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ZBIGNIEW SZAFRANIEC

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ON THE DIVISION OF FUNCTIONS OF CLASS C^r
BY REAL ANALYTIC FUNCTIONS

BY

ZBIGNIEW SZAFRANIEC (*)

RÉSUMÉ. — Soit $(X, 0)$ un germe d'ensemble analytique cohérent. Supposons que les fonctions analytiques g_1, \dots, g_p engendrent un idéal $I(X)_0$. Il existe une fonction croissante $e: N \rightarrow N$ telle que, si une fonction f de classe $C^{e(r)}$ s'annule sur X , on a $f = \varphi_1 \cdot g_1 + \dots + \varphi_p \cdot g_p$ (φ_i étant des fonctions de classe C^r). Dans cet article nous démontrons une estimation de $e(r)$ dans des cas spéciaux.

ABSTRACT. — Let $(X, 0)$ be a germ of an analytic coherent set in R^n . Assume that analytic functions g_1, \dots, g_p generate ideal $I(X)_0$. There exists an increasing function $e: N \rightarrow N$ such that, for any function f of class $C^{e(r)}$ vanishing on X , there exist C^r -functions $\varphi_1, \dots, \varphi_p$ such that $f = \varphi_1 \cdot g_1 + \dots + \varphi_p \cdot g_p$. In this paper we investigate the problem of the estimation of $e(r)$ in some special cases.

Let $(X, 0)$ be a germ of an analytic coherent set in R^n . Assume that analytic functions g_1, \dots, g_p generate the ideal

$$I(X)_0 = \{g \in \mathcal{O}_{n,0} \mid g|_X \equiv 0\}.$$

J. Cl. TOUGERON in [7] showed that there exists an increasing function $e: N \rightarrow N$ such that, for any $C^{e(r)}$ -function f vanishing on X , there exist C^r -functions $\varphi_1, \dots, \varphi_p$ such that

$$f = \varphi_1 \cdot g_1 + \dots + \varphi_p \cdot g_p.$$

J. J. RISLER in [5] estimated precisely the function $e(r)$ in the case of plane curves.

In this paper we investigate the problem of the estimation of $e(r)$ in some special cases.

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Z. SZAFRANIEC, Institute of Mathematics, 80-952 Gdansk, Wita Stwosza 57, Poland.

1. Strongly irreducible polynomials

For any $x \in \mathbb{R}^n \subset \mathbb{C}^n$, let us denote by $\mathcal{O}_{n,x}(\tilde{\mathcal{O}}_{n,x})$ the ring of germs of real analytic (holomorphic) functions at x . We denote by $\mathfrak{m}_{n,x}(\tilde{\mathfrak{m}}_{n,x})$ the maximal ideal of $\mathcal{O}_{n,x}(\tilde{\mathcal{O}}_{n,x})$.

DEFINITION 1. — Let:

$$P(X', X) = X^p + a_1(X')X^{p-1} + \dots + a_p(X') \in \tilde{\mathcal{O}}_{n,0}[X]$$

be a distinguished polynomial. Let $\delta \in \tilde{\mathcal{O}}_{n,0}$ be the discriminant of the polynomial P . Assume that $\delta \neq 0$. Denote by ω the initial form of δ at 0.

We say that P is *strongly irreducible* if there exist a constant $\varepsilon > 0$ and a set W such that the following conditions are satisfied:

$$(1.0) \quad W \subset \{(X', X) \in \mathbb{C}^{n+1} \mid 0 < \|X'\| < \varepsilon, \\ P(X', X) = 0, \delta(X') \neq 0, \omega(X') \neq 0\},$$

(1.1) W is a nonempty, connected and open subset of

$$\tilde{V}(P) = \{(X', X) \in \mathbb{C}^{n+1} \mid P(X', X) = 0\},$$

(1.2) If $w \in W$, $t \in \mathbb{C}$ and $0 < |t| \leq 1$ then

$$\pi^{-1}(t \cdot \pi(w)) \cap \tilde{V}(P) \subset W,$$

where $\pi : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$ is the projection.

LEMMA. — 1 Let $P \in \tilde{\mathcal{O}}_{n,0}[X]$ be a distinguished polynomial. Let δ be the discriminant of P . Assume that $\delta \neq 0$. Denote by ω the initial form of δ at the origin.

We require $H \subset \mathbb{C}^n$ to be a complex hyperplane such that:

- (i) $\dim_{\mathbb{C}} H \geq 1$, $0 \in H$,
- (ii) $P|_{H \times \mathbb{C}}$ is irreducible in $\tilde{\mathcal{O}}_{H \times \mathbb{C}, 0}$,
- (iii) $\omega|_H$ has no critical points, except possibly for the origin itself.

