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MAXIMAL S -EXPANSIONS ARE BERNOULLI SHIFTS

BY

CORNELIS KRAAIKAMP (*)

RÉSUMÉ. — Nous montrons dans cette note que les systèmes sous-jacent à une classe de développement en fraction continue (les « S -expansions maximales ») sont tous isomorphes, ce qui entraîne que ces systèmes sont de Bernoulli. En particulier, le système associé à la fraction continue optimale, qui est une S -expansion maximale, est de Bernoulli, donc K , ce qui répond à une question de Pierre Liardet [L].

ABSTRACT. — In this paper it is shown that the systems underlying any two maximal S -expansions are isomorphic, and from this it follows that these systems are Bernoulli. This answers a question, recently posed by Pierre Liardet [L], whether the « underlying » ergodic system of the Optimal Continued Fraction (OCF) forms a K -system, since the OCF is a maximal S -expansion.

1. Introduction

Let x be an irrational number between 0 and 1. The expansion of x as a regular continued fraction (RCF) is denoted by

$$(1) \quad x = [0; B_1, B_2, \dots, B_n, \dots],$$

where $B_n \in \mathbb{N}$, $n \geq 1$. Finite truncation in (1) yields the corresponding sequence of regular convergents $(P_n/Q_n)_{n \geq -1}$.

Define the RCF-operator $T : [0, 1) \rightarrow [0, 1)$ by :

$$Tx := \frac{1}{x} - \left[\frac{1}{x} \right], \quad x \neq 0; \quad T0 := 0.$$

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Here $[\cdot]$ denotes the so-called *entier* (or *floor*) function. Furthermore, if we define the function $B : [0, 1) \longrightarrow \mathbb{N} \cup \{\infty\}$ by

$$B(x) := \left[\frac{1}{x} \right], \quad x \neq 0; \quad B(0) := \infty,$$

then the regular continued fraction is the process (in the sense of [OW]) associated with T and B , i.e. $B_{n+1} = B(T^n(x))$. Hence, if x has the expansion (1), then $T(x) = [0; B_2, \dots, B_n, \dots]$. Put

$$T_n := T^n(x), \quad n \geq 0; \quad V_n := \frac{Q_{n-1}}{Q_n}, \quad n \geq 0;$$

then $T_n = [0; B_{n+1}, B_{n+2}, \dots]$ and a simple calculation shows that

$$V_n = [0; B_n, \dots, B_1], \quad n \geq 1; \quad V_0 = 0.$$

Moreover, $(T_n, V_n)_{n \geq 0}$ is a sequence in $\Omega := ([0, 1) \setminus \mathbb{Q}) \times [0, 1]$.

Fundamental in the theory of S -expansions is the following theorem :

THEOREM 1 ([NIT], 1977; [N], 1981). — *Let \mathcal{B} be the collection of Borel-sets of Ω and let μ be the probability measure on (Ω, \mathcal{B}) with density $(\log 2)^{-1}(1 + xy)^{-2}$. Define the operator $\mathcal{T} : \Omega \longrightarrow \Omega$ by :*

$$\mathcal{T}(x, y) := \left(Tx, \frac{1}{[1/x] + y} \right), \quad (x, y) \in \Omega.$$

Then $(\Omega, \mathcal{B}, \mu, \mathcal{T})$ forms an ergodic system.

Notice that for each irrational number x one has :

$$\mathcal{T}^n(x, 0) = (T_n, V_n), \quad n \geq 0.$$

Here and in the following

$$[a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \dots, \varepsilon_n a_n, \dots]$$

is the abbreviation of

$$(2) \quad a_0 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \dots + \frac{\varepsilon_n}{a_n + \dots}}},$$

where $a_0 \in \mathbb{Z}$; $a_i \in \mathbb{N}$, $\varepsilon_i \in \{\pm 1\}$, for $i \geq 1$. We call $[a_0; \varepsilon_1 a_1, \dots]$ a *semi-regular continued fraction* (SRCF) in case $\varepsilon_i + a_i \geq 1$, $\varepsilon_{i+1} + a_i \geq 1$, for $i \geq 1$ and infinitely often $\varepsilon_{i+1} + a_i \geq 2$. Finite truncation in (2) yields the sequence of convergents $(r_n/s_n)_{n \geq -1}$, which converges to a unique irrational number x in case $[a_0; \varepsilon_1 a_1, \dots]$ is semi-regular.

