BULLETIN DE LA S. M. F.

CORNELIS KRAAIKAMP

# Maximal $S$-expansions are Bernoulli shifts

*Bulletin de la S. M. F.*, tome 121, nº 1 (1993), p. 117-131

<http://www.numdam.org/item?id=BSMF_1993__121_1_117_0>

*Bull. Soc. math. France,*
121, 1993, p. 117–131.

# MAXIMAL $S$-EXPANSIONS ARE BERNOULLI SHIFTS

BY

CORNELIS KRAAIKAMP (*)

RÉSUMÉ. — Nous montrons dans cette note que les systèmes sous-jacent à une classe de développement en fraction continue (les « $S$-expansions maximales ») sont tous isomorphes, ce qui entraîne que ces systèmes sont de Bernoulli. En particulier, le système associé à la fraction continue optimale,qui est une $S$-expansion maximale, est de Bernoulli, donc $K$, ce qui répond à une question de Pierre Liardet [L].

ABSTRACT. — In this paper it is shown that the systems underlying any two maximal $S$-expansions are isomorphic, and from this it follows that these systems are Bernoulli. This answers a question, recently posed by Pierre Liardet [L], whether the « underlying » ergodic system of the Optimal Continued Fraction (OCF) forms a $K$-system, since the OCF is a maximal $S$-expansion.

## 1. Introduction

Let $x$ be an irrational number between 0 and 1. The expansion of $x$ as a regular continued fraction (RCF) is denoted by

$$x = \big[\,0; B_1, B_2, \ldots, B_n, \ldots\big], \tag{1}$$

where $B_n \in \mathbb{N}$, $n \geq 1$. Finite truncation in (1) yields the corresponding sequence of regular convergents $(P_n/Q_n)_{n\geq -1}$.

Define the RCF-operator $T : [0,1) \longrightarrow [0,1)$ by :

$$Tx := \frac{1}{x} - \Big[\frac{1}{x}\Big], \quad x \neq 0\,; \quad T0 := 0.$$

(*) Texte reçu le 17 juin 1991, révisé le 4 mai 1992.

Cornelis KRAAIKAMP, Technische Universiteit Delft, Fac. TWI, vagroep SSOR, Mekelweg 4, 2628CD Delft, Netherlands.

Research supported by the Netherlands Organization for Scientific Research (NWO).

Classification AMS : 11K50, 28D05.

Here $[\,\cdot\,]$ denotes the so-called *entier* (or *floor*) function. Furthermore, if we define the function $B : [0,1) \longrightarrow \mathbb{N} \cup \{\infty\}$ by

$$B(x) := \left[\frac{1}{x}\right], \quad x \neq 0\,; \quad B(0) := \infty,$$

then the regular continued fraction is the process (in the sence of [OW]) associated with $T$ and $B$, i.e. $B_{n+1} = B(T^n(x))$. Hence, if $x$ has the expansion (1), then $T(x) = [0; B_2, \ldots, B_n, \ldots]$. Put

$$T_n := T^n(x),\ n \geq 0\,; \quad V_n := \frac{Q_{n-1}}{Q_n}\,,\ n \geq 0\,;$$

then $T_n = [0; B_{n+1}, B_{n+2}, \ldots]$ and a simple calculation shows that

$$V_n = [0; B_n, \ldots, B_1], \quad n \geq 1\,; \quad V_0 = 0.$$

Moreover, $(T_n, V_n)_{n\geq 0}$ is a sequence in $\Omega := \big([0,1) \setminus \mathbb{Q}\big) \times [0,1]$.

Fundamental in the theory of $S$-expansions is the following theorem :

THEOREM 1 ([NIT], 1977; [N], 1981). — *Let $\mathcal{B}$ be the collection of Borel-sets of $\Omega$ and let $\mu$ be the probability measure on $(\Omega, \mathcal{B})$ with density $(\log 2)^{-1}(1+xy)^{-2}$. Define the operator $\mathcal{T} : \Omega \longrightarrow \Omega$ by :*

$$\mathcal{T}(x,y) := \Big(Tx, \frac{1}{[1/x] + y}\Big), \qquad (x,y) \in \Omega.$$

*Then $(\Omega, \mathcal{B}, \mu, \mathcal{T})$ forms an ergodic system.*

Notice that for each irrational number $x$ one has :

$$\mathcal{T}^n(x,0) = (T_n, V_n), \qquad n \geq 0.$$

Here and in the following

$$[a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots, \varepsilon_n a_n, \ldots]$$

is the abreviation of

$$a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cdots + \cfrac{\varepsilon_n}{a_n + \cdots}}}, \tag{2}$$

where $a_0 \in \mathbb{Z}$; $a_i \in \mathbb{N}$, $\varepsilon_i \in \{\pm 1\}$, for $i \geq 1$. We call $[a_0; \varepsilon_1 a_1, \ldots]$ a *semi-regular continued fraction* (SRCF) in case $\varepsilon_i + a_i \geq 1$, $\varepsilon_{i+1} + a_i \geq 1$, for $i \geq 1$ and infinitely often $\varepsilon_{i+1} + a_i \geq 2$. Finite truncation in (2) yields the sequence of convergents $(r_n/s_n)_{n\geq -1}$, which converges to a unique irrational number $x$ in case $[a_0; \varepsilon_1 a_1, \ldots]$ is semi-regular.

