Conjugate $\bar{w}_0$ to the right. It is not hard to check that the groups of all matrices of the form

$$u = \bar{w}_0(0,-\tfrac{1}{2}A^*,B)\begin{pmatrix} I_{k-n} & & & \\ & I_n & \frac{1}{2}T & \\ & & I_n & \\ & & & I_{k-n} \end{pmatrix}\bar{w}_0^{-1}$$

equals $U_{2k,k}$ and that $\psi(b_{k-n,k-n}) = \psi_k(u)$. Recall that $C = (\mathrm{e}^t,y)$. Using the definition of the group R and changing variables $e\to eJ_n$, we obtain

$$\int \varphi_{\sigma',\zeta}(w_n g)\omega_\psi(g)\phi(2e)\tilde{f}_{\sigma,s,\chi}(w_k u j'(r\tau((e,0,0))g)\bar{w}_0)\psi_k(u)\,\mathrm{d}u\,\mathrm{d}e\,\mathrm{d}r\,\mathrm{d}g.$$

Here g is integrated over $V_{2n,n}\setminus\mathrm{Sp}_{2n}$, r over R, u over $U_{2k,k}$ and e over F^n. From this the proposition follows. $\square$

A similar identity holds for $J_2(\widetilde{W},\phi,f_s)$. More precisely, we shall assume that

$$\widetilde{\pi} = \mathrm{Ind}_{\overline{P}_{2n,n}}^{\widetilde{\mathrm{Sp}}_{2n}}\left(\sigma'\otimes\delta_{\overline{P}_{2n,n}}^{\frac{1}{2}+\zeta}\otimes\gamma^{-1}\right),$$

where σ' is an admissible generic representation of GL_n. Given $\widetilde{\varphi}_{\sigma',\zeta}$ in $\mathrm{Ind}_{\overline{P}_{2n,n}}^{\mathrm{Sp}_{2n}}\left(\mathcal{W}(\sigma',\psi)\otimes\delta_{\overline{P}_{2n,n}}^{\frac{1}{2}+\zeta}\otimes\gamma^{-1}\right)$ we get

$$W(g) = \int_{U_{2n,n}}\widetilde{\varphi}_{\sigma',\zeta}(ug)\widetilde{\psi}_n(u)\,\mathrm{d}u.$$

We have :

PROPOSITION 6.2. — *With $\widetilde{W}(g)$ as above, we have :*

$$\omega_{\sigma'}(-1)^{k-2}\gamma\big(\sigma\times\sigma',(k+1)s-(n+1)\zeta-\tfrac{1}{2}k,\psi^{-1}\big)J_2(\widetilde{W},\phi,f_{\sigma,s})$$
$$=\int\widetilde{\varphi}_{\sigma',\zeta}(w_ng)\omega_\psi(g)\phi(2x)f_{\sigma,s,\chi}\big(w_kuj'(r\tau((x,0,0))g)\bar{w}_0\big)$$
$$\psi_k(u)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g.$$

Here all variables are integrated as in Proposition 6.1. Each integral converges in some (s,ζ) domain and admits a meromorphic continuation to all values of (s,ζ). The above equality is understood in the analytic continuation sense. $\square$

6.2 The Unramified Computations ($k>n$). — We keep the notations of Section 3.1. Following [S1], we shall use the identities established in Section 6.1 to compute the local unramified integrals.

THEOREM 6.3. — *Let p be an odd prime. For all unramified data and for* $\mathrm{Re}(s)$ *large*

$$J_1(W,\phi,\tilde{f}_{\sigma,s})=\frac{L(\pi\otimes\sigma,s(k+1)-\frac{1}{2}k)}{L(\sigma,V^2,2s(k+1)-k)}.$$

Proof. — We may assume that $\pi=\mathrm{Ind}_{\overline{P}_{2n}}^{\mathrm{Sp}_{2n}}(\sigma'\otimes\delta_{\overline{P}_{2n,n}}^{\frac{1}{2}+\zeta})$ and σ' is a generic unramified representation of GL_n. To use Proposition 6.1 we first need to normalize identity (6.1). It follows from [GS], Section 2.3 that

$$\int_{U_{2n,n}}\varphi_{\sigma',\zeta}(u)\widetilde{\psi}_n(u)\,\mathrm{d}u=L\big(\widehat{\sigma'},(\tfrac{1}{2}+\zeta)(n+1)-\tfrac{1}{2}(n-1)\big)^{-1}$$
$$L\big(\widehat{\sigma'},\Lambda^2,(1+2\zeta)(n+1)-n\big)^{-1}$$

Here $\widehat{\sigma'}$ denotes the contragredient representation of σ'. Hence we can write

$$W(g)=L\big(\widehat{\sigma'},(\tfrac{1}{2}+\zeta)(n+1)-\tfrac{1}{2}(n-1)\big)L\big(\widehat{\sigma'},\Lambda^2,(1+2\zeta)(n+1)-n\big)$$
$$\int_{U_{2n,n}}\varphi_{\sigma',\zeta}(ug)\widetilde{\psi}_n(u)\,\mathrm{d}u$$

and with this normalization $W(e)=1$. Note that we obtain the L functions of $\widehat{\sigma'}$ since we induce from a lower parabolic. In a similar way the integral

$$\int_{U_{2k,k}}\tilde{f}_{\sigma,s,\chi}(w_kuh)\psi_k(u)\,\mathrm{d}u,\quad h\in\widetilde{\mathrm{Sp}_{2k}},\tag{6.4}$$

defines a Whittaker functional for the induced space $\tilde{I}(\mathcal{W}(\sigma\otimes\chi,\psi^{-1}),s)$. Again, it follows as in [GS], Section 2.3 that the normalizing factor for (6.4) is $L(\sigma,V^2,2s(k+1)-k)^{-1}$. Thus the function

$$\widetilde{W}_{\sigma,s,\chi}(h)=L\big(\sigma,V^2,2s(k+1)-k\big)\int_{U_{2k,k}}\tilde{f}_{\sigma,s,\chi}(w_k uh)\psi_k(u)\,\mathrm{d}u$$

is the normalized unramified vector in $\tilde{I}(\mathcal{W}(\sigma\otimes\chi,\psi^{-1}),s)$. Applying Proposition 6.1 we have,

$$\begin{aligned}\gamma\big(\sigma\times\sigma',(k+1)s-(n+1)\zeta-\tfrac{1}{2}k,\psi^{-1}\big)&J_1(W,\phi,\tilde{f}_{\sigma,s})\\ =L\big(\widehat{\sigma}',(\tfrac{1}{2}+\zeta)(n+1)-\tfrac{1}{2}(n-1)\big)&L\big(\widehat{\sigma}',\Lambda^2,(1+2\zeta)(n+1)-n\big)\\ \int\varphi_{\sigma',\zeta}(w_ng)\omega_\psi(g)\phi(2x)&\tilde{f}_{\sigma,s,\chi}\big(w_kuj'\big(r\tau((x,0,0))g\big)\bar{w}_0\big)\\ &\psi_k(u)\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}g,\end{aligned}$$

where the domain of integration is as in Proposition 6.1. Since $\tilde{f}_{\sigma,s,\chi}$ is unramified we may ignore $\bar{w}_0$ and using Iwasa decomposition for $V_{2n,n}\backslash\mathrm{Sp}_{2n}$, we may replace j' by j and replace $\phi(2x)$ by $\phi(x)$. Multiplying the above identity by $L(\sigma,V^2,2s(k+1)-k)$ we obtain

$$\begin{aligned}(6.5)\quad &\gamma\big(\sigma\times\sigma',(k+1)s-(n+1)\zeta-\tfrac{1}{2}h,\psi^{-1}\big)J_1(W,\phi,\tilde{f}_{\sigma,s})\\ &=\frac{L(\widehat{\sigma}',(\frac{1}{2}+\zeta)(n+1)-\frac{1}{2}(n-1))L(\widehat{\sigma}',\Lambda^2,(1+2\zeta)(n+1)-n)}{L(\sigma,V^2,2s(k+1)-k)}\\ &\qquad\int\varphi_{\sigma',\zeta}(w_ng)\omega_\psi(g)\phi(x)\widetilde{W}_{\sigma,s,\chi}\big(j\big(r\tau((x,0,0))g\big)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}g,\end{aligned}$$

where r,x and g are integrated as before. Notice that

$$F_{\sigma',\zeta}(g)=\varphi_{\sigma',\zeta}(w_ng)\in\operatorname{Ind}_{P_{2n,n}}^{\mathrm{Sp}_{2n}}\big(\widehat{\sigma}'\otimes\delta_{P_{2n,n}}^{\frac{1}{2}+\zeta}\big).$$

