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HERVÉ JACQUET
YANGBO YE

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RELATIVE KLOOSTERMAN INTEGRALS FOR GL(3)

BY

HERVÉ JACQUET AND YANGBO YE (*)

RÉSUMÉ. — Nous démontrons, dans le cas particulier du groupe GL(3), l'égalité de deux intégrales locales. L'une est une intégrale de Kloosterman, c'est-à-dire l'analogue local d'une somme de Kloosterman ; l'autre est un nouveau type d'intégrale de Kloosterman ; c'est une intégrale de Kloosterman "relative". Nous conjecturons cette égalité pour tout GL(m). L'égalité aura des applications à la théorie du changement de base dans le cas quadratique.

ABSTRACT. — We prove, in the case of the group GL(3), the equality of two local integrals. One is a Kloosterman integral, that is to say, the local analogue of a Kloosterman sum ; the other is a new kind of Kloosterman integral, a "relative" Kloosterman integral. We conjecture that the equality is true for GL(m). The equality will have applications to the theory of the base change in the quadratic case.

1. Introduction

We would like to present evidence for a conjecture of importance to representation theory. This conjecture may be of independent interest. We consider a local non Archimedean field F of odd residual characteristic and a non ramified quadratic extension E of F ; we denote by $z \mapsto \bar{z}$ or σ the conjugation with respect to F, by ζ the quadratic character of $F^\times$ attached to E. We denote by R_F and R_E the rings of integers, by P_F and P_E the maximal ideals and by q the cardinality of R_F/P_F ; we choose an additive character ψ_F of F of conductor R_F and set $\psi_E(z) = \psi_F(z+\bar{z})$. We denote by $|\cdot|_F$ and $|\cdot|_E$ the absolute values on E and F. We drop the indices if this does not create confusion. We choose for additive Haar measures on F and E the measures self-dual for ψ_F and ψ_E. The multiplicative Haar measures on $F^\times$ and $E^\times$ are the ones for which the groups of units have volume 1.

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H. JACQUET, Columbia University, Dept. of Mathematics, New York, N.Y. 10027, USA. Partially supported by NSF Grant DMS-88-01759.
Y. YE, The University of Iowa, Dept. of Mathematics, Iowa City, Iowa 52242, USA.

If s is any matrix we set $s^* = {}^t\bar{s}$; we denote by S_m^E or simply S_m or even S the set of invertible $m \times m$ matrices s such that $s^* = s$. We denote by G the group $\mathrm{GL}(m, E)$ regarded as an algebraic group over F; thus $G(F) = \mathrm{GL}(m, E)$. Let N be the group of upper triangular matrices with unit diagonal. The group $G(F)$ operates on S by $s \mapsto g^*sg$. The orbits of $N(E)$ can easily be described but, for the moment, we consider only the orbits of maximal dimension. Their union is an open set. Let A be the group of diagonal matrices. Then each orbit of maximum dimension has exactly one representative in $A(F)$. Let K be $\mathrm{GL}(m, R_E)$ and Φ the characteristic function of $K \cap S$; we recall that $K \cap S$ is identical to the orbit of 1_m under K. We use the character ψ_E to define a generic character θ_E of $N(E)$:

$$\theta_E(n) = \psi_E\Big(\sum_{1\le i\le m-1} n_{i,i+1}\Big).$$

For $a \in F$ we define an "orbital integral" :

$$J(a) = \int_{N(E)} \Phi(n^*an)\theta_E(n)\,\mathrm{d}n \tag{1}$$

On the other hand, let G' be the group $\mathrm{GL}(m, F)$, regarded as an algebraic group defined over F. Then $N(F) \times N(F)$ operates on $G'(F)$ by $g' \mapsto {}^tn_1g'n_2$. We define a generic character θ_F of $N(F)$:

$$\theta_F(n) = \psi_F\Big(\sum_{1\le i\le m-1} n_{i,i+1}\Big).$$

Let $K' = \mathrm{GL}(m, R_F)$ and f' be the characteristic function of K'. We define an orbital integral :

$$I(a) = \int_{N(F)\times N(F)} f'({}^tnan_2)\theta_F(nn_2)\,\mathrm{d}n\mathrm{d}n_2 \tag{2}$$

The measures are such that the volumes of $N(E) \cap K$ and $N(F) \cap K'$ are one. The conjecture asserts that

$$I(a) = \gamma(a)J(a)\,,$$

where γ is a quadratic character of $A(F)$. More precisely, define the fundamental dominant weights Λ_i, $1 \le i \le n$, by :

$$\Lambda_i(a) = \prod_{1\le j\le i} a_j$$

then :

$$\gamma(a) = \prod_{1\leq i\leq n} \zeta\big(\Lambda_i(a)\big).$$

The purpose of this paper is to prove the conjecture for $m = 3$. For $m = 2$ it has been proved in [Y] or [J-Y]. ZAGIER proves the same identity (expressed in classical language) in [Z].

The conjecture is interesting in its own right. The integral on the left is a Kloosterman integral, the local analogue of a Kloosterman sum. The integral on the right is then a new kind of Kloosterman integral : we call it a *relative Kloosterman integral.* We refer the reader to [Z], [G] and the bibliography therein for background material on Kloosterman sums.

The applications to representation theory can be roughly described as follows. We go to a Global situation assuming E and F to be number fields. Let H be the unitary group attached to some $\epsilon \in S$. An automorphic representation π of $\mathrm{GL}(m, E_{\mathbb{A}})$ is said to be distinguished with respect to H if there is a form ϕ in the space of π such that

$$\int_{H(F)\backslash H(F_{\mathbb{A}})} \phi(h)\,\mathrm{d}h$$

is non zero. This notion was first introduced in [H-L-R] . An argument which is essentially in [H-L-R] shows that if π is H-distinguished then it is invariant under σ and thus is a base change ([A-C]). We conjecture the converse. An approach to the proof of the converse

$$\iint K(h,n)\,\mathrm{d}h\,\theta_E(n)\,\mathrm{d}n = \iint K'(n_1,n_2)\theta_F(n_1n_2)\,\mathrm{d}n_1\mathrm{d}n_2$$

where K and K' are the kind of kernel one considers in the trace formula. They are attached to functions f and f' on $G(F_{\mathbb{A}})$ and $G'(F_{\mathbb{A}})$ respectively. The integral on the left depends only on the integral

$$\int_{H(F_{\mathbb{A}})} f(hg)\,\mathrm{d}h\,.$$

This may viewed as a function Φ on $S(F_{\mathbb{A}})$ and the relation between Φ and f' is of the type we have just considered. The identity we conjecture is then the "fundamental lemma for the unit element of the Hecke algebra" for the trace formula considered. Thus our result is a first step in Ye's research program, as outlined in [J-Y]. See also [Y], [J]. For a more classical interpretation, see [Z].

We will write $\mathrm{diag}(a_1, a_2, \dots, a_m)$ for the diagonal matrix with eigenvalues a_i and

$$I(a_1, a_2, \dots, a_n)$$

for the value of I on it and likewise for J. Recall that $I(a) \neq 0$ implies $|\det a| = 1$. Likewise for J. Moreover, if $m = 1$, we have then $I(a) = J(a) = 1$. If u is a unit in F then $I(ua) = I(a)$ and likewise for J.

For $m = 2$, the conjecture reads :

PROPOSITION. — $I(a,b) = \zeta(a)J(a,b) = \zeta(b)J(a,b)$.

In [Y] and [J-Y] the relation is formulated as an integral formula :

$$\int J(ab, b)\,\mathrm{d}^\times b = \int I(ab, b)\zeta(b)\,\mathrm{d}^\times b.$$

It is easy to see this implies the proposition. Indeed, fix a. Then each integrand vanishes unless $|b^2 a| = 1$. It follows that

$$J(ab, b) = \zeta(b) I(ab, b)\,.$$

or more simply :

$$I(a, b) = \zeta(a) J(a, b)\,.$$

Our purpose is to establish the conjecture for $\mathrm{GL}(3)$:

THEOREM. — $I(a,b,c) = \zeta(b)J(a,b,c)$.

We proceed as follows. In section 1, we establish a reduction formula. In sections 2 and 3, we find an explicit expression for the functions I and J in the $\mathrm{GL}(3)$ case. In sections 5 and 6, we compute the formal Mellin transform of the function I. In sections 7 and 8, we compute the formal Mellin transform of J. We compare them in section 9 and arrive then at our relation; just as in the $\mathrm{GL}(2)$ case ([J-Y]), the proof ultimately depends on a classical equality of Gaussian sums. The proof is elementary but intricate; clearly, new methods will have to be brought in to prove the equality in higher dimensions. Nonetheless, we feel our result is worth publishing. Finally, in section 10, we put our result in the context of Ye's research program.