A SRCF-expansion (2) is a *fastest* expansion of $x$ in case the growth-rate of the denominators $s_n$ is maximal. One can show that this means that these denominators grow asymptotically as fast as the denominators of the *nearest integer continued fraction* (NICF) expansion convergents of $x$, (see also [B, sect. 3]). *Closest* expansions are those expansions of $x$ for which $\sup\{\theta_k; \theta_k = s_k \mid s_k x - r_k|, k \geq 0\}$ is minimal. Every irrational number $x$ admits an expansion for which $\theta_k < \frac{1}{2}$ and $k \geq 1$, given by *Minkowski's diagonal continued fraction* (DCF). In general the NICF does not yield closest expansions, while the DCF does not yield fastest expansions. An expansion which is always both fastest and closest for all irrational numbers $x$ is the *Optimal Continued Fraction* (OCF), see [BK1], [BK2].

Now let $x$ be an irrational number, and let (2) be some SRCF-expansion of $x$. Suppose that we have for a certain $k \geq 0$ :

$$a_{k+1} = 1, \quad \varepsilon_{k+1} = \varepsilon_{k+2} = 1.$$

The operation by which the continued fraction (2) is replaced by†

$$\big[a_0; \varepsilon_1 a_1, \ldots, \varepsilon_{k-1} a_{k-1}, \varepsilon_k (a_k + 1), -(a_{k+2} + 1), \varepsilon_{k+3} a_{k+3}, \ldots\big],$$

which again is a SRCF-expansion of $x$, with convergents, say, $(c_n/d_n)_{n\geq -1}$, is called *the singularization of the partial quotient* $a_{k+1}$ *equal to* 1. One easily shows that $(c_n/d_n)_{n\geq -1}$ is obtained from $(r_n/s_n)_{n\geq -1}$ by skipping the term $r_k/s_k$. See also [K1, sect. 2 and 4].

## 2. $S$-expansions

A simple way to derive a strategy for singularization is given by a *singularization area* $S$.

DEFINITION 1. — A subset $S$ from $\Omega$ is called a *singularization area* if it satisfies :

(i) $S \in \mathcal{B}$ and $\mu(\partial S) = 0$;
(ii) $S \subset \big([\frac{1}{2}, 1) \setminus \mathbb{Q}\big) \times [0, 1]$;
(iii) $\mathcal{T}_S \cap S = \emptyset$.

† In case $k = 0$ this comes down to replacing (2) by $[a_0 + 1; -(a_2 + 1), \varepsilon_3 a_3, \varepsilon_4 a_4, \ldots]$.

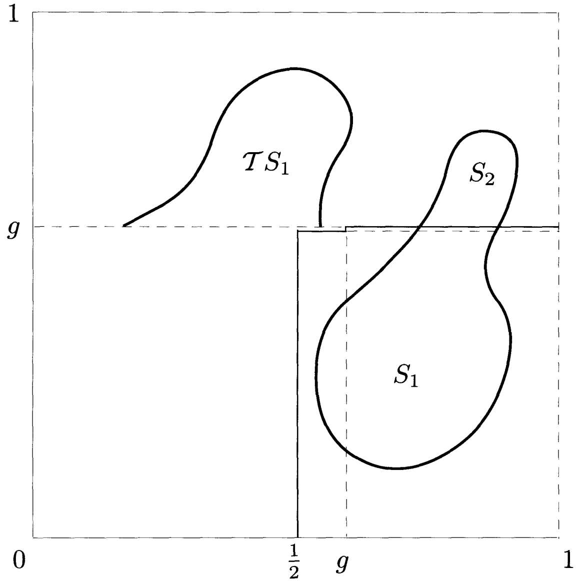


*Figure 1.*

Here and in the sequel we put :

$$g := \tfrac{1}{2}(\sqrt{5}-1), \quad G := g+1 = g^{-1}.$$

REMARK 1. — Consider the following singularization area $S^*$, where

$$S^* := \left[\tfrac{1}{2}, g\right] \times [0,g] \cup (g,1) \times (0,g),$$

and let $S$ be some singularization area. Put $S_1 := S \cap S^*$, $S_2 := S \setminus S_1$ (see also figure 1). Then by invariance of $\mu$, due to definition 1 (iii) and from the fact that $S^* \cup \mathcal{T}S^*$ covers the rectangle containing any singularization area it now follows that :

$$\begin{aligned}\mu(S) &= \mu(\mathcal{T}S_1) + \mu(S_2) = \mu(\mathcal{T}S_1 \cup S_2)\\ &\leq \mu(\mathcal{T}S^*) = \mu(S^*) = 1 - \frac{\log G}{\log 2}\cdot\end{aligned}$$

Thus we see, that

$$0 \leq \mu(S) \leq 1 - \frac{\log G}{\log 2} = 0.3057\ldots,$$

see also [K1, thm (4.7)]. A singularization area is called *maximal* in case

$$\mu(S) = 1 - \frac{\log G}{\log 2}\cdot$$

DEFINITION 2. — Let $S$ be a singularization area and let $x$ be a real irrational number. The *$S$-expansion* of $x$ is that semi-regular continued fraction expansion converging to $x$, which is obtained from the RCF-expansion of $x$ by singularizing $B_{n+1}$ if and only if $(T_n, V_n) \in S$, $n \geq 0$.