Hence the above integral equals $I_2(\widetilde{W}_{\sigma,s,\chi},\phi,F_{\sigma',\zeta},\frac{1}{2}+\zeta)$. Applying Theorem 3.2 for the representations $\widehat{\sigma}'$ and

$$\widetilde{\pi}=\operatorname{Ind}_{P_{2k,k}}^{\widetilde{\mathrm{Sp}}_{2k}}\big(\sigma\otimes\delta_{P_{2k,k}}^{s}\otimes\chi\otimes\gamma_{\det}^{-1}\big)$$

the above integral equals

$$\begin{aligned}L_{\psi^{-1}}\big(&\operatorname{Ind}_{P_{2k,k}}^{\widetilde{\mathrm{Sp}}_{2k}}(\sigma\otimes\delta_{P_{2k,k}}^{s}\otimes\chi\otimes\gamma_{\det}^{-1})\otimes\widehat{\sigma}',(n+1)(\tfrac{1}{2}+\zeta)-\tfrac{1}{2}n\big)\\ &=L\big(\sigma\otimes\delta_{P_{2k,k}}^{s-\frac{1}{2}}\otimes\widehat{\sigma}',(n+1)(\tfrac{1}{2}+\zeta)-\tfrac{1}{2}n\big)\\ &\qquad\times L\big(\widehat{\sigma}\otimes\delta_{P_{2k,k}}^{-s+\frac{1}{2}}\otimes\widehat{\sigma}',(n+1)(\tfrac{1}{2}+\zeta)-\tfrac{1}{2}n\big)\\ &\qquad\qquad\times L\big(\widehat{\sigma}',(n+1)(\tfrac{1}{2}+\zeta)-\tfrac{1}{2}(n-1)\big)^{-1}\\ &\qquad\qquad\qquad\times L\big(\widehat{\sigma}',\Lambda^2,2(n+1)(\tfrac{1}{2}+\zeta)-n\big)^{-1}.\end{aligned}$$

Plugging this to (6.5) we obtain

$$\gamma\big(\sigma\times\sigma',(k+1)s-(n+1)\zeta-\tfrac{1}{2}k,\psi^{-1}\big)J_1(W,\phi,\tilde f_{\sigma,s})$$
$$=\frac{1}{L(\sigma,V^2,2s(k+1)-k)}L\big(\sigma\otimes\widehat{\sigma}',(k+1)s+(n+1)\zeta-\tfrac{1}{2}k\big)$$
$$\times L\big(\widehat{\sigma}\otimes\widehat{\sigma}',-(k+1)s+(n+1)\zeta+\tfrac{1}{2}k+1\big).$$

It follows from [JPSS] that for $\alpha\in\mathbb{C}$

$$\gamma(\sigma\times\sigma',\alpha,\psi^{-1})=\frac{L(\widehat{\sigma}\times\widehat{\sigma}',1-\alpha)}{L(\sigma\times\sigma',\alpha)}.$$

Hence

$$J_1(W,\phi,\tilde f_s)$$
$$=\frac{1}{L(\sigma,V^2,2s(k+1)-k)}\,L\big(\sigma\otimes\widehat{\sigma}',(k+1)s+(n+1)\zeta-\tfrac{1}{2}k\big)$$
$$\times L\big(\sigma\otimes\sigma',(k+1)s-(n+1)\zeta-\tfrac{1}{2}k\big)$$
$$=\frac{1}{L(\sigma,V^2,2s(k+1)-k)}L\big(\sigma\otimes\widehat{\sigma}'\otimes\delta_{\overline{P}_{2n,n}}^{-\zeta},(k+1)s-\tfrac{1}{2}k\big)$$
$$\times L\big(\sigma\otimes\sigma'\otimes\delta_{\overline{P}_{2n,n}}^{\zeta},(k+1)s-\tfrac{1}{2}k\big)$$
$$=\frac{L(\pi\otimes\sigma,(k+1)s-\frac{1}{2}k)}{L(\sigma,V^2,2s(k+1)-k)}.\quad\square$$

In a similar way, using Proposition 6.2 and Theorem 3.1, we prove :

THEOREM 6.4. — *Let p be an odd prime. For all unramified data and for* $\mathrm{Re}(s)$ *large*

$$J_2(\widetilde{W},\phi,f_{\sigma,s})=\frac{L_\psi(\widetilde{\pi}\otimes\sigma,s(k+1)-\frac{1}{2}k)}{L(\sigma,s(k+1)-\frac{1}{2}(k-1))L(\sigma,\Lambda^2,2s(k+1)-k)}.$$

6.3. Justifications (for Section 6.1). — We first establish the convergence, in a right half plane, of the integrals J_1,J_2. Since their structure is similar, it is enough to consider one family of integrals, say J_1.

PROPOSITION 6.5. — *The integrals $J_1(W,\phi,\tilde f_{\sigma,s})$ converge absolutely in a right half plane* $\mathrm{Re}(s)\geq s_0$.

Proof. — Let us write J_1 in explicit coordinates (performing already the conjugation $\gamma v\tau(h)\gamma^{-1}$)

$$(6.5)\qquad J_1(W,\phi,\tilde f_{\sigma,s})=\int_{V_{2n,n}\backslash \mathrm{Sp}_{2n}}\int_{\overline{X}_{k,n}} W(g)\omega_\psi(g)\phi(\xi_0+u_{k-n})\\ \tilde f_{\sigma,s}\left(\begin{pmatrix} I_n &&&\\ 0 & I_{k-n} &&\\ u & v & I_{k-n} &\\ 0 & u' & 0 & I_n\end{pmatrix}\gamma g\right)\psi(v_{k-n,1})\,\mathrm{d}\bar x\,\mathrm{d}g,$$

where

$$(6.6.)\qquad \overline{X}_{k,n}=\left\{\bar x=\begin{pmatrix} I_n &&&\\ 0 & I_{k-n} &&\\ u & v & I_{k-n} &\\ 0 & u' & 0 & I_n\end{pmatrix}\in \mathrm{Sp}_{2k}\right\}.$$

Assume, first, that the local field F is non-archimedean. Using the Iwasawa decomposition in $\mathrm{Sp}_{2n}(F)$, it is enough to establish the convergence of

$$(6.7)\qquad \int\limits_{A_n}\int\limits_{\overline{X}_{k,n}} W(a)\delta_{B_n}^{-1}(a)\omega_\psi(a)\phi(\xi_0+u_{k-n})\tilde f_{\sigma,s}(\bar x\gamma a)\psi(v_{k-n,1})\,\mathrm{d}\bar x\,\mathrm{d}g.$$