A SRCF-expansion (2) is a *fastest* expansion of x in case the growth-rate of the denominators s_n is maximal. One can show that this means that these denominators grow asymptotically as fast as the denominators of the *nearest integer continued fraction* (NICF) expansion convergents of x , (see also [B, sect. 3]). *Closest* expansions are those expansions of x for which $\sup\{\theta_k; \theta_k = s_k | s_k x - r_k|, k \geq 0\}$ is minimal. Every irrational number x admits an expansion for which $\theta_k < \frac{1}{2}$ and $k \geq 1$, given by *Minkowski's diagonal continued fraction* (DCF). In general the NICF does not yield closest expansions, while the DCF does not yield fastest expansions. An expansion which is always both fastest and closest for all irrational numbers x is the *Optimal Continued Fraction* (OCF), see [BK1], [BK2].

Now let x be an irrational number, and let (2) be some SRCF-expansion of x . Suppose that we have for a certain $k \geq 0$:

$$a_{k+1} = 1, \quad \varepsilon_{k+1} = \varepsilon_{k+2} = 1.$$

The operation by which the continued fraction (2) is replaced by[†]

$$[a_0; \varepsilon_1 a_1, \dots, \varepsilon_{k-1} a_{k-1}, \varepsilon_k (a_k + 1), -(a_{k+2} + 1), \varepsilon_{k+3} a_{k+3}, \dots],$$

which again is a SRCF-expansion of x , with convergents, say, $(c_n/d_n)_{n \geq -1}$, is called *the singularization of the partial quotient a_{k+1} equal to 1*. One easily shows that $(c_n/d_n)_{n \geq -1}$ is obtained from $(r_n/s_n)_{n \geq -1}$ by skipping the term r_k/s_k . See also [K1, sect. 2 and 4].

2. S -expansions

A simple way to derive a strategy for singularization is given by a *singularization area* S .

DEFINITION 1. — A subset S from Ω is called a *singularization area* if it satisfies :

- (i) $S \in \mathcal{B}$ and $\mu(\partial S) = 0$;
- (ii) $S \subset ([\frac{1}{2}, 1) \setminus \mathbb{Q}) \times [0, 1]$;
- (iii) $\mathcal{T}_S \cap S = \emptyset$.

[†] In case $k = 0$ this comes down to replacing (2) by $[a_0 + 1; -(a_2 + 1), \varepsilon_3 a_3, \varepsilon_4 a_4, \dots]$.

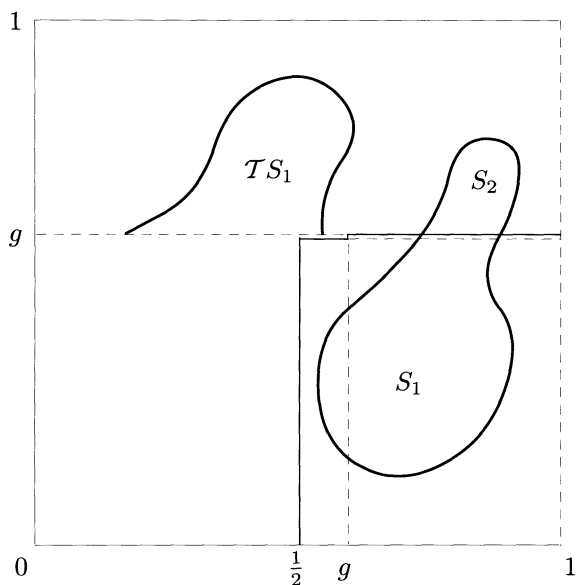


Figure 1.