REMARKS 2.

(i) We need the condition $\mu(\partial S) = 0$ on $S$ to draw the following conclusion. Let $x$ be an irrational number, with RCF-expansion (1), and let $A(S, N)$ be defined by :

$$A(S,N) := \sharp\{0 \leq j \leq N\,;\ (T_j, V_j) \in S\}.$$

Then we have for almost all $x$ (see also [K1, 4.6 (ii)]) :

$$\lim_{N\to\infty} \frac{1}{N} A(S,N) = \mu(S).$$

(ii) It is impossible to singularize in the RCF-expansion (1) of an irrational number $x$ a partial quotient greater than 1, and still obtain a SRCF which converges to $x$, (see [K1, cor. 1.10]). It is for this that each singularization area $S$ must satisfy $S \subset [\frac{1}{2}, 1) \times [0, 1]$.

Some examples of singularization areas are :

1. — $S_{\text{nicf}} := [\frac{1}{2}, 1) \times [0, g]$; this area, which needs some minor modifications in order to satisfy the above definition 1, see $S^*$ from remark 1, yields the *nearest integer continued fraction* (NICF). The area $S_{\text{nicf}}$ is maximal; see also [K1, sect. 4].

2. — $S_{\text{dcf}} := \left\{(T,V) \in \Omega\,;\ \dfrac{T}{1+TV} > \dfrac{1}{2}\right\}$; this area yields the *diagonal continued fraction* (DCF) of Minkowski; it is not maximal, see [K2].

3. — $S_{\text{ocf}} := \left\{(T,V) \in \Omega\,;\ V < \min\left(T, \dfrac{2T-1}{1-T}\right)\right\}$; this area yields the OCF and is maximal, see also [K1], [BK1].

Let $S$ be a singularization area and let $x$ be a real irrational number, with RCF-expansion (1) and RCF-convergents $(P_n/Q_n)_{n\geq -1}$. Furthermore, let $[a_0; \varepsilon_1 a_1, \ldots, \varepsilon_k a_k, \ldots]$ be the $S$-expansion of $x$, with convergents $r_k/s_k$ for $k \geq -1$. Define the shift $t$ by :

$$t(x - a_0) := [0; \varepsilon_2 a_2, \ldots, \varepsilon_k a_k, \ldots].$$

For a fixed $x$ and for $k \geq 0$ we put :

$$t_k := t^k(x - a_0) = \big[0; \varepsilon_{k+1}a_{k+1}, \varepsilon_{k+2}a_{k+2}, \ldots\big],$$
$$v_k := s_{k-1}/s_k.$$

One easily shows, see also [K1], (1.4) and (5.1), that

$$v_k = \big[0; a_k, \varepsilon_k a_{k-1}, \ldots, \varepsilon_2 a_1\big], \quad k \geq 1\,; \;\; v_0 = 0.$$

We have the following theorem :

THEOREM 2. — *Let $S$ be a singularization area and put :*

$$\Delta_S := \Omega \setminus S, \quad \Delta_S^- := \mathcal{T}S, \quad \Delta_S^+ := \Delta_S \setminus \Delta_S^-.$$

*Then one has :*

(1) *The system $(\Delta_S, \mathcal{B}, \rho_S, \mathcal{O}_S)$ forms an ergodic system. Here $\rho_S$ is the probability measure on $(\Delta_S, \mathcal{B})$ with density*

$$\big((1 - \mu(S)) \log 2\big)^{-1}(1 + xy)^{-2}$$

*and the map $\mathcal{O}_S$ is induced by $\mathcal{T}$ on $\Delta_S$.*

(2) $(T_n, V_n) \in S \Leftrightarrow P_n/Q_n$ *is not an $S$-convergent.*

(3) *If $P_n/Q_n$ is not an $S$-convergent, then both $P_{n-1}/Q_{n-1}$ and $P_{n+1}/Q_{n+1}$ are $S$-convergents.*

(4) $(T_n, V_n) \in \Delta_S^+$ *if and only if*

$$\exists\, k : \begin{cases} r_{k-1} = P_{n-1}, & r_k = P_n, \\ s_{k-1} = Q_{n-1}, & s_k = Q_n \end{cases} \quad \textit{and} \quad t_k = T_n,\; v_k = V_n.$$

(5) $(T_n, V_n) \in \Delta_S^-$ *if and only if*

$$\exists\, k : \begin{cases} r_{k-1} = P_{n-2}, & r_k = P_n, \\ s_{k-1} = Q_{n-2}, & s_k = Q_n \end{cases} \quad \textit{and} \quad t_k = \frac{-T_n}{1 + T_n},\; v_k = 1 - V_n.$$

(See also [K1, thm (5.3)].)