Here A_n is the diagonal subgroup of Sp_{2n} and $\bar x$ is the element appearing in (6.6). B_n is the standard Borel subgroup of Sp_{2n}. Write $a=\begin{pmatrix} b & \\ & b^*\end{pmatrix}$, where $b\in \mathrm{GL}_n(F)$ is diagonal. Then (6.7) equals

$$(6.8)\qquad \int\limits_{(F^*)^n}\int\limits_{\overline{X}_{k,n}} W\begin{pmatrix} b & \\ & b^*\end{pmatrix}\delta_{B_n}^{-1}\begin{pmatrix} b & \\ & b^*\end{pmatrix}\phi(\xi_0+u_{k-n})\tilde f_{\sigma,s}\left(\bar x\gamma;\begin{pmatrix} b & \\ & I_{k-n}\end{pmatrix}\right)\\ |\det b|^{(k+1)s+n-k+\frac12}\psi(v_{k-n,1})\,\mathrm{d}\bar x\,\mathrm{d}b.$$

In general, we have (as in [S1, p. 22, Lemma 4.4.])

$$(6.9)\qquad \left|\tilde f_{\sigma,s}\left(u\begin{pmatrix} t & \\ & t^*\end{pmatrix}r\right)\right|\le |\det b|^{(k+1)\,\mathrm{Re}(s)}\sum c_{j,s}\eta_j(t)$$

for $u\in V_{2k,k}$, $r\in K(\mathrm{Sp}_{2k})$. Here t is diagonal in $\mathrm{GL}_n(F)$ and lies in the support of a gauge on $\mathrm{GL}_n(F)$, which is independent of u and r. The η_j are positive quasi-characters which depend on σ. Thus, (6.8) is majorized by

$$(6.10)\qquad c_\phi\sum c_{j,s}\int\limits_{(F^*)^n}\delta_{B_n}^{-1}\cdot\xi\begin{pmatrix} b & \\ & b^*\end{pmatrix}|\det b|^{(k+1)\,\mathrm{Re}(s)+n-k+\frac12}\eta_j(b)\,\mathrm{d}b\\ \int\limits_{\overline{X}_{k,n}} H(\bar x)^{(k+1)\,\mathrm{Re}(s)}E_j(\bar x)\,\mathrm{d}\bar x.$$

Here ξ is a gauge on $\mathrm{Sp}_{2n}(F)$, which majorizes W. c_ϕ is a bound for ϕ, and H and E_j are defined on $\mathrm{Sp}_{2k}(F)$ as follows. In the notation of (6.9)

$$H\left(u\begin{pmatrix} t & \\ & t^* \end{pmatrix} r\right) = |\det t|, \tag{6.11}$$

$$E_j\left(u\begin{pmatrix} t & \\ & t^* \end{pmatrix} r\right) = \eta_j(t). \tag{6.12}$$

The $\mathrm{d}\bar{x}$-integral in (6.10) converges for $\mathrm{Re}(s) \gg 0$ (as in [S1, Lemma 4.5]). Each $\mathrm{d}b$-integral in (6.10) is a linear combination of integrals of the form

$$\int_{(F^*)^n} \phi(b_1, \dots, b_n)\chi_1(b_1)\cdots\chi_n(b_n) \, |b_1 b_2^2 \cdots b_n^n|^{(k+1)\,\mathrm{Re}(s)}\, \mathrm{d}^*(b_1, \dots, b_n), \tag{6.13}$$

where $\phi \in S(F^n)$ (is positive) and $\chi_1, \dots, \chi_n$ are positive quasi-characters of F^* (depending on π, σ). The integrals (6.13) converge in a right half plane (which depends on $\chi_1, \dots, \chi_n, k$).

Now assume that F is archimedean. The absolute convergence in a right half plane is obtained similarly, only that in the Iwasawa decomposition, which leads to (6.7), we get rid of the compact integration, not by K-finiteness, which we do not assume here, but rather by the majorizations

$$|W(a \cdot h)| \leq \xi(a), \quad a \in A_n,\ h \in K(\mathrm{Sp}_{2n}), \tag{6.14}$$

$$\left|\tilde{f}_{\sigma,s}\left(u\begin{pmatrix} t & \\ & t^* \end{pmatrix} r\right)\right| \leq c_s |\det t|^{(k+1)\,\mathrm{Re}(s)} |\omega_\sigma(t_n)| \cdot \|t_n^{-1}t\|^N. \tag{6.15}$$

In (6.14), ξ is a gauge on $\mathrm{Sp}_{2n}(F)$. In (6.15), c_s is a constant which depends on s. Here t has the form $\mathrm{diag}(t_1t_2\cdots t_n, t_2\cdots t_n, \dots, t_{n-1}t_n)$ and ω_σ is the central character of σ. The integer N depends on σ. Finally,

$$\|\mathrm{diag}(a_1, a_2, \dots, a_{n-1}, 1)\| = 1 + \sum_{i=1}^{n-1} |a_i|^2 + \sum_{i=1}^{n-1} |a_i|^{-2}.$$

With these majorizations the proof now continues as before without change. □

Next, we show that the integrals J_1, J_2 can be made to be identically 1 (for all s), for a choice of data $(W, \phi, \tilde{f}_{\sigma,s})$ in case F is non-archimedean.

PROPOSITION 6.6. — *Let F be non-archimedean. There is a choice of $W \in W(\pi,\psi)$, $\phi \in S(F^n)$ and a section $\tilde{f}_{\sigma,s}$, such that*

$$J_1(W,\phi,\tilde{f}_{\sigma,s}) = 1, \quad \forall s \in \mathbb{C}.$$

(*A similar proposition holds for J_2, with exactly the same proof.*)

Proof. — Write the integral (6.5) as follows

$$(6.16) \quad J_1(W,\phi,\tilde{f}_{\sigma,s}) = \int\limits_{A_n \times \overline{V}_{2n,n}} \int\limits_{\overline{X}_{k,n}} W(a\bar{z})\delta_{B_n}^{-1}(a)\omega_\psi(a\bar{z})\phi(\xi_0 + u_{k-n})$$
$$\tilde{f}_{\sigma,s}\left(\begin{pmatrix} I_n & & & \\ 0 & I_{k-n} & & \\ u & v & I_{k-n} & \\ 0 & u' & 0 & I_n \end{pmatrix}\gamma a\bar{z}\right)\psi(v_{k-n,1})\,\mathrm{d}\bar{x}\,\mathrm{d}a\,\mathrm{d}\bar{z}.$$

Here

$$\overline{V}_{2n,n} = \left\{\bar{z} = \begin{pmatrix} \bar{z}_1 & 0 \\ y & \bar{z}_1^* \end{pmatrix} \in \mathrm{Sp}_{2n}\ ;\ \bar{z}_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \in \mathrm{GL}_n\right\}.$$

Write again $a = \begin{pmatrix} b & \\ & b^* \end{pmatrix}$, b-diagonal. Then (6.16) equals

$$(6.17) \quad \int W\left(\begin{pmatrix} b\bar{z}_1 & \\ & b^*\bar{z}_n^* \end{pmatrix}\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}\right)\delta_{B_n}^{-1}\begin{pmatrix} b & \\ & b^* \end{pmatrix}|\det b|^{(k+1)\,\mathrm{Re}(s)+n-k+\frac{1}{2}}$$
$$\omega_\psi\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}\phi(b_n\xi_0\bar{z}_1 + u_{k-n})\psi(v_{k-n,1})$$
$$\tilde{f}_{\sigma,s}\left(\begin{pmatrix} I_n & & & \\ 0 & I_{k-n} & & \\ u & v & I_{k-n} & \\ y & u' & 0 & I_n \end{pmatrix}\gamma,\begin{pmatrix} b\bar{z}_1 & \\ & I_{k-n} \end{pmatrix}\right)\mathrm{d}(\cdots).$$