2. A reduction formula

We give another formula for $I(a)$: in (2), the integrand vanishes unless ${}^t na$ is in $K'N(F)$; assuming this is the case, let us write :

$$ {}^t n = kn', $$

with $k \in K'$ and $n' \in N(F)$, and change n_2 to $n'^{-1}n_2$. The integral takes the form :

$$ \int f'(n_2)\theta_F(n_2)\,\mathrm{d}n_1 \int \theta_F(nn'^{-1})\,\mathrm{d}n. $$

The first integral is 1 and we find :

$$ I(a) = \int \theta_F(nn'^{-1})\,\mathrm{d}n. \tag{3} $$

For instance, for $m = 2$, we find :

$$ I(a,b) = \begin{cases} 1 & \text{if } \ |a| = 1\,; \\ \displaystyle\int_{|ax|=1} \psi_F\big(x - b/(ax)\big)\,\mathrm{d}x & \text{if } \ |a| < 1. \end{cases} $$

Let $e_i, 1 \le i \le m$, be the elements of the canonical basis of the space of column vectors. Recall that a matrix g is in $K'N(F)$ if and only if the following vectors

$$ ge_1,\ ge_1 \wedge ge_2,\ ge_1 \wedge ge_2 \wedge ge_3 \cdots \wedge ge_r, \dots,\ ge_1 \wedge ge_2 \wedge \cdots ge_m $$

have all norms 1. This already implies that if a is in compact set and ${}^t na \in K'N(F)$, then n is in compact set and

$$ |a_1 a_2 \cdots a_r| \le 1, \quad \text{for} \quad 1 \le r \le m-1, \ |\det a| = 1. \tag{4} $$

Thus $I(a)$ converges and defines a smooth function of a the support of which is contained in the set defined by the above inequalities.

Now suppose a is a diagonal matrix of the form

$$ a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, $$

where a_1 and a_2 have determinant one and size m_1 and m_2. Let us write an element $n \in N(F)$ in the form :

$$ n = \begin{pmatrix} n_1 & v \\ 0 & n_2 \end{pmatrix}; $$

then

$$ {}^t na = \begin{pmatrix} {}^t n_1 a_1 & 0 \\ v a_1 & {}^t n_2 a_2 \end{pmatrix}; $$

the above conditions already imply that the entries of v are integers; thus

$$ k_0 = \begin{pmatrix} 1_{m_1} & 0 \\ v & 1_{m_2} \end{pmatrix} $$

is in K' and

$$ {}^t na = k_0 \begin{pmatrix} {}^t n_1 a_1 & 0 \\ 0 & {}^t n_2 a_2 \end{pmatrix}; $$

it follows that the elements ${}^t n_i a_i$ have decompositions ${}^t n_i a_i = k_i n_i'$; finally we find

$$ {}^t na = k \begin{pmatrix} n_1' & 0 \\ 0 & n_2' \end{pmatrix}. $$

Thus we have :

$$ I(a) = \int \mathrm{d}v \int \theta_F(n_1 n_1'^{-1}) \, \mathrm{d}n_1 \int \theta_F(n_2 n_2'^{-1}) \, \mathrm{d}n_2, $$

or

$$ (5) \qquad I(a) = I(a_1) I(a_2) \int \mathrm{d}v = I(a_1)\, I(a_2). $$

Of course this is not a new relation (see [G]).

We now study the other integral. In the integral (1), the integrand vanishes unless $n^* a$ is in $KN(E)$. Assuming this is the case we write :

$$ n^* a = kn' $$

with $k \in K$ and $n' \in N(F)$, and then the integrand is zero unless $n'n$ is in K, in which case it is equal to $\theta_E(n)$. Thus we find :

$$ (6) \qquad J(a) = \int \theta_E(n) \, \mathrm{d}n $$

with domain

$$ n^* a = kn', \quad k \in K, \quad n' \in N(E), \quad n'n \in K. $$

For instance, for $m = 2$, we find :

$$ J(a,b) = 1 \quad \text{if} \quad |a| = 1; $$

for $|a|<1$, we find :

$$J(a,b)=\int \psi_E(x)\,\mathrm{d}x,$$

with domain :

$$|ax|=1,\quad \left|x+\frac{b}{a\,\bar{x}}\right|\leq 1.$$

Just as before, the integral converges and defines a smooth function J of a, with support in the set defined by 4. Moreover, if a is a diagonal matrix of the form

$$a=\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},$$

where a_1 and a_2 have determinant one, then the same argument as before shows that :

$$J(a)=J(a_1)J(a_2).$$

Let us apply this to the case $m=3$. If $|a|=1$ then

$$\begin{aligned} I(a,b,c) &= I(a)I(b,c)=I(b,c),\\ J(a,b,c) &= J(a)J(b,c)=J(b,c). \end{aligned}$$

In this case the theorem follows from the case $m=2$. Likewise if $|ab|=1$ then

$$I(a,b,c)=I(a,b)I(c)=I(a,b),$$
$$J(a,b,c)=J(a,b)J(c)=J(a,b).$$

The theorem follows again from the case $m=2$. We will let I_0 be the function equal to $I(a,b,c)$ if $|a|<1$, $|ab|<1$ and to 0 otherwise. We will define J_0 similarly. It will suffice to prove the relation

$$I_0(a,b,c)=\zeta(b)J_0(a,b,c). \tag{7}$$

3. Computation of I_0

Suppose

$$|a|<1,\quad |ab|<1,\quad |abc|=1.$$

We set $\alpha=\mathrm{diag}(a,b,c)$ and

$$n=\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then :

$$ {}^t n\alpha = \begin{pmatrix} a & 0 & 0 \\ ax & b & 0 \\ az & by & c \end{pmatrix}. $$

The conditions for this element to be in $K'N(F)$ are :

$$ \begin{cases} \sup\bigl(|a|,|ax|,|az|\bigr) = 1, \\ \sup\bigl(|ab|,|aby|,\bigl|ab(z-xy)\bigr|\bigr) = 1, \\ |abc| = 1. \end{cases} $$

It remains to compute the Iwasawa decomposition of the above element and the corresponding element n'; this depends on the absolute values of the elements listed above. We arrive at the following possibilities : if

$$ |az| = 1, \quad \bigl|ab(xy-z)\bigr| = 1, \quad |ax| < 1, \quad |aby| < 1 \tag{8} $$

then

$$ n' = \begin{pmatrix} 1 & (by)/(az) & c/(az) \\ 0 & 1 & (cx)/(b(xy-z)) \\ 0 & 0 & 1 \end{pmatrix}. $$

If

$$ |ax| = 1, \quad \bigl|ab(xy-z)\bigr| = 1, \quad |az| \le 1, \quad |aby| \le 1 \tag{9} $$

then

$$ n' = \begin{pmatrix} 1 & b/(ax) & 0 \\ 0 & 1 & (cx)/(b(xy-z)) \\ 0 & 0 & 1 \end{pmatrix}. $$

If

$$ |az| = 1, \quad |aby| = 1, \quad |ax| < 1, \quad \bigl|ab(xy-z)\bigr| \le 1 \tag{10} $$

then

$$ n' = \begin{pmatrix} 1 & (by)/(az) & c/(az) \\ 0 & 1 & c/(by) \\ 0 & 0 & 1 \end{pmatrix}. $$

There are no other possibilites. In particular, the case

$$ |ax| = 1, \quad |aby| = 1, \quad |az| \le 1, \quad |ab(xy-z)| < 1 $$

cannot occur as the relations are incompatible. We also remark that in case (9) the three first conditions imply the last one : in fact, multiplying the second relation by $|a|$ we get at once $|aby| < 1$. Similarly, in (11) the inequality $|ax| < 1$ is actually a consequence of the other relations.