In view of THEOREM 2 we define the map $\mathcal{M} : \Delta_S \to \mathbb{R}^2$ by :

$$\mathcal{M}(T, V) := \begin{cases} (T, V) & \text{if } (T, V) \in \Delta_S^+\,; \\[2mm] \Big(\dfrac{-T}{1+T}, 1 - V\Big) & \text{if } (T, V) \in \Delta_S^-. \end{cases}$$

We have the following theorem :

THEOREM 3. — *Let $S$ be a singularization area and put $\Omega_S := \mathcal{M}(\Delta)$. Let $\mathcal{B}$ be the collection of Borel subsets of $\Omega_S$ and let $\mu_S$ be the probability measure on $(\Omega_S, \mathcal{B})$, defined by :*

$$\mu_S(E) := \rho_S\big(\mathcal{M}^{-1}(E)\big), \qquad E \in \mathcal{B}.$$

*Furthermore, if we define the map $\mathcal{T}_S : \Omega_S \to \Omega_S$ by*

$$\mathcal{T}_S(t,v) := \mathcal{M}\big(\mathcal{O}_S(\mathcal{M}^{-1}(t,v))\big), \qquad (t,v) \in \Omega_S,$$

*then $\mathcal{T}_S$ is conjugate to $\mathcal{O}_S$ by $\mathcal{M}$ and we have :*

(1) *For each irrational number $x$ and for each $k \geq 0$*

$$(t_k, v_k) \in \Omega_S \quad \text{and} \quad \mathcal{T}_S(t_k, v_k) = \big(t_{k+1}, v_{k+1}\big).$$

(2) *$(\Omega_S, \mathcal{B}, \mu_S, \mathcal{T}_S)$ forms an ergodic system. The entropy of $\mathcal{T}_S$ equals*

$$h(\mathcal{T}_S) = \frac{h(\mathcal{T})}{1-\mu(S)} = \frac{1}{1-\mu(S)}\,\frac{\pi^2}{6\log 2}\,.$$

(3) *$\rho_S$ has density $\big((1-\mu(S))\log 2\big)^{-1}(1+tv)^{-2}$.*

REMARKS 3.

(i) Due to the way in which it is constructed it follows that $(\Omega_S, \mathcal{B}, \mu_S, \mathcal{T}_S)$ is the two-dimensional ergodic system underlying the corresponding $S$-expansion. Now let the map $f_S : \Omega_S \to \mathbb{R} \cup \{\infty\}$ be defined by :

$$f_S(t,v) := \left|\frac{1}{t}\right| - \tau_1(t,v), \qquad (t,v) \in \Omega_S,$$

where $\tau_1$ is the first coordinate function of $\mathcal{T}_S$. Among other things we then have (see [K1, thm (5.11) and cor. (5.12)]) :

$$f_S(t,v) \in \mathbb{N} \ \text{ for } \ (t,v) \in \Omega_S \text{ and } t \neq 0\,;$$

$$\mathcal{T}_S(t,v) = \left(\left|\frac{1}{t}\right| - f_S(t,v),\ \frac{1}{\operatorname{sgn}(t).v + f_S(t,v)}\right) \ \text{ for } \ (t,v) \in \Omega_S\,;$$

$$a_{k+1} = f_S(t_k, v_k) \ \text{ for } \ k \geq 0, \ \text{ where } \ (t_0, v_0) = (x - a_0, 0).$$

Thus we see that the $S$-expansion is the process associated with $\mathcal{T}_S$ and $f_S$.

(ii) It is not always possible to give a closed expression in $t$ and $s$ of the function $f_S$, but in some cases, when $\partial S$ is sufficiently smooth, this turns out to be possible. For example, let the singularization area $S_\alpha$ for $\frac{1}{2} \le \alpha \le 1$ be given by[‡]

$$S_\alpha := \big[\alpha, (1-\alpha)/\alpha\big) \times [0, g) \cup \big[(1-\alpha)/\alpha, 1\big) \times [0, 1]$$

for $\frac{1}{2} \le \alpha < g$ and

$$S_\alpha := [\alpha, 1] \times [0, 1], \quad g \le \alpha \le 1.$$

For each $\alpha \in \big[\frac{1}{2}, 1\big]$ we now have, see also [K1, sec. 6] :

$$f_{S_\alpha}(t, v) = \Big[\Big|\frac{1}{t}\Big| + 1 - \alpha\Big], \qquad (t, v) \in \Omega_{S_\alpha}.$$

The $S$-expansions generated by the singularization areas $S_\alpha$ are the so-called *$\alpha$-expansions.* These $\alpha$-expansions were introduced and studied by H. NAKADA in [N]. For a closed expression in $t$ and $v$ of $f_{\mathrm{dcf}}$ (resp. $f_{\mathrm{ocf}}$) the reader is referred to [K2] (resp. [BK1]).

## 3. Each maximal $S$-expansion is Bernoulli

For some $S$-expansions it is known that properties stronger than ergodicity hold; In [N], H. NAKADA showed that for each $\alpha$-expansion, with $\frac{1}{2} \le \alpha \le 1$, the «underlying» system $(\Omega_{S(\alpha)}, \mathcal{B}, \mu_{S(\alpha)}, \mathcal{T}_{S(\alpha)})$ is Kolmogorov.