Choose the right γ-translate of $\tilde{f}_{\sigma,s}$ to have support in $P_{2k,k}\cdot\Omega$, where Ω is a small neighbourhood of I_{2k}, such that $\tilde{f}_{\sigma,s}(\omega\gamma;m) = W'(m)$, for $\omega \in \Omega$. W' is a given function in the Whittaker model of σ. With this choice (u,v,y) in (6.17) must lie in a small neighbourhood $\mathcal{V}$ of zero, and we can choose Ω so small that $\omega_\psi\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}\phi = \phi$ and $W\left(g\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}\right) = W(g)$, for y, such that (u,v,y) lies in $\mathcal{V}$. Up to a constant, (6.17) becomes

$$(6.18) \quad \int W\begin{pmatrix} b\bar{z}_1 & \\ & b^*\bar{z}_1^* \end{pmatrix}\delta_{B_n}^{-1}\begin{pmatrix} b & \\ & b^* \end{pmatrix}\phi(\xi_0 b\bar{z}_1)$$
$$|\det b|^{s'}W'\begin{pmatrix} b\bar{z}_1 & \\ & I_{k-1} \end{pmatrix}\mathrm{d}\bar{z}_1\,\mathrm{d}b.$$

Here $s' = (k+1)\operatorname{Re}(s) + n - k + \frac{1}{2}$. Also v above is so small that $\phi(\xi + u_{k-n}) = \phi(\xi)$, if $(u,v,y) \in \mathcal{V}$. Now, it is easy to choose data in (6.18), to make it constant. We can simply choose W', such that the function $(b, \bar{z}_1) \mapsto W'\begin{pmatrix} b\bar{z}_1 & \\ & I_{k-n} \end{pmatrix}$ is the characteristic function of any given small neighbourhood of (I_n, I_n). $\square$

PROPOSITION 6.7. — *Assume that F is archimedean. For each complex number s_0, there are choices of data $(W_j, \phi_j, \tilde{f}^{(j)}_{\sigma,s})$, such that $\sum_j J_1(W_j, \phi_j, \tilde{f}^{(j)}_{\sigma,s})$ is meromorphic and nonzero at s_0. (Similar proposition for J_2).*

Proof. — Write J_1 in the form (6.16). Let the right γ-translate of $\tilde{f}_{\sigma,s}$ have support in $P_{2k,k} \cdot \overline{U}_{2k,k}$ and assume

$$\tilde{f}_{\sigma,s}\Big(v\begin{pmatrix} m & \\ & m^* \end{pmatrix}\bar{u}; e\Big) = \gamma^{-1}(\det m)|\det m|^{(k+1)s}\varphi(\bar{u})W'(em),$$

where $v \in U_{2k,k}$, $\bar{u} \in \overline{U}_{2k,k}$ (the opposite to $U_{2k,k}$); $e, m \in \mathrm{GL}_n(F)$, and $\varphi \in C_c^\infty(\overline{U}_{2k,k})$. With this choice

$$J_1(W, \phi, \tilde{f}_{\sigma,s}) = \int W\Big(\begin{pmatrix} b\bar{z}_1 & \\ & b^*\bar{z}_1^* \end{pmatrix}\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}\Big) \omega_\psi\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}\phi(\xi_0 b\bar{z}_1 + u_{k-n})\varphi(u,v,y)\delta_{B_n}^{-1}\begin{pmatrix} b & \\ & b^* \end{pmatrix} W'\begin{pmatrix} b\bar{z}_1 & \\ & I_{k-n} \end{pmatrix}\psi(V_{k-n,1})|\det b|^{(k+1)s+n-k+\frac{1}{2}}\,\mathrm{d}(\cdots). \tag{6.19}$$

Notation being as in Proposition 6.6. $\varphi(u,v,y)$ is short for

$$\varphi\begin{pmatrix} I_n & & & \\ & I_{k-n} & & \\ u & v & I_{k-n} & \\ y & u' & & I_n \end{pmatrix}.$$

Choose φ of the form $\varphi(u,v,y) = \varphi_1(u)\varphi_2(v)\varphi_3(y)$. The $\mathrm{d}v$-integration in (6.19) is carried separately and gives a constant $\int \varphi_2(v)\psi(v_{k-n,1})\,\mathrm{d}v$. Consider the $\mathrm{d}y$-integration

$$\int \varphi_3(y)\pi\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}W \otimes \omega_\psi\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}\phi\,\mathrm{d}y.$$

This is a convolution of φ_3 against $W \otimes \phi \in W(\pi, \psi) \otimes \omega_\psi$. By [DM], this represents, up to linear combinations, a general element of $W(\pi, \psi) \otimes \omega_\psi$. Thus, a suitable linear combination of integrals of the form (6.19) gives

$$\int W\begin{pmatrix} b\bar{z}_1 & \\ & b^*\bar{z}_1^* \end{pmatrix} \phi(\xi_0 b\bar{z}_1 + u_{k-n})\varphi_1(u) \\ W'\begin{pmatrix} b\bar{z}_1 & \\ & I_{k-n} \end{pmatrix} \delta_{B_n}^{-1}\begin{pmatrix} b & \\ & b^* \end{pmatrix} |\det b| \,\mathrm{d}u\,\mathrm{d}b\,\mathrm{d}\bar{z}_1. \tag{6.20}$$

Here $s' = (k+1)s + n - k + \frac{1}{2}$. Again, by [DM],

$$\int \varphi_1(u)\phi(\xi + u_{k-n})\,\mathrm{d}u$$

represents, up to linear combinations, a general element of $S(F^n)$. Thus, a suitable linear combination of integrals of the form (6.20) becomes

$$\int W\begin{pmatrix} b\bar{z}_1 & \\ & b^*\bar{z}_1^* \end{pmatrix} W'\begin{pmatrix} b\bar{z}_1 & \\ & I_{k-1} \end{pmatrix} \phi(\xi_0 b\bar{z}_1)\delta_{B_n}^{-1}\begin{pmatrix} b & \\ & b^* \end{pmatrix} |\det b|^{s'}\,\mathrm{d}b\,\mathrm{d}\bar{z}_1.$$

Note that since $\delta_{B_n}^{-1}\begin{pmatrix} b & \\ & b^* \end{pmatrix} = \delta_{B_{\mathrm{GL}_n}}^{-1}(b)|\det b|^{-(n+1)}$, the last integral becomes

$$\int_{N_n\backslash \mathrm{GL}_n} W\begin{pmatrix} m & \\ & m^* \end{pmatrix} W'\begin{pmatrix} m & \\ & I_{k-n} \end{pmatrix} \phi(\xi_0 m)|\det m|^{s'-(n+1)}\,\mathrm{d}m \tag{6.21}$$

and for

$$W'' = \int \alpha(y)\sigma\begin{pmatrix} I_n & y & \\ & 1 & \\ & & I_{k-n-1} \end{pmatrix} W'\,\mathrm{d}y,$$

where α is a C_c^∞-function,

$$W''\begin{pmatrix} m & \\ & I_{k-n} \end{pmatrix} = \widehat{\alpha}(\xi_0 m) W'\begin{pmatrix} m & \\ & I_{k-n} \end{pmatrix}. \tag{6.22}$$

Choosing ϕ, such that $\phi = \widehat{\alpha}$, we see that (6.21) becomes

$$\int_{N_n\backslash \mathrm{GL}_n} W\begin{pmatrix} m & \\ & m^* \end{pmatrix} W''\begin{pmatrix} m & \\ & I_{k-n} \end{pmatrix} |\det m|^{s'-(n+1)}\,\mathrm{d}m.$$