Correspondingly, we find that $I(a,b,c)$ is the sum of three integrals I_1, I_2, I_3.

$$I_1 = \int \psi_F\Big(x + y - \frac{by}{az} - \frac{cx}{b(xy-z)}\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{11}$$

the domain of integration is defined by :

$$|ax| < 1, \quad |aby| < 1, \quad |az| = 1, \quad \big|ab(xy-z)\big| = 1\,;$$

$$I_2 = \int \psi_F\Big(x + y - \frac{b}{ax} - \frac{cx}{b(xy-z)}\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{12}$$

the domain of integration is defined by :

$$|ax| = 1, \quad |az| \le 1, \quad \big|ab(xy-z)\big| = 1\,;$$

$$I_3 = \int \psi_F\Big(x + y - \frac{by}{az} - \frac{c}{by}\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{13}$$

the domain of integration is defined by :

$$|az| = 1, \quad |aby| = 1, \quad \big|ab(xy-z)\big| \le 1.$$

To compute I_2, we replace y by $t = xy - z$. Then

$$I_2 = |a| \int \psi(x + \frac{t}{x} + \frac{z}{x} - \frac{b}{ax} - \frac{cx}{bt})\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}z\,;$$

the integration is over

$$|ax| = 1, \quad |az| \le 1, \quad |abt| = 1.$$

In particular, the integral in z disappears leaving :

$$I_2 = \int \psi\Big(x + \frac{t}{x} - \frac{b}{ax} - \frac{cx}{bt}\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

the integration is over

$$|ax| = 1, \quad |abt| = 1.$$

After changing t to tx, we get :

$$|a|^{-1}\int \psi\Big(x-\frac{b}{ax}+t-\frac{c}{bt}\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

the integration over

$$|ax|=1,\quad |bt|=1.$$

It will be convenient to introduce the notation :

$$K(u,v)=\int_{|ux|=1}\psi\Big(x-\frac{v}{ux}\Big)\,\mathrm{d}x.$$

We see that

$$I_2=|a|^{-1}K(a,b)K(b,c).$$

Similarly, in I_3, we can change x into $x+z/y$ to find that

$$I_3=\int \psi_F\Big(x+\frac{z}{y}+y-\frac{by}{az}-\frac{c}{by}\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

the domain of integration is now

$$|x|\le 1,\quad |az|=1,\quad |aby|=1.$$

The integration in x disappears and after changing z to zy we get :

$$I_3=|ab|^{-1}K(b^{-1},a^{-1})K(ab,ac).$$

Now, after a change of variables, we see that the function $K(u,v)$ has the following formal property :

$$|uv|K(u,v)=K(v^{-1},u^{-1}).$$

We easily conclude from this that :

$$I_2+I_3=2K(a,b)K(ab,ac). \tag{14}$$

At this point, we have to use another simple property of the function $K(u,v)$. Assume $|uv|<1$ and $|u|<1$. Then $K(u,v)=0$ if $|v|>1$. If $|v|\le 1$, then $K(u,v)=0$ unless $|u|=q^{-1}$ and then $K(u,v)=-1$. This follows from the orthogonality relations for characters.

We conclude that

$$I_2+I_3=2\quad\text{if}\quad |a|=q^{-1},\ |b|=1,\ |c|=q, \tag{15}$$

and is zero in all other cases. We denote this function of (a,b,c) by I_s ("supplementary term").

4. Computation of J_0

Let us compute $J_0(a,b,c)$. Recall that $|a|<1$, $|ab|<1$ and $|abc|=1$. We use the notations of the previous section. Our method is to compute ${}^t\bar{n}\alpha$, to write down the conditions for this matrix to be in $KN(E)$, to compute the corresponding decompostion

$$ {}^t\bar{n}\alpha = kn' $$

and to impose the condition that $n'n$ be in K. We find that J_0 is the sum of 3 integrals, J_1, J_2, J_3, with the same integrand but three different domains of integration.

The first integral is

$$ J_1 = \iint \psi_E(x+y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{16} $$

it is extended to the set :

$$ \begin{cases} |ax|<1, \quad |aby|<1, \quad |az|=1, \quad |ab(xy-z)|=1, \\ \left|x+\dfrac{b\bar{y}}{a\bar{z}}\right| \le 1, \quad \left|y+\dfrac{c\bar{x}}{b(\bar{x}\bar{y}-\bar{z})}\right| \le 1, \quad \left|z+\dfrac{c}{a\bar{z}}+\dfrac{by\bar{y}}{a\bar{z}}\right| \le 1. \end{cases} \tag{17} $$

The second integral J_2 is given by the same integrand but the domain of integration is now :

$$ \begin{cases} |ax|=1, \quad |ab(xy-z)|=1, \quad |az|\le 1, \\ |x+\dfrac{b}{a\bar{x}}| \le 1, \quad |z+\dfrac{by}{a\bar{x}}| \le 1, \quad \left|y+\dfrac{ac\bar{x}}{ab(\bar{x}\bar{y}-\bar{z})}\right| \le 1. \end{cases} $$

Recall that the three first conditions imply in fact that :

$$ |aby|<1. $$

If we change z to

$$ z-\frac{by}{a\bar{x}} $$

we find two new conditions :

$$ |z|\le 1, \quad \left|abxy-abz+\frac{ab^2y}{a\bar{x}}\right| = 1. $$

Since $|ab|<1$, we find :

$$ |aby|\cdot\left|x+\frac{b}{a\bar{x}}\right| = 1. $$

Since in fact $|aby| < 1$, this is incompatible with the requirement that

$$\left| x + \frac{b}{a\bar{x}} \right| \leq 1.$$

The domain of integration is thus empty and the integral zero.

Similarly, the domain of integration for the third integral J_3 is given by :

$$\begin{cases} |az| = 1, \quad |aby| = 1, \quad \big|ab(xy - z)\big| \leq 1, \\ \left| x + \dfrac{b\bar{y}}{a\bar{z}} \right| \leq 1, \quad \left| y + \dfrac{c}{b\bar{y}} \right| \leq 1, \quad \left| z + \dfrac{c}{a\bar{z}} + \dfrac{by\bar{y}}{a\bar{z}} \right| \leq 1. \end{cases}$$

If we multiply the last relation by $a^3 b\bar{z}$ we get

$$\big| abaza\bar{z} + a^2abc + a^2b^2y\bar{y} \big| \leq |a| \cdot |ab| < 1.$$

This in turn implies $|aby| < 1$, a contradiction. So J_3 is zero also and we conclude that J_0 is in fact equal to J_1.

5. Mellin transform of I_0

We denote by $\chi_i, 1 \leq i \leq 3$, three (quasi-)characters of $F^\times$ and by $\hat{I}_0$ the formal Mellin transform :

$$\text{(18)} \qquad \iiint I_0(abc, bc, c)\chi_1(a)\chi_2(b)\chi_3(c)\, \mathrm{d}^\times a\, \mathrm{d}^\times b\, \mathrm{d}^\times c.$$

The character χ_3 is fixed, unramified, trivial or quadratic. We set

$$\chi_3(x) = |x|^{-t}, \quad q^{-t} = Z.$$

Thus $Z^2 = 1$, and $Z = -1$ if $\chi_3 = \zeta$. We also set :

$$\chi_i(x) = |x|^{t_i}\chi_{i,0}(x), \quad q^{-t_i} = X_i.$$

The characters $\chi_{i,0}$ are unitary and fixed; we take them trivial if they are unramified. We call $P_F^{m_i}$ the conductor of χ_i. The integral in (a, b) does not converge, but the Mellin transform is to be interpreted as a formal Laurent series in the 2 variables X_1, X_2. Recall that I_0 is the sum of an integral I_1 and a supplementary term I_s ; we define similarly $\hat{I}_1$ and $\hat{I}_s$.