In [FO], N.A. FRIEDMAN and D.S. ORNSTEIN proved that each invertible transformation on a probability space, which has a weakly Bernoulli generator, is a Bernoulli-shift. Here we apply this result to the transformation

$$T_{1/2} : \big[-\tfrac{1}{2}, \tfrac{1}{2}\big) \to \big[-\tfrac{1}{2}, \tfrac{1}{2}\big),$$

defined by

$$T_{1/2}(x) := \begin{cases} |1/x| - \big[|1/x| + \frac{1}{2}\big] & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

‡ In case $\frac{1}{2} \le \alpha \le g$ the definition of $S_\alpha$ needs some minor modification in order to satisfy definition 1. Notice that $S_{1/2} = S_{\mathrm{nicf}}$.

For each irrational number $x$ this transformation yields a unique SRCF-expansion of $x$, the *nearest integer continued fraction* (NICF) expansion of $x$.

In 1979, G.J. RIEGER [R] showed that the generator of this NICF-transformation $T_{1/2}$, equipped with the probability measure with density $d(x)$, where

$$d(x) = \begin{cases} (\log G)^{-1}(x+G+1)^{-1} & \text{if } x \in \left[-\frac{1}{2}, 0\right], \\ (\log G)^{-1}(x+G)^{-1} & \text{if } x \in \left[0, \frac{1}{2}\right), \end{cases}$$

is weakly Bernoulli. Hence, the natural extension of this transformation to an invertible one is a Bernoulli-shift. In view of THEOREM 3 we therefore see that :

(3) *the dynamical system* $\left(\Delta_{\mathrm{nicf}}, \mathcal{B}, \rho_{\mathrm{nicf}}, \mathcal{O}_{\mathrm{nicf}}\right)$ *is Bernoulli.*

RIEGER's result can easily be obtained for any $\alpha$-expansion; thus we see that the systems $(\Omega_{S(\alpha)}, \mathcal{B}, \mu_{S(\alpha)}, \mathcal{T}_{S(\alpha)})$ are all Bernoulli. This raises the natural question whether more generally properties stronger than ergodicity can be obtained; for instance, P. LIARDET [L] recently posed the question whether the OCF is Kolmogorov, or even Bernoulli.

We have the following theorem :

THEOREM 4. — *Each maximal $S$-expansion is a Bernoulli-shift.*

This theorem is now an immediate consequence of (3) and of the following isomorphism theorem :

THEOREM 5. — *The systems* $(\Delta_S, \mathcal{B}, \rho_S, \mathcal{O}_S)$ *and* $(\Delta_{\mathrm{nicf}}, \mathcal{B}, \rho_{\mathrm{nicf}}, \mathcal{O}_{\mathrm{nicf}})$ *are isomorphic for each maximal singularization area $S$.*

*Proof.* — In [K1, sect. 4] (for the NICF) and in [BK2] (for the OCF) it is shown, that in order to obtain the NICF (resp. the OCF), one must singularize (in a certain manner) exactly $\left[\frac{1}{2}(m+1)\right]$ partial quotients in each block

$$\ldots,\ B_n \neq 1,\ B_{n+1} = 1, \ldots,\ B_{n+m} = 1,\ B_{n+m+1} \neq 1, \ldots$$

of $m$ consecutive regular partial quotients equal to 1 (here $m \in \mathbb{N}$; in case $n = 0$ we do not need to assume that $B_n \neq 1$). We will show here, that after removing a certain set of measure zero from $\Omega$, the same property holds for any $S$-expansion with a maximal singularization area $S$. Once this property is established, an isomorphism follows in a natural way.

Define for each $m \in \mathbb{N}$ a sequence of $m$ consecutive rectangles $\mathcal{R}_m$, $\mathcal{T}\mathcal{R}_m, \dots, \mathcal{T}^{m-1}\mathcal{R}_m$, where :

$$\mathcal{R}_m := \begin{cases} \big[F_m/F_{m+1}, F_{m+2}/F_{m+3}\big) \times \big[0, \frac{1}{2}\big) & \text{if } m \text{ is even,} \\ \big[F_{m+2}/F_{m+3}, F_m/F_{m+1}\big) \times \big[0, \frac{1}{2}\big) & \text{if } m \text{ is odd.} \end{cases}$$

Here $(F_k)_{k\geq 0}$ is the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, \dots$$

The rectangles $\mathcal{R}_m$ (with $m \in \mathbb{N}$) are introduced here to characterize points at the beginning of a sequence of $m$ consecutive partial quotients equal to 1. The intervals $\big[F_m/F_{m+1}, F_{m+2}/F_{m+3}\big)$ (in case $m$ is even) and $\big[F_{m+2}/F_{m+3}, F_m/F_{m+1}\big)$ (in case $m$ is odd) give the number of 1's, while the interval $[0, \frac{1}{2})$ expresses the fact that we are at the beginning of such a string of $m$ consecutive 1's. Put

$$\mathcal{R}_{m,i} := \mathcal{T}^i \mathcal{R}_m \qquad (0 \leq i \leq m-1),$$

and notice that by invariance of $\mu$ one has :

$$\mu(\mathcal{R}_{m,i}) = \mu(\mathcal{R}_{m,j}) \qquad \big(0 \leq i, j \leq m-1\big).$$