Now, as in [S2, Section 4], this last integral is meromorphic in s. Fix now $s = s_0$. If the integral (6.20) is identically zero for all data, then, since ϕ is arbitrary, it follows that the following integral is identically zero

$$\int W\begin{pmatrix} b\bar{z} & & & \\ & 1 & & \\ & & 1 & \\ & & & b^*\bar{z}^* \end{pmatrix} W'\begin{pmatrix} b\bar{z} & \\ & I_{k-n+1} \end{pmatrix} \\ \delta_{B_n}^{-1}\begin{pmatrix} b & & & \\ & 1 & & \\ & & 1 & \\ & & & b^* \end{pmatrix} |\det b|^{s'}\,\mathrm{d}b\,\mathrm{d}\bar{z}, \tag{6.23}$$

where $b = \operatorname{diag}(b_1, \dots, b_{n-1})$ and $\bar{z} = \begin{pmatrix} 1 & & \\ & \ddots & \\ * & & 1 \end{pmatrix} \in \mathrm{GL}_{n-1}$. Replace W' by $W'' = \int \alpha(y)\sigma\begin{pmatrix} I_{n-1} & y & \\ & 1 & \\ & & I_{k-n} \end{pmatrix} \mathrm{d}y$, where α is a C_c^∞-function. As in (6.22),

$$W''\begin{pmatrix} m & \\ & I_{k-n+1} \end{pmatrix} = \widehat{\alpha}(\xi_0 m) W'\begin{pmatrix} m & \\ & I_{k-n+1} \end{pmatrix}.$$

Thus

$$\int W\begin{pmatrix} b\bar{z} & & & \\ & 1 & & \\ & & 1 & \\ & & & b^*\bar{z}^* \end{pmatrix} W'\begin{pmatrix} b\bar{z} & \\ & I_{k-n+1} \end{pmatrix} \widehat{\alpha}(\xi_0 b\bar{z}) \\ \delta_{B_n}^{-1}\begin{pmatrix} b & & & \\ & 1 & & \\ & & 1 & \\ & & & b^* \end{pmatrix} |\det b|^{s'} \,\mathrm{d}b\,\mathrm{d}\bar{z} \equiv 0.$$

Since α is arbitrary, we conclude that

$$\int W\begin{pmatrix} b\bar{z} & & \\ & I_4 & \\ & & b^*\bar{z}^* \end{pmatrix} W'\begin{pmatrix} b\bar{z} & \\ & I_{k-n+2} \end{pmatrix} \delta_{B_n}^{-1}\begin{pmatrix} b & & \\ & I_4 & \\ & & b^* \end{pmatrix} |\det b|^{s'} \,\mathrm{d}b\,\mathrm{d}\bar{z} \equiv 0,$$

etc. Finally we get that $W(I_{2n})W'(I_n)$ is identically zero, which is absurd. ∎

REMARK. — In case F is nonarchimedean, with residue field having q elements, the integrals J_1, J_2 are rational functions in q^{-s}. They satisfy certain equivariance properties for trilinear forms which guarantee their uniqueness up to scalar multiples for almost all values of q^{-s}. This will appear in greater generality in a work of Baruch and Rallis (compare [S1, Sec. 8]). A general principle of Bernstein implies the rationality of J_1, J_2 (See [GPS, 1.2.3]). Here is a description of these equivariance properties.

Let π be an irreducible, $\widetilde{\psi}_n$-generic representation of $\mathrm{Sp}_{2n}(F)$ (resp. $\widetilde{\mathrm{Sp}}_{2n}(F)$) and σ an irreducible, generic representation of $\mathrm{GL}_k(F)$. The integral J_1 (resp. J_2) belongs to the space E of trilinear forms J on $N(\pi, \widetilde{\psi}_n) \times S(F^n) \times V_{I(\sigma,s)}$ (resp. $W(\pi, \widetilde{\psi}_n) \times S(F^n) \times V_{\tilde{I}(\sigma,s)}$), which satisfy

$$J\big(W, \phi, I(\sigma,s)(v) f_{\sigma,s}\big) = \psi_{k-n-1}^{-1}(v) J(W, \phi, f_{\sigma,s}),$$

$$J\big(W, \omega_\psi(h)\phi, I(\sigma,s)(\tau(h)) f_{\sigma,s}\big) = J(W, \phi, f_{\sigma,s}),$$

$$J\big(\pi(g)W, \omega_\psi(g)\phi, I(\sigma,s)(g) f_{\sigma,s}\big) = J(W, \phi, f_{\sigma,s})$$

for $W \in W(\pi, \widetilde{\psi}_n)$, (resp. $W \in W(\widetilde{\pi}, \widetilde{\psi}_n)$), $\phi \in S(F^n)$, $f_{\sigma,s} \in V_{I(\sigma,s)}$, (resp. $\tilde{f}_{\sigma,s} \in \tilde{I}(\sigma,s)$), $v \in V_{k-n-1}(F)$, $h \in H_n(F)$ and $g \in \widetilde{Sp}_{2n}(F)$ (resp. $g \in \mathrm{Sp}_{2n}(F)$). We may also replace π by a representation fully induced from a parabolic subgroup and an irreducible (generic) representation of the Levi part. The proof of the one-dimensionality of the space of E, for almost all values of q^{-s}, follows closely the proof of Theorem 5.1.

The following three propositions justify the formal steps taken in Proposition 6.1. From now on, F is assumed to be non-archimedean.

PROPOSITION 6.8. — *The integral*

$$\int_{N_n\backslash \mathrm{Sp}_{2n}} \int \varphi_{\sigma',\zeta}(g)\omega_\psi((x,0,z)g)\phi(\xi_0)\tilde{f}_{\sigma,s}\big(\gamma(A,0,B)g\big)\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}g \tag{6.24}$$

converges absolutely in a domain of the form

$$\begin{cases} (n+1)\,\mathrm{Re}(\zeta+\frac12)+C<(k+1)\,\mathrm{Re}(s), \\ (k+1)\,\mathrm{Re}(s)<(1+\varepsilon_0)(n+1)\,\mathrm{Re}(\zeta+\frac12)+Q, \\ R<\mathrm{Re}(\zeta), \end{cases} \tag{6.25}$$

where Q, C, R, ε_0 are constants which depend on σ, σ', n, k and ε_0 is positive.

Proof. — Using the Iwasawa decomposition in $\mathrm{Sp}_{2n}(F)$ (and K-finiteness). It is enough to consider

$$\begin{aligned}\int \Big|\varphi_{\sigma',\zeta}\Big(p\begin{pmatrix} b & \\ & b^*\end{pmatrix}\Big)\Big|\,\delta_{\overline{B}_n}^{-1}\begin{pmatrix} b & \\ & b^*\end{pmatrix} \\ \Big|\omega_\psi\Big((x,0,z)\begin{pmatrix} I_n & \\ y & I_n\end{pmatrix}\begin{pmatrix} b & \\ & b^*\end{pmatrix}\Big)\phi(\xi_0)\Big| \\ \left|\tilde{f}_{\sigma,s}\left(\gamma(A,0,B)\begin{pmatrix} I_{k-n} & & \\ & \begin{pmatrix} I_n & \\ y & I_n\end{pmatrix}\begin{pmatrix} b & \\ & b^*\end{pmatrix} & \\ & & I_{k-n}\end{pmatrix}\right)\right|\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}b\,\mathrm{d}y.\end{aligned} \tag{6.26}$$