Then the polynomial P is strongly irreducible.

The sketch of the proof.

Let $h = \dim_{\mathbb{C}} H$. We may assume that

$$H = \{X \in \mathbb{C}^n \mid X_{h+1} = \dots = X_n = 0\}.$$

Denote by M the linear space of all complex $(n-h) \times h$ -matrices. Let

$$\gamma = \{(L, v) \in \mathbb{C}P(h-1) \times \mathbb{C}^h \mid v \in L\},$$

be the canonical line bundle of $\mathbb{C}P(h-1)$.

We define a holomorphic map $\theta : M \times \gamma \rightarrow \mathbb{C}^n$ by $\theta(A, (L, v)) = (v, A(v))$.

Of course : $\theta(0 \times \gamma) = H$.

We use the notation:

$$G_1 = \{(A, (L, v)) \in M \times \gamma \mid A=0, v=0\},$$

$$G_2 = \{(A, (L, v)) \in M \times \gamma \mid A=0\},$$

$$S = \{(A, (L, v)) \in M \times \gamma \mid v=0\}.$$

The homogeneous form ω_{1H} has an isolated singular point at the origin. Then there exist an open set $U_1 \subset S$ and a closed complex manifold $N_1 \subset U_1$ such that:

- (1) $G_1 \subset U_1,$
- (2) N_1 is transverse to G_1 in $S,$
- (3) $\{(A, (L, v)) \in M \times \gamma \mid \omega \circ \theta(A, (L, v)) = 0, (A, (L, 0)) \in U_1\}$
 $= \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in N_1\} \cup U_1.$

The form ω_{1H} has an isolated singular point at the origin, so δ_{1H} has an isolated singular point at the origin.

It follows that there exists an open set $U_2 \subset M \times \gamma$ and a closed complex manifold $N_2 \subset U_2$ such that:

- (4) $G_1 \subset U_2,$
- (5) $U_2 \cap S \subset U_1,$
- (6) N_2 is transverse to G_1 in $U_2,$
- (7) $\{(A, (L, v)) \in M \times \gamma \mid \delta \circ \theta(A, (L, v)) = 0\} \cap U_2 = N_2 \cup (S \cap U_2),$
- (8) $N_2 \cap S = U_2 \cap N_1.$

Then there exist open sets $V_1 \subset S, V_2 \subset G_1$ and a constant $\varepsilon > 0$ such that:

- (9) $N_1 \cap G_1 \subset V_2,$
- (10) $G_1 \subset V_1,$

$$(11) \quad \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, (0, (L, 0)) \notin V_2, 0 < \|v\| < \varepsilon\},$$

is a deformation retract of

$$\{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, (A, (L, v)) \notin N_2 \cup S, 0 < \|v\| < \varepsilon\}.$$

Denote

$$Z = \{(A, (L, v), X) \in M \times \gamma \times \mathbb{C} \mid P(\theta(A, (L, v)), X) = 0, \\ (A, (L, 0)) \in V_1, (A, (L, v)) \notin N_2 \cup S, 0 < \|v\| < \varepsilon\}.$$

By (7) the projection

$$\pi: Z \rightarrow \{(A, (L, v)) \in M \times \gamma \mid (A, (L, 0)) \in V_1, \\ (A, (L, v)) \notin N_2 \cup S, 0 < \|v\| < \varepsilon\}$$

is a covering map.

Set

$$Z_1 = \{(A, (L, v), X) \in M \times \gamma \times \mathbb{C} \mid P(\theta(A, (L, v)), X) = 0, \\ (A, (L, 0)) \in V_1, (0, (L, 0)) \notin V_2, 0 < \|v\| < \varepsilon\}.$$

By (11) Z_1 is a deformation retract of Z . Set

$$Z'_1 = \{(0, (L, v), X) \in (G_2 \setminus (N_2 \cup S)) \times \mathbb{C} \mid \\ P(\theta(0, (L, v)), X) = 0, 0 < \|v\| < \varepsilon\}.$$

The germ of $P|_{H \times \mathbb{C}}$ at 0 is irreducible, so, by ([4], Proposition 11, p. 55), we may assume that Z'_1 is connected. Then, if ε is sufficiently small, the sets Z and Z_1 are connected.

Denote $W = (\theta \times \text{id}_{\mathbb{C}})(Z_1) \subset \mathbb{C}^n \times \mathbb{C}$. Then W is an open, connected subset of $\tilde{V}(P)$.

