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Smoothness and irreducibility of varieties of plane curves with nodes and cusps


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SMOOTHNESS AND IRREDUCIBILITY OF VARIETIES
OF PLANE CURVES WITH NODES AND CUSPS

BY

EUGENII SHUSTIN (*)

0. Introduction

In the present article we deal with plane projective algebraic curves
over an algebraically closed field of characteristic 0.

It is well-known that the variety of irreducible curves of a given degree
with a given number of nodes is non-singular [9], irreducible [2], and
that each germ of this variety is a transversal intersection of germs of
equisingular strata corresponding to all singular points [9] (from now on,
speaking of a variety with the last property, we shall write T-variety, or
variety with property T).

Our goal is a similar result for curves with nodes and ordinary cusps.
Let \( V(d, m, k) \) denote the set of irreducible curves of degree \( d \) with \( m \)

nodes and \( k \) cusps as their only singularities.

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It is known (see [5], [6]) that
\[ \frac{9}{8} m + 2k > \frac{5}{8} d^2. \]

On the other hand (see [13]), \( V(d, m, k) \neq \emptyset \) when
\[ m + 2k \leq \frac{1}{2} d^2 + O(d). \]

Our result is

\textbf{THEOREM 0.2.}

If \( m + 2k \leq \alpha_0 d^2 \), where
\[ \alpha_0 = \frac{7 - \sqrt{13}}{81} \approx 0.0419, \]
then \( V(d, m, k) \) is a non-empty non-singular \( T \)-variety of dimension
\[ \frac{1}{2} d(d + 3) - m - 2k. \]

If \( m + 2k \leq \alpha_1 d^2 \), where
\[ \alpha_1 = \frac{2}{225} \approx 0.0089, \]
then \( V(d, m, k) \) is irreducible.

Let us make some comments.

First, (0.3) implies (0.1), and then \( V(d, m, k) \neq \emptyset \).

Let \( \mathbb{P}^N \), with \( N = \frac{1}{2} d(d + 3) \), be the space of plane curves of degree \( d \).

Let \( z \) be a singular point of \( F \in \mathbb{P}^N \). It is well-known (see [2, [9]]) that:

1. if \( z \) is a node then the germ at \( F \) of the variety of curves \( \Phi \in \mathbb{P}^N \), having a node in some neighbourhood of \( z \), is smooth, has codimension 1, and its tangent space is open in \( \{ \Phi \in \mathbb{P}^N \mid z \in \Phi \} \);

2. if \( z \) is a cusp then the germ at \( F \) of the variety of curves \( \Phi \in \mathbb{P}^N \), having a cusp in some neighbourhood of \( z \), is smooth, has codimension 2, and its tangent space is open in \( \{ \Phi \in \mathbb{P}^N \mid (\Phi \cdot F)(z) \geq 3 \} \) (here and further on the notation \( (F \cdot G)(z) \) means the intersection number of the curves \( F, G \) at the point \( z \)).

Hence the property \( T \) implies the smoothness of \( V(d, m, k) \) and the expected value of its dimension given in \textbf{THEOREM 0.2}. Further, it is well-known [15] that \( V(d, m, k) \) is a non-singular \( T \)-variety, when
\[ k < 3d. \]
Generalizations of this fact to arbitrary singularities, given in [1], [12] are based — in fact — on the same idea. The following conditions are sufficient for the smoothness of \( V(d, m, k) \) and the property \( T \):

\[
m = 0, \quad 2k < \frac{(7 - \sqrt{13})d^2}{81} \approx 0.0418 d^2 \quad \text{(see [10], [11])},
\]

and for the irreducibility of \( V(d, m, k) \):

\[
m + 2k < \frac{3}{2} d \quad \text{(see [10], [11])},
\]

\[
k \leq 3 \quad \text{(see [7])},
\]

\[
\frac{1}{2} (d^2 - 4d + 1) \leq m \leq \frac{1}{2} (d^2 - 3d + 2) \quad \text{(see [8])}.
\]

The main idea of our proof is as follows. We have to prove the property \( T \) and the irreducibility for \( V(d, m, k) \). To any curve \( F \in V(d, m, k) \) with nodes \( z_1, \ldots, z_m \) and cusps \( w_1, \ldots, w_k \) we assign two linear systems of curves of degree \( n \):

\[
\Lambda_1(n, F) = \{ \Phi \mid z_1, \ldots, z_m \in \Phi, \ (\Phi \cdot F)(w_i) \geq 3, \ i = 1, \ldots, k \},
\]

\[
\Lambda_2(n, F) = \{ \Phi \mid z_1, \ldots, z_m \in \text{Sing}(\Phi), \ (\Phi \cdot F)(w_i) \geq 6, \ i = 1, \ldots, k \}.
\]

First we show the non-speciality of \( \Lambda_1(d, F) \) for any \( F \in V(d, m, k) \), which means according to the Riemann-Roch theorem that

\[
(0.6) \quad \dim \Lambda_1(d, F) = \frac{1}{2} d(d + 3) - m - 2k.
\]

On the other hand, \( \Lambda_1(d, F) \) is the intersection of the tangent spaces to germs of equisingular strata at \( F \) in the space of curves of degree \( d \), and (0.6) gives us the transversality of this intersection, or the desired property \( T \). Then we show that, for any \( F \) from some open dense subset \( U \subset V(d, m, k) \), the system \( \Lambda_2(d, F) \) is non-special. That implies the irreducibility. Indeed, first we show that an open dense subset of \( \Lambda_2(d, F) \) is contained in \( V(d, m, k) \); more precisely, it consists of curves of degree \( d \) having \( m \) nodes in a fixed position and \( k \) cusps in a fixed position with fixed tangents. Then from the non-speciality we derive that \( \dim \Lambda_2(d, F) = \text{const} \), \( F \in U \), and that conditions imposed by fixed singular points on curves of degree \( d \) are independent. Afterwards we represent \( U \) as an open dense subset of the space of some linear bundle, whose fibres are \( \Lambda_2(d, F) \), \( F \in U \), and whose base is an open dense subset of \( \text{Sym}^m(\mathbb{P}^2) \times \text{Sym}^k(P(T\mathbb{P}^2)) \), where \( P(T\mathbb{P}^2) \) is the projectivization of the tangent bundle of the plane.
The text is divided into five parts: in section 1 there are some preliminary notions and results; in section 2 we present examples of reducible varieties or ones without property $T$; in section 3 we construct irreducible curves in $\Lambda_1(n, F)$, where $n < d$; in section 4 we prove the property $T$; and in section 5 — the irreducibility.