(Recall that $B_{k-n,k-n}=z$, $A_{k-n}=x$). Here b is integrated over the diagonal subgroup of $\mathrm{GL}(n,F)$. We have

$$\varphi_{\sigma,\zeta}\begin{pmatrix} b & \\ & b^*\end{pmatrix} = |\det b|^{-(n+1)(\zeta+\frac12)}W_{\sigma'}(b),$$

where $W_{\sigma'} \in W(\sigma', \psi)$. Thus (6.26) has the form

$$(6.27)\quad \int \left|W_{\sigma'}(b)\right| \cdot |\det b|^{s'-\zeta'} \chi_0(b) \left|\omega_\psi\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix}\phi(\xi_0 b + x)\right| \left| f'_{\sigma,s}\left(\begin{pmatrix} I_n & & & \\ & I_{k-n} & & \\ A & B & I_{k-n} & \\ y & A' & & I_n \end{pmatrix}; \begin{pmatrix} b & \\ & I_{k-n} \end{pmatrix}\right)\right| \mathrm{d}A\,\mathrm{d}B\,\mathrm{d}b\,\mathrm{d}y,$$

where $\zeta' = (n+1)\,\mathrm{Re}(\zeta + \frac{1}{2})$, $s' = (k+1)\,\mathrm{Re}(s)$ and χ_0 is a certain positive quasi-character (obtained from δ and change of variable) $f'_{\sigma,s}$ is the right γ-translate of $\tilde{f}_{\sigma,s}$. Write the Iwasawa decomposition

$$(6.28)\quad \begin{pmatrix} I_n & & & \\ & I_{k-n} & & \\ A & B & I_{k-n} & \\ y & A' & & I_n \end{pmatrix} = v \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_k & & & \\ & & & t_k^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} r,$$

where $v \in V_{2n,n}$, $k \in K(\mathrm{Sp}_{2k})$. As in [S1, p. 81], we have

$$(6.29)\quad \begin{cases} [z]^{-2j} \le \left|\dfrac{t_j}{t_{j+1}}\right| \le [z]^{2j}, & j = 1, \dots, k-1, \\ [z]^{-k} \le |\det t| \le [z]^{-1}, & \end{cases}$$

where $z = \begin{pmatrix} A & B \\ y & A' \end{pmatrix}$ and $[z] = \max\{1, \|z\|\}$, where $\|z\|$ is the sup-norm of the coordinates of z. As in (6.9), we have a majorization

$$(6.30)\quad \left| f'_{\sigma,s}\left(\begin{pmatrix} I_k & \\ z & I_k \end{pmatrix}; \begin{pmatrix} b & \\ & I_{k-n} \end{pmatrix}\right)\right| \le |\det t|^{s'} \sum \alpha_j \eta_j\left(\begin{pmatrix} b & \\ & I_{k-n} \end{pmatrix} t\right),$$

where $t = \mathrm{diag}(t_1, \dots, t_n)$ and η_j are positive quasi-characters and, moreover, $\begin{pmatrix} b & \\ & I_{k-n} \end{pmatrix} t$ lies in the support of a gauge on $\mathrm{GL}_k(F)$. Assume that $s' \ge 0$. Then, by (6.29)

$$(6.31)\quad |\det t|^{s'} \le [z]^{-s'}$$

(if $s' < 0$, then, by (6.29), $|\det|^{s'} \le [z]^{-ks'}$). Inequalities (6.29) also implies that there is an integer K_1, such that

$$\eta_j(t) \le [z]^{K_1}, \qquad \forall j. \tag{6.32}$$

Since $\begin{pmatrix} b & \\ & I_{k-n} \end{pmatrix} t$ lies in the support of a gauge, then, writing

$$b = \mathrm{diag}(b_1 b_2 \cdots b_n, b_2 \cdots b_n, \ldots, b_{n-1}b_n, b_n),$$

there are positive constants c_j, such that

$$\left|\frac{b_j t_j}{t_{j+1}}\right| \le c_j, \qquad j = 1, \ldots, k-1,$$

and hence

$$|b_j| \le c_j \left|\frac{t_{j+1}}{t_j}\right| \le c_j [z]^{2j} \tag{6.33}$$

(put $b_{n+1} = \cdots = b_k = 1$). Using (6.30)–(6.33) (for $s' > 0$) we get a majorization

$$\left| f'_{\sigma,s}\left(\begin{pmatrix} I_k & \\ z & I_k \end{pmatrix}; \begin{pmatrix} b & \\ & I_{k-n} \end{pmatrix}\right)\right| \le c'[z]^{s'+L_1}. \tag{6.34}$$

Let $\xi_{\sigma'}$ be a gauge on $\mathrm{GL}_n(F)$, majorizing $W_{\sigma'}$. Thus (6.27) is majorized by a sum of constant multiples of integrals of the form

$$\int_{|b_n| \le c_n [z]^{2n}} \xi_{\sigma'}(b) |\det b|^{s'-\zeta'} \widetilde{\chi}(b) \left|\omega_\psi \begin{pmatrix} I_n & \\ y & I_n \end{pmatrix} \phi(\xi_0 b + x)\right| [z]^{-s'+L_1}\, \mathrm{d}b\, \mathrm{d}z. \tag{6.35}$$

(Recall that $z = \begin{pmatrix} A & B \\ y & A' \end{pmatrix}$, and $A_{k-n} = x$). $\widetilde{\chi}$ is of the form $\chi_0 \eta_j$. We used (6.33) and (6.34). Now write the Iwasawa decomposition

$$\begin{pmatrix} I_n & \\ y & I_n \end{pmatrix} = \begin{pmatrix} I_n & T \\ & I_n \end{pmatrix} \begin{pmatrix} m_y & \\ & m_y^* \end{pmatrix} r_y,$$

where $r_y \in K(\mathrm{Sp}_{2n})$ and $m_y \in \mathrm{GL}_n(F)$. As in (6.29), we find

$$[y]^{-n} \le |\det m_y| \le [y]^{-1} \le 1. \tag{6.36}$$

From (1.3) and (1.4),

$$\left|\omega_\psi\begin{pmatrix} I_n & \\ y & I_n\end{pmatrix}\phi(u)\right| = |\det m_y|^{\frac{1}{2}}\cdot\left|\omega_\psi(r_y)\phi(u\cdot m_y)\right| \le \left|\omega_\psi(r_y)\phi(um_y)\right|.$$

Thus there is a bound c_ϕ, such that $\left|\omega_\psi\begin{pmatrix} I_n & \\ y & I_n\end{pmatrix}\phi(u)\right| \le c_\phi$, for all y and u, and hence (6.35) is majorized by a constant multiple of

$$(6.37)\qquad \int_{|b_n|\le c_n[z]^{2n}} \xi_{\sigma'}(b)|\det b|^{s'-\zeta'}\chi(b)[z]^{-s'+L_1}\,\mathrm{d}b\,\mathrm{d}z.$$

This integral is considered in [S1, p. 86] and converges in the indicated domain. ▯

PROPOSITION 6.9. — *The integral.*

$$(6.38)\qquad \int \varphi_{\sigma,\zeta}(g,m)\widehat{\omega_\psi((x,0,z)g)\phi}(e)$$
$$\tilde f_{\sigma,s}\left(\gamma_0(A,0,B)g;\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & I_{k-n-1}\\ m & 0 & y\end{pmatrix}\begin{pmatrix} I_n & {}^te & 0\\ & 1 & 0\\ & & I_{k-n-1}\end{pmatrix}\right)$$
$$|\det m|^{(k+1)s-(n+1)(\zeta+\frac{1}{2})-k}\,\mathrm{d}e\,\mathrm{d}y\,\mathrm{d}m\,\mathrm{d}(A,B)\,\mathrm{d}g.$$

converges absolutely in a domain of the form

$$(6.39)\qquad \begin{cases} (1-\varepsilon_1)(n+1)\operatorname{Re}(\zeta+\frac{1}{2})+E<(k+1)\operatorname{Re}(s),\\ (k+1)\operatorname{Re}(s)<(n+1)\operatorname{Re}(\zeta+\frac{1}{2})+D,\\ M<\operatorname{Re}(\zeta),\end{cases}$$

where D,E,M,ε_1 are constants which depend on (σ,σ',k,n) and ε_1 is positive. In (6.38), *g is integrated over $\mathrm{GL}_n\setminus\mathrm{Sp}_{2n}$, where GL_n is identified with the Levi part of $P_{2n,n}$. The variable m is integrated over $N_n\setminus\mathrm{GL}_n$.*