If V_1 is a sufficiently small neighbourhood of G_1 in S then, by (8) and (9), we have:

$$\pi(W) = \{(X', X) \in \mathbb{C}^n \times \mathbb{C} \mid 0 < \|X'\| < \varepsilon, P(X', X) = 0, \\ \delta(X') \neq 0, \omega(X') \neq 0\}.$$

By definition of W , if $w \in W$, $t \in \mathbb{C}$ and $0 < |t| \leq 1$ then

$$\pi^{-1}(t \cdot \pi(w)) \cap \tilde{V}(P) \subset W.$$

This completes the proof. ■

Example 1. — Let $P \in \tilde{\mathcal{O}}_{1,0}[X]$ be a distinguished irreducible polynomial. Then P is strongly irreducible.

Example 1. — Let $P \in \tilde{\mathcal{O}}_{1,0}[X]$ be a distinguished irreducible polynomial. Then P is strongly irreducible.

Example 2. — Let $P(X', X) = X^2 + X_1^2 + X_2^2 + f(X_3, \dots, X_n)$, where $f \in \tilde{\mathfrak{m}}_{n-2,0}$, $df(0) = 0$.

Set $H = \{X' \in \mathbb{C}^n \mid X_3 = \dots = X_n = 0\}$. Then $P|_{H \times \mathbb{C}} = X^2 + X_1^2 + X_2^2$ is irreducible in $\tilde{\mathcal{O}}_{H \times \mathbb{C}, 0}$ and $\omega|_H = -4(X_1^2 + X_2^2)$ has an isolated singular point at the origin.

Hence P is strongly irreducible.

COROLLARY 1. — Let $P \in \tilde{\mathcal{O}}_{n,0}[X]$ be a distinguished polynomial. Let δ be the discriminant of P . Assume that there exists a function $\Delta \in \tilde{\mathcal{O}}_{n,0}$ such that $\Delta \neq 0$ and $\tilde{V}(\Delta) = \tilde{V}(\delta)$. Denote by ω' the initial form of Δ at 0.

We require $H \subset \mathbb{C}^n$ to be a complex hyperplane such that:

- (i) $\dim_{\mathbb{C}} H \geq 1, 0 \in H$,
- (ii) the germ of $P|_{H \times \mathbb{C}}$ is irreducible in $\tilde{\mathcal{O}}_{H \times \mathbb{C}, 0}$,
- (iii) $\omega'|_H$ has an isolated singular point at 0.

Then P is strongly irreducible.

Example 3. — Let $P(X', X) = X^3 + X_1^2 + X_2^2 + f(X_3, \dots, X_n)$, where $f \in \tilde{\mathfrak{m}}_{n-2,0}$ and $df(0) = 0$.

Then $\delta(X') = -27(X_1^2 + X_2^2 + f(X_3, \dots, X_n))^2$.

Set $\Delta(X') = X_1^2 + X_2^2 + f(X_3, \dots, X_n)$. Of course $\tilde{V}(\Delta) = \tilde{V}(\delta)$.

Set

$$H = \{X' \in \mathbb{C}^n \mid X_3 = \dots = X_n = 0\}.$$

The germ of $P|_{H \times \mathbb{C}} = X^3 + X_1^2 + X_2^2$ at 0 is irreducible and $\omega'|_H = X_1^2 + X_2^2$ has an isolated singular point at the origin. Hence P is strongly irreducible.

LEMMA 2. — Let $P(X', X) = X^p + a_1(X')X^{p-1} + \dots + a_p(X') \in \tilde{\mathcal{O}}_{n,0}[X]$ be a distinguished strongly irreducible polynomial.

Denote by d the degree of the form ω .

Let $f_1, f_2 \in \tilde{\mathcal{O}}_{n+1,0}$ be germs such that $f_1 \cdot f_2 \in \tilde{\mathcal{O}}_{n+1,0} \cdot P + \tilde{\mathfrak{m}}_{n+1,0}^{e(r)}$, where $r \in \mathbb{N}$, $e(r) = 2p(r + [d/2] + 1)$, $[d/2]$ is the integer part of $d/2$.

Then $f_1 \in \tilde{\mathcal{O}}_{n+1,0} \cdot P + \tilde{\mathfrak{m}}_{n+1,0}$ or $f_2 \in \tilde{\mathcal{O}}_{n+1,0} \cdot P + \tilde{\mathfrak{m}}_{n+1,0}$.