Here and in the sequel we put :

$$g := \frac{1}{2}(\sqrt{5} - 1), \quad G := g + 1 = g^{-1}.$$

REMARK 1. — Consider the following singularization area S^* , where

$$S^* := \left[\frac{1}{2}, g\right] \times [0, g] \cup (g, 1) \times (0, g),$$

and let S be some singularization area. Put $S_1 := S \cap S^*$, $S_2 := S \setminus S_1$ (see also figure 1). Then by invariance of μ , due to definition 1 (iii) and from the fact that $S^* \cup TS^*$ covers the rectangle containing any singularization area it now follows that :

$$\begin{aligned} \mu(S) &= \mu(TS_1) + \mu(S_2) = \mu(TS_1 \cup S_2) \\ &\leq \mu(TS^*) = \mu(S^*) = 1 - \frac{\log G}{\log 2}. \end{aligned}$$

Thus we see, that

$$0 \leq \mu(S) \leq 1 - \frac{\log G}{\log 2} = 0.3057 \dots,$$

see also [K1, thm (4.7)]. A singularization area is called *maximal* in case

$$\mu(S) = 1 - \frac{\log G}{\log 2}.$$

DEFINITION 2. — Let S be a singularization area and let x be a real irrational number. The S -*expansion* of x is that semi-regular continued fraction expansion converging to x , which is obtained from the RCF-expansion of x by singularizing B_{n+1} if and only if $(T_n, V_n) \in S$, $n \geq 0$.

REMARKS 2.

(i) We need the condition $\mu(\partial S) = 0$ on S to draw the following conclusion. Let x be an irrational number, with RCF-expansion (1), and let $A(S, N)$ be defined by :

$$A(S, N) := \# \{0 \leq j \leq N ; (T_j, V_j) \in S\}.$$

Then we have for almost all x (see also [K1, 4.6 (ii)]) :

$$\lim_{N \rightarrow \infty} \frac{1}{N} A(S, N) = \mu(S).$$

(ii) It is impossible to singularize in the RCF-expansion (1) of an irrational number x a partial quotient greater than 1, and still obtain a SRCF which converges to x , (see [K1, cor. 1.10]). It is for this that each singularization area S must satisfy $S \subset [\frac{1}{2}, 1) \times [0, 1]$.

Some examples of singularization areas are :

1. — $S_{\text{nicf}} := [\frac{1}{2}, 1) \times [0, g]$; this area, which needs some minor modifications in order to satisfy the above definition 1, see S^* from remark 1, yields the *nearest integer continued fraction* (NICF). The area S_{nicf} is maximal; see also [K1, sect. 4].

2. — $S_{\text{dcf}} := \left\{ (T, V) \in \Omega ; \frac{T}{1 + TV} > \frac{1}{2} \right\}$; this area yields the *diagonal continued fraction* (DCF) of Minkowski; it is not maximal, see [K2].

3. — $S_{\text{ocf}} := \left\{ (T, V) \in \Omega ; V < \min \left(T, \frac{2T - 1}{1 - T} \right) \right\}$; this area yields the OCF and is maximal, see also [K1], [BK1].

Let S be a singularization area and let x be a real irrational number, with RCF-expansion (1) and RCF-convergents $(P_n/Q_n)_{n \geq -1}$. Furthermore, let $[a_0; \varepsilon_1 a_1, \dots, \varepsilon_k a_k, \dots]$ be the S -expansion of x , with convergents r_k/s_k for $k \geq -1$. Define the shift t by :

$$t(x - a_0) := [0; \varepsilon_2 a_2, \dots, \varepsilon_k a_k, \dots].$$

For a fixed x and for $k \geq 0$ we put :

$$t_k := t^k(x - a_0) = [0; \varepsilon_{k+1}a_{k+1}, \varepsilon_{k+2}a_{k+2}, \dots],$$

$$v_k := s_{k-1}/s_k.$$

One easily shows, see also [K1], (1.4) and (5.1), that

$$v_k = [0; a_k, \varepsilon_k a_{k-1}, \dots, \varepsilon_2 a_1], \quad k \geq 1; \quad v_0 = 0.$$

We have the following theorem :

THEOREM 2. — *Let S be a singularization area and put :*

$$\Delta_S := \Omega \setminus S, \quad \Delta_S^- := \mathcal{T}S, \quad \Delta_S^+ := \Delta_S \setminus \Delta_S^-.$$

Then one has :

- (1) *The system $(\Delta_S, \mathcal{B}, \rho_S, \mathcal{O}_S)$ forms an ergodic system. Here ρ_S is the probability measure on (Δ_S, \mathcal{B}) with density*

$$((1 - \mu(S)) \log 2)^{-1} (1 + xy)^{-2}$$

and the map \mathcal{O}_S is induced by \mathcal{T} on Δ_S .