We first modify the expression for I_1. We write it :

$$I_1 = \int \psi_F\Big(x + y - \frac{by}{az} - \frac{acx}{abz'}\Big)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

The domain of integration is defined by :

$$\begin{cases} z+z'=xy, \\ |ax|<1, \quad |aby|<1, \quad |a|<1, \quad |ab|<1, \\ |az|=1, \quad |abz'|=1, \quad |abc|=1. \end{cases}$$

It will be more convenient to write these conditions as follows :

$$\begin{cases} |az|=1, \quad |z'|=|c|, \quad |abc|=1, \\ |ax|<1, \quad |y|<|c|, \quad |a|<1, \quad 1<|c|. \end{cases}$$

We now replace (a,b,c) by (abc,bc,c); the integral takes the form :

$$\int \psi_F\Big(x+y-\frac{y}{az}-\frac{x}{bz'}\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

The domain of integration (and definition of the function) is now :

$$\begin{cases} |abcz|=1, \quad |z'|=|c|, \quad |ab^2c^3|=1, \\ |abcx|<1, \quad |y|<|c|, \quad |abc|<1, \quad 1<|c|. \end{cases}$$

Finally, we change z to zxy (and z' to $z'xy$). The integral takes the from :

$$\int \psi_F\Big(x+y-\frac{1}{axz}-\frac{1}{byz'}\Big)|xy|\,\mathrm{d}x\,\mathrm{d}y\mathrm{d}z.$$

The range of the integration is now :

$$\begin{cases} z+z'=1, \\ |abcxyz|=1, \quad |xyz'|=|c|, \quad |ab^2c^3|=1, \\ 1<|zy|, \quad 1<|zxy|, \quad 1<|z'x|, \quad 1<|z'xy|. \end{cases}$$

We substitute this in the expression for $\hat{I}_1$. We first compute formally and then justify our steps. In the expression for $\hat{I}_1$ we have just obtained we change :

$$a \text{ to } \frac{a}{zx}, \quad b \text{ to } \frac{b}{z'y}.$$

The integral takes the form :

$$\hat{I}_1=\int \psi(x+y-a^{-1}-b^{-1})\,|x|\chi_1^{-1}(x)|\,y|\chi_2^{-1}(y)$$
$$\chi_1(a)\chi_2(b)\chi_3(c)\chi_1^{-1}(z)\chi_2^{-1}(z')\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}z\,\mathrm{d}^\times c.$$

The range of integration is now :

$$\begin{cases} |abxy| = 1, & \left|\dfrac{bc}{y}\right| = |z|, \quad |abc| = |z'|, \\ 1 < |zy|, & 1 < |zxy|, \quad 1 < |z'x|, \quad 1 < |z'xy|. \end{cases}$$

Finally, we change c to $c/(ab)$ and a, b to minus their inverses. We get :

$$\hat{I}_1 = \chi_1\chi_2(-1) \int \psi_F(x+y+a+b)|x|\chi_1^{-1}(x)\,|y|\chi_2^{-1}(y) \\ \chi_3^{-1}\chi_1^{-1}(a)\chi_3^{-1}\chi_2^{-1}(b)\chi_1^{-1}(z)\chi_2^{-1}(z')\chi_3(c) \\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}z\,\mathrm{d}^\times c.$$

The domain of integration is

$$\begin{cases} z + z' = 1, \quad |c| = |z'|, \\ \left|\dfrac{z}{z'}\right| = \left|\dfrac{a}{y}\right| = \left|\dfrac{x}{b}\right|, \\ 1 < |zy|, \quad 1 < |z'x|, \quad 1 < |zxy|, \quad 1 < |xyz'|. \end{cases}$$

We see that the integration in c disappears; so our final result is the integral :

$$\hat{I}_1 = \chi_1\chi_2(-1) \int \psi_F(x+y+a+b)|x|\chi_1^{-1}(x)|y|\chi_2^{-1}(y) \\ \chi_3^{-1}\chi_1^{-1}(a)\chi_3^{-1}\chi_2^{-1}(b)\chi_1^{-1}(z)\chi_3\chi_2^{-1}(z') \\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}z. \tag{19}$$

The domain of integration is

$$\begin{cases} z + z' = 1, \quad \left|\dfrac{z}{z'}\right| = \left|\dfrac{a}{y}\right| = \left|\dfrac{x}{b}\right|, \\ 1 < |zy|, \quad 1 < |z'x|, \quad 1 < |zxy|, \quad 1 < |xyz'|. \end{cases} \tag{20}$$

To give a meaning to the integral, we consider the integral

$$\int I_1(abc, bc, c)\chi_1(a)\chi_2(b)\chi_3(c)\phi_1(a)\phi_2(b)\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}^\times c,$$

where ϕ is the characteristic function of the set $q^{-A} \leq |x| \leq q^A$. The integrand has then compact support, thus represents a Laurent polynomial in X_1, X_2. As A tends to infinity, this Laurent polynomial tends to the

Laurent series which is the formal Mellin transform : here the topology on the space of Laurent series is the pointwise convergence of the coefficients.

We can perform our sequences of manipulations on this convergent integral; we obtain the same integrand as before, with an extra condition on the domain of integration, namely,

$$q^{-A} \leq |azx| \leq q^{A}, \quad q^{-A} \leq |bz'y| \leq q^{A}. \tag{21}$$

The integral is then convergent and represents a Laurent polynomial. As A tends to infinity, this polynomial approaches the Mellin transform. We keep this interpretation in mind while computing our integrals.

Finally, we recall that an integral of the form :

$$\int \psi_F(x)\chi(x)\,\mathrm{d}x$$

can be interpreted as a formal Laurent series. If χ is ramified of conductor P_F^m then the integral is equal to the integral

$$\int_{|x|=q^m} \psi_F(x)\chi(x)\,\mathrm{d}x.$$

If χ is unramified, then the integral is equal to the integral

$$\int_{|x|\leq q} \psi_F(x)\chi(x)\,\mathrm{d}x.$$

We recall also that

$$\int_{|x|=q} \psi_F(x)\,\mathrm{d}x = -1.$$

Thus we may integrate over $|x|$, $|y|$, $|a|$, $|b|$ less than a fixed constant depending only on the ramification of the characters.

6. Computation of $\hat{I}_0$

In this section we compute $\hat{I}_0$. Recall the function I_0 is equal to the sum $I_1 + I_s$ where I_s is defined by :

$$I_s(a,b,c) = 2 \quad \text{if} \quad |a| = q^{-1}, \ |b| = 1, \ |c| = q,$$

and $I_s = 0$ otherwise. Equivalently :

$$I_s(abc,bc,c) = 2 \quad \text{if} \quad |a| = q^{-1}, \ |b| = q^{-1}, \quad |c| = q,$$

and $I_s = 0$ otherwise. Of course the Mellin transform $\hat{I}_s$ is zero unless χ_1, χ_2 are unramified. Then it is equal to

$$2X_1X_2Z.$$

In the other cases, $\hat{I}_0 = \hat{I}_1$. The computation of $\hat{I}_1$ will depend on the conductors; recall $P_F^{m_i}$ denotes the conductor of χ_i.

6.1. The case $m_1 > 0$, $m_2 > 0$.

If $m_1 > 0, m_2 > 0$ we can take the integration in (20) over the set :

$$|a| = |x| = q^{m_1}, \quad |b| = |y| = q^{m_2}.$$

Suppose first that $m_1 = m_2$. We recall (20). We have then $|z| = |1-z|$. This implies in particular $|z| \geq 1$ so the inequalities in (20) are satisfied. After converting the multiplicative Haar measures to additive ones, we find :

$$(22)\quad \hat{I}_1 = (1-q^{-1})^{-2}\chi_1\chi_2(-1)\int_{|x|=q^{m_1}} \chi_1^{-1}(x)\psi_F(x)\,\mathrm{d}x$$
$$\int_{|a|=q^{m_1}} \chi_1^{-1}\chi_3^{-1}(a)\psi_F(a)\,\mathrm{d}a \int_{|y|=q^{m_2}} \chi_2^{-1}(y)\psi_F(y)\,\mathrm{d}y$$
$$\int_{|b|=q^{m_2}} \chi_2^{-1}\chi_3^{-1}(b)\psi_F(b)\,\mathrm{d}b \int_{|z|=|1-z|} \chi_1^{-1}(z)\chi_3\chi_2^{-1}(1-z)\,\mathrm{d}z$$

the variable z is restricted by the condition $|z| = |1-z|$. To make the integral converge we impose condition (21) and then let A tend to infinity; we conclude that our Mellin transform is given by the above expression, where the last integral is interpreted as a formal Laurent series.