This lemma is analogous to Lemma 1.7 in [6].

Proof. — We define a map $h_1 : W \times \mathbb{C} \rightarrow \mathbb{C}^n$ by $h_1(w, t) = t^{p-1} \cdot \pi(w)$. By (1.2), if $0 < |t| \leq 1$ then $\pi^{-1}(h_1(w, t)) \cap \tilde{V}(P) \subset W$. Denote $D = \{t \in \mathbb{C} \mid |t| \leq 1\}$. Since $\pi(W) \subset \mathbb{C}^n \setminus \tilde{V}(\delta)$, so there exists a holomor-

phic function $h_2 : W \times (D \setminus \{0\}) \rightarrow \mathbb{C}$ such that:

- (1) $P(h_1(w, t), h_2(w, t)) \equiv 0,$
 (2) $(h_1(w, 1), h_2(w, 1)) \equiv w.$

The polynomial $P(X', X)$ is distinguished, so h_2 is bounded. Then there exists a holomorphic extension $h_2 : W \times D \rightarrow \mathbb{C}$. Hence $h = h_1 \times h_2 : W \times \mathbb{C} \rightarrow \mathbb{C}^n \times \mathbb{C}$ is holomorphic. Then, by Proposition 2.2 ([2], p. 55), there exists a constant $C_1 > 0$ such that:

- (3) $|h_2(w, t)| \leq C_1 \cdot \|h_1(w, t)\|^{1/p}$ for $(w, t) \in W \times \mathbb{C}$.

Then

- (4) $|h_2(w, t)| \leq C_1 \cdot |t|^{(p-1)!} \|\pi(w)\|^{1/p}.$

By (1) and (4) there exist constants $C_2, C_3 > 0$ such that, for any $(w, t) \in W \times D$,

$$|(f_1 \cdot f_2) \circ h(w, t)| \leq C_2 \cdot \|h(w, t)\|^{e(r)} \leq C_3 \cdot |t|^{2(r+[d/2]+1)p!}.$$

The set W is connected, so $W \times \{0\}$ is a connected complex submanifold of $W \times D$.

It follows that, for example,

$$f_1 \circ h(w, 0) \equiv \dots \equiv \frac{\partial^{(r+[d/2]+1)p!-1}}{\partial t^{(r+[d/2]+1)p!-1}} (f_1 \circ h)(w, 0) \equiv 0.$$

Then there exists a continuous function $k : W \rightarrow \mathbb{R}_+$ such that:

- (5) $|f_1 \circ h(w, t)| \leq k(w) \cdot |t|^{(r+[d/2]+1)p!}$
 $= k(w) \cdot \|\pi(w)\|^{-1} \cdot \|t^{p!} \cdot \pi(w)\|^{(r+[d/2]+1)}$
 $= k(w) \cdot \|\pi(w)\|^{-1} \cdot \|h_1(w, t)\|^{(r+[d/2]+1)}.$

From the preparation theorem we have:

$$f_1 = Q \cdot P + \sum_{j=1}^p b_j(X') \cdot X^{p-j}, \quad \text{where } b_j \in \tilde{m}_n, 0.$$

Let $w_0 \in W, t \in D$. Denote by $\xi_1(t), \dots, \xi_p(t)$ the roots of the polynomial $P(t^{p!} \cdot \pi(w_0), X)$.

Then

$$f_1(t^{p^1} \cdot \pi(w_0), \xi_i(t)) = \sum_{j=1}^n b_j(t^{p^1} \cdot \pi(w_0)) \cdot \xi_i^{p^{-j}}(t).$$

By Cramer's rule

$$b_j(t^{p^1} \cdot \pi(w_0)) = (\det [s_{kl}(t)]) / (\prod_{1 \leq n < m \leq p} (\xi_n(t) - \xi_m(t))),$$

where if $l \neq j$ then

$$s_{kl}(t) = \xi_k^{p^{-l}}(t), \quad s_{kj}(t) = f_1(t^{p^1} \cdot \pi(w_0), \xi_k(t)).$$

Of course

$$|\prod_{1 \leq n < m \leq p} (\xi_n(t) - \xi_m(t))| = |\delta(t^{p^1} \cdot \pi(w_0))|^{1/2}.$$

By (1.0) $\pi(W) \subset C^n \setminus \tilde{V}(w)$. By (5) there exist constants $C_4, C_5 > 0$ such that:

$$|\delta(t^{p^1} \cdot \pi(w_0))|^{1/2} > C_4 \cdot |t^{p^1}|^{d/2}, \quad |\det [s_{kl}(t)]| < C_5 \cdot |t^{p^1}|^{(r+[d/2]+1)}.$$

Then

$$|b_j(t^{p^1} \cdot \pi(w_0))| < (C_5/C_4) \cdot |t^{p^1}|^r.$$

The set $\pi(W)$ is open in C^n , so $b_j \in \tilde{m}'_{n,0}$.