- (2) *$(T_n, V_n) \in S \Leftrightarrow P_n/Q_n$ is not an S -convergent.*
 (3) *If P_n/Q_n is not an S -convergent, then both P_{n-1}/Q_{n-1} and P_{n+1}/Q_{n+1} are S -convergents.*
 (4) *$(T_n, V_n) \in \Delta_S^+$ if and only if*

$$\exists k : \begin{cases} r_{k-1} = P_{n-1}, & r_k = P_n, \\ s_{k-1} = Q_{n-1}, & s_k = Q_n \end{cases} \quad \text{and} \quad t_k = T_n, \quad v_k = V_n.$$

- (5) *$(T_n, V_n) \in \Delta_S^-$ if and only if*

$$\exists k : \begin{cases} r_{k-1} = P_{n-2}, & r_k = P_n, \\ s_{k-1} = Q_{n-2}, & s_k = Q_n \end{cases} \quad \text{and} \quad t_k = \frac{-T_n}{1 + T_n}, \quad v_k = 1 - V_n.$$

(See also [K1, thm (5.3)].)

In view of THEOREM 2 we define the map $\mathcal{M} : \Delta_S \rightarrow \mathbb{R}^2$ by :

$$\mathcal{M}(T, V) := \begin{cases} (T, V) & \text{if } (T, V) \in \Delta_S^+; \\ \left(\frac{-T}{1 + T}, 1 - V \right) & \text{if } (T, V) \in \Delta_S^-. \end{cases}$$

We have the following theorem :

THEOREM 3. — *Let S be a singularization area and put $\Omega_S := \mathcal{M}(\Delta)$. Let \mathcal{B} be the collection of Borel subsets of Ω_S and let μ_S be the probability measure on (Ω_S, \mathcal{B}) , defined by :*

$$\mu_S(E) := \rho_S(\mathcal{M}^{-1}(E)), \quad E \in \mathcal{B}.$$

Furthermore, if we define the map $\mathcal{T}_S : \Omega_S \rightarrow \Omega_S$ by

$$\mathcal{T}_S(t, v) := \mathcal{M}(\mathcal{O}_S(\mathcal{M}^{-1}(t, v))), \quad (t, v) \in \Omega_S,$$

then \mathcal{T}_S is conjugate to \mathcal{O}_S by \mathcal{M} and we have :

(1) For each irrational number x and for each $k \geq 0$

$$(t_k, v_k) \in \Omega_S \quad \text{and} \quad \mathcal{T}_S(t_k, v_k) = (t_{k+1}, v_{k+1}).$$

(2) $(\Omega_S, \mathcal{B}, \mu_S, \mathcal{T}_S)$ forms an ergodic system. The entropy of \mathcal{T}_S equals

$$h(\mathcal{T}_S) = \frac{h(\mathcal{T})}{1 - \mu(S)} = \frac{1}{1 - \mu(S)} \frac{\pi^2}{6 \log 2}.$$

(3) ρ_S has density $((1 - \mu(S)) \log 2)^{-1} (1 + tv)^{-2}$.

REMARKS 3.

(i) Due to the way in which it is constructed it follows that $(\Omega_S, \mathcal{B}, \mu_S, \mathcal{T}_S)$ is the two-dimensional ergodic system underlying the corresponding S -expansion. Now let the map $f_S : \Omega_S \rightarrow \mathbb{R} \cup \{\infty\}$ be defined by :

$$f_S(t, v) := \left\lfloor \frac{1}{t} \right\rfloor - \tau_1(t, v), \quad (t, v) \in \Omega_S,$$

where τ_1 is the first coordinate function of \mathcal{T}_S . Among other things we then have (see [K1, thm (5.11) and cor. (5.12)]) :

$$f_S(t, v) \in \mathbb{N} \quad \text{for} \quad (t, v) \in \Omega_S \quad \text{and} \quad t \neq 0;$$

$$\mathcal{T}_S(t, v) = \left(\left\lfloor \frac{1}{t} \right\rfloor - f_S(t, v), \frac{1}{\operatorname{sgn}(t) \cdot v + f_S(t, v)} \right) \quad \text{for} \quad (t, v) \in \Omega_S;$$

$$a_{k+1} = f_S(t_k, v_k) \quad \text{for} \quad k \geq 0, \quad \text{where} \quad (t_0, v_0) = (x - a_0, 0).$$

Thus we see that the S -expansion is the process associated with \mathcal{T}_S and f_S .