Proof. — Using the Iwasawa decomposition for g, in (6.38), it is enough to take g of the form $\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}$. Also, using the Iwasawa decomposition for m, in (6.38), it is enough to take m of the form $b\cdot r$, where b is diagonal and $r\in K(\mathrm{GL}_n)$. As before, let $\xi_{\sigma'}$ be a gauge majorizing $\varphi_{\sigma',\zeta}(I;m)$. Thus, it is enough to consider

$$(6.40)\qquad \int \xi_{\sigma'}(b)\left|\widehat{\omega_\psi\Big((x,0,z)g\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\Big)\phi}(e)\right|\beta_0(b)|\det b|^{s'-\zeta'}$$
$$\left|f'_{\sigma,s}\left(\begin{pmatrix} I_n & & & \\ & I_{k-n} & & \\ A & B & I_{k-n} & \\ u & A' & & I_n\end{pmatrix};\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & I_{k-n-1}\\ br & 0 & y\end{pmatrix}\begin{pmatrix} I_n & {}^te & 0\\ & 1 & 0\\ & & I_{k-n-1}\end{pmatrix}\right)\right|d(\ldots).$$

Here β_0 is a certain fixed positive quasi-character and $f'_{\sigma,s}$ is the right γ_0-translate of $\tilde{f}_{\sigma,s}$. We have

$$\widehat{\omega_\psi\Big((x,0,z)g\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\Big)\phi}(e) = \psi(z-2x\cdot{}^te)\widehat{\omega_\psi\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\phi}(e)$$

and hence

$$\Big|\widehat{\omega_\psi\Big((x,0,z)g\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\Big)\phi}(e)\Big| = \Big|\widehat{\omega_\psi\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\phi}(e)\Big|$$

We have

$$\begin{aligned}
&\Big|\widehat{\omega_\psi\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\phi}(e)\Big| \\
&\qquad = \Big|\omega_\psi\Big(\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\Big)\phi(2e)\Big| \\
&\qquad = \Big|\omega_\psi\Big(\begin{pmatrix} I_n & -J_n\,uJ_n \\ & I_n\end{pmatrix}\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}\Big)\phi(2e)\Big| \\
&\qquad = \Big|\omega_\psi\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}\phi(2e)\Big| = \tilde{\phi}(e).
\end{aligned}$$

Here $\tilde{\phi}$ is a positive Schwartz function. Thus (6.40) becomes

$$(6.41)\quad \int \xi_{\sigma'}(b)\beta_0(b)|\det b|^{s'-\zeta'}|\tilde{\phi}(e)| \\
\Big|f'_{\sigma,s}\Big(\begin{pmatrix} I_k & \\ W & I_k\end{pmatrix};\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ br & 0 & y\end{pmatrix}\begin{pmatrix} I_n & {}^te & 0 \\ & 1 & 0 \\ & & I_{k-n-1}\end{pmatrix}\Big)\Big| \\
\mathrm{d}b\,\mathrm{d}(W,e,y)\,\mathrm{d}r.$$

"Move" $e' = \begin{pmatrix} I_n & {}^te & 0 \\ & 1 & 0 \\ & & I_{k-n-1}\end{pmatrix}$ in (6.41) to the left in $f'_{\sigma,s}$ and conjugate it via $\begin{pmatrix} I_k & \\ W & I_k\end{pmatrix}$. Denote

$$\begin{pmatrix} e' & \\ & e'^* \end{pmatrix}\begin{pmatrix} I_k & \\ W & I_k\end{pmatrix}\begin{pmatrix} e' & \\ & e'^* \end{pmatrix}^{-1} = \begin{pmatrix} I_k & \\ W' & I_k\end{pmatrix}$$

and change variable $W \mapsto W'$ ($\mathrm{d}W = \mathrm{d}W'$). Then (6.41) becomes

$$(6.42)\quad \int \xi_{\sigma'}(b)\beta_0(b)|\det b|^{s'-\zeta'}\tilde{\phi}(e)$$
$$\left|f'_{\sigma,s}\left(\begin{pmatrix} I_k & \\ W & I_k\end{pmatrix}\begin{pmatrix} e' & \\ & e'^{*}\end{pmatrix};\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ br & 0 & y\end{pmatrix}\right)\right| \mathrm{d}b\,\mathrm{d}(W,e,y)\,\mathrm{d}r.$$

Since $\tilde{\phi}$ is a linear combination of characteristic functions of small neighbourhoods, it is enough to consider, instead of (6.44), integrals of the form

$$(6.43)\quad \int \xi_{\sigma'}(b)\beta_0(b)|\det b|^{s'-\zeta'}$$
$$\left|f''_{\sigma,s}\left(\begin{pmatrix} I_k & \\ W & I_k\end{pmatrix};\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ br & 0 & y\end{pmatrix}\right)\right| \mathrm{d}b\,\mathrm{d}(W,y)\,\mathrm{d}r.$$

Change variable $y \mapsto$ by (this changes $\beta_0(b)$ to $\beta_1(b)$, by a fixed power of $|\det b|$). We have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ br & 0 & by\end{pmatrix} = \begin{pmatrix} I_{k-n} & \\ & b\end{pmatrix}\begin{pmatrix} 1 & & \\ & I_{k-n-1} & \\ & y & I_n\end{pmatrix}\begin{pmatrix} & 1 & \\ & & I_{k-n-1} \\ r & & \end{pmatrix}.$$

Denote $r' = \begin{pmatrix} & 1 & \\ & & I_{k-n-1} \\ r & & \end{pmatrix}$. In (6.43), "move" r' to the left in $f''_{\sigma,s}$, *i.e.*

$$(6.44)\quad f''_{\sigma,s}\left(\begin{pmatrix} I_k & \\ W & I_k\end{pmatrix};\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ br & 0 & by\end{pmatrix}\right)$$
$$= f''_{\sigma,s}\left(\begin{pmatrix} I_k & \\ r'^{*}Wr'^{-1} & I_k\end{pmatrix}\begin{pmatrix} r' & \\ & r'^{*}\end{pmatrix};\begin{pmatrix} I_{k-n} & \\ & b\end{pmatrix}\begin{pmatrix} 1 & & \\ & I_{k-n-1} & \\ & y & I_n\end{pmatrix}\right).$$

Change variable $r'^{*}Wr'^{-1} \mapsto W$. Now, it suffices to consider the integral of the form (we use the K-finiteness of $f''_{\sigma,s}$)

$$(6.45)\quad \int \xi_{\sigma'}(b)\beta_1(b)|\det b|^{s'-\zeta'}$$
$$\left|f_{\sigma,s}\left(\begin{pmatrix} I_n & \\ W & I_k\end{pmatrix};\begin{pmatrix} I_{k-n} & \\ & b\end{pmatrix}\begin{pmatrix} 1 & & \\ & I_{k-n-1} & \\ & y & I_n\end{pmatrix}\right)\right| \mathrm{d}b\,\mathrm{d}(W,y).$$