Define moreover

$$\mathcal{U}_{m,i} := \mathcal{R}_{m,i} \cap S, \quad \mathcal{W}_{m,i} := \mathcal{R}_{m,i} \cap S^c \qquad \big(0 \leq i \leq m-1\big)\,;$$

then $\{\mathcal{U}_{m,i}, \mathcal{W}_{m,i}\}$ forms a «partition» of $\mathcal{R}_{m,i}$ (notice that one of $\mathcal{U}_{m,i}$, $\mathcal{W}_{m,i}$ might be empty). Now let $i \in \{0, 1, \dots, m-1\}$, then each one of

$$\big\{\mathcal{T}^i\mathcal{U}_{m,0}, \mathcal{T}^i\mathcal{W}_{m,0}\big\}, \dots, \big\{\mathcal{U}_{m,i}, \mathcal{W}_{m,i}\big\}, \dots$$
$$\dots, \big\{\mathcal{T}^{i-(m-1)}\mathcal{U}_{m,m-1}, \mathcal{T}^{i-(m-1)}\mathcal{W}_{m,m-1}\big\}$$

forms a «partition» of $\mathcal{R}_{m,i}$. Put :

$$\mathcal{P}_{m,i} := \big\{\mathcal{T}^i\mathcal{U}_{m,0}, \mathcal{T}^i\mathcal{W}_{m,0}\big\} \vee \dots \vee \{\mathcal{U}_{m,i}, \mathcal{W}_{m,i}\}$$
$$\vee \dots \vee \big\{\mathcal{T}^{i-(m-1)}\mathcal{U}_{m,m-1}, \mathcal{T}^{i-(m-1)}\mathcal{W}_{m,m-1}\big\}.$$

Then $\mathcal{P}_{m,i}$ forms a finite partition of $\mathcal{R}_{m,i}$, and for each $i, j$ in the set $\{0, 1, \dots, m-1\}$ one has that (with a slight abuse of language)

$$\mathcal{T}^{j-i} : \mathcal{P}_{m,i} \to \mathcal{P}_{m,j}$$

forms a bijection. Now let $A \in \mathcal{P}_{m,0}$ be such that $\mu(A) > 0$ and put :

$$A_k := \begin{cases} \mathcal{T}^k A & \text{if } 0 \le k \le m-1, \\ \emptyset & \text{if } k \ge m. \end{cases}$$

Notice that the definition of $\mathcal{P}_{m,0}$ implies that $A_k \cap S \neq \emptyset$ is equivalent with $A_k \subset S$. But then it follows from definition 2 that we have, in case $(T_n, V_n) \in A$ and $0 \le k \le m-1$ :

$$\textit{singularize } B_{n+k+1} \textit{ if and only if } A_k \cap S \neq \emptyset.$$

Now suppose that $\kappa_A < \left[\frac{1}{2}(m+1)\right]$, where

$$\kappa_A := \#\{k\,;\ 0 \le k \le m-1,\ A_k \cap S \neq \emptyset\}.$$

Putting

$$S^* := \Big(S \setminus \bigcup_{k=0}^{m-1} A_k\Big) \cup \Big(\bigcup_{k=0}^{\infty} A_{2k}\Big),$$

one easily verifies that $S^*$ satisfies all three conditions of definition 1, i.e. $S^*$ is also a singularization area. Due to $\mu(A) > 0$ we moreover have, that

$$\mu(S^*) - \mu(S) = \Big(\left[\tfrac{1}{2}(m+1)\right] - \kappa_A\Big)\mu(A) > 0,$$

which is impossible, since $S$ is a maximal singularization area. Thus we see, that for each $m \in \mathbb{N}$ :

$$A \in \mathcal{P}_{m,0},\ \ \mu(A) > 0 \ \Rightarrow\ \kappa_A = \left[\tfrac{1}{2}(m+1)\right].$$

(Inequality $\kappa_A > \left[\frac{1}{2}(m+1)\right]$ is impossible due to condition (iii) from definition 1.) Put :

$$E_\infty := \{g\} \times [0,1],$$

$$E_m := \{\mathcal{T}^k A\,;\ k \in \mathbb{Z},\ A \in \mathcal{P}_{m,0},\ \mu(A) = 0\} \quad (m \in \mathbb{N}),$$

$$E := E_\infty \cup \Big(\bigcup_{m=1}^{\infty} E_m\Big).$$

Clearly one has $E \in \mathcal{B}$, $\mu(E) = 0$.