(ii) It is not always possible to give a closed expression in t and s of the function f_S , but in some cases, when ∂S is sufficiently smooth, this turns out to be possible. For example, let the singularization area S_α for $\frac{1}{2} \leq \alpha \leq 1$ be given by[‡]

$$S_\alpha := [\alpha, (1-\alpha)/\alpha) \times [0, g) \cup [(1-\alpha)/\alpha, 1) \times [0, 1]$$

for $\frac{1}{2} \leq \alpha < g$ and

$$S_\alpha := [\alpha, 1] \times [0, 1], \quad g \leq \alpha \leq 1.$$

For each $\alpha \in [\frac{1}{2}, 1]$ we now have, see also [K1, sec. 6] :

$$f_{S_\alpha}(t, v) = \left[\left\lfloor \frac{1}{t} \right\rfloor + 1 - \alpha \right], \quad (t, v) \in \Omega_{S_\alpha}.$$

The S -expansions generated by the singularization areas S_α are the so-called α -*expansions*. These α -expansions were introduced and studied by H. NAKADA in [N]. For a closed expression in t and v of f_{def} (resp. f_{ocf}) the reader is referred to [K2] (resp. [BK1]).

3. Each maximal S -expansion is Bernoulli

For some S -expansions it is known that properties stronger than ergodicity hold; In [N], H. NAKADA showed that for each α -expansion, with $\frac{1}{2} \leq \alpha \leq 1$, the «underlying» system $(\Omega_{S(\alpha)}, \mathcal{B}, \mu_{S(\alpha)}, \mathcal{T}_{S(\alpha)})$ is Kolmogorov.

In [FO], N.A. FRIEDMAN and D.S. ORNSTEIN proved that each invertible transformation on a probability space, which has a weakly Bernoulli generator, is a Bernoulli-shift. Here we apply this result to the transformation

$$T_{1/2} : \left[-\frac{1}{2}, \frac{1}{2}\right) \rightarrow \left[-\frac{1}{2}, \frac{1}{2}\right),$$

defined by

$$T_{1/2}(x) := \begin{cases} |1/x| - \left[|1/x| + \frac{1}{2}\right] & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

[‡] In case $\frac{1}{2} \leq \alpha \leq g$ the definition of S_α needs some minor modification in order to satisfy definition 1. Notice that $S_{1/2} = S_{\text{nief}}$.

For each irrational number x this transformation yields a unique SRCF-expansion of x , the *nearest integer continued fraction* (NICF) expansion of x .

In 1979, G.J. RIEGER [R] showed that the generator of this NICF-transformation $T_{1/2}$, equipped with the probability measure with density $d(x)$, where

$$d(x) = \begin{cases} (\log G)^{-1}(x + G + 1)^{-1} & \text{if } x \in [-\frac{1}{2}, 0], \\ (\log G)^{-1}(x + G)^{-1} & \text{if } x \in [0, \frac{1}{2}], \end{cases}$$

is weakly Bernoulli. Hence, the natural extension of this transformation to an invertible one is a Bernoulli-shift. In view of THEOREM 3 we therefore see that :

(3) *the dynamical system $(\Delta_{\text{nicf}}, \mathcal{B}, \rho_{\text{nicf}}, \mathcal{O}_{\text{nicf}})$ is Bernoulli.*

RIEGER's result can easily be obtained for any α -expansion; thus we see that the systems $(\Omega_{S(\alpha)}, \mathcal{B}, \mu_{S(\alpha)}, \mathcal{T}_{S(\alpha)})$ are all Bernoulli. This raises the natural question whether more generally properties stronger than ergodicity can be obtained; for instance, P. LIARDET [L] recently posed the question whether the OCF is Kolmogorov, or even Bernoulli.