Suppose that $m_1 > m_2$. Then

$$\left|\frac{z}{1-z}\right| = q^{m_1-m_2}.$$

This amounts to

$$|1-z| = q^{m_2-m_1}.$$

The other inequalities are then satisfied and as before we find :

$$(23)\quad \hat{I}_1 = (1-q^{-1})^{-2}\chi_1\chi_2(-1)\int_{|x|=q^{m_1}} \chi_1^{-1}(x)\psi_F(x)\,\mathrm{d}x$$
$$\int_{|a|=q^{m_2}} \chi_1^{-1}\chi_3^{-1}(a)\psi_F(a)\,\mathrm{d}a \int_{|y\ =q^{m_2}} \chi_2^{-1}(y)\psi_F(y)\,\mathrm{d}y$$
$$\int_{|b|=q^{m_2}} \chi_2^{-1}\chi_3^{-1}(b)\psi_F(b)\,\mathrm{d}b \int_{|1-z|=q^{m_2-m_1}} \chi_1^{-1}(z)\chi_3\chi_2^{-1}(1-z)\,\mathrm{d}z$$

If $m_2 > m_1$ we find :

$$(24)\quad \hat{I}_1 = (1-q^{-1})^{-2}\chi_1\chi_2(-1)\int_{|x|=q^{m_1}} \chi_1^{-1}(x)\psi_F(x)\,\mathrm{d}x$$
$$\int_{|a|=q^{m_1}} \chi_1^{-1}\chi_3^{-1}(a)\psi_F(a)\,\mathrm{d}a \int_{|y|=q^{m_2}} \chi_2^{-1}(y)\psi_F(y)\,\mathrm{d}y$$
$$\int_{|b|=q^{m_2}} \chi_2^{-1}\chi_3^{-1}(b)\psi_F(b)\,\mathrm{d}b \int_{|z|=q^{m_1-m_2}} \chi_1^{-1}(z)\chi_3\chi_2^{-1}(1-z)\,\mathrm{d}z.$$

6.2. The case : $m_1 > 0$, $m_2 = 0$.

We now assume $m_1 > 0, m_2 = 0$. We can take the integration over the set :

$$|a| = |x| = q^{m_1}.$$

Thus we find (see (20)) :

$$|y| = |b|.$$

In addition we have $|b| = |y| \leq q$. The contribution of the term $|y| = |b| = q$ can be computed with the formulas (22) to (24), where $P_F^{m_1}$ is the conductor of χ_1 but m_2 of the formulas is replaced by 1, the character χ_2 being unramified.

We now discuss the contribution of a term $|b| = |y| = q^{-r}$, $r \geq 0$. We have then

$$\left|\frac{z}{1-z}\right| = q^{m_1+r}.$$

This implies

$$|z| = 1.$$

But then the required inequality $1 < |zy|$ cannot be met. Thus there is no contribution and the integral $\hat{I}_1$ is given by the formulas (22) to (24), where the m_2 of of the formulas is replaced by 1.

Similarly, if $m_2 > 0$ and $m_1 = 0$, we find the integral $\hat{I}_1$ is given by the formulas (22) to (24), where the m_1 of of the formula is replaced by 1.

6.3. The case $m_1 = m_2 = 0$.

We now assume χ_1 and χ_2 unramified. In our integral we can take

$$|a|,\ |b|,\ |x|,\ |y|$$

all less than or equal to q. From this and (20), we derive that $|z| \geq 1$ and $|1-z| \geq 1$; this implies $|z| = |1-z|$ and thus $|a| = |y|$ and $|x| = |b|$.

The term corresponding to $|a| = |y| = |x| = |b| = q$ will be called the top term and noted $\hat{I}_t$. It is again given by formula (22) with m_1, m_2 of the formula replaced by 1.

Consider now the contribution to the integral of the set defined by :

$$|a| = |y| = q, \quad |x| = |b| \leq 1.$$

Formally, it can be written :

$$(1 - q^{-1})^{-2} X_1 X_2 Z \int_{|y|=q} \psi_F(y)\,\mathrm{d}y \int_{|a|=q} \psi_F(a)\,\mathrm{d}a$$
$$\iiint \chi_1^{-1}(x)\chi_3^{-1}\chi_2^{-1}(b)\chi_1^{-1}(z)\chi_3\chi_2^{-1}(1-z)\,\mathrm{d}b\,\mathrm{d}x\,\mathrm{d}z\,;$$

the two first integrals are equal to -1 so their product is one. To evaluate the remaining integral we write

$$|z| = q^r$$

with $r \geq 0$; for $r = 0$ we demand that $|1-z| = 1$. We have then $|x| = q^{-m}$ with $0 \leq m < r$. Thus $r > 0$. We get then :

$$(1 - q^{-1}) X_1 X_2 Z \sum_{r,m} X_1^{r-m} X_2^{r-m} Z^{-r-m} q^{-2m+r},$$

the sum over :

$$0 \leq m < r.$$

Recall that to sum this series, we must impose condition (21) and then let A tend to infinity. That amounts to take $r - m < B$ for an appropriate B and then let B tend to infinity. We are led to set $s = r - m$; the above expression can then be written :

$$(1 - q^{-1}) X_1 X_2 Z \sum_{s>0,\, m\geq 0} (X_1 X_2)^s q^{s-m} Z^{-s-2m}.$$

This is thus a product :

$$(1 - q^{-1}) \sum_m q^{-m} Z^{-2m} X_1 X_2 Z \sum_s (X_1 X_2 Z^{-1})^s q^s.$$

At this point, we see that the meaning we give to this expression is just the usual one as a formal Laurent series and we find :

$$\frac{1 - q^{-1}}{1 - q^{-1} Z^{-2}} \frac{X_1^2 X_2^2 q}{1 - X_1 X_2 q Z^{-1}}.$$

Since $Z^2 = 1$ this reduces to :

$$\frac{X_1^2 X_2^2 q}{1 - X_1 X_2 q Z}. \tag{25}$$

The contribution of the set $|x| = |b| = q$ is the same.

It remains to find the contribution of the set :

$$|a| = |y| \leq 1, \quad |x| = |b| \leq 1.$$

Formally, it can be written :

$$(1 - q^{-1})^{-2} \int \chi_1^{-1}(x)\chi_1^{-1}\chi_3^{-1}(a)\chi_2^{-1}(y)\chi_3^{-1}\chi_2^{-1}(b)$$
$$\chi_1^{-1}(z)\chi_3\chi_2^{-1}(1 - z)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

We write :

$$|a| = |y| = q^{-n_1}, \quad |x| = |b| = q^{-n_2}, \quad |z| = q^r,$$

where

$$0 \leq n_1, \quad 0 \leq n_2, \quad n_1 + n_2 < r.$$

We find

$$(1 - q^{-1})^3 \sum_{r, n_1, n_2} (X_1 X_2)^{r-(n_1+n_2)} q^{r-2n_1-2n_2} Z^{-r-n_1-n_2}.$$

To sum this series we must again impose condition (21) and let A tend to infinity. This amount to demand that

$$r - n_1 - n_2 \leq B$$

for a suitable B and then let B tend to infinity. We are led to set :

$$s = r - n_1 - n_2.$$

The above expression takes then the form :

$$(1 - q^{-1})^3 \sum (X_1 X_2)^s q^{s-n_1-n_2} Z^{-s-2n_1-2n_2},$$

the sum over

$$0 < s, \; 0 \leq n_1, \; 0 \leq n_2.$$

Again, this decomposes into a product :

$$(1-q^{-1})^3 \sum_{n_1,n_2\geq 0} q^{-n_1-n_2} Z^{-2n_1-2n_2} \sum_{s>0} (X_1X_2)^s q^s Z^{-s}.$$

The meaning of this expression is again the usual one and we find :

$$\frac{(1-q^{-1})^3}{(1-q^{-1}Z^{-2})^2} \frac{X_1X_2qZ^{-1}}{1-X_1X_2qZ^{-1}}.$$

Since $Z^2 = 1$ this reduces to :

(26) $$(1-q^{-1})\frac{X_1X_2qZ}{1-X_1X_2qZ}.$$

To find the Mellin transform of I_0 we must add the term

$$\hat{I}_s = 2X_1X_2Z.$$

Summing up, we see that the Mellin transform of I_0 is then equal to the top term $\hat{I}_t$ plus :

$$\frac{2(X_1X_2)^2q}{1-X_1X_2Zq} + 2X_1X_2Z + \frac{(1-q^{-1})X_1X_2Zq}{1-X_1X_2Zq}.$$

After simplification, we find :

(27) $$\widehat{I_0} = \hat{I}_t + (1+q^{-1})\frac{X_1X_2Zq}{1-X_1X_2Zq}.$$

7. The Mellin transform of J_0

We denote by $\widehat{J_0}$ the formal Mellin transform :

$$\iint J_0(abc, bc, c)\chi_1(a)\chi_2(b)\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}^\times c,$$

where the χ_i are as before.