Then $f_1 - Q \cdot P \in \tilde{m}'_{n+1,0}$. ■

COROLLARY 2. — *If $P \in \tilde{\mathcal{O}}_{n,0}[X]$ is strongly irreducible then P is irreducible in $\tilde{\mathcal{O}}_{n+1,0}$.*

2. Functions vanishing on an analytic set

DEFINITION 2. — Let $I \subset \mathcal{O}_{n,0}$ be an ideal. We denote by $\sqrt[R]{I}$ the ideal of germs vanishing on $V(I)_0$.

We say that I is *real* if $I = \sqrt[R]{I}$.

Let $\mathfrak{p} \subset \mathcal{O}_{n,0}$ be a prime ideal, $\{0\} \neq \mathfrak{p} \neq \mathcal{O}_{n,0}$. By [4] there exists, after a linear change of coordinates in R^n , an integer $k, 0 < k \leq n$, such that $\mathcal{O}_{k,0} \rightarrow A = \mathcal{O}_{n,0}/\mathfrak{p}$ is an injection which makes A a finite $\mathcal{O}_{k,0}$ -module.

Further, if K is the quotient field of $\mathcal{O}_{k,0}$, L that of A , we have $L = K(X_{k+1} \text{ mod } \mathfrak{p})$, and for any $i \in [k+1, n]$, the minimal polynomial P_i

of X_i over K is in $\mathcal{O}_{k,0}[X]$ and is distinguished, so that there is a distinguished polynomial

$$(2.0) \quad P_i(X', X_i) = X_i^{p_i} + \sum_{j=1}^{p_i} a_{ij}(X') X_i^{p_i-j}, \quad X' = (X_1, \dots, X_k),$$

with $P_i(X', X_i) \in \mathfrak{p}$.

Let $\delta(X') \in \mathcal{O}_{k,0}$ be the discriminant of the polynomial P_{k+1} . Then $\delta \notin \mathfrak{p}$.

Let $p = p_{k+1}$. There are polynomials Q_i of degree $< p$ in $\mathcal{O}_{k,0}[X]$ such that, for $i \in [k+2, n]$ we have $\delta \cdot X_i - Q_i(X_{k+1}) \in \mathfrak{p}$.

Let $\pi: R^n = R^k \times R^{n-k} \rightarrow R^k$ be the natural projection. There exists a fundamental system of neighbourhoods $\Omega = \Omega' \times \Omega''$ of 0 in $R^n = R^k \times R^{n-k}$ such that

$$(2.1) \quad \pi|_{V(\mathfrak{p}) \cap \Omega} \rightarrow \Omega' \text{ is proper.}$$

LEMMA 3 (see [4]). — *There exists a constant $N \leq p^{n-k}$ such that for any point $x \in V(\mathfrak{p}) \cap \Omega$ and any $f \in \mathcal{O}_{n,x}$:*

$$\delta^N \cdot f \equiv g \pmod{P_{k+1}, \delta \cdot X_{k+2} - Q_{k+2}, \dots, \delta \cdot X_n - Q_n},$$

where g is an element in $\mathcal{O}_{k,\pi(x)}[X_{k+1}]$.

LEMMA 4 (see [7]). — *There exists a constant $\alpha \in N$, $\alpha \geq 1$, such that for any point $x' \in V(\delta) \cap \Omega'$ and any connected component U of $\Omega' \setminus V(\delta)$, if $x' \in \bar{U}$, then there exists a sequence (y^i) of points of U such that*

$$\lim y^i = x' \quad \text{and} \quad \{y \in \Omega' \mid \|y - y^i\| < \|x' - y^i\|^\alpha\} \subset U.$$

LEMMA 5 (see [7]). — *There exists a constant $v \in N$ such that for any $x \in V(\mathfrak{p}) \cap \Omega$ and any germs $f_0, \dots, f_{n-k} \in \mathcal{O}_{n,x}$ if*

$$h = f_0 \cdot \delta^N + f_1 \cdot P_{k+1} + \sum_{i=k+2}^n f_{i-k} \cdot (\delta \cdot X_i - Q_i) \in \mathfrak{m}_{n,x}^{N+v}, \quad r \in N,$$

then there exist germs $g_0, \dots, g_{n-k} \in \mathfrak{m}_{n,x}$ such that:

$$h = g_0 \cdot \delta^N + g_1 \cdot P_{k+1} + \sum_{i=k+2}^n g_{i-k} \cdot (\delta \cdot X_i - Q_i).$$

