Odd values of the Ramanujan $\tau$-function


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ODD VALUES OF THE RAMANUJAN τ-FUNCTION

BY

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Ramanujan's τ-function is defined by the relation

\[ q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n. \]

It is conjectured by ATKIN and SERRE [6, equation 4.11 k] that for any \( \varepsilon > 0 \),

\[ |\tau(p)| > p^{(0.21-\varepsilon)}. \]

In particular this implies that for any \( a \), there are only finitely many primes \( p \) such that \( \tau(p) = a \). In this note, we study a related, though simpler, question. Our main result is the following.

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Theorem. — There exists an effectively computable absolute constant $c > 0$, such that for all positive integers $n$ for which $\tau(n)$ is odd, we have

$$|\tau(n)| \geq (\log n)^c.$$ 

It follows from the theorem that for an odd integer $a$, the equation

$$\tau(n) = a$$

has only finitely many solutions.

As $\tau(p)$ is even, all integers satisfying (1) are squarefull (i.e. every prime divisor of $n$ appears to at least the second power). We apply the theory of linear forms in logarithms to obtain lower bounds for $\tau(p^m)$, $p$ a prime and $m \geq 2$, which, in particular gives the theorem.

We require several lemmas.

Lemma 1. — $\tau(p^m) = 0$ if and only if $m$ is odd and $\tau(p) = 0$.

Proof. — Write $\tau(p) = x_p + \bar{x}_p$, $x_p = p^{1/2} e^{i \theta_p}$, $0 \leq \theta_p \leq \pi$. Set

$$\gamma_m(p) = \begin{cases} 1, & \text{if } m \text{ is even} \\ \tau(p), & \text{if } m \text{ is odd} \end{cases}$$

and $\zeta = \exp(2\pi i/(m + 1))$. Then, as in Ramanujan [4],

$$\tau(p^m) = (x_p^{m+1} - \bar{x}_p^{m+1})/(x_p - \bar{x}_p)$$

$$= \gamma_m(p) \prod_{r=1}^{[m/2]} (x_p - \zeta^r \bar{x}_p)(x_p - \zeta^{-r} \bar{x}_p)$$

$$= \gamma_m(p) \prod_{r=1}^{[m/2]} (\tau(p)^2 - 4p^{11} \cos^2(\pi r/(m + 1))).$$

If the $r$-th factor is zero,

$$4 \cos^2(\pi r/(m + 1)) = \zeta^r + \zeta^{-r} + 2 = \tau(p)^2/p^{11},$$

is both an algebraic integer and a rational number. Thus it is a rational integer and so must be one of 1, 2 or 3. But none of $\tau(p)^2 - p^{11}$, $\tau(p)^2 - 2p^{11}$, $\tau(p)^2 - 3p^{11}$ can be zero, since $\tau(2) = -24 \neq \pm 2^6$ and $\tau(3) = 252 \neq \pm 3^6$. Thus $\tau(p^m) = 0$ if and only if $\gamma_m(p) = 0$.

The next three lemmas depend on the theory of linear forms in logarithms. They are stronger than needed for the proof of the theorem. They may be of independent interest.
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**Lemma 2.** There is an effectively computable absolute constant $C_1 > 0$ such that for all $m \geq 2$, we have

$$|\tau(p^m)| \geq |\gamma_m(p)| p^{(\frac{11}{2}) \left(m - C_1 \log m\right)}.$$

**Proof.** Suppose that $m$ is odd. If $\tau(p) = 0$, there is nothing to prove. If $\tau(p) \neq 0$, then we see from (2) that $\tau(p^m) \neq 0$. Then

$$|\tau(p^m)/\gamma_m(p)| = \frac{|x_p^{m+1} - \bar{x}_p^{m+1}|}{\|x_p^2 - \bar{x}_p^2\|^{-1}} \geq \frac{1}{2} p^{(\frac{11}{2}) \left(m - C_1 \log m\right)} \left| (x_p - \bar{x}_p)^{m+1} - 1 \right| \geq p^{(\frac{11}{2}) \left(m - C_1 \log m\right)}.$$

where in the final step, we used the fact that the height of $x_p - \bar{x}_p$ is bounded by a power of $p$ and estimate of Baker [1] on linear forms. If $m$ is even, the required estimate follows similarly.