1. Preliminaries

Here we shall recall some notions and well-known classical results [3], [14], and also present some simple technical results needed below. Namely, we introduce a certain class of linear systems of plane curves and show how to compute their dimensions by means of linear series on curves.

Let $\Sigma = \bigoplus_{t \geq 0} \Sigma(t)$ be the graded ring of polynomials in three homogeneous variables over the base field. We think of the space of plane curves of degree $t$ as the projectivization $P(\Sigma(t))$. A linear system of plane curves of degree $t$ is a subspace of $P(\Sigma(t))$. Let

$$I = \bigoplus_{t \geq 0} I(t) \subset \Sigma$$

be a homogeneous ideal, defining a zero-dimensional subscheme $Z \subset \mathbb{P}^2$. This ideal determines a sequence of linear systems $\Lambda(t) = P(I(t))$, $t \geq 1$. Denote this class of linear systems by $C$. In other words, these are linear systems defined by linear conditions associated to finite many base points. It is well-known [3] that

$$\dim \Lambda(t) = \dim P(\Sigma(t)) - \deg Z + i(\Lambda(t)),$$

where $i(\Lambda(t)) \geq 0$ is called the *speciality index* of $\Lambda(t)$. If $i(\Lambda(t)) = 0$ then the linear system $\Lambda(t)$ is called *non-special*. For a given ideal $I$, $\Lambda(t)$ is non-special when $t$ is big enough (see [3]).

**Proposition 1.2.** — Let $\Lambda(t), \Lambda'(t)$ belong to $C$, and, for all $t \geq 1$,

$$\Lambda(t) \subset \Lambda'(t).$$

If, for some $n \geq 1$, the system $\Lambda(n)$ is non-special then $\Lambda'(n)$ is non-special.

**Proof.** — The systems $\Lambda(t), \Lambda'(t)$ are non-special for $t$ big enough. Take a straight line $L \in P(\Sigma(1))$ not intersecting the zero-dimensional schemes $Z, Z'$ associated to our linear systems. Let us embed the space $P(\Sigma(n))$ into $P(\Sigma(t))$, multiplying by $L^{t-n}$. Then:

$$\Lambda(n) = \Lambda(t) \cap P(\Sigma(n)), \quad \Lambda'(n) = \Lambda'(t) \cap P(\Sigma(n)).$$
So, the non-speciality of $A(n)$ means the transversality of the intersection of $A(t)$ and $P(\Sigma(n))$ in $P(\Sigma(t))$. But this implies that $A'(t)$ and $P(\Sigma(n))$ intersect transversally in $P(\Sigma(t))$, hence
\[
\text{codim}(A'(n), P(\Sigma(n))) = \text{codim}(A'(t), P(\Sigma(t))) = \deg Z',
\]
what is equivalent to the desired non-speciality. \[\square\]

From now on, divisor will always mean an effective Cartier divisor on a curve.

Let $F$ be a reduced plane curve. For any divisor $D$ on $F$ and any component $H \subset F$ the symbol $D|_H$ means the restriction of $D$ on $H$. For any curve $G$ the symbol $G_F$ means the formal expression $\sum n(P) \cdot P$, where the sum is taken over all local branches of $F$ and $n(P)$ is the intersection number of $P$ and $G$. If $F, G$ have no common components, $G_F$ is the divisor on $F$ cut out by $G$, otherwise we admit infinite coefficients in the above expression.

By $D(F)$ we denote the double point divisor of the curve $F$. We omit its exact definition (see, for example, [14]), but only list the properties used in the sequel.

**Proposition 1.3 (see [14]).**

1. The divisor $D(F)$ can be expressed as
\[
D(F) = \sum n(P) \cdot P,
\]
where $P$ runs through all the local branches of $F$ centered at singular points, and the coefficients $n(P)$ are positive integers. In particular, $n(P) = 1$ for both branches centered at a node, and $n(P) = 2$ for a branch centered at a cusp.

2. Let $z$ be a singular point of the curve $F$ and a non-singular point of some curve $G$, then
   (i) for any singular local branch $P$ of $F$ centered at $z$,
   \[
   (G \cdot P)(z) \leq n(P) + 1,
   \]
   (ii) for any pair $P_1, P_2$ of local branches of $F$ centered at $z$,
   \[
   (G \cdot P_1) \leq n(P_1), \quad \text{or} \quad (G \cdot P_2) \leq n(P_2).
   \]

3. If $F$ is an irreducible curve of degree $d$ and geometric genus $g(F)$ then:
\[
\deg D(F) = d(d - 3) + 2 - 2g(F).
\]
If a reduced curve \( G \) has no common components with \( F \) then:

\[
D(FG)\mid_F = D(F) + G_F.
\]

For any divisor \( D \) on \( F \), the symbol \( \mathcal{L}_F(n, D) \) denotes the linear system of plane curves of degree \( n \)

\[
\{ \Phi \mid \Phi_F \geq D + D(F) \}.
\]

It is clear from the definition and Proposition 1.3 that \( \Lambda_1(n, F), \Lambda_2(n, F) \) belong to this class. Also these systems belong to \( \mathcal{C} \).