From now on we make the assumptions:

(2.2) $V(\mathfrak{p})$ is coherent in a neighbourhood of 0,

(2.3) the set $V(\mathfrak{p}) \cap \Omega \setminus V(\delta) \times R^{n-k}$ is dense in $V(\mathfrak{p}) \cap \Omega$,

(2.4) If $(x', x_{k+1}) \in V(P_{k+1}) \cap (\Omega' \times R)$, then there exist polynomials

$$R_1, \dots, R_s(x', x_{k+1}), \quad Q \in \mathcal{O}_{k, x'}[X_{k+1} - x_{k+1}]$$

such that

$$P_{k+1} = R_1 \dots R_s(x', x_{k+1}) \cdot Q \text{ in } \mathcal{O}_{k, x'}[X_{k+1} - x_{k+1}],$$

(2.5) polynomials R_i are distinguished and strongly irreducible in $\tilde{\mathcal{O}}_{k, x'}[X_{k+1} - x_{k+1}]$,

$$(2.6) \quad Q(x', x_{k+1}) \neq 0,$$

(2.7) for any $i \in [1, s(x', x_{k+1})]$

$$(x', x_{k+1}) \in \overline{V(R_i) \setminus (\cup_{j \neq i} V(R_j) \cup (V(\delta) \times R))}.$$

Example 4. - Assume that $f \in \mathfrak{m}_{n-3,0}$, $df(0) = 0$ and

$$(1) \quad \{x \in R^{n-3} \mid f(x) \leq 0\} = \overline{\{x \in R^{n-3} \mid f(x) < 0\}}.$$

Define

$$P(X_1, \dots, X_n) = f(X_1, \dots, X_{n-3}) + X_{n-2}^2 + X_{n-1}^2 + X_n^2 \in \mathcal{O}_{n-1,0}[X_n].$$

The germ of P at 0 is irreducible, so $\mathfrak{p} = \mathcal{O}_{n,0} \cdot P$ is prime in $\mathcal{O}_{n,0}$.

If $x \in V(\mathfrak{p}) \setminus R^{n-3} \times \{0\}$, then $dP(x) \neq 0$, so the germ of P at x generates $I(V(\mathfrak{p}))_x$.

If $x = (x', x'') \in V(\mathfrak{p}) \cap R^{n-3} \times \{0\}$, where $x' \in R^{n-3}$ and $x'' \in R^3$ then $f(x') = 0$. Hence the germ of P at x is irreducible. By (1) the germ of $V(\mathfrak{p})$ at x contains regular points. From Lemma 2.5 ([3], p. 14), the germ of P at x generates $I(V(\mathfrak{p}))_x$. So $V(\mathfrak{p})$ is coherent.

Let $\delta = -4(f(X_1, \dots, X_{n-3}) + X_{n-2}^2 + X_{n-1}^2)$ be the discriminant of P .

By (1), $V(\mathfrak{p}) \setminus (V(\delta) \times R)$ is dense in $V(\mathfrak{p})$ in some neighbourhood of the origin.

If $x = (x', x'') \in V(\mathfrak{p}) \cap (R^{n-3} \times \{0\})$ then, by Example 2, the germ of P at x is strongly irreducible.

If $x \in V(\mathfrak{p}) \setminus (R^{n-3} \times \{0\})$, then $\delta(\pi(x)) \neq 0$ or $d\delta(\pi(x)) \neq 0$. Hence, by Definition 1 or lemma 1, the polynomial P is strongly irreducible.

So the conditions (2.2)-(2.7) are satisfied.

Let $d = \text{degree } \omega$, where ω is the initial form of δ at 0. By induction we can define functions $e_i : N \rightarrow N$.

Set

$$e_0(r) = p \cdot (r + v + [d/2] + 1),$$

$$e_1(r) = p \cdot \alpha \cdot (e_0(r) - 1),$$

⋮

$$e_i(r) = p \cdot \alpha \cdot (2p \cdot (e_{i-1}(r) + [d/2] + 1) - 1),$$

Set $e'(r) = e_p(r)$.

THEOREM 1. — Assume that $\mathfrak{p} \subset \mathcal{O}_{n,0}$ is a prime ideal satisfying the conditions (2.2)-(2.7).