We have the following theorem :

THEOREM 4. — *Each maximal S -expansion is a Bernoulli-shift.*

This theorem is now an immediate consequence of (3) and of the following isomorphism theorem :

THEOREM 5. — *The systems $(\Delta_S, \mathcal{B}, \rho_S, \mathcal{O}_S)$ and $(\Delta_{\text{nicf}}, \mathcal{B}, \rho_{\text{nicf}}, \mathcal{O}_{\text{nicf}})$ are isomorphic for each maximal singularization area S .*

Proof. — In [K1, sect. 4] (for the NICF) and in [BK2] (for the OCF) it is shown, that in order to obtain the NICF (resp. the OCF), one must singularize (in a certain manner) exactly $[\frac{1}{2}(m+1)]$ partial quotients in each block

$$\dots, B_n \neq 1, B_{n+1} = 1, \dots, B_{n+m} = 1, B_{n+m+1} \neq 1, \dots$$

of m consecutive regular partial quotients equal to 1 (here $m \in \mathbb{N}$; in case $n = 0$ we do not need to assume that $B_n \neq 1$). We will show here, that after removing a certain set of measure zero from Ω , the same property holds for any S -expansion with a maximal singularization area S . Once this property is established, an isomorphism follows in a natural way.

Define for each $m \in \mathbb{N}$ a sequence of m consecutive rectangles \mathcal{R}_m , $\mathcal{T}\mathcal{R}_m, \dots, \mathcal{T}^{m-1}\mathcal{R}_m$, where :

$$\mathcal{R}_m := \begin{cases} [F_m/F_{m+1}, F_{m+2}/F_{m+3}) \times [0, \frac{1}{2}) & \text{if } m \text{ is even,} \\ [F_{m+2}/F_{m+3}, F_m/F_{m+1}) \times [0, \frac{1}{2}) & \text{if } m \text{ is odd.} \end{cases}$$

Here $(F_k)_{k \geq 0}$ is the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, \dots$$

The rectangles \mathcal{R}_m (with $m \in \mathbb{N}$) are introduced here to characterize points at the beginning of a sequence of m consecutive partial quotients equal to 1. The intervals $[F_m/F_{m+1}, F_{m+2}/F_{m+3})$ (in case m is even) and $[F_{m+2}/F_{m+3}, F_m/F_{m+1})$ (in case m is odd) give the number of 1's, while the interval $[0, \frac{1}{2})$ expresses the fact that we are at the beginning of such a string of m consecutive 1's. Put

$$\mathcal{R}_{m,i} := \mathcal{T}^i \mathcal{R}_m \quad (0 \leq i \leq m-1),$$

and notice that by invariance of μ one has :

$$\mu(\mathcal{R}_{m,i}) = \mu(\mathcal{R}_{m,j}) \quad (0 \leq i, j \leq m-1).$$

Define moreover

$$\mathcal{U}_{m,i} := \mathcal{R}_{m,i} \cap S, \quad \mathcal{W}_{m,i} := \mathcal{R}_{m,i} \cap S^c \quad (0 \leq i \leq m-1);$$

then $\{\mathcal{U}_{m,i}, \mathcal{W}_{m,i}\}$ forms a «partition» of $\mathcal{R}_{m,i}$ (notice that one of $\mathcal{U}_{m,i}, \mathcal{W}_{m,i}$ might be empty). Now let $i \in \{0, 1, \dots, m-1\}$, then each one of

$$\begin{aligned} & \{\mathcal{T}^i \mathcal{U}_{m,0}, \mathcal{T}^i \mathcal{W}_{m,0}\}, \dots, \{\mathcal{U}_{m,i}, \mathcal{W}_{m,i}\}, \dots \\ & \dots, \{\mathcal{T}^{i-(m-1)} \mathcal{U}_{m,m-1}, \mathcal{T}^{i-(m-1)} \mathcal{W}_{m,m-1}\} \end{aligned}$$

forms a «partition» of $\mathcal{R}_{m,i}$. Put :

$$\begin{aligned} \mathcal{P}_{m,i} := & \{\mathcal{T}^i \mathcal{U}_{m,0}, \mathcal{T}^i \mathcal{W}_{m,0}\} \vee \dots \vee \{\mathcal{U}_{m,i}, \mathcal{W}_{m,i}\} \\ & \vee \dots \vee \{\mathcal{T}^{i-(m-1)} \mathcal{U}_{m,m-1}, \mathcal{T}^{i-(m-1)} \mathcal{W}_{m,m-1}\}. \end{aligned}$$