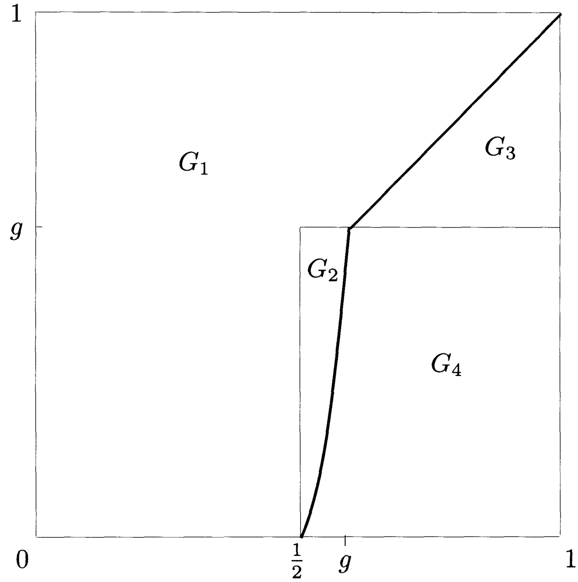


*Figure 2.* (*We have depicted* $G_1, \ldots, G_4$ *in case* $S = S_{\text{nicf}}$.)

Now define the bijection $\psi : \Delta_S \setminus E \to \Delta_{\text{nicf}} \setminus E$ by

$$\psi(\xi,\eta) := \begin{cases} (\xi,\eta) & \text{if } (\xi,\eta) \in G_1 := (\Delta_S \cap \Delta_{\text{nicf}}) \setminus E, \\ \mathcal{T}(\xi,\eta) & \text{if } (\xi,\eta) \in G_2 := (\Delta_S \setminus \Delta_{\text{nicf}}) \setminus E, \end{cases}$$

and define moreover (see figure 2) :

$$G_3 := \big(\Delta_{\text{nicf}} \setminus \Delta_S\big) \setminus E, \quad G_4 := \big(S \cap S_{\text{nicf}}\big) \setminus E.$$

Notice that due to the assumption that $S$ is maximal and the definition of $E$ one has that $\mathcal{T}G_2 = G_3$.

With the above notations we have the following lemma :

LEMMA. — $\psi\big(\mathcal{O}_S(\xi,\eta)\big) = \mathcal{O}_{\text{nicf}}\big(\psi(\xi,\eta)\big)$, $(\xi,\eta) \in \Delta_S \setminus E$.

*Proof.* — We discern the following five cases :

- $(\xi,\eta) \in G_1$ and (i) : $\mathcal{T}(\xi,\eta) \in G_1$, (ii) : $\mathcal{T}(\xi,\eta) \in G_2$, (iii) : $\mathcal{T}(\xi,\eta) \in G_4$;
- $(\xi,\eta) \in G_2$ and (j) : $\mathcal{O}_S(\xi,\eta) \in G_1$, (jj) : $\mathcal{O}_S(\xi,\eta) \in G_2$.

We will show here only the case (jj); the other cases are proved in the same vein.

Let $(\xi,\eta) \in G_2$, then $\mathcal{O}_S(\xi,\eta) = \mathcal{T}^2(\xi,\eta)$, and due to $\mathcal{O}_S(\xi,\eta) \in G_2$ one has by definition of $\psi$ that

$$\psi(\mathcal{O}_S(\xi,\eta)) = \mathcal{T}\big(\mathcal{O}_S(\xi,\eta)\big) = \mathcal{T}^3(\xi,\eta).$$

Moreover $(\xi,\eta) \in G_2$ implies that $\psi(\xi,\eta) = \mathcal{T}(\xi,\eta) \in S$, hence

$$\mathcal{T}\big(\psi(\xi,\eta)\big) = \mathcal{T}^2(\xi,\eta) = \mathcal{O}_S(\xi,\eta) \in G_2 \subset S_{\text{nicf}},$$

and one finds

$$\mathcal{O}_{\text{nicf}}\big(\psi(\xi,\eta)\big) = \mathcal{T}^3(\xi,\eta) = \psi\big(\mathcal{O}_S(\xi,\eta)\big). \quad \square$$

Since $\psi : \Delta_S \setminus E \to \Delta_{\text{nicf}} \setminus E$ is a bijection, it at once follows from the Lemma and from $\mu(E) = 0$ that $(\Delta_S, \mathcal{B}, \rho_S, \mathcal{O}_S)$ and $(\Delta_{\text{nicf}}, \mathcal{B}, \rho_{\text{nicf}}, \mathcal{O}_{\text{nicf}})$ are isomorphic. $\square$

## 4. Some corollaries of the proof of Theorem 5

An easy calculation shows, that for $m \geq 1$, $0 \leq i \leq m-1$,

$$\mu(\mathcal{R}_{m,i}) = \frac{1}{\log 2}\left|\log\left(\frac{F_{m+1}F_{m+5}}{F_{m+3}^2}\right)\right|,$$

where $(F_k)_{k\geq 0}$ is again the sequence of Fibonacci numbers. For $i \geq m$, put

$$\mathcal{R}_{m,i} := \emptyset.$$

Now let $S$ be some singularization area. Then the set $B_S \in \mathcal{B}$, defined by

$$B_S := \big([\tfrac{1}{2},1) \times [0,1] \cap \Omega\big) \setminus \big(S \cup \mathcal{T}^{-1}S \cup \mathcal{T}S\big), \tag{4}$$

is called the *area of the preservation* of 1's. It at once follows from definitions 1 and 2 that for any number $x$ with R.C.F.-expansion (1) one has :