Write the Iwasawa decomposition

$$(6.46)\quad \begin{pmatrix} 1 & & \\ & I_{k-n-1} & \\ & y & I_n\end{pmatrix} = v\cdot t'\cdot r'',$$

where $v \in N_k$, $t'_y = \mathrm{diag}(t'_1, \ldots, t'_k)$, $r'' \in K(\mathrm{GL}_k)$. As before, by "moving" r'' back, changing variable $r''^{*}Wr''^{-1} \mapsto W''$, *etc.* It is enough to consider

$$(6.47) \quad \int \xi_{\sigma'}(b)\beta_1(b)|\det b|^{s'-\zeta'} \left|f_{\sigma,s}\left(\begin{pmatrix} I_k & \\ W & I_k \end{pmatrix}; \begin{pmatrix} I_{k-n} & \\ & b \end{pmatrix} t'_y\right)\right| \mathrm{d}b\,\mathrm{d}(W,y).$$

As in (6.30) and (6.31) we have, for $s' > 0$, the estimate

$$(6.48) \quad \left|f_{\sigma,s}\left(\begin{pmatrix} I_k & \\ W & I_k \end{pmatrix}; \begin{pmatrix} I_{k-n} & \\ & b \end{pmatrix} t'y\right)\right| \leq [W]^{-s'}\xi_\sigma\left(\begin{pmatrix} I_{k-n} & \\ & b \end{pmatrix} t'ytw\right),$$

where ξ_σ is a gauge on $\mathrm{GL}_n(F)$ (majorizing the Whittaker functions in σ, $m \mapsto f_{\sigma,s}(\rho; m)$ for $p \in K(\mathrm{Sp}_{2k})$). The matrix t_w is obtained from the Iwasawa decomposition

$$(6.49) \qquad \begin{pmatrix} I_k & \\ W & I_k \end{pmatrix} = u\begin{pmatrix} t_w & \\ & t_w^* \end{pmatrix}\rho,$$

$u \in V_{2k,k}$, $t_w = \mathrm{diag}(t_1, \ldots, t_k), \rho \in K(\mathrm{Sp}_{2k})$. So now consider

$$(6.50) \quad \int \xi_{\sigma'}(b)\beta_1(b)|\det b|^{s'-\zeta'}[W]^{-s'}\xi_\sigma\left(\begin{pmatrix} I_{k-n} & \\ & b \end{pmatrix} t'_y t_w\right) \mathrm{d}b\,\mathrm{d}(W,y).$$

Clearly, in (6.46), we have $t'_1 = 1$. Now, we are at the situation of the proof of Proposition 11.16 (11.16.1) of [S1]. We conclude that the integral (6.38) converges absolutely in a domain of the form (6.39). ▯

PROPOSITION 6.10. — *The integral* (6.38) *is equal* (*in the domain* (6.39)) *to*

$$\omega_{\sigma'}(-1)^{k-2}\gamma\left(\sigma \times \sigma', (k+1)s - (n+1)\zeta - \tfrac{1}{2}k, \psi^{-1}\right) J_1(W, \phi, \tilde{f}_{\sigma,s})$$

(*and hence the analytic continuations of* (6.38) *and* J_1 *are related by the same functional equation.*)

Proof. — By the remark following Proposition (6.7) we know that the integral (6.38) and $J_1(W, \phi, \tilde{f}_{\sigma,s})$ are proportional (in the domain (6.39)). (The equivariance properties, mentioned in the remark, which guarantee the proportionality are easily seen to be satisfied since (6.38) is obtained from J_1 by formal manipulations.)

We will find the proportionality factor by a special substitution of data. Choose $\varphi_{\sigma',\zeta}$ to have support in $\overline{P_{2n,n}} \cdot \Omega$, where Ω is a small neighbourhood of I_{2n}, and such that $\varphi_{\sigma',\zeta}$, ϕ and $\tilde{f}_{\sigma,s}$ are right Ω-invariant. Let

$$\varphi_{\sigma',\zeta}\Big(\begin{pmatrix} m & \\ * & m^* \end{pmatrix}; I_n\Big) = |\det m|^{-(n+1)(\zeta+\frac{1}{2})} W_{\sigma'}(m).$$

Then, using the form (6.2) of J_1, we get (for this substitution)

$$\begin{aligned}(6.51)\quad J_1(W,\phi,\tilde{f}_{\sigma,s}) = c(\Omega)\int & W_{\sigma'}(m)\omega_\psi\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\phi(\xi_0 m + x)\\ & \psi(z)\tilde{f}_{\sigma,s}\left(\begin{pmatrix} I_n & & & \\ & I_{k-n} & & \\ A & B & I_{k-n} & \\ u & A' & & I_n\end{pmatrix}\gamma; \begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}\right)\\ & |\det m|^{(k+1)s-(n+1)\zeta+\frac{1}{2}n-k}\, dm\, d(u,A,B).\end{aligned}$$

Here m is integrated over $N_n \setminus \mathcal{G}L_n$, $c(\Omega) \equiv c$ is the measure of Ω. Recall that $A_{k-n} = x$ and $B_{k-n,1} = z$. Next choose $\tilde{f}_{\sigma,s}$ so that its right σ-translate has support in $P_{2k,k}\cdot\Omega_1$, and is also right Ω_1-invariant, where Ω_1 is a small neighbourhood of I_{2k}. Denote $\tilde{f}_{\sigma,s}(\gamma; r) = W_\sigma(r)$. Take Ω_1 so small that

$$\begin{pmatrix} I_n & & & \\ & I_{k-n} & & \\ A & B & I_{k-n} & \\ u & A' & & I_n\end{pmatrix} \in \Omega_1 \implies \begin{cases} \omega_\psi\begin{pmatrix} I_n & \\ u & I_n\end{pmatrix}\phi = \phi, \\ \phi(e+x) = \phi,\ \psi(z) = 1 \\ (x = A_{k-n},\ z = B_{k-n,1}).\end{cases}$$

We get from (6.51) (and this substitution) that

$$\begin{aligned}(6.52)\quad J_1(W,\phi,\tilde{f}_{\sigma,s}) = c_1 c\,\omega_{\sigma'}(-1)\int_{N_n\setminus \mathrm{GL}_n} & W_{\sigma'}(m)\sigma(\widehat{\phi})\\ & W'_\sigma\begin{pmatrix} m & \\ & I_{k-n}\end{pmatrix}|\det m|^{(k+1)s-(n+1)\zeta+\frac{1}{2}n-k}\, dm.\end{aligned}$$

Here $W'_\sigma = \sigma\begin{pmatrix} -I_n & \\ & I_{k-n}\end{pmatrix} W_\sigma$ and $c_1 = c(\Omega_1)$ is the measure of Ω_1. Apply now the same substitutions to the integral (6.38). We get

$$(6.53)\qquad c\int W_{\sigma'}(m)\omega_\psi\widehat{\Big((x,0,z)\begin{pmatrix} I_n & \\ & uI_n\end{pmatrix}\Big)}\phi(e)$$

$$| \det m|^{(k+1)s-(n+1)\zeta-k+\frac{1}{2}n}$$

$$\tilde{f}_{\sigma,s}\left(\begin{pmatrix} I_n & & & \\ & I_{k-n} & & \\ A & B & I_{k-n} & \\ u & A' & & I_n \end{pmatrix}\gamma; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ m & 0 & y \end{pmatrix}\right.$$
$$\left.\begin{pmatrix} I_n & {}^t e & \\ & 1 & \\ & & I_{k-n-1} \end{pmatrix}\begin{pmatrix} -I_n & & \\ & 1 & \\ & & I_{k-n-1} \end{pmatrix}\right) \mathrm{d}(\ldots)$$

$$= cc_1 \int_{N_n\backslash \mathrm{GL}_n} W_{\sigma'}(m)\sigma(\widehat{\phi})W'_\sigma\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-n-1} \\ m & 0 & y \end{pmatrix}$$
$$| \det m|^{(k+1)s-(n+1)\zeta+\frac{1}{2}n-k}\, \mathrm{d}m.$$

The integrals in (6.52) and (6.53) are related by the local functional equation of [JPSS] by the stated gamma factor. ∎

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