Let f be a function of class $C^{e'(r)}$ vanishing on $V(\mathfrak{p})$ in a sufficiently small neighbourhood Ω of 0.

Then, for any $x \in \Omega$, the Taylor expansion $T_x^{e'(r)} f \in I(V(\mathfrak{p}))_x + \mathfrak{m}_{n,x}^r$

(where $T_x^{e'(r)} f = \sum_{|\beta| \leq e'(r)} (1/\beta!) \cdot (\partial^\beta f / \partial X^\beta)(x) (X-x)^\beta \in \mathcal{O}_{n,x}$).

Proof. — Let $x = (x', x_{k+1}, x'') \in V(\mathfrak{p}) \cap \Omega$. Let $Y_{k+1} = X_{k+1} - x_{k+1}$. By (2.4) $P_{k+1} = R_1 \cdot \dots \cdot R_s \cdot Q$ in $\mathcal{O}_{k,x'}[Y_{k+1}]$.

Denote by $\delta_i(\omega_i)$ the discriminant of the polynomial R_i (the initial form of δ_i at x').

From Lemma 3, there exists $g \in \mathcal{O}_{k,x'}[Y_{k+1}]$ such that

$$\delta^N \cdot (T_x^{e'(r)} f) \equiv g \pmod{I(V(\mathfrak{p}))_x}.$$

The function f vanishes on $V(\mathfrak{p})_x$, so $T_x^{e'(r)} f$ and g are $e'(r)$ -flat on $V(\mathfrak{p})_x$ (see [7]).

Every polynomial R_i has degree $\leq p$ and, by Corollary 2 and (2.5), is irreducible.

From Proposition 5.6 ([2], p. 50) there exist $a_{i1}, \dots, a_{ip} \in \mathcal{O}_{k,x'}$ such that

$$g^p + a_{i1} \cdot g^{p-1} + \dots + a_{ip} \in \mathcal{O}_{k+1,(x',x_{k+1})} \cdot R_i.$$

Then a_{ip} is $e'(r)$ -flat on $V(\mathfrak{p}) \cap (V(R_i) \times R^{n-k-1})$.

By (2.0), (2.1) and Proposition 2.2 ([2], p. 55), a_{ip} is $(e'(r)/p)$ -flat on $\pi(V(R_i))_x$.

By (2.7) there exists a connected component U of

$$\{y' \in R^k \mid \|x' - y'\| < \varepsilon, \delta(y') \neq 0\}$$

such that $x' \in \bar{U}$ and $U \subset \pi(V(R_i))_x$.

Then, from Lemma 5.11, [7] and Lemma 4

$$a_{ip} \in \mathfrak{m}_{k,x}^{2p(e_{p-1}(r)+[d/2]+1)}.$$

We have

$$g(g^{p-1} + a_{i1} \cdot g^{p-2} + \dots + a_{i,p-1}) \in \mathcal{O}_{k+1,x} \cdot R_i + \mathfrak{m}_{k,x}^{2p(e_{p-1}(r)+[d/2]+1)}.$$

If x is sufficiently close to 0 then $\text{degree } \omega_i \leq d$.

Then, from Lemma 2,

$$g \in \mathcal{O}_{k+1,x} \cdot R_i + \mathfrak{m}_{k+1,x}^{e_{p-1}(r)}$$

or

$$g^{p-1} + \dots + a_{i,p-1} \in \mathcal{O}_{k+1,x} \cdot R_i + \mathfrak{m}_{k+1,x}^{e_{p-1}(r)}.$$

In the second case, repeating this process $p-1$ times, we can prove that g is $e_0(r)$ -flat on

$$\tilde{V}(R_i) = \{ (y', y_{k+1}) \in \mathbb{C}^k \times \mathbb{C} \mid R_i(y', y_{k+1}) = 0 \} \text{ at } (x', x_{k+1}).$$

Then g is $e_0(r)$ -flat on $\tilde{V}(R_1, \dots, R_s) = \tilde{V}(R_1) \cup \dots \cup \tilde{V}(R_s)$ at (x', x_{k+1}) .

From the preparation theorem we have

$$g = S \cdot R_1 \dots R_s + \sum_{j=1}^p b_j \cdot Y_{k+1}^{-j}, \quad \text{where } b_j \in \mathcal{O}_{k,x}.$$

By Cramer's rule $b_j \in \mathfrak{m}_{k,x}^{+v}$. The arguments are the same as in the proof of Lemma 2.