We first modify the expression for $J_0 = J_1$. We first change z to zxy. We recall (17). We obtain :

$$J_1 = \int \psi_E(x+y)|xy|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

The domain of integration is now :

$$\begin{cases} z+z'=1, \quad 1<|yz|, \quad 1<|xz'|, \quad 1<|xyz|, 1<|xyz'|, \\ |axyz|=1, \quad |xyz'|=|c|, \quad |abc|=1, \\ \left|x+\dfrac{b}{a\bar{x}\bar{z}}\right|\leq 1, \left|y+\dfrac{c}{b\bar{y}\bar{z}'}\right|\leq 1, \quad \left|az\bar{z}x\bar{x}y\bar{y}+c+by\bar{y}\right|\leq 1. \end{cases}$$

We remark that $|\cdot|=|\cdot|_E$ is the absolute value on E. Next, we replace (a,b,c) by (abc,bc,c). The new domain of integration is :

$$\begin{cases} |a|=\left|\dfrac{z'}{z^2xy}\right|, \quad |b|=\left|\dfrac{z}{z'^2xy}\right|, \\ |c|=|xyz'|, \quad 1<|yz|, \quad 1<|xz'|, \quad 1<|xyz|, 1<|xyz'|, \\ |ax\bar{x}\bar{z}+1|\leq\left|\dfrac{z'}{zy}\right|, \quad |by\bar{y}\bar{z}'+1|\leq\left|\dfrac{z}{z'x}\right|, \\ |abcx\bar{x}y\bar{y}z\bar{z}+c+bcy\bar{y}|\leq 1. \end{cases}$$

We replace J_0 by this integral in the expression for $\widehat{J_0}$. We compute formally and then justify our steps. In the expression for $\widehat{J_0}$ we have just obtained we change

$$a \quad \text{to} \quad \frac{a}{x\bar{x}z\bar{z}}, \quad b \quad \text{to} \quad \frac{b}{y\bar{y}z'\bar{z}'},$$

the integral for $\widehat{J_0}$ takes the form :

$$\int \psi_E(x+y)|xy|\chi_1^{-1}(x\bar{x})\chi_2^{-1}(y\bar{y})\chi_1(a)\chi_2(b) \\ \chi_1^{-1}(z\bar{z})\chi_2^{-1}(z'\bar{z}')\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}^\times c\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

The domain of integration is now :

$$\begin{cases} |c|=|xyz'|, \quad 1<|yz|, \quad 1<|xz'|, \quad 1<|xyz|, \quad 1<|xyz'| \\ \left|a+z\right|\leq\left|\dfrac{z'}{y}\right|=\left|\dfrac{a}{x}\right|, \quad \left|b+z'\right|\leq\left|\dfrac{z}{x}\right|=\left|\dfrac{b}{y}\right|, \quad |ab+z'\bar{z}'+b|\leq\left|\dfrac{z'}{xy}\right|. \end{cases}$$

In particular, the integration for c disappears leaving our final expression for $\widehat{J_0}$:

$$(28) \quad \widehat{J_0}=\int \psi_E(x+y)|xy|\chi_1^{-1}(x\bar{x})\chi_2^{-1}(y\bar{y})\chi_1(a)\chi_2(b) \\ \chi_1^{-1}(z\bar{z})\chi_2^{-1}(z'\bar{z}')\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

The domain of integration is :

$$(29)\qquad \begin{cases} z+z'=1, \quad 1<|yz|, \quad 1<|xz'|, \quad 1<|xyz|, \quad 1<|xyz'| \\ \left|a+z\right| \le \left|\dfrac{z'}{y}\right| = \left|\dfrac{a}{x}\right|, \quad \left|b+z'\right| \le \left|\dfrac{z}{x}\right| = \left|\dfrac{b}{y}\right|, \\ |ab+z'\bar{z}'+b| \le \left|\dfrac{z'}{xy}\right|. \end{cases}$$

To give a meaning to this integral, we consider again the characteristic function ϕ of the set

$$q^{-A} \le |x| \le q^{A}$$

in $F^\times$. We consider the integral :

$$\iint J_0(abc, bc, c)\chi_1(a)\chi_2(b)\phi(a)\phi(b)\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}^\times c.$$

It converges and represents a Laurent polynomial in X_1, X_2. We apply our sequence of formal manipulations to this new integral. We obtain the same expression with an extra condition on the domain of integration :

$$(30)\qquad q^{-A} \le \left|\frac{a}{x\bar{x}z\bar{z}}\right|_F \le q^A, \quad q^{-A} \le \left|\frac{b}{y\bar{y}z'\bar{z}'}\right|_F \le q^A.$$

The integral converges and represents a Laurent polynomial. To obtain the formal Mellin transform we let A tend to infinity.

It will be convenient to discuss some simple consequences of the inequalities (29). We claim that the inequalities

$$|x|>1, \quad |y|\le 1$$

are not compatible with them. Indeed we have then

$$|z| > |y|^{-1} > 1.$$

Hence

$$|z| = |1-z| > 1.$$

On the other hand,

$$|b| \le \left|\frac{b}{y}\right|;$$

thus the inequality

$$|b+1-z| \le \left|\frac{z}{x}\right|$$

is equivalent to

$$|1-z| \leq \left|\frac{z}{x}\right|$$

or

$$|x| \leq 1,$$

a contradiction. Similarly, the inequalities

$$|x| \leq 1, \quad |y| > 1$$

are not compatible with the inequalities (29).

Now suppose $|x| \leq 1$ and $|y| \leq 1$. The required inequalities

$$1 < |yz|, \quad 1 < |xz'|, \quad 1 < |xyz|, \quad 1 < |xyz'|$$

reduce to the single inequality

$$|z| > |xy|^{-1}.$$

We have then $|z| = |1-z| > 1$. Next, assume

$$|a| = \left|\frac{zx}{y}\right|, \quad |b| = \left|\frac{zy}{x}\right|.$$

Then the required inequalities

$$\left|a+z\right| \leq \left|\frac{z'}{y}\right| = \left|\frac{a}{x}\right|, \quad \left|b+z'\right| \leq \left|\frac{z}{x}\right| = \left|\frac{b}{y}\right|$$

are implied by the previous conditions. The remaining required inequality is then equivalent to

$$\left|\frac{ab}{(1-z)(1-\bar{z})}+1\right| \leq \frac{1}{|zxy|}.$$

In summary then, the domain of integration in this case is defined by the following conditions :

$$(31) \qquad \begin{cases} |x| \leq 1, \quad |y| \leq 1, \quad |z| > |xy|^{-1} \\ |a| = \left|\dfrac{zx}{y}\right|, \quad |b| = \left|\dfrac{zy}{x}\right|, \\ \left|\dfrac{ab}{(1-z)(1-\bar{z})}+1\right| \leq \dfrac{1}{|zxy|}. \end{cases}$$

Now assume $|x| > 1$, $|y| > 1$. The required inequality

$$|a+z| \leq \left|\frac{a}{x}\right|$$

will be written

$$z = -a(1+u)$$

where

$$|u| \leq \frac{1}{|x|}.$$

Similarly, the required inequality

$$|b+(1-z)| \leq \left|\frac{b}{y}\right|$$

will be written

$$1-z = -b(1+v)$$

where

$$|v| \leq \frac{1}{|y|}.$$

We have then

$$|z| = |a|, \quad |1-z| = |b|.$$

The required equalities :

$$|a| = \left|\frac{(1-z)x}{y}\right|, \quad |b| = \left|\frac{zy}{x}\right|,$$

reduce to

$$\left|\frac{a}{b}\right| = \left|\frac{x}{y}\right|.$$

The required inequality

$$|ab + z'\bar{z}' + b| \leq \left|\frac{z'}{xy}\right|$$

reduces to

$$|a + b(1+v)(1+\bar{v}) + 1| \leq \frac{1}{|xy|}.$$

The domain of integration is defined in this case by the following conditions :

$$
(32)\qquad \begin{cases} |x| > 1, \quad |y| > 1, \\ z = -a(1+u), \quad |u| \le \dfrac{1}{|x|}, \quad 1 - z = -b(1+v), \quad |v| \le \dfrac{1}{|y|}, \\ 1 = -a(1+u) - b(1+v), \\ \left|\dfrac{a}{b}\right| = \left|\dfrac{x}{y}\right|, \quad \left|a + b(1+v)(1+\bar v) + 1\right| \le \dfrac{1}{|xy|}, \\ 1 < |bx|, \quad 1 < |ay|, \quad 1 < |bxy|, \quad 1 < |axy|. \end{cases}
$$

Of course we must verify that these equations are consistent : if v is given satisfying $|v| \le |y|^{-1}$ then we must verify the element u obtained by solving the third relation satisfies $|u| \le |x|^{-1}$. This is so if and only if

$$
\left|a + b(1+v) + 1\right| \le \left|\frac{a}{x}\right| = \left|\frac{b}{y}\right|.
$$

But this inequality is implied by the others relations. Thus we can use v instead of z as variable of integration in our integral. Similarly, we could use u.