**Theorem 1.4** (Brill-Noether (see [14])). — If \( F \) is irreducible then curves from \( \mathcal{L}_F(n, D) \) cut out on \( F \) the linear series \( |nL_F - D - D(F)| \), where \( L \) is a general straight line.

**Theorem 1.5** (Noether (see [14])). — Let \( F_1, \ldots, F_k \) be different irreducible curves of degrees \( n_1, \ldots, n_k \), and \( F = F_1 \cdots F_k \), \( \deg F = d \). Then:

\[
\mathcal{L}_F(n, D) = \sum_{i=1}^k \mathcal{L}_{F_i}(n + n_i - d, D\mid_{F_i}) \cdot F_1 \cdots F_i-1F_{i+1} \cdots F_k.
\]

**Theorem 1.7** (Riemann-Roch for curves (see [3], [14])). — For any divisor \( D \) on an irreducible curve \( F \) the dimension of the linear series \( |D| \) is

\[
dim |D| = \deg D - g(F) + i(D),
\]

where \( i(D) \) is non-negative. If \( \deg D > 2g(F) - 2 \) then \( i(D) = 0 \).

**Proposition 1.8.** — For any reduced curve \( F \) of degree \( d \leq n \),

\[
\dim \mathcal{L}_F(n, D) \geq \frac{1}{2} n(n + 3) - \frac{1}{2} \deg D(F) - \deg D.
\]

The non-speciality of \( \mathcal{L}_F(n, D) \) is equivalent to the equality in (1.9).

**Proof.** — Assume that \( F \) is irreducible. Representing \( \mathcal{L}_F(n, D) \) as the span of \( |nL_F - D - D(F)| \) and \( F \cdot P(\Sigma(n - d)) \), we obtain

\[
dim \mathcal{L}_F(n, D) = \dim |nL_F - D - D(F)| + \dim \Sigma(n - d),
\]

hence according to Theorem 1.7 and Proposition 1.3,

\[
\dim \mathcal{L}_F(n, D) \geq nd - \deg D - \deg D(F) - g(F)
\]

\[
+ \frac{1}{2} (n - d + 1)(n - d + 2)
\]

\[
= nd - \frac{1}{2} d(d - 3) - 1 - \deg D - \frac{1}{2} \deg D(F)
\]

\[
+ \frac{1}{2} (n - d + 1)(n - d + 2),
\]
which is equivalent to (1.9). Also we obtain that the equality in (1.9) means \( i(D) = 0 \). Therefore, for all \( t \geq d \),

\[(1.10) \quad \text{codim}(\mathcal{L}_F(t, D), P(\Sigma(t))) = \frac{1}{2} \deg D(F) + \deg D.\]

On the other hand, for \( t \) big enough, \( \mathcal{L}_F(t, D) \) is non-special. Comparing this with (1.1) and (1.10), we get that the equality in (1.9) means the non-speciality of \( \mathcal{L}_F(n, D) \).

If \( F \) is reducible, combine the previous computation with (1.6).

**Proposition 1.11.** — Let \( F \in V(d, m, k) \).

(1) If \( G \in \Lambda_1(n, F) \) is reduced, then there is a divisor \( D \) on \( G \) of degree \( \leq m + 2k \) such that, for all \( t \geq 1 \),

\[(1.12) \quad \Lambda_1(t, F) \supset \mathcal{L}_G(t, D).\]

(2) Let \( G \in \Lambda_1(n, F) \) be irreducible, let \( S \) be a subset of \( \text{Sing}(F) \), and let \( H \) be a reduced curve containing \( S \) but not \( G \). Let \( \Lambda_3(t, F, S) \) be a linear system of curves \( \Phi \in \Lambda_1(t, F) \) such that \( S \subset \text{Sing}(\Phi) \), and \( \Phi \) meets \( F \) at each cusp from \( S \) with multiplicity \( \geq 5 \). Then there is a divisor \( D \) on \( GH \) such that

\[
\deg D|_G \leq m + 2k, \quad \deg D|_K \leq \text{card}(S \cap K),
\]

for each component \( K \subset H \), and, for all \( t \geq 1 \),

\[
\Lambda_3(t, F, S) \supset \mathcal{L}_{GH}(t, D).
\]

**Proof.** — We will construct the divisor \( D = \sum n(P) \cdot P \) explicitly.

(1) We have to find a divisor \( D \) on \( G \) such that any curve from \( \mathcal{L}_G(t, D) \) goes through each node and each cusp of \( F \), and intersects a tangent line to \( F \) at any cusp with multiplicity \( \geq 2 \).

Let \( z \) be a node of \( F \). Since \( G \in \Lambda_1(n, F) \), then \( G \) goes through \( z \). If \( G \) is non-singular at \( z \) we can put \( n(P) = 1 \) for the local branch \( P \) of \( G \) centered at \( z \). If \( G \) is singular at \( z \) then we can put \( n(P) = 0 \) for all local branches of \( G \) centered at \( z \), because in this case, according to **Proposition 1.3**, curves from \( \mathcal{L}_G(t, 0) \) go through \( z \).

Let \( z \) be a cusp of \( F \). Analogously, \( G \) goes through \( z \). If \( G \) is non-singular at \( z \), then the local branch \( P \) of \( G \) at \( z \) is tangent to the tangent line \( L \) to the curve \( F \) at \( z \). Put \( n(P) = 2 \). Now, since any curve from \( \mathcal{L}_G(p, D) \) intersects \( P \) with multiplicity \( \geq 2 \), the same holds for \( L \). If \( G \) is singular
at $z$, then either there is a singular local branch $P$ of $G$ centered at $z$, or there are at least two local branches $P_1, P_2$ of $G$ centered at $z$. In the first case we put $n(P) = 2$, in the second case we put $n(P_1) = n(P_2) = 1$. According to Proposition 1.3 any curve from $\mathcal{L}_G(t, D)$ is singular at $z$, and thereby intersects $L$ with multiplicity $\geq 2$.