By (2.4)

$$\delta^N \cdot (T_x^{e'(r)} f) \in (P_{k+1}, \delta \cdot X_{k+2} - Q_{k+2}, \dots, \delta \cdot X_n - Q_n) \mathcal{O}_{n,x} + \mathfrak{m}_{n,x}^{+v}.$$

From Lemma 5, there exists $h \in \mathfrak{m}_{n,x}^r$ such that

$$\delta^N \cdot (T_x^{e'(r)} f - h) \in I(V(\mathfrak{p}))_x.$$

By (2.3), $T_x^{e'(r)} f - h \in I(V(\mathfrak{p}))_x$.

Then $T_x^{e'(r)} f \in I(V(\mathfrak{p}))_x + \mathfrak{m}_{n,x}^r$. ■

THEOREM 2 (see [1]). — Let $g_1, \dots, g_m \in \mathcal{O}_{n,0}$.

There exist a linear function $N \ni r \mapsto e''(r) = a'' \cdot r + b'' \in N$ and an open neighbourhood Ω of 0 such that:

if $f: \Omega \rightarrow \mathbb{R}$ is a function of class $C^{e''(r)}$ and, for any $x \in \Omega$, $T_x^{e''(r)} f \in (g_1, \dots, g_m) \cdot \mathcal{O}_{n, x} + \mathfrak{m}_{n, x}^{e''(r)}$, then there exist functions $\varphi_1, \dots, \varphi_m: \Omega \rightarrow \mathbb{R}$ of class C^r such that:

$$f = \varphi_1 \cdot g_1 + \dots + \varphi_m \cdot g_m.$$

THEOREM 3. — Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a germ of an analytic coherent set. Then $I(X)_0 = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are prime ideals in $\mathcal{O}_{n, 0}$.

Suppose that every ideal \mathfrak{p}_i satisfies assumptions (2.2)-(2.7). Then there exists a linear function $N \ni r \mapsto e(r) = a \cdot r + b \in N$ such that for any function f of class $C^{e(r)}$ vanishing on X :

$$f = \varphi_1 \cdot g_1 + \dots + \varphi_m \cdot g_m,$$

where $g_1, \dots, g_m \in I(X)_0$ and $\varphi_1, \dots, \varphi_m$ are germs of function of class C^r .

This theorem is a sharpened version of the result of J.-Cl. TOUGERON (see Theorem 5.12, [7]).

Proof. — We have $I(X)_0 \subset \sqrt{I(X)_0} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$, where

$$\mathfrak{p}_1, \dots, \mathfrak{p}_k \subset \mathcal{O}_{n, 0}$$

are prime ideals. The germ of X at 0 is coherent, so the ideal $I(X)_0$ is real. Then $\sqrt{I(X)_0} \subset \sqrt[{\mathbb{R}}]{I(X)_0} = I(X)_0$. Hence $I(X)_0 = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$.

From Theorem 1 there exists a linear function $N \ni r \mapsto e'(r) = a' \cdot r + b' \in N$ such that, for any function f of class $C^{e'(r)}$ vanishing on X and any $x \in X$ we have

$$T_x^{e'(r)} f \in \bigcap_{i=1}^k (I(V(\mathfrak{p}_i))_x + \mathfrak{m}_{n, x}^r).$$

By (2.3) and ([7], Theorem 3.8), there exists a constant $v' \in N$ such that, for any $x \in X$ we have

$$\bigcap_{i=1}^k (I(V(\mathfrak{p}_i))_x + \mathfrak{m}_{n, x}^{r+v'}) \subset \bigcap_{i=1}^k I(V(\mathfrak{p}_i))_x + \mathfrak{m}_{n, x}^r \subset I(X)_x + \mathfrak{m}_{n, x}^r.$$

Let g_1, \dots, g_m be generators of $I(X)_0$. Let $e''(r)$ be a function as in Theorem 2.

Define $e(r) = e'(e''(r) + v') = a \cdot r + b$.

Let f be a germ of class $C^{e(r)}$ vanishing on X .

Then, for any $x \in X$ in some neighbourhood of 0 we have

$$T_x^{e(r)} f \in \bigcap_{i=1}^k (I(V(p_i)))_x + \mathfrak{m}_{n,x}^{e''(r)+v'} \subset I(X)_x + \mathfrak{m}_{n,x}^{e''(r)}.$$

From Theorem 2 there exist functions $\varphi_1, \dots, \varphi_m$ of class C^r such that

$$f = \varphi_1 \cdot g_1 + \dots + \varphi_m \cdot g_m.$$

This completes the proof. ■

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