8. Computation of $\widehat{J}_0$

In this section, we compute $\widehat{J}_0$. Recall $P_F^{m_i}$ denotes the conductor of χ_i. Thus $P_E^{m_i}$ is the conductor of the character $z \mapsto \chi_i(z\bar z)$.

8.1. The case $m_1 > 0$, $m_2 > 0$.

In the integral (28) we can take :

$$
(33)\qquad |x| = q^{2m_1}, \quad |y| = q^{2m_2}.
$$

and the domain of integration to be defined by (32). Suppose first $m_1 = m_2$. Then $|x| = |y|$ so $|a| = |b|$. This implies that $|z| = |1 - z|$. So $|z| \ge 1$, that is, $|a| = |b| \ge 1$. Furthemore

$$
\chi_1^{-1}(z\bar z) = \chi_1^{-2}(a), \chi_2^{-1}(z'\bar z') = \chi_2^{-2}(b).
$$

If we use v for variable of integration, we have

$$
\mathrm{d}z = |b|_E \,\mathrm{d}v = |a|_F |b|_F \,\mathrm{d}v.
$$

Hence the integral takes the form :

$$q^{2m_1+2m_2}\int \chi_1^{-1}(x\bar{x})\psi_E(x)\,\mathrm{d}x\int \chi_2^{-1}(y\bar{y})\psi_E(y)\,\mathrm{d}y$$
$$\chi_1^{-1}(a)\chi_2^{-1}(b)|a|_F|b|_F\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}v.$$

The domain of integration is

$$\begin{cases} \big|a+b(1+v)(1+\bar{v})+1\big| \le \dfrac{1}{|xy|}, \\ |a|=|b|\ge 1, \quad |v|\le q^{-2m_1}. \end{cases}$$

We can change b to

$$\frac{b}{(1+v)(1+\bar{v})}$$

and integrate over v :

$$\int \mathrm{d}v = q^{-2m_1}.$$

Converting the multiplicative Haar measures to additive ones, we see the integral takes the form :

$$(34)\quad (1-q^{-1})^{-2}q^{2m_2}\int \chi_1^{-1}(x\bar{x})|x|\psi_E(x)\,\mathrm{d}x$$
$$\int \chi_2^{-1}(y\bar{y})|y|\psi_E(y)\,\mathrm{d}y\int \chi_1^{-1}(a)\chi_2^{-1}(b)\,\mathrm{d}a\,\mathrm{d}b.$$

The domain of integration is now :

$$|a|=|b|\ge 1, \quad |a+b+1|_F\le q^{-2m_1}.$$

We have then

$$\chi_2^{-1}(b)=\chi_2^{-1}\big(-(a+1)\big).$$

We can then integrate over b :

$$\int \chi_2^{-1}(b)\,\mathrm{d}b = q^{-2m_2}\chi_2^{-1}\big(-(a+1)\big).$$

We can change a to $-a$ to arrive at the final form for our integral :

$$(35)\quad \widehat{J}_0=\chi_1\chi_2(-1)(1-q^{-1})^{-2}$$
$$\int_{|x|_E=q^{2m_1}}\chi_1^{-1}(x\bar{x})\psi_E(x)\,\mathrm{d}x\int_{|y|_E=q^{2m_2}}\chi_2^{-1}(y\bar{y})\psi_E(y)\,\mathrm{d}y$$
$$\int_{|a|_F=|1-a|_F}\chi_1^{-1}(a)\chi_2^{-1}(1-a)\,\mathrm{d}a.$$

To give a meaning to the integral, we must impose the conditions (30) and let A tend to infinity. We arrive at the same final integral with an extra condition on a of the form :

$$q^{-B} \leq |a|_F \leq q^B.$$

As A tends to infinity so does B and we obtain the above integral in which the a integral defines a formal Laurent series.

Now we assume $m_1 > m_2$. This time

$$\left|\frac{z}{1-z}\right| = \left|\frac{a}{b}\right| = \left|\frac{x}{y}\right| = q^{2m_1-2m_2}.$$

Thus

$$|z| = |a| = 1, \quad |1-z| = |b| = q^{2m_2-2m_1}.$$

We proceed as before, taking v for variable with

$$\mathrm{d}z = q^{2m_2-2m_1}\,\mathrm{d}v.$$

We change b to

$$\frac{b}{(1+v)(1+\bar{v})}$$

and integrate over v :

$$\int \mathrm{d}v = q^{-2m_2}.$$

Our integral takes the form :

$$(1-q^{-1})^{-2} q^{m_2+m_1} \int \chi_1^{-1}(x\bar{x})\psi_E(x)\,\mathrm{d}x \int \chi_2^{-1}(y\bar{y})\psi_E(y)\,\mathrm{d}y \\ \chi_1^{-1}(a)\chi_2^{-1}(b)\,\mathrm{d}a\,\mathrm{d}b.$$

This time the domain of integration is :

$$|a+b+1| \leq q^{-2m_1-2m_2}, \quad |a|_F = 1, \quad |b|_F = q^{m_2-m_1}.$$

This implies

$$|1+a|_F = q^{m_2-m_1}$$

and

$$\chi_2^{-1}(b) = \chi_2^{-1}\big(-(1+a)\big).$$

We can then integrate over b :

$$\int \chi_2^{-1}(b)\,\mathrm{d}b = q^{-m_1-m_2}\chi_2^{-1}\bigl(-(1+a)\bigr).$$

Finally, we change a to $-a$ and arrive at :

$$(36)\quad \widehat{J}_0 = \chi_1\chi_2(-1)(1-q^{-1})^{-2} \int_{|x|_E=q^{2m_1}} \chi_1^{-1}(x\bar{x})\psi_E(x)\,\mathrm{d}x \int_{|y|_E=q^{2m_2}} \chi_2^{-1}(y\bar{y})\psi_E(y)\,\mathrm{d}y \int_{|1-a|_F=q^{m_2-m_1}} \chi_1^{-1}(a)\chi_2^{-1}(1-a)\,\mathrm{d}a.$$

If $m_1 < m_2$ we arrive similarly at :

$$(37)\quad \widehat{J}_0 = \chi_1\chi_2(-1)(1-q^{-1})^{-2} \int_{|x|_E=q^{2m_1}} \chi_1^{-1}(x\bar{x})\psi_E(x)\,\mathrm{d}x \int_{|y|_E=q^{2m_2}} \chi_2^{-1}(y\bar{y})\psi_E(y)\,\mathrm{d}y \int_{|a|=q^{m_1-m_2}} \chi_1^{-1}(a)\chi_2^{-1}(1-a)\,\mathrm{d}a.$$

8.2. The case $m_1 > 0$, $m_2 = 0$.

In the integral (28) we can take :

$$|x| = q^{2m_1}, \quad |y| \leq q^2.$$

However, we have seen that we cannot have $|x| > 1$ and $|y| \leq 1$. Thus, in fact,

$$|y| = q^2$$

and the integral is given by the formulas (35) to (37) where $P_F^{m_1}$ is the conductor of χ_1 and the m_2 of the formulas is replaced by 1. Similarly, if $m_1 = 0$ and $m_2 > 0$, then the Mellin transform is given by the same formulas with m_1 in the formulas replaced by 1.

8.3. The case $m_1 = 0, m_2 = 0$.

Finally, we assume χ_1 and χ_2 unramified. In (28) we can take :

$$|x| \leq q^2, \quad |y| \leq q^2.$$

Since we must have :

$$|x| > 1, \quad |y| > 1$$

or

$$|x| \leq 1, \quad |y| \leq 1.$$

We have in fact either

(38) $$|x| = |y| = q^2$$

or

(39) $$|x| \leq 1, \quad |y| \leq 1.$$

The contribution of the set (38), we call the top term in $\widehat{J}$ and denote by $\widehat{J}_t$. It is given by formula (35) with m_1, m_2 replaced by 1.