*the partial quotient $B_{n+1}$ of $x$, equals 1 and is unchanged by the $S$-singularization* $\Big\} \Leftrightarrow (T_n, V_n) \in B_S.$

We have the following corollary of the proof of THEOREM 5, see also [K1, thm (4.11)] :

COROLLARY. — *Let $S$ be a singularization area, and let $B_S$ be defined as in* (4). *Then :*

$$S \text{ is maximal } \Rightarrow \mu(B_S) = 0.$$

One could wonder whether the converse of this corollary holds; i.e. does $\mu(B_S) = 0$ («with probability 1 no partial quotient equal to 1 survives») imply that $S$ is maximal? We have the following proposition, which easily follows from the tools developed in the proof of THEOREM 5.

PROPOSITION. — *Let $\mathcal{R}_{m,i}$, $m \geq 1$, $i \geq 0$, be defined as before. Put*

$$S^* := \bigcup_{m=1}^{\infty} \Big\{ \Big( \bigcup_{i=0}^{\infty} \mathcal{R}_{m,3i+1} \Big) \cup \mathcal{R}_{m,m-m^*} \Big\},$$

*where*

$$m^* = m^*(m) := \Big[ \frac{1}{3}(m+2) \Big] - \Big[ \frac{1}{3}(m+1) \Big], \quad \textit{for } m \geq 1.$$

*Then $S^*$ forms a non-maximal singularization area such that $\mu(B_{S^*}) = 0$ and one has :*

$$\text{(5)} \qquad \varrho := \mu(S^*) = \frac{1}{\log 2} \sum_{m=1}^{\infty} \Big[ \frac{1}{3}(m+2) \Big] \Big| \log\Big( \frac{F_{m+1}F_{m+5}}{F_{m+3}^2} \Big) \Big|.$$

*Moreover, if $S \in \mathcal{B}$ is a singularization area for which $\mu(B_S) = 0$, then :*

$$\mu(S) \geq \varrho.$$

REMARK 4. — Using (5) one finds with the aid of a computer, that :

$$\varrho = 0.2776\ldots$$

Apart from this nothing is known about the constant $\varrho$. Compare this with the case $S$ is maximal. One has :

$$S_{\text{nicf}} \doteq \bigcup_{m=1}^{\infty} \Big( \bigcup_{i=0}^{\infty} \mathcal{R}_{m,2i} \Big),$$

which yields that for $S$ maximal one has

$$\begin{aligned} \mu(S) &= \frac{1}{\log 2} \sum_{m=1}^{\infty} \big[ \tfrac{1}{2}(m+1) \big] \Big| \log\Big( F_{m+1}F_{m+5}/F_{m+3}^2 \Big) \Big| \\ &= 1 - \log G / \log 2 = 0.3057\ldots \end{aligned}$$

ACKNOWLEDGEMENTS. — This paper was partly written during a stay of six months at the University of Washington, Seattle. I would like to thank the Department of Mathematics of the University of Washington for their hospitality. I also would like to thank the referee for many helpful suggestions concerning the presentation of this paper.

## BIBLIOGRAPHY

[B] BOSMA (W.). — *Optimal Continued Fractions*, Indag. Math., t. **50**, 1988, p. 353–379.

[BK1] BOSMA (W.) and KRAAIKAMP (C.). — *Metrical Theory for Optimal Continued Fractions*, J. Number Theory, t. **34**, 1990, p. 251–270.

[BK2] BOSMA (W.) and KRAAIKAMP (C.). — *Optimal approximation by continued fractions*, J. Austral. Math. Soc. Ser. A, t. **50**, 1991, p. 481–504.

[FO] FRIEDMAN (N.A.) and ORNSTEIN (D.S.). — *On isomorphisms of weak Bernoulli transformations*, Adv. in Math., t. **5**, 1970, p. 365–394.

[K1] KRAAIKAMP (C.). — *A new class of continued fraction expansions*, Acta Math., t. LVII, 1991, p. 1–39.

[K2] KRAAIKAMP (C.). — *Statistic and ergodic properties of Minkowski's Diagonal Continued Fraction*, Theoret. Comput. Sci., t. **65**, 1989, p. 197–212.

[L] LIARDET (P.). — *Question posed at the public defence of the dissertation of C. Kraaikamp*, Amsterdam, 1990, April 23.

[NIT] NAKADA (H.), ITO (Sh.) and TANAKA (S.). — *On the invariant measure for the transformation associated with some real continued fraction*, Keio Eng. Rep., t. **30**, 1977, p. 159–175.

[N] NAKADA (H.). — *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math., t. **4**, 1981, p. 399–426.

[OW] ORNSTEIN (D.S.) and WEISS (B.). — *Statistical Properties of Chaotic Systems*, Bull. Amer. Math. Soc., t. **24**, 1991, p. 11–116.

[R] RIEGER (G.J.). — *Mischung und Ergodizität bei Kettenbrüchen nach nächsten Ganzen*, J. Reine Angew. Math., t. **310**, 1979, p. 171–181.