(2) We can obtain the second statement easily by combining the previous arguments with the Neother theorem. []

2. Non-transversality and reducibility

The upper bounds in the sufficient conditions (0.3), (0.4) are the best possible as far as the exponent of $d$ is concerned. The slightly modified classical examples [15] presented below give an upper bound for the allowable coefficient of $d^2$ in (0.3), (0.4).

Theorem 2.1. — The set $V(6p, 0, 6p^2)$ is reducible if $p = 1, 2$, and has components with different dimensions if $p \geq 3$.

Proof. — The case $p = 1$ is well-known [15]. Let $p \geq 2$. It is easy to see that the curves

$$H = F_{2p}^2 + G_{3p}^2$$

belong to $V(6p, 0, 6p^2)$, where $F_{2p}, G_{3p}$ are general curves of degrees $2p, 3p$ respectively. A simple computation gives us:

$$\dim \{ H \in V(6p, 0, 6p^2) \mid H = F_{2p}^2 + G_{3p}^2 \}$$

$$= \frac{1}{2} 6p(6p + 3) - 12p^2 + \frac{1}{2} (p - 1)(p - 2).$$

According to [13], for $p \geq 2$, there is a component of $V(6p, 0, 6p^2)$ with dimension:

$$\frac{1}{2} 6p(6p + 3) - 12p^2.$$

If $p \geq 3$ we obtain at least two components of $V(6p, 0, 6p^2)$ with different dimensions.

Let $p = 2$. According to (0.5), $V(12, 0, 24)$ is a T-variety, and hence has dimension 42. According to (2.2) curves $H = F_4^2 + G_6^2$ form a component $\tilde{V}$ of $V(12, 0, 24)$. Assume that $\tilde{V} = V(12, 0, 24)$.

Let $J$ be an irreducible curve of degree 12 with 28 cusps constructed in [13]. Since $V(12, 0, 28)$ is a T-variety (see (0.5)), we can smooth out any four cusps of $J$, preserving the others, by means of a variation of $J$ in the space $P(\Sigma(12))$. Indeed, since all 28 equisingular strata intersect transversally at $J$, we can leave four of them by moving $J$ along the intersection of the others. So we obtain that $J$ belongs to the closure of $\tilde{V}$,
and hence to any set $s_{24}$ of 24 cusps of $J$ there correspond a quartic $F_4$ and a sextic $G_6$, passing through $s_{24}$. Distinct 24-tuples of cusps correspond to distinct quartics, because, according to Bézout’s theorem, a quartic cannot contain more than 24 cusps of $J$. On the other hand, two 24-tuples $s_{24}, s'_{24}$ with 23 common cusps give quartics $F_4, F'_4$ with 23 common points. Therefore $F_4, F'_4$ have a common component $C_i$ of degree $i = 1, 2$, or 3. If $i = 3$ then $F_4 = C_3C_1, F'_4 = C_3C'_1$. Since $C_3$ passes through at most 18 cusps of $J$, then the straight lines $C_1, C'_1$ have at least 5 common points, that means they coincide. The cases $i = 1$ or 2 lead analogously to contradictions, which prove that $V(12, 0, 24)$ is reducible.

**Theorem 2.3.** — The set $V(7p - 3, 0, 6p^2)$ contains a component without property $T$ when $p \geq 3$.

**Proof.** — Obviously, the curve $H = A_{p-3}F_{2p}^3 + B_{p-3}G_{3p}^2$ belongs to $V(7p - 3, 0, 6p^2)$, if $A_{p-3}, F_{2p}, B_{p-3}, G_{3p}$ are general curves of degrees $p - 3, 2p, p - 3, 3p$ respectively. The property $T$ is equivalent to the non-speciality of $\Lambda_1(7p - 3, H)$. From Theorem 1.5 it is not difficult to deduce that

$$\Lambda_1(7p - 3, H) = \{ \Phi \mid \Phi = R_{3p-3}E_{2p}^3 + S_{4p-3}G_{3p}^2 \}$$

with arbitrary curves $R_{3p-3}, S_{4p-3}$ of degrees $3p - 3, 4p - 3$. Further, a trivial computation gives

$$\dim \Lambda_1(7p - 3, H) = \frac{1}{2} (7p - 3) \cdot 7p - 12p^2 + 1,$$

that means $\Lambda_1(7p - 3, H)$ is special.

**Corollary 2.4.** — The allowable coefficient at $d^2$ in the right hand side of (0.3) cannot exceed $\frac{12}{49}$, and in the right hand side of (0.4) cannot exceed $\frac{1}{3}$.

### 3. Main lemma

**Lemma 3.1.** — For any curve $F \in V(d, m, k)$ and real $\alpha \geq (m + 2k)/d^2$, there is an irreducible curve $\Phi \in \Lambda_1(n, F)$, where $n = \left[ (\sqrt{2}\alpha + 2/3)d \right]$.

**Proof.** — Let $z_1, \ldots, z_m$ be the nodes of $F$, and let $w_1, \ldots, w_k$ be the cusps of $F$. Let $h$ be the minimal integer such that $\Lambda_1(h, F) \neq \emptyset$. Then, $\Lambda_1(h - 1, F) = \emptyset$ implies

$$m + 2k > \frac{1}{2} (h - 1)(h + 2),$$

and hence

$$h < \sqrt{2(m + 2k)} \leq \sqrt{2\alpha d}.$$
Take a general curve $H \in \Lambda_1(h, F)$. Assume that $H = H^i_1 \cdots H^i_r$, where $H_1, \ldots, H_r$ are irreducible components of degrees $h_1, \ldots, h_r$ respectively. Since $h$ is minimal, 
\[
\max\{i_1, \ldots, i_r\} \leq 2.
\]

We shall construct the curve $\Phi$ as follows. First we will construct, for each $s = 1, \ldots, r$, a curve $C_s$ of degree $\ell_s = i_sh_s + \frac{2}{3}d$ such that $C_s$ does not contain $H_s$ and the curve
\[
R_s \overset{\text{def}}{=} H^i_1 \cdots H^i_{s-1} C_s H^i_{s+1} \cdots H^i_r
\]
belongs to $\Lambda_1(h + \ell_s - i_sh_s, F)$. After that we obtain the desired curve $\Phi$ in the form
\[
G_0H + G_1R_1 + \cdots + G_rR_r,
\]
where $G_0, G_1, \ldots, G_r$ are generic curves of suitable degrees.

The rest of the proof is divided into five steps: in steps 1, 2, 3 and 4 we construct the curves $C_1, \ldots, C_r$, in the fifth step we construct the curve $\Phi$.

Let us do the construction of $C_1$. Let $H_1$ pass through $z_1, \ldots, z_p$, $w_1, \ldots, w_q$ and meet $F$ at $w_{q+1}, \ldots, w_{q+t}$ with multiplicities $\geq 3$. Let $\deg H_1 = h_1$. The Bézout theorem gives:
\[
2p + 2q + 3t \leq h_1d.
\]

**Step 1.** — Assume $i_1 = 1$. Let us find a curve $C_1$ passing through $z_1, \ldots, z_p, w_1, \ldots, w_q$, meeting $F$ at $w_{q+1}, \ldots, w_{q+t}$ with multiplicities $\geq 3$, and not containing $H_1$. This can be done under the following sufficient condition on $\ell_1 = \deg C_1$
\[
\frac{1}{2} \ell_1(\ell_1 + 3) - \frac{1}{2}(\ell_1 - h_1)(\ell_1 - h_1 + 3) > p + q + 2t,
\]
which is equivalent to
\[
\ell_1 > \frac{1}{2} h_1 - \frac{3}{2} + \frac{p + q + 2t}{h_1},
\]
and, using (3.4), we can take
\[
\ell_1 \leq \frac{1}{2} h_1 + \frac{p + q + 2t}{h_1} \leq \frac{1}{2} h_1 + \frac{p + q + 2t}{2p + 2q + 3t}d \leq \frac{1}{2} h_1 + \frac{2}{3}d.
\]

**Step 2.** — From now on assume $i_1 = 2$. First we look for a curve $C'_1$ of degree $\ell'_1$ passing through $z_1, \ldots, z_p$, meeting $F$ at $w_{q+1}, \ldots, w_{q+t}$ with
multiplicities $\geq 3$ and not containing $H_1$. As in the first step we have the following sufficient condition for the existence of such a curve
\[ \ell_1' > \frac{1}{2}h_1 - \frac{3}{2} + \frac{p + 2t}{h_1}, \]
and then we can take
\[ (3.6) \quad \ell_1' \leq \frac{1}{2}h_1 + \frac{p + 2t}{h_1}. \]

**Step 3.** — Assume that there is a curve $H_1' \neq H_1$ of degree $h_1' \leq h_1$, passing through $w_1, \ldots, w_q$, and that $h_1'$ is the minimal such degree. This minimality implies:
\[ (3.7) \quad h_1'^2 \leq 2q. \]
Then we put $C_1 = C_1'(H_1')^2$. According to (3.6)
\[ \ell_1 = \deg C_1 \leq \frac{1}{2}h_1 + \frac{p + 2t}{h_1} + 2h_1'. \]
Hence, using (3.4), (3.7) and the initial assumption $h_1' \leq h_1$, it is easy to compute
\[ (3.8) \quad \ell_1 \leq \frac{1}{2}h_1 + \frac{p + q + 2t}{2p + 2q + 3t} - \frac{q}{h_1} + 2h_1' \]
\[ \leq \frac{1}{2}h_1 + \frac{2}{3}d - \frac{q}{h_1} + 2h_1' \leq \frac{2}{3}d + 2h_1. \]

**Step 4.** — Now assume that $H_1$ is the unique curve of degree $\leq h_1$, passing through $w_1, \ldots, w_q$. In particular, that means
\[ (3.9) \quad \frac{1}{2}h_1(h_1 + 3) \leq q. \]
Let $C_1'$ be the curve of degree $\ell_1'$ constructed in step 2. Consider two situations.

- Assume first $q \leq h_1^2$. Then $2q < 2h_1(2h_1 + 3)/2$. That means the set $M$ of curves of degree $2h_1$, meeting $F$ at $w_1, \ldots, w_q$ with multiplicities $\geq 3$, is infinite. The only curve in $M$ containing $H_1$ is $H_1^2$. Now take $H_1' \in M$ different from $H_1^2$, and put $C_1 = C_1'H_1'$. Here:
\[ \ell_1 = \deg C_1 \leq \frac{1}{2}h_1 + \frac{p + 2t}{h_1} + 2h_1. \]
Finally, from (3.4), (3.9) we get:

(3.10) \[ \ell_1 \leq \frac{2}{3} d + 2h_1. \]

- Now assume:

(3.11) \[ q \geq h_1^2 + 1. \]

Introduce the integer

\[ h'_1 = \max \{ \nu \in \mathbb{N} | h_1 (\nu - h_1) + 1 \leq q \}. \]

According to (3.11), \( h'_1 \geq 2h_1 \). Denote by \( M \) the set of curves of degree \( h'_1 \) meeting \( F \) with multiplicity \( \geq 3 \) at each point \( w_1, \ldots, w_\pi \), where \( \pi = h_1 (h'_1 - h_1) + 1 \). If \( G \in M \) contains \( H_1 \) then \( G = G_1 H_1 \), and \( G_1 \) goes through \( w_1, \ldots, w_\pi \), because

\[ (G_1 \cdot F)(w_i) = (G \cdot F)(w_i) - (H_1 \cdot F)(w_i) \geq 3 - 2 = 1, \quad i = 1, \ldots, \pi. \]