We pass to the contribution of the set (39). It can be written :

$$\int \chi_1^{-1}(x\bar{x})|x|\chi_2^{-1}(y\bar{y})|y|\chi_1(a)\chi_2(b) \\ \chi_1^{-1}(z\bar{z})\chi_2^{-1}(z'\bar{z}')\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}z,$$

with domain of integration defined by (31). According to (31), we must take :

$$|x| = q^{-2n_1}, \quad |y| = q^{-2n_2}, \quad |z| = |1-z| = q^{2m},$$

with

$$n_1 \geq 0, \quad n_2 \geq 0, \quad m > n_1 + n_2.$$

Also :

$$|a|_F = q^{n_2 - n_1 + m}, \quad |b|_F = q^{n_1 - n_2 + m},$$
$$\Big|\frac{ab}{(1-z)(1-\bar{z})} + 1\Big|_F \leq q^{n_1 + n_2 - m}.$$

The Mellin transform can be computed as :

$$(1 - q^{-2})^2 \sum_{n_1, n_2, m} (X_1 X_2)^{m - n_1 - n_2} q^{-4n_1 - 4n_2} \int \mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}z.$$

In this formula, the integral is over the set

$$\begin{cases} |z| = q^{2m}, \quad |a|_F = q^{n_2 - n_1 + m}, \quad |b|_F = q^{n_1 - n_2 + m}, \\ \Big|\dfrac{ab}{(1-z)(1-\bar{z})} + 1\Big|_F \leq q^{n_1 + n_2 - m}. \end{cases}$$

After a change of variables, the integral can be written as :

$$\int \mathrm{d}^\times a\, \mathrm{d}^\times b\, \mathrm{d}z,$$

taken over the set

$$|z| = q^{2m}, \quad |a|_F = |b|_F = 1, \quad \Big|ab + 1\Big| \leq q^{n_1+n_2-m}.$$

After integrating over z and a say, it reduces to

$$(1 - q^{-2})q^{2m} \int \mathrm{d}^\times b$$

taken over

$$|b + 1| \leq q^{n_1+n_2-m}.$$

So its value is :

$$(1 + q^{-1})q^{m+n_1+n_2}.$$

Thus the Mellin transform of J_0 is given by :

$$(1 - q^{-2})^2(1 + q^{-1}) \sum (X_1X_2)^{m-n_1-n_2} q^{m-3n_1-3n_2}.$$

Now we remember our interpretation for the integral : we impose condition (30) ; it amounts to demand that

$$m - n_1 - n_2 < B$$

for some B and let B tend to infinity. As before, we are led to set

$$s = m - n_1 - n_2.$$

Then the above expression becomes :

$$(1 - q^{-2})^2(1 + q^{-1}) \sum (X_1X_2)^s q^s \sum q^{-2n_1} q^{-2n_2}.$$

The sums are for $s > 0$ and $n_1 \geq 0, n_2 \geq 0$. As before the meaning of this expression is just the usual one and we find :

$$(1 + q^{-1}) \sum (X_1X_2)^s q^s,$$

or

$$\widehat{J_0} = \widehat{J_t} + (1 + q^{-1}) \frac{X_1X_2q}{1 - X_1X_2q} \tag{40}$$

9. Comparison

Recall the Hasse-Davenport equality of Gaussian sums :

$$\int_{|z|_E=q^{2m}} \psi_E(z)\chi(z\bar{z})\,\mathrm{d}z = \int_{|x|_F=q^m} \psi_F(x)\chi(x)\,\mathrm{d}x$$

$$\int_{|x|_F=q^m} \psi_F(x)\zeta(x)\chi(x)\,\mathrm{d}x,$$

where χ is a character of conductor P_F^m. If χ is unramified, it is true if $m=1$.

Using this formula we can prove :

$$\int J_0(abc,bc,c)\chi_1(a)\chi_2(b)\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}^\times c =$$
$$\int I_0(abc,bc,c)\chi_1(a)\chi_2\zeta(b)\zeta(c)\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}^\times c.$$

Indeed this follows at once from the formulas (35) to (37) and the formulas (22) to (24) when χ_1 or χ_2 is ramified. If χ_1 and χ_2 are unramified, it follows similarly that the top terms $\hat{I}_t, \widehat{J}_t$ coincide. Comparing (27) to (40) we see the other terms coincide as well. So we are done. Thus we find :

$$\int I_0(abc,bc,c)\zeta(c)\zeta(b)\,\mathrm{d}^\times c = \int J_0(abc,bc,c)\,\mathrm{d}^\times c.$$

Since both functions are 0 unless the product of the entries has absolute value 1, we find :

$$\zeta(bc)I_0(abc,bc,c) = J_0(abc,bc,c)$$

or :

$$\zeta(b)I_0(a,b,c) = J_0(a,b,c).$$

This concludes the proof of the theorem.

10. Concluding remarks

A more general statement concerns arbitrary functions of the Hecke algebras : let f be a bi$-K-$invariant compactly supported function on $G(F)$ and f' the bi$-K'-$invariant function on $G'(F)$ which is its image under the base change homomorphism. Let Φ_f be the function on S defined by :

$$\Phi_f(s) = \int f(g)\Phi({}^t\bar{g}^{-1}sg^{-1})\,\mathrm{d}g.$$

Let $I(a,\Phi_f)$ and $J(a,f')$ be the orbital integrals of Φ_f and f' respectively. Then the general fundamental lemma asserts that :

$$I(a,\Phi_f)=\gamma(a)J(a,f').$$

More generally, let us consider the case of an arbitrary quadratic extension of local fields. Then we conjecture the existence of a transfer factor γ with the following property : for every smooth function of compact support Φ on S there is a smooth function of compact support f' on G' such that

$$I(a,\Phi)=\gamma(a)J(a,f'), \tag{41}$$

and conversely. We will say that Φ and f' match if this relation is true.

The existence of matching pairs cannot be established without considering the other orbits as well. Let W be the Weyl group of A identified to the group of permutation matrices. Then, every orbit of $N(E)$ contains exactly one element of the form wa, where $w^2=1$ and $waw^{-1}=\bar{a}$ (see [S]). We say the orbit is relevant if the character θ_E is trivial on the fixator of wa. It is so if and only if there is a standard Levi-subgroup M of G such that w is the longest element of $W\cap M$ and a is in the center of M. Similarly, we consider the orbits of $N(F)\times N(F)$ on $G'(F)$ and define the notion of relevant orbits. Remarkably, the relevant orbits have the same set of representatives (see [G]). For such a wa we can consider the orbital integrals :

$$I(wa,\Phi)=\int\Phi(n^*wan)\theta_E(n)\,\mathrm{d}n,$$

the integral over the quotient of $N(E)$ by the fixator of wa. Similarly, we can consider the orbital integrals :

$$I(wa,f')=\int f'({}^tn_1wan_2)\theta_F(n_1n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.$$

The behavior at infinity of the orbital integrals associated to the orbits of maximum dimension should determine the other orbital integrals, as is the case for GL(2); this is why we did not investigate the other orbits. We hope to come back on this question in the near future. Finally, it is natural to conjecture the existence of transfer factors $\gamma(wa)$, defined for a general relevant element wa, such that if the relation (41) is satisfied then

$$I(wa,\Phi)=\gamma(wa)I(wa,f').$$

We caution however that the true relation may be somewhat more complicated.

BIBLIOGRAPHY

[A–C] ARTHUR (J.) and CLOZEL (L.). — *Simple algebras, base change and the advanced theory of the trace formula, Princeton University Press,* 1989, Annals of Mathematics Studies, **120**.

[G] GOLDFELD (D.). — *Kloosterman zeta functions for* $\mathrm{GL}(n,\mathbb{Z})$, Proc. Intern. Cong. Math. Berkeley, t. **1**, 1986, p. 417—424.

[H–L–R] HARDER (G.), LANGLANDS (R.P.) and RAPOPORT (M.). — *Algebraische Zyklen auf Hilbert-Blumenthal Fläachen*, J. Math., t. **366**, 1986, p. 53—120.

[J] JACQUET (H.). — *On the non vanishing of some L-functions*, Proc. Indian Acad. Sci. Math. Sci., t. **97**, 1987, p. 117—155.

[J–Y] JACQUET (H.) et YE (Y.). — *Une remarque sur le changement de base quadratique*, C.R. Acad. Sci. Paris, t. **311**, 1990, p. 671—676.

[S] SPRINGER (T.). — *Some results on algebraic groups with involutions*, Adv. Stud. Math., Tokyo, t. **6**, 1984, p. 323—343.

[Y] YE (Y.). — *Kloosterman integrals and base change for* GL(2), J. Reine Angew. Math., t. **400**, 1989, p. 57—121.

[Z] ZAGIER (D.). — *Modular forms associted to real quadratic fields*, Invent. Math., t. **30**, 1975, p. 1—46.