Then $\mathcal{P}_{m,i}$ forms a finite partition of $\mathcal{R}_{m,i}$, and for each i, j in the set $\{0, 1, \dots, m-1\}$ one has that (with a slight abuse of language)

$$\mathcal{T}^{j-i} : \mathcal{P}_{m,i} \rightarrow \mathcal{P}_{m,j}$$

forms a bijection. Now let $A \in \mathcal{P}_{m,0}$ be such that $\mu(A) > 0$ and put :

$$A_k := \begin{cases} \mathcal{T}^k A & \text{if } 0 \leq k \leq m-1, \\ \emptyset & \text{if } k \geq m. \end{cases}$$

Notice that the definition of $\mathcal{P}_{m,0}$ implies that $A_k \cap S \neq \emptyset$ is equivalent with $A_k \subset S$. But then it follows from definition 2 that we have, in case $(T_n, V_n) \in A$ and $0 \leq k \leq m-1$:

$$\text{singularize } B_{n+k+1} \text{ if and only if } A_k \cap S \neq \emptyset.$$

Now suppose that $\kappa_A < \left\lceil \frac{1}{2}(m+1) \right\rceil$, where

$$\kappa_A := \#\{k; 0 \leq k \leq m-1, A_k \cap S \neq \emptyset\}.$$

Putting

$$S^* := \left(S \setminus \bigcup_{k=0}^{m-1} A_k \right) \cup \left(\bigcup_{k=0}^{\infty} A_{2k} \right),$$

one easily verifies that S^* satisfies all three conditions of definition 1, i.e. S^* is also a singularization area. Due to $\mu(A) > 0$ we moreover have, that

$$\mu(S^*) - \mu(S) = \left(\left\lceil \frac{1}{2}(m+1) \right\rceil - \kappa_A \right) \mu(A) > 0,$$

which is impossible, since S is a maximal singularization area. Thus we see, that for each $m \in \mathbb{N}$:

$$A \in \mathcal{P}_{m,0}, \mu(A) > 0 \Rightarrow \kappa_A = \left\lceil \frac{1}{2}(m+1) \right\rceil.$$

(Inequality $\kappa_A > \left\lceil \frac{1}{2}(m+1) \right\rceil$ is impossible due to condition (iii) from definition 1.) Put :

$$E_{\infty} := \{g\} \times [0, 1],$$

$$E_m := \{\mathcal{T}^k A; k \in \mathbb{Z}, A \in \mathcal{P}_{m,0}, \mu(A) = 0\} \quad (m \in \mathbb{N}),$$

$$E := E_{\infty} \cup \left(\bigcup_{m=1}^{\infty} E_m \right).$$

Clearly one has $E \in \mathcal{B}$, $\mu(E) = 0$.

Let $(\xi, \eta) \in G_2$, then $\mathcal{O}_S(\xi, \eta) = T^2(\xi, \eta)$, and due to $\mathcal{O}_S(\xi, \eta) \in G_2$ one has by definition of ψ that

$$\psi(\mathcal{O}_S(\xi, \eta)) = T(\mathcal{O}_S(\xi, \eta)) = T^3(\xi, \eta).$$

Moreover $(\xi, \eta) \in G_2$ implies that $\psi(\xi, \eta) = T(\xi, \eta) \in S$, hence

$$T(\psi(\xi, \eta)) = T^2(\xi, \eta) = \mathcal{O}_S(\xi, \eta) \in G_2 \subset S_{\text{nicf}},$$

and one finds

$$\mathcal{O}_{\text{nicf}}(\psi(\xi, \eta)) = T^3(\xi, \eta) = \psi(\mathcal{O}_S(\xi, \eta)). \quad \square$$