Hence \( G_1 \) contains \( H_1 \), because these curves meet at \( \pi > \deg H_1 \cdot \deg G_1 \) points. Then there is a curve \( H'_1 \in M \) not containing \( H_1 \) as component, because the sufficient condition for this existence is

\[ \frac{1}{2} h'_1 (h'_1 + 3) - 2 (h_1 (h'_1 - h_1) + 1) > \frac{1}{2} (h'_1 - 2h_1) (h'_1 - 2h_1 + 3), \]

which is equivalent to

\[ 3h_1 > 2. \]

Finally, let \( H''_1 \) be a curve of the smallest degree \( h''_1 \) meeting \( F \) at \( w_i, \)
\[ i = h_1(h'_1 - h_1) + 2, \ldots, q, \] with multiplicities \( \geq 3 \). By definition of \( h'_1 \), the number of these points is \( \leq h_1 - 1 \). If \( h_1 \leq 3 \) then, obviously, \( h''_1 \leq h_1 - 1 \). If \( h_1 \geq 4 \), since

\[ \frac{1}{2} (h''_1 - 1) (h''_1 + 2) \leq 2h_1 - 2, \]

we have:

(3.12) \[ h''_1 \leq \frac{5}{6} h_1. \]

The last inequality is true in the case \( h_1 \leq 3 \) too. In particular, \( H''_1 \) does not contain \( H_1 \). Now put \( C_1 = C''_1 H'_1 H''_1 \). Here we have from (3.4), (3.6), and (3.12)

\[ \ell_1 = \deg C_1 \leq \frac{1}{2} h_1 + \frac{p + 2t}{h_1} + h'_1 + h''_1 \]

(3.13) \[ = \frac{1}{2} h_1 + \frac{p + \frac{4}{3} q + 2t}{h_1} + h'_1 + h''_1 - \frac{4q}{3h_1} \]

\[ \leq \frac{1}{2} h_1 + \frac{2}{3} d + h'_1 + h''_1 - \frac{4}{3} (h'_1 - h_1) \]

\[ \leq \frac{5}{6} h_1 + \frac{2}{3} d - \frac{1}{3} h'_1 \leq 2h_1 + \frac{2}{3} d, \]

because \( h'_1 \geq 2h_1 \) as it was mentioned above.
Step 5. — Inequalities (3.2), (3.5), (3.8), (3.10), (3.13) and the definitions of $n$ and $\alpha$ imply that the degree of any curve $R_j$, $j = 1, \ldots, r$, defined by (3.3), is less than $n$. Now consider the linear system

$$
\lambda_0 G_0 H + \lambda_1 G_1 R_1 + \cdots + \lambda_r G_r R_r, \quad (\lambda_0, \ldots, \lambda_r) \in \mathbb{P}^r,
$$

where $G_0, \ldots, G_r$ are generic curves of positive degrees $n - h$, $n - \deg R_1$, $\ldots$, $n - \deg R_r$ respectively. According to the construction of $H, C_1, \ldots, C_r$, this is a subsystem of $\Lambda_1(n, F)$. Also note that the curves $G_0, \ldots, G_r$ do not go through base points of our linear system. Then we obtain immediately from the Bertini theorem (see [3], [14]) that a generic member in the linear system (3.14) is reduced and irreducible. Thereby we can take this generic member as the desired curve $\Phi$.

4. The property T

Let $F \in V(d, m, k)$. As said above, the property T and the smoothness of $V(d, m, k)$ at $F$ follow from:

**Proposition 4.1.** — Under condition (0.3) the linear system $\Lambda_1(d, F)$ is non-special.

**Proof.** — According to Lemma 3.1, under condition (0.3) there is an irreducible curve $\Phi \in \Lambda_1(n, F)$, $n = \lfloor \sqrt{2\alpha_0 + \frac{2}{3}d} \rfloor$. According to Proposition 1.11 there is a divisor $D$ on $\Phi$ of degree

$$
\deg D \leq m + 2k
$$

such that

$$
\Lambda_1(p, F) \supset \mathcal{L}_\Phi(p, D), \quad p \geq 1.
$$

Therefore, according to Proposition 1.2, it is enough to establish the non-speciality of $\mathcal{L}_\Phi(d, D)$, which will follow from

$$
\deg(G_{\Phi} - D(\Phi) - D) > 2g(\Phi) - 2
$$

where $G \in \mathcal{L}_\Phi(d, D)$, and $g(\Phi)$ is the geometric genus of $\Phi$. Indeed, we have by Proposition 1.3 and (4.2):

$$
\deg(G_{\Phi} - D(\Phi) - D) = nd - n(n - 3) - 2 + 2g(\Phi) - \deg D
$$

$$
\geq n(d - n + 3) - (m + 2k) + 2g(\Phi) - 2
$$

$$
> n(d - n + 3) - \alpha_0 d^2 + 2g(\Phi) - 2.
$$

Since $n = \lfloor (\sqrt{2\alpha_0 + \frac{2}{3}d}) \rfloor$ and $\alpha_0$ is the positive root of the equation

$$
(\sqrt{2\alpha} + \frac{2}{3})(\frac{1}{3} - \sqrt{2\alpha}) = \alpha,
$$
hence
\[ n(d - n + 3) \geq \alpha_0 d^2. \]
Then (4.4) implies (4.3) and completes the proof.

5. Irreducibility

We prove the irreducibility along the plan mentioned in introduction.

**Proposition 5.1.** — For any \( F \in V(d, m, k) \), the intersection of \( V(d, m, k) \) with \( \Lambda_2(d, F) \) contains an open dense subset of \( \Lambda_2(d, F) \), and consists exactly of curves from \( V(d, m, k) \) with the same nodes, and the same cusps with the same tangents as \( F \).

**Proof.** — Since \( F \in \Lambda_2(d, F) \) and any curve \( G \in \Lambda_2(d, F) \) is singular at \( z_1, \ldots, z_m \), then the Bertini theorem implies that almost all curves in \( \Lambda_2(d, F) \) have nodes at \( z_1, \ldots, z_m \), and are non-singular outside \( \text{Sing}(F) \). Consider the cusp \( w_1 \in F \). In some affine neighbourhood of \( w_1 \) we fix an affine coordinate system \((x, y)\) such that \( w_1 = (0; 0) \), and \( y = 0 \) is a tangent to \( F \) at \( w_1 \). Then in this neighbourhood, \( F \) is defined by polynomial

\[ Ay^2 + Bx^3 + \sum_{2i + 3j > 6} A_{ij} x^i y^j, \quad AB \neq 0, \]

and can be locally parametrized analytically (see [3], [14]) by

\[ x = \tau^2, \quad y = \lambda \tau^3 + O(\tau^4). \]

To determine \((G \cdot F)(w_1)\) we plug (5.3) into the affine equation

\[ \sum a_{ij} x^i y^j = 0 \]

of \( G \), and then compute the order of vanishing at \( \tau = 0 \) (see [14]). Thus we obtain for curves \( G \in \Lambda_2(d, F) \) that:

\[ a_{00} = a_{01} = a_{10} = a_{11} = a_{20} = 0. \]

Since \( F \in \Lambda_2(d, F) \), almost all curves in \( \Lambda_2(d, F) \) have affine equations like (5.2), that means they have a cusp at \( w_1 \) with the tangent \( y = 0 \). Now to complete the proof we should note that any curve \( G \in V(d, m, k) \) with nodes \( z_1, \ldots, z_m \), cusps \( w_1, \ldots, w_k \) and tangents \( T_{w_1} F, \ldots, T_{w_k} F \), belongs to \( \Lambda_2(d, F) \).
PROPOSITION 5.4. — Under condition (0.4), $V(d, m, k)$ contains an open dense subset $\tilde{V}$ consisting of curves $F$ with non-special linear system $A^d(F)$.

REMARK 5.5. — Even under condition (0.4), $V(d, m, k)$ may contain curves $G$ with special system $A^d(G)$. This holds for $V(2p, 3p, 0)$, $p \geq 3$ (see [12]).

Proof of Proposition 5.4. — It is enough to prove the statement for any irreducible component $V_0$ of $V(d, m, k)$. For any curve $G \in V_0$, let $\mathcal{H}(G)$ denote the linear system of curves of the smallest degree $h$, passing through $\text{Sing}(G)$. Evidently, any $H \in \mathcal{H}(G)$ is reduced, and

$$h = \deg H \leq \sqrt{2(m+k)} < \sqrt{2a_1 d}.$$  

Denote by $V_1$ the set of curves $G \in V_0$ with maximal $h$. This is an open dense irreducible subset of $V_0$ (see [3]). Now denote by $V_2$ the set of curves $G \in V_1$ with minimal $\dim \mathcal{H}(G)$. Similarly this is an open dense irreducible subset of $V_1$. Then

$$W = \bigcup_{G \in V_2} \mathcal{H}(G)$$

is irreducible as the image in $P(\Sigma(h))$ of the space of a projective bundle with base $V_2$ and fibres $\mathcal{H}(G)$, $G \in V_2$. Denote by $W_0$ the set of curves $H \in W$ with minimal number of irreducible components. It is irreducible, open and dense in $W$. In particular, that means all the curves $H \in W_0$ determine the same sequence of degrees of their components (up to permutation). Moreover, if $H$ runs through $W_0$, then any of its irreducible components $K$ runs through some irreducible set

$$W(K) \subset P(\Sigma(\deg K)).$$

For $H \in W_0 \cap \mathcal{H}(G)$ define $N(H)$ to be $\sum N(K, H)$, where $K$ runs through all components of $H$, and $N(K, H) = \text{card}(K \cap \text{Sing}(G))$. Denote by $W_1$ the set of curves $H \in W_0$ with minimal $N(H)$. First, it is an open dense irreducible subset of $W_0$, and, second, for any component $K$ of $H \in W_1$,

$$N(K, H) = \min_{K' \in W(K)} N(K', H').$$

At last, introduce

$$V_3 = \{G \in V_2 \mid \mathcal{H}(G) \cap W_1 \neq \emptyset\}.$$  

According to the above construction, this is an open dense subset of $V_0$. Now we will show that $V_3$ contains an open subset consisting of curves satisfying the conditions of PROPOSITION 5.4, in three steps.
Step 1. — Fix $G \in V_3$ and $H \in \mathcal{H}(G) \cap W_1$. Show that any component $K$ of $H$ of degree $\delta$ contains at most $\frac{1}{2} \delta(\delta + 3)$ points from $\text{Sing}(G)$.

Indeed, let $K$ contain $\frac{1}{2} \delta(\delta + 3) + 1$ points from $\text{Sing}(G)$. Denote the set of these points by $S$, and consider the linear system $\Lambda_3(d, G, S)$. Let us take an irreducible curve $\Phi \in \Lambda_1(n, G)$, $n = \lfloor (\frac{2}{3} + \sqrt{2\alpha_1})d \rfloor$, from Lemma 3.1. According to Proposition 1.11, for all $t \geq 1$,

$$\Lambda_3(t, G, S) \supset \mathcal{L}_{\Phi,K}(t, D),$$

where

$$\deg D|_{\Phi} \leq m + 2k, \quad \deg D|_{K} \leq \frac{1}{2} \delta(\delta + 3) + 1.$$  

We shall show that $\mathcal{L}_{\Phi,K}(d, D)$ is non-special. According to Proposition 1.8 and arguments from its proof, this is equivalent to

$$i((d - \delta)L - D\Phi - D|_{\Phi}) = i((d - n)L_K - D(K) - D|_{K}) = 0.$$  

According to Theorem 1.7 these equalities follow from :

$$\begin{align*}
(5.8) \quad & (d - \delta)n - \deg D\Phi - \deg D|_{\Phi} > 2g(\Phi) - 2, \\
(5.9) \quad & \delta(d - n) - \deg D(K) - \deg D|_{K} > 2g(K) - 2.
\end{align*}$$

According to Proposition 1.3 and (5.7), inequality (5.8) follows from :

$$m + 2k < n(d - \delta - n + 3).$$

This inequality can be easily deduced from the definition of $n$ and (5.6), because $\alpha_1 = \frac{2}{225}$ satisfies the inequality :

$$\alpha_1 < (\frac{2}{3} + \sqrt{2\alpha_1})(\frac{1}{3} - 2\sqrt{2\alpha_1}).$$

Analogously, by Proposition 1.3 and (5.7) the inequality (5.9) follows from :

$$d - n \geq \frac{3}{2} \delta.$$  

This can be easily deduced from the definition of $n$ and (5.6), because $\alpha_1$ is the root of the equation

$$1 - (\frac{2}{3} + \sqrt{2\alpha}) = \frac{3}{2} \sqrt{2\alpha}.$$  

So, according to Proposition 1.2, $\Lambda_3(d, G, S)$ is non-special, and according to Propositions 1.3 and 1.8

$$\dim \Lambda_3(d, G, S) = \frac{1}{2} d(d + 3) - m - 2k - 2 \cdot \text{card} S.$$
If \( G \) runs through \( V_3 \), then \( S \) runs through some subset of 
\[ \text{Sym}^{6(\delta + 3)/2 + 1}(\mathbb{P}^2), \]
thereby defining a morphism 
\[ \nu : \tilde{V}_3 \longrightarrow \text{Sym}^{6(\delta + 3)/2 + 1}(\mathbb{P}^2), \]
where \( \tilde{V}_3 \) is the finite covering of \( V_3 \) corresponding to different choices of \( \frac{1}{2} \delta(\delta + 3) + 1 \) points from the set \( \text{Sing}(G) \cap K \) (which might contain more than \( \frac{1}{2} \delta(\delta + 3) + 1 \) points). The tangent space to the fibre \( \nu^{-1}(S) \) at the point \((G, S) \in \tilde{V}_3 \) is contained in \( \Lambda_3(d, G, S) \) (see [2], [12]). Therefore (5.10) implies:
\[ \dim \nu^{-1}(S) = \dim V_3 - 2 \left( \frac{1}{2} \delta(\delta + 3) + 1 \right), \]
hence \( \nu(\tilde{V}_3) \) is dense in \( \text{Sym}^{6(\delta + 3)/2 + 1}(\mathbb{P}^2) \). Therefore there is a curve \( G \in V_3 \) such that, for any set \( S \) of \( \frac{1}{2} \delta(\delta + 3) + 1 \) points in \( \text{Sing}(G) \cap K \), no more than \( \frac{1}{2} \delta(\delta + 3) \) points of \( S \) lie on a curve of degree \( \delta \). But this contradicts the definition of the set \( V_3 \) and the initial assumption that \( K \cap \text{Sing}(G) \) contains more than \( \frac{1}{2} \delta(\delta + 3) \) points, and thus completes the proof.

**Step 2.** Consider the linear system \( \Lambda_3(d, G, \text{Sing}(G)) \). As in the previous step, for any curve \( H \in \mathcal{H}(G) \cap W_1 \) and all \( t \geq 1 \) we have 
\[ \Lambda_3(t, G, \text{Sing}(G)) \supset \mathcal{L}_{\Phi H}(t, D), \]
where \( D \) satisfies (5.7) for any component \( K \subset H \), hence \( \Lambda_3(d, G, \text{Sing}(G)) \) is non-special.

**Step 3.** As in the first step, the non-speciality of \( \Lambda_3(d, G, \text{Sing}(G)) \), \( G \in V_3 \), implies that the image of \( V_3 \) by the morphism 
\[ \mu : V_3 \rightarrow \text{Sym}^{m+k}(\mathbb{P}^2), \quad \mu(G) = \text{Sing}(G), \]
contains an open dense subset \( U \) of \( \text{Sym}^{m+k}(\mathbb{P}^2) \). According to [4], under condition (0.4), for any \( Z \) from some open subset \( U' \subset U \) the linear system \( \Lambda(d, Z) \) of curves of degree \( d \), having multiplicity \( \geq 3 \) at each point \( z \in Z \), is non-special. It is easy to see that for any \( G \in V(d, m, k) \) and \( t \geq 1 \) :
\[ \Lambda_2(t, G) \supset \Lambda(t, \text{Sing}(G)). \]
Therefore \( \Lambda_2(d, F) \) is non-special for any curve \( F \in \mu^{-1}(U') \cap V_3 \).
Now we can finish the proof of the irreducibility of $V(d, m, k)$, showing that $\tilde{V}$ is irreducible. To any curve $F \in \tilde{V}$ we assign the set $\text{Sing}(F)$ and the set of tangents at its cusps. Thereby we obtain a morphism 

$$\pi : \tilde{V} \rightarrow \text{Sym}^m(\mathbb{P}^2) \times \text{Sym}^k(P(T\mathbb{P}^2)),$$

where $P(T\mathbb{P}^2)$ is the projectivization of the tangent bundle of the plane. According to Proposition 5.1 any fibre of $\pi$ is an open subset of some linear system $\Lambda_2(d, F)$, $F \in \tilde{V}$, hence is irreducible. The non-speciality of these linear systems, Propositions 1.2 and 1.8 imply immediately that all the fibres have the same dimension:

$$\frac{1}{2} d(d + 3) - 3m - 5k = \dim \tilde{V} - \dim (\text{Sym}^m(\mathbb{P}^2) \times \text{Sym}^k(P(T\mathbb{P}^2))).$$

Finally, this equality means that $\pi(\tilde{V})$ is dense in 

$$\text{Sym}^m(\mathbb{P}^2) \times \text{Sym}^k(P(T\mathbb{P}^2)),$$

hence is irreducible. This completes the proof.

**BIBLIOGRAPHY**