The constant $C_1$ above is quite large and so the bound is non-trivial only for large $m$. The next lemma gives a bound which is non-trivial for bounded $m$.

**Lemma 3.** Let $m \geq 6$. There is an effectively computable number $C_2 > 0$ depending only on $m$ such that either $\tau(p^m) = 0$ or

$$|\tau(p^m)| > p^{C_2}.$$

**Proof.** Let $m \geq 6$ and $\tau(p^m) \neq 0$. Observe that $\tau(p^m)/\gamma_m(p)$ is a binary form in $\tau(p^2)$ and $p^{11}$ with at least three distinct linear factors. We apply an estimate of Feldman [3] or Baker [2] on the magnitude of integral solutions of Thue's equation to obtain the assertion of the lemma.

**Remark.** In fact, we could have applied a theorem of Roth [5] on the approximations of algebraic numbers by rationals to obtain the following stronger, but ineffective, version of Lemma 3: for every $\epsilon > 0$, $\tau(p^m) = 0$ or

$$|\tau(p^m)| \gg_{\epsilon, m} p^{(\frac{11}{2}) \left(m - d\right) - \epsilon}.$$

**Lemma 4.** There is an effectively computable absolute constant $C_3 > 0$ such that

$$|\tau(p^m)| \geq (\log p)^{C_3}, \quad m = 2, 4.$$
Further.

\[
\min(\left|\tau(p^3)\right|, \left|\tau(p^5)\right|) \geq \frac{1}{2} p^{11/2}
\]

whenever \(\tau(p) \neq 0\).

Proof. — Observe that

\[
\tau(p)^2 = p^{11} + \tau(p^2)
\]

and

\[
(2 \tau(p)^2 - 3 p^{11})^2 = 5 p^{22} + 4 \tau(p^4).
\]

Now we apply an estimate of Sprindzuk [7] on the magnitude of integral solutions of hyperelliptic equations to obtain the first inequality of the lemma. The second inequality follows immediately from the relations.

\[
\tau(p^3) = \tau(p)(\tau(p)^2 - 2 p^{11})
\]

and

\[
\tau(p^5) = \tau(p)(\tau(p)^2 - 3 p^{11})(\tau(p)^2 - p^{11}).
\]

Proof of theorem. — Let \( n \) be such that \( \tau(n) \) is odd. As remarked in the beginning, we see that \( n \) is squarefull. Therefore, if \( p \) and \( m \) are such that \( p^m \| n \), it follows from our lemmas that

\[
\left|\tau(p^m)\right| \geq (\log p^m)^{C_\Delta}
\]

where \( C_\Delta > 0 \) is an effectively computable absolute constant. Hence,

\[
\left|\tau(n)\right| = \prod_{p^m \| n} \left|\tau(p^m)\right| \geq (\log n)^{C_\Delta}
\]

which implies the assertion of the theorem.

The same method can be used to study the Fourier coefficients of other modular forms. Indeed, let \( f \) be a cusp form of weight \( k \geq 4 \) for \( \Gamma_0(N) \) and write

\[
f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i nz}
\]

for the Fourier expansion at \( i \infty \). Suppose that:

(i) \( f \) is a normalized eigenform for all the Hecke operators \( T_p \) for \( (p, N) = 1 \);

(ii) \( f \) does not have complex multiplication:

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(iii) the $a_n$ are rational integers;
(iv) $a_2 \neq \pm 2^{k/2}$ and $a_3 \neq \pm 3^{k/2}$.

Then, for $n$ squarefull, $a_n = 0$ or 

$$|a_n| \geq (\log n)^D$$

for some effectively computable constant $D > 0$ which depends only on $f$.

REFERENCES