Since $\psi : \Delta_S \setminus E \rightarrow \Delta_{\text{nicf}} \setminus E$ is a bijection, it at once follows from the Lemma and from $\mu(E) = 0$ that $(\Delta_S, \mathcal{B}, \rho_S, \mathcal{O}_S)$ and $(\Delta_{\text{nicf}}, \mathcal{B}, \rho_{\text{nicf}}, \mathcal{O}_{\text{nicf}})$ are isomorphic. \square

4. Some corollaries of the proof of Theorem 5

An easy calculation shows, that for $m \geq 1$, $0 \leq i \leq m-1$,

$$\mu(\mathcal{R}_{m,i}) = \frac{1}{\log 2} \left| \log \left(\frac{F_{m+1} F_{m+5}}{F_{m+3}^2} \right) \right|,$$

where $(F_k)_{k \geq 0}$ is again the sequence of Fibonacci numbers. For $i \geq m$, put

$$\mathcal{R}_{m,i} := \emptyset.$$

Now let S be some singularization area. Then the set $B_S \in \mathcal{B}$, defined by

$$(4) \quad B_S := \left(\left[\frac{1}{2}, 1 \right) \times [0, 1] \cap \Omega \right) \setminus (S \cup T^{-1}S \cup TS),$$

is called the *area of the preservation* of 1's. It at once follows from definitions 1 and 2 that for any number x with R.C.F.-expansion (1) one has :

$$\left. \begin{array}{l} \text{the partial quotient } B_{n+1} \text{ of } x, \text{ equals 1 and} \\ \text{is unchanged by the } S\text{-singularization} \end{array} \right\} \Leftrightarrow (T_n, V_n) \in B_S.$$

We have the following corollary of the proof of THEOREM 5, see also [K1, thm (4.11)] :

COROLLARY. — *Let S be a singularization area, and let B_S be defined as in (4). Then :*

$$S \text{ is maximal} \Rightarrow \mu(B_S) = 0.$$

One could wonder whether the converse of this corollary holds ; i.e. does $\mu(B_S) = 0$ («with probability 1 no partial quotient equal to 1 survives») imply that S is maximal ? We have the following proposition, which easily follows from the tools developed in the proof of THEOREM 5.

PROPOSITION. — *Let $\mathcal{R}_{m,i}$, $m \geq 1$, $i \geq 0$, be defined as before. Put*

$$S^* := \bigcup_{m=1}^{\infty} \left\{ \left(\bigcup_{i=0}^{\infty} \mathcal{R}_{m,3i+1} \right) \cup \mathcal{R}_{m,m-m^*} \right\},$$

where

$$m^* = m^*(m) := \left[\frac{1}{3}(m+2) \right] - \left[\frac{1}{3}(m+1) \right], \quad \text{for } m \geq 1.$$

Then S^* forms a non-maximal singularization area such that $\mu(B_{S^*}) = 0$ and one has :

$$(5) \quad \varrho := \mu(S^*) = \frac{1}{\log 2} \sum_{m=1}^{\infty} \left[\frac{1}{3}(m+2) \right] \left| \log \left(\frac{F_{m+1} F_{m+5}}{F_{m+3}^2} \right) \right|.$$

Moreover, if $S \in \mathcal{B}$ is a singularization area for which $\mu(B_S) = 0$, then :

$$\mu(S) \geq \varrho.$$

REMARK 4. — Using (5) one finds with the aid of a computer, that :

$$\varrho = 0.2776 \dots$$

Apart from this nothing is known about the constant ϱ . Compare this with the case S is maximal. One has :

$$S_{\text{nief}} \doteq \bigcup_{m=1}^{\infty} \left(\bigcup_{i=0}^{\infty} \mathcal{R}_{m,2i} \right),$$

which yields that for S maximal one has

$$\begin{aligned} \mu(S) &= \frac{1}{\log 2} \sum_{m=1}^{\infty} \left[\frac{1}{2}(m+1) \right] \left| \log \left(F_{m+1} F_{m+5} / F_{m+3}^2 \right) \right| \\ &= 1 - \log G / \log 2 = 0.3057 \dots \end{aligned